Conformal Geometry of the Visual Cortex

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Geometry of Neuroscience

Functional Architecture of the V1 visual cortex

Filtering of optical signals by visual neurons and local differential data; integration of local differential data and global geometry, through global coherence of functional architecture of visual areas

This lecture is based on:

References:

Pe Jeat Petitot, *Neurogéométrie de la vision*, Les Éditions de l'École Polytechnique, 2008

TSBBLW Duyan Taa, Jie Shia, Brian Bartonb, Alyssa Brewerb, Zhong-Lin Luc, Yalin Wang, *Characterizing human retinotopic mapping with conformal geometry: A preliminary study*, 2014

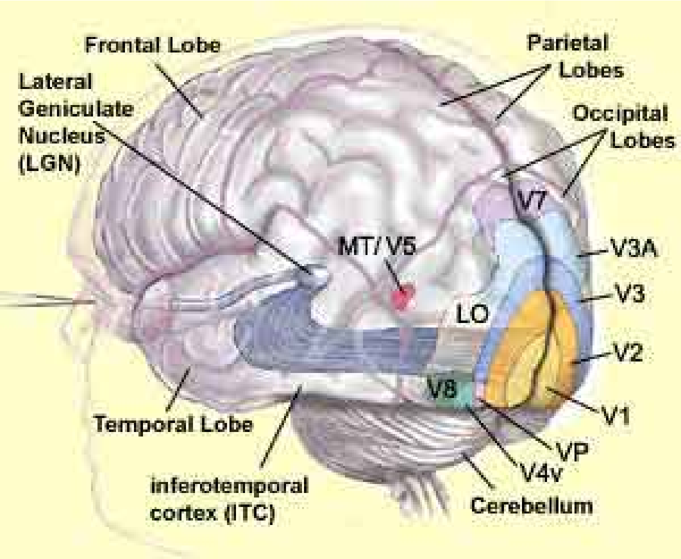
WGCTY Yalin Wang, Xianfeng Gu, Tony F. Chan, Paul M. Thompson, Shing-Tung Yau, *Intrinsic Brain Surface Conformal Mapping using a Variational Method*, Proceedings of SPIE Vol. 5370, 2004

Areas of the visual cortex

• V1: the first of the primary visual areas (numerous feedbacks of successive areas like V2 and V4: here focus only on the geometry of V1)

• high-resolution buffer hypothesis of Lee–Mumford: V1 not just a bottom-up early-module but participating in all visual processes that require fine resolution

- Lee, T.S., Mumford, D., Romero, R., Lamme, V.A.F., *The role of primary visual cortex in higher level vision*, Vision Research, 38 (1998) 2429–2454.



Location of the Visual Areas

Structures in V1

- neurophysiology identifies three types of structures in primate V1

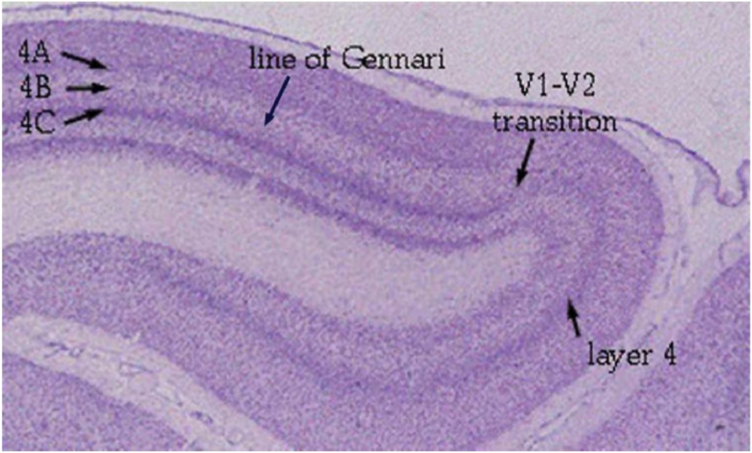
1. laminar
2. retinotopic (retinal mapping)
3. (hyper)columnar

Laminar Structure

• organized in 6 distinct horizontal layers (parallel to the surface of the cortex)

• look in particular at layer 4 (and sublayer $4C$): main target of thalamocortical afferents and intra-hemispheric corticocortical afferents

• contains different types of stellate and pyramidal neurons



laminar structure and the 4th layer

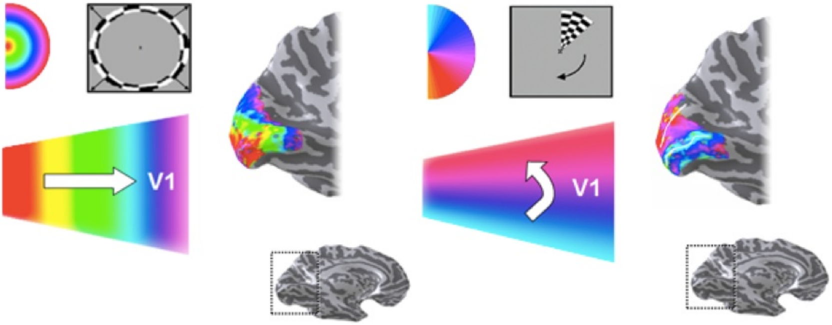
Retinotopy

• adjacent neurons with receptive fields covering overlapping portions of the visual field

• mapping of the visual input from the retina of the visual cortex are conformal maps (preserving local shape and local angles, but not distances and sizes)

• logarithmic conformal mapping from the retina to the sublayer $4C$ of layer 4 of the laminar structure

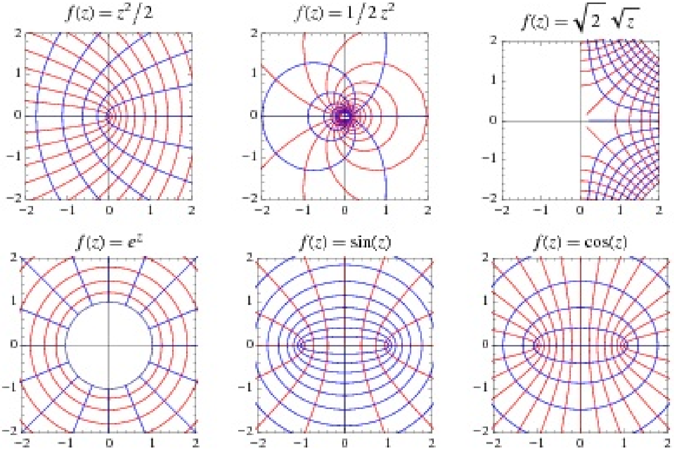
• Note: in cortical areas other than V1 adjacent points of the visual field may be mapped to non-adjacent regions

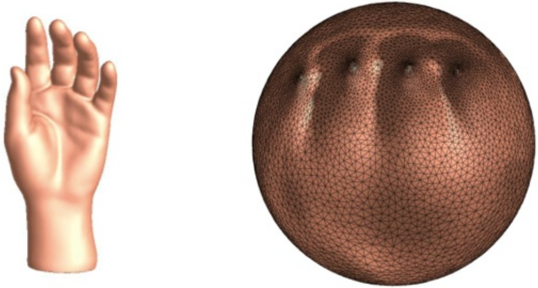


eccentricity and polar angle data (from [TSBBLW])

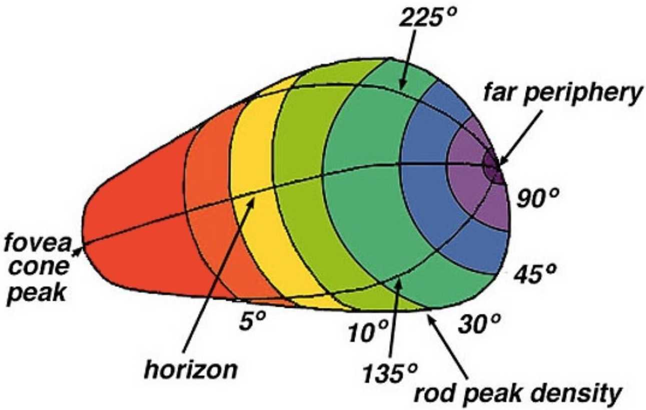
Conformal maps

biholomorphic maps $w = f(z)$ where $f'(z) \neq 0$





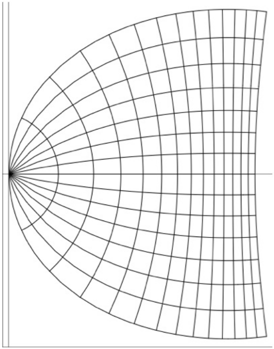
genus zero surface conformally mapped to S^2 (from [WGCTY])



the unfolded striate cortex with the mapping of the visual field

Models of retinotopy conformal mapping

- the $\log(z + a)$ model (also referred to as "monopole model")



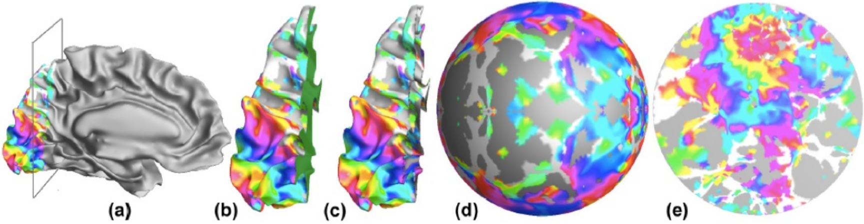
- more general $\log(\frac{w(z)+a}{w(z)+b})$ model (also known as "wedge-dipole model")

TSBBLW Duyan Taa, Jie Shia, Brian Bartonb, Alyssa Brewerb, Zhong-Lin Luc, Yalin Wang, *Characterizing human retinotopic mapping with conformal geometry: A preliminary study*, 2014

- two step procedure to modeling retinotopy by conformal mapping

1. conformal map from brain visual cortex to the unit disk
2. conformal map from visual field to the unit disk

From the visual cortex to the unit disk: conformal flattening



(1) slice along plane to isolate visual cortex regions; (b) visual regions after slicing; (c) double covering; (d) projection of double covering to a sphere; (e) stereographic projection to the unit disk (from [TSBBLW])

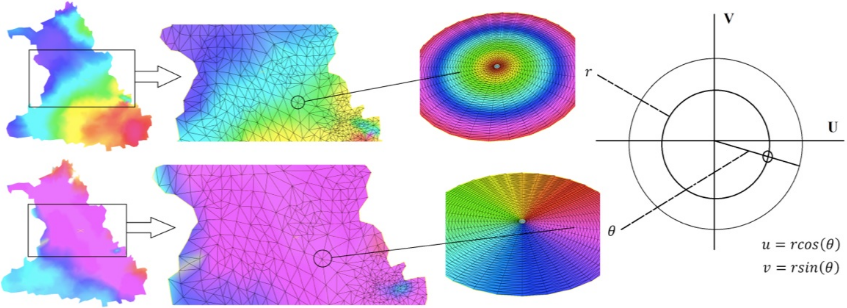
Mesh and u, v-coordinates

Data collection provides:

- simplicial complex (mesh triangulation) K of cortical area
- color gradient data for eccentricity and polar angle: parameterization of visual stimulus in the visual field as $u = r\cos(\theta)$ and $v = r\sin(\theta)$

general technique for constructing conformal mapping from

WGCTY Yalin Wang, Xianfeng Gu, Tony F. Chan, Paul M. Thompson, Shing-Tung Yau, *Intrinsic Brain Surface Conformal Mapping using a Variational Method*, Proceedings of SPIE Vol. 5370, 2004



mesh K with color gradient data for eccentricity and polar angle determining u, v-coordinates at each vertex of the mesh (from [TSBBLW])

Constructing the conformal maps: energy minimizing [WGCTY]

- piecewise linear functions $\mathcal{C}^{PL}(K)$, quadratic form

$$\langle f_1, f_2 \rangle = \frac{1}{2} \sum_{e \in E(K)} k_e \left(f_1(s(e)) - f_1(t(e)) \right) \left(f_2(s(e)) - f_2(t(e)) \right)$$

$e \in E(K)$ edges, $s(e), t(e) \in V(K)$ source and target vertices; $k_e > 0$ parameters

- Energy functional

$$E(f) = \langle f, f \rangle = \sum_e k_e \left\| f(s(e)) - f(t(e)) \right\|^2$$

when all $k_e = 1$: Tutte energy

• discrete Laplacian

$$\Delta(f) = \sum_e k_e(f(t(e)) - f(s(e))$$

energy minimizing f satisfies $\Delta(f) = 0$

• for vector valued functions: apply Δ componentwise

• $f : K_1 \to K_2$ map between two meshes (embedded in Euclidean spaces $\mathbb{E}^3$)

$$(\Delta f(v))^\perp = \langle \Delta f(v), \vec{n}(f(v)) \rangle \ \ \vec{n}(f(v))$$

normal component, with $\vec{n}(f(v))$ normal vector to K_2 at $f(v)$

• harmonic map $f : K_1 \to K_2$ iff $\Delta f(v) = (\Delta f(v))^\perp$ (only normal no tangential component)

• vanishing of absolute derivative

$$Df(v) = \Delta f(v) - (\Delta f(v))^\perp$$

conformal maps to S^2 by steepest descent [WGCTY]

- non-uniqueness of solutions: action of Möbius transformations on $S^2 = \mathbb{P}^1(\mathbb{C})$

$$\mathrm{GL}_2(\mathbb{C}) \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

- constraints to obtain a unique solutions:
  - zero-mass constraint: $f : K_1 \to K_2$

$$\int f \, d\sigma_{K_1} = 0$$

  - landmark constraints: manually labelled set of curves or point sets, optimal Möbius transformation that reduces distance between images of landmarks in the sphere S^2

Algorithm 1 [WGCTY] (steepest descent with Tutte energy)

1. start with mesh K and Gauss map $\tau : K \to S^2$ with $N(v) = n(v)$ normal to $K \subset \mathbb{E}^3$
2. compute Tutte energy $E_0 = E(\tau)$
3. compute absolute derivative $D\tau(v)$
4. update τ by $\delta\tau = -D\tau(v) \cdot \delta t$ (fixed increment length δt)
5. compute Tutte energy: if $E_{new} < E_0 + \delta E$ (fixed threshold δE) output, else update E_0 to E and repeat

Unique minimum, convergence to Tutte embedding of graph (1-skeleton of K) in the sphere S^2

Algorithm 2 [WGCTY] (from Tutte embedding to conformal map)

1. compute Tutte embedding τ as before and its Tutte energy E_0
2. compute absolute derivative $D\tau(v)$ and update $\delta\tau(v) = -D\tau(v)\delta t$
3. compute Möbius transformation $\gamma_0 : S^2 \to S^2$ that minimizes norm of the mass center

$$\gamma_0 = \mathrm{argmin}_\gamma \left\| \int \gamma \circ \tau \, d\sigma_K \right\|^2$$

4. compute harmonic energy: where coefficients $k_e = a_e^\alpha + a_e^\beta$ (for edge e in boundary of faces F_α and F_β)

$$a_e^\alpha = \frac{1}{2} \frac{(s(e) - v) \cdot (t(e) - v)}{|(s(e) - v) \times (t(e) - v)|}$$

where v third vertex in triangle face F_α

5. if $E < E_0 + \delta E$ output current function; otherwise update E_0 to E and repeat

• used minimization of mass center norm by Möbius transformations, but also want to evaluate how good conformal parameterization is, with respect to some given landmarks

• suppose obtained two parameterizations $f_i : S^2 \to S$, compare them in terms of given landmarks

• formulate again in terms of an energy functional

$$E(f_1, f_2) = \int_{S^2} \|f_1(u,v) - f_2(u,v)\|^2 \, du \, dv$$

look for Möbius transformation $\gamma_\star$ that minimizes this energy

$$\gamma_\star = \mathrm{argmin}_\gamma \, E(f_1, f_2 \circ \gamma)$$

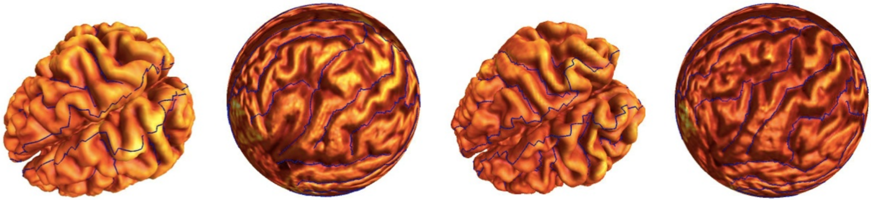
• using landmarks to only compare over a finite set of points (or over some assigned curves)

• say landmarks are finite sets of points $\mathcal{P} \subset S_1$ and $\mathcal{Q} \subset S_2$ with bijection $p_i \leftrightarrow q_i$, $i = 1, \dots, n$ between their preimages on S^2

• look for Möbius transformation γ that minimizes

$$E(\gamma) = \sum_{i=1}^{n} \|p_i - \gamma(q_i)\|^2$$

non-linear problem, but assuming $\gamma(\infty) = \infty$ by stereographic projection transform into a least square problem



landmark constraints: matching along preassigned curves, minimize landmark mismatch for representations from different subjects (from [WGCTY])

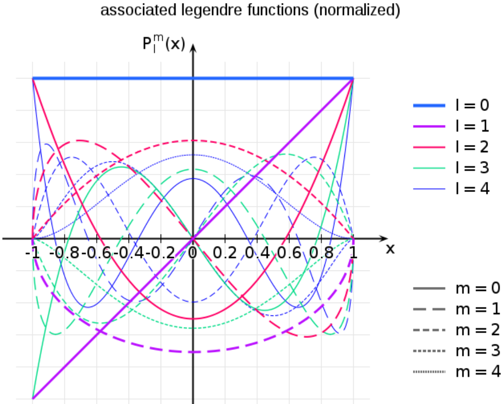
Spherical harmonics orthonormal basis for $L^2(S^2)$

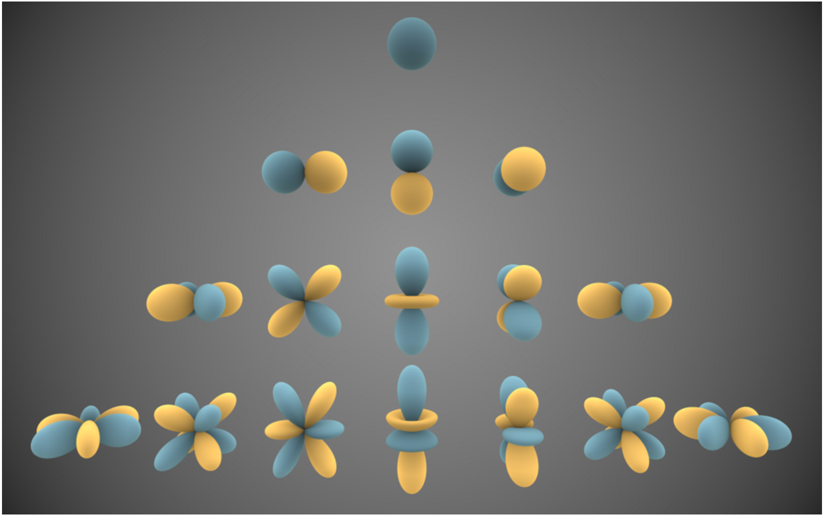
- $\ell \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $|m| \leq \ell$ (degree and order)

$$Y_\ell^m(\theta, \phi) = k_{\ell,m} \, P_\ell^m(\cos(\theta)) \, e^{im\phi}$$

P_ℓ^m associated Legendre polynomials

$$\frac{d}{dx}((1-x^2)\frac{d}{dx}P_\ell^m(x)) + (\ell(\ell+1) - \frac{m^2}{1-x^2})P_\ell^m = 0$$





Real spherical harmonics $\ell = 0, \ldots, 3$, yellow=negative, blue=positive, distance from origin=value in angular direction

• Expansion in spherical harmonics $f \in L^2(S^2)$

$$f = \sum_{\ell \geq 0} \sum_{m:\, |m| \leq \ell} \langle f, Y_\ell^m \rangle \, Y_\ell^m$$

• suppose constructed conformal mapping of visual cortex to S^2, have coordinates on the cortex surface (embedded in $\mathbb{E}^3$)

$$x^0(\theta, \phi), \quad x^1(\theta, \phi), \quad x^2(\theta, \phi)$$

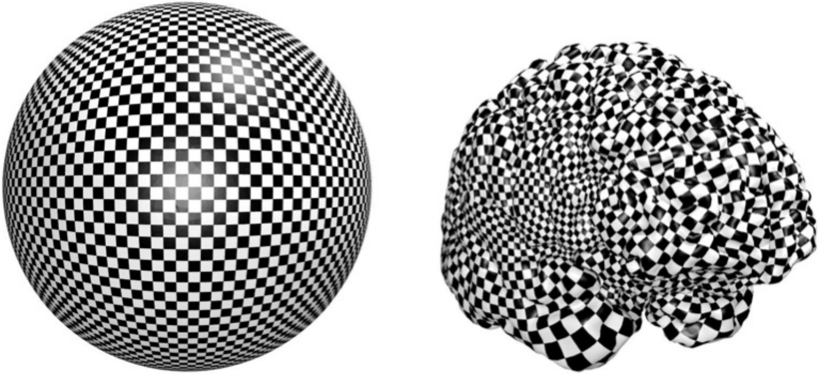
with (θ, ϕ) angle coordinates on S^2

$$x^i(\theta, \phi) \in L^2(S^2), \qquad \text{with} \quad \hat{x}^i(\ell, m) = \langle x^i, Y_\ell^m \rangle$$

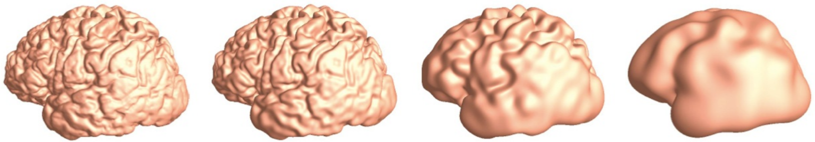
coefficients of expansion in harmonic forms

• *Fast Spherical Harmonic Transform* to compute $\hat{x}^i(\ell, m)$

• compression, denoising, feature detection, shape analysis: more efficiently performed on the Fourier modes $\hat{x}^i(\ell, m)$

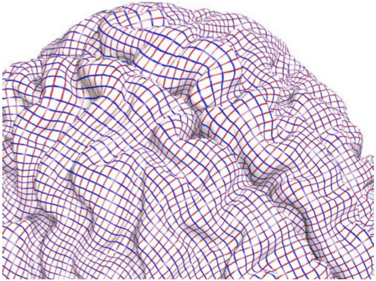


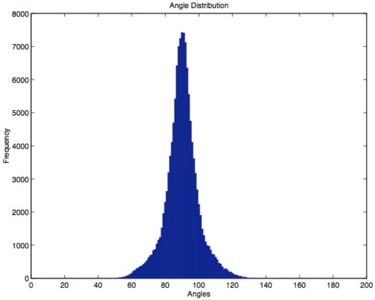
a conformal map from S^2 to the brain surface (from [WGCTY])



geometric compression using low spherical harmonics and rescaling to smaller low frequencies coefficients (from [WGCTY])

How good is modeling by conformal maps?





measuring deviation from conformality by deviation from right angle through inverse mapping from S^2 to cortex surface (from [WGCTY])

Beltrami equation and Beltrami coefficient

- a conformal structure at a point $z \in \mathbb{C}$ is determined by a complex dilatation $\mu(z)$ with $|\mu(z)| < 1$

- intuitively, a conformal structure picks an ellipse centered at the origin as the new circle

- notation: for $z = x + iy$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right)$$

- if $\mu(z) = \mu$ constant, the function $h(z) = z + \mu\bar{z}$ satisfies Beltrami equation

$$\frac{\partial h}{\partial \bar{z}} = \mu(z)\,\frac{\partial h}{\partial z}$$

- for constant $\mu(z) = \mu$ round circle in h-plane corresponds to ellipse with constant $|z + \mu\bar{z}|$ in z-plane: direction of axes from argument of μ eccentricity from $|\mu|$

• for $\mu(z)$ real analytic: Gauss isothermal coordinates $\exists$ local solution $h(z)$ to Beltrami equation; Morrey for measurable $\mu(z)$

• a solution $h(z)$ on a local open set U is a quasi-conformal mapping with complex dilatation $\mu(z)$

• conformal structure on a Riemann surface S: section of a disk D bundle over S

$$\mu_\beta(z_\beta) = \mu_\alpha(z_\alpha) \frac{\partial z_\beta / \partial z_\alpha}{\partial \bar{z}_\beta / \partial \bar{z}_\alpha}$$

gluing of local $\mu_\alpha : U_\alpha \to D$ on overlaps

• Beltrami differential on a Riemann surface S is antilinear homomorphism of tangent spaces $T_z S$

• local solutions h_α of Beltrami equation determine conformal coordinates for a Riemann surface S_μ topologically equivalent to S but with a new conformal structure.

• in genus zero case: by Uniformization Theorem S_μ is conformally equivalent to $S = \mathbb{P}^1(\mathbb{C})$ with unique conformal isomorphism h that fixes $\{0, 1, \infty\}$

• view h as quasiconformal isomorphism with dilatation $\mu(z)$

$$h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$

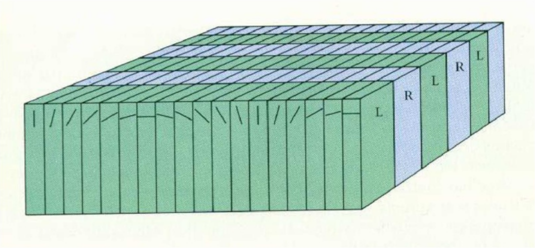
• conclusion from [TSBBLW]: compute Beltrami coefficient μ for regions of V1 and V2 where reasonably smooth eccentricity and polar angle data: conformal map is very good approximation

• from the neuroscience point of view: why conformal maps?

Columnar Structure

- another type of geometric structure present in visual cortex V1
- Hubel–Wiesel: columnar structures in V1: neurons sensitive to orientation record data (z, ℓ)
  - z = a position on the retina
  - ℓ = a line in the plane
- local product structure

$$\pi : \mathcal{R} \times \mathbb{P}^1 \twoheadrightarrow \mathcal{R}$$



Fiber bundles

• topological space (or smooth differentiable manifold) E with base B and fiber F with

- surjection $\pi : E \twoheadrightarrow B$
- fibers $E_x = \pi^{-1}(x) \simeq F$ for all $x \in B$
- open covering $\mathcal{U} = \{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha) \simeq U_\alpha \times F$ with π restricted to $\pi^{-1}(U_\alpha)$ projection $(x, s) \mapsto x$ on $U_\alpha \times F$

• sections $s : B \to E$ with $\pi \circ s = id$; locally on U_α

$$s|_{U_\alpha}(x) = (x, s_\alpha(x)), \quad \text{with } s_\alpha : U_\alpha \to F$$

• model of V1: bundle $\mathcal{E}$ with base $\mathcal{R}$ the retinal surface, fiber $\mathbb{P}^1$ the set of lines in the plane

• topologically $\mathbb{P}^1(\mathbb{R}) = S^1$ (circle) so locally V1 product $\mathbb{R}^2 \times S^1$

• We will see this leads to a geometric models of V1 based on Contact Geometry