Classical and Quantum Information

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Some references

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- M. Marcolli, *Gamma spaces and information*. J. Geom. Phys. 140 (2019), 26–55

Space

- Classical: sample space $\Omega = \{1, \ldots, N\}$ category of (finite) sets
- Quantum: complex Hilbert space $\mathcal{H}$ of dimension $\dim \mathcal{H} = n$: (finite dimensional) Hilbert spaces

Events

- Classical: set $\mathcal{P}(\Omega)$ of subsets of Ω, Boolean algebra with $\cup$, $\cap$ and complement (OR, AND, NOT)
- Quantum: set $\mathcal{P}(\mathcal{H})$ of orthogonal projections in $\mathcal{H}$ with operations $\bigvee$ (max) and $\bigwedge$ (min), $\perp$ complement, but

$$E \wedge (F_1 \vee F_2) \neq (E \wedge F_1) \vee (E \wedge F_2)$$

unless E, F_1, F_2 mutually commute

Observables (random variables)

- Classical: $C(\Omega) = \{f : \Omega \to \mathbb{C}\} = \mathbb{C}^{\#\Omega}$ commutative C^*-algebra; real valued random variables $f : \Omega \to \mathbb{R}$
- Quantum: $\mathcal{B}(\mathcal{H})$ the noncommutative C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ (all linear operators since $\mathcal{H}$ finite dim: sum of matrix algebras); real valued random variables are *hermitian* operators $A = A^*$, which have $\mathrm{Spec}(A) \subset \mathbb{R}$ and

$$A = \sum_{\lambda \in \mathrm{Spec}(A)} \lambda \, E_\lambda$$

Characteristic functions

- Classical: set $E \in \mathcal{P}(\Omega)$ and $\chi_E(x) = 1$ if $x \in E$ and zero otherwise; for $f : \Omega \to \mathbb{C}$

$$f(x) = \sum_{y \in f(\Omega)} y\, \chi_{f^{-1}(y)}(x)$$

$$\chi_{f^{-1}(y)} \cdot \chi_{f^{-1}(y')} = 0, \quad \text{for } y \neq y' \quad \text{and} \sum_{y \in f(\Omega)} \chi_{f^{-1}(y)}(x) = 1, \quad \forall x \in \Omega$$

$$f(x)^r = \sum_{y \in f(\Omega)} y^r\, \chi_{f^{-1}(y)}(x), \qquad \text{and } \varphi(f) = \sum_{y \in f(\Omega)} \varphi(y)\, \chi_{f^{-1}(y)}(x)$$

for $r \in \mathbb{N}$ and $\varphi : \mathbb{C} \to \mathbb{C}$

- Quantum: $\{E_\lambda\}$ spectral projections of A

$$E_\lambda E_{\lambda'} = 0, \text{ for } \lambda \neq \lambda', \qquad \text{and} \sum_{\lambda \in \mathrm{Spec}(A)} E_\lambda = 1$$

$$A^r = \sum_\lambda \lambda^r E_\lambda, \quad \varphi(A) = \sum_\lambda \varphi(\lambda) E_\lambda$$

spectral theorem for $A = A^*$ and $\varphi : \mathbb{R} \to \mathbb{R}$

Probability distributions and states

- Classical: $P:\Omega \to \mathbb{R}_+$ with $P=(p_x)_{x\in\Omega}$ with $p_x \geq 0$ and $\sum_{x\in\Omega} p_x = 1$, simplex $\Delta_\Omega \ni P$

$$\mathbb{P}(E,P) = \sum_{x\in E} p_x, \quad E \in \mathcal{P}(\Omega)$$

$$\mathbb{P}(f=\lambda) = \mathbb{P}(f^{-1}(\lambda), P)$$

- Quantum: instead of $P=(p_x)$ have a density matrix ρ non-negative and self-adjoint with $\mathrm{Tr}(\rho)=1$

$$\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}, \quad \varphi(A) = \mathrm{Tr}(\rho A)$$

$$\varphi(A^*A) \geq 0 \quad (\text{as } \rho = \eta^*\eta \geq 0) \quad \varphi(1)=1 \quad (\text{as } \mathrm{Tr}(\rho)=1)$$

$$\mathbb{P}(A=\lambda) := \mathrm{Tr}(\rho E_\lambda), \text{ for } \lambda \in \mathrm{Spec}(A), \text{ zero otherwise}$$

Expectation values

- Classical: random variable $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$)

$$\mathbb{E}_P(f) = \sum_{x\in\Omega} f(x)\, p_x$$

k-th moment of f = expectation of f^k

$$M_k(f,P) = \mathbb{E}_P(f^k) = \sum_{x\in\Omega} f(x)^k\, p_x = \sum_{\lambda\in f(\Omega)} \lambda^k\, P(f^{-1}(k))$$

$$\mathbb{E}_P(e^{itf}) = \sum_{\lambda\in f(\Omega)} e^{it\lambda}\, P(f^{-1}(k))$$

- Quantum: expectation of an observable $A = A^*$, state evaluation

$$\mathbb{E}_\rho(A) = \mathrm{Tr}(\rho A) = \sum_{\lambda\in\mathrm{Spec}(A)} \lambda \mathrm{Tr}(\rho E_\lambda)$$

$$\mathbb{E}_\rho(e^{itA}) = \mathrm{Tr}(\rho e^{itA}) = \sum_{\lambda\in\mathrm{Spec}(A)} e^{it\lambda}\mathrm{Tr}(\rho E_\lambda)$$

Variance

- Classical: random variable $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$)

$$\mathrm{Var}_P(f) = \mathbb{E}_P(f - \mathbb{E}_P(f))^2 \geq 0$$

zero if all mass distribution of f concentrated at $\mathbb{E}_P(f)$

- Quantum: observable $A = A^*$

$$\mathrm{Var}_\rho(A) = \mathrm{Tr}(\rho(A - \mathrm{Tr}(\rho A))^2) \geq 0$$

zero if operator range of ρ contains in eigenspace of A with eigenvalue $\mathrm{Tr}(\rho A)$

Extreme points

- Classical: simplex Δ_Ω has $N = \#\Omega$ extremal points given by probabilities delta functions

$$\delta_\omega(x) = \begin{cases} 1 & x = \omega \\ 0 & x \neq \omega \end{cases}$$

- Quantum: set of all density matrices ρ is a convex set (and eigenvalues of ρ are $\lambda \geq 0$)

$$\rho = \sum_{\lambda \in \mathrm{Spec}(\rho)} \lambda E_\lambda, \quad \text{with } \sum_\lambda \lambda \dim E_\lambda = 1$$

one-dimensional projections:

$$E_\lambda = \sum_i E_{\lambda,i}$$

one-dimensional projections cannot be further decomposed (not convex combinations of other states): extreme points

$$\rho = u\, u^* = |u\rangle\langle u|, \quad u \in \mathcal{H}, \quad \|u\| = 1$$

$$\mathrm{Tr}(u\, u^* A) = \mathrm{Tr}(u^* A u) = \langle u, Au\rangle$$

Variance

- with respect to pure state $\rho = u\, u^*$

$$\mathrm{Var}_\rho(A) = \mathrm{Tr}(uu^*(A - \langle u, Au\rangle)^2) = \|(A - \langle u, Au\rangle)u\|^2$$

zero when u eigenvector of A

Product spaces

- Classical: (Ω_1, P_1) and (Ω_2, P_2)

$$(\Omega_1 \times \Omega_2, P_1 P_2), \quad P_1\, P_2(x,y) = P_1(x) P_2(y)$$

  independent systems (Note: not a categorical product)

- Quantum: $(\mathcal{H}_1, \rho_1)$ and $(\mathcal{H}_2, \rho_2)$

$$(\mathcal{H}_1 \otimes \mathcal{H}_2, \rho_1 \otimes \rho_2)$$

Dynamics

- Classical: $T : \Omega \to \Omega$ invertible transformation, evolve functions $f : \Omega \to \mathbb{C}$ or equivalently evolve states $P \in \Delta_\Omega$

$$f \mapsto f \circ T, \qquad P \mapsto P \circ T^{-1}$$

(opposite transformations: change of variable in integration)

- Quantum: unitary linear operator $U : \mathcal{H} \to \mathcal{H}$
    1. *Heisenberg picture*: evolve observables/operators

$$A \mapsto U^* A U$$

    2. *Schrödinger picture*: evolve states

$$\rho \mapsto U \rho U^*$$

compatible via trace $\mathrm{Tr}(\rho U^* A U) = \mathrm{Tr}(U \rho U^* A)$

Pure and mixed states

- pure states: nonzero vectors ψ in $\mathcal{H} = \mathbb{C}^{n+1}$, only up to scale $\lambda \in \mathbb{C}^*$

$$\rho = \frac{1}{\langle\psi|\psi\rangle} |\psi\rangle\langle\psi| = \frac{1}{\langle\lambda\psi|\lambda\psi\rangle} |\lambda\psi\rangle\langle\lambda\psi|$$

- so pure states = points in $\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^*$
- mixed states: convex combinations $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ density matrices
- Schrödinger equation:

$$i\hbar\partial_t|\psi\rangle = H\,|\psi\rangle$$

$$i\hbar\,\dot{\rho} = [H,\rho]$$

- in projective coordinates ψ is $(z_0 : \ldots : z_n)$ with

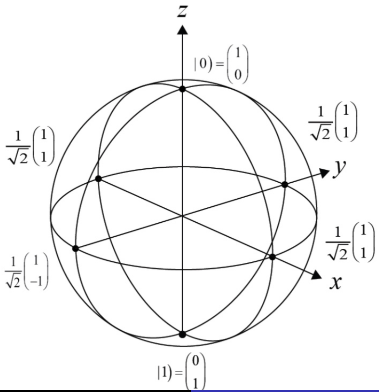
$$i\hbar\,\dot{z}_\alpha = \sum_\beta H_{\alpha\beta}\,z_\beta$$

Projections and probabilities: (quantum logic)

- (closed) subspaces of Hilbert space $\mathcal{H}$ and their projections P
- partially ordered by inclusions
- $\wedge$ intersection of subspaces, $\vee$ join (span of union)
- not distributive
- $\mathcal{H}_1 \subset \mathcal{H}$ has ∞-many complementary $\mathcal{H}_2 \cap \mathcal{H}_1 = \{0\}$ but only one orthogonal $\mathcal{H}_1^\perp$ with $P_1 P_1^\perp = P_1^\perp P_1 = 0$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$
- only commuting observable are simultaneously measurable in quantum mechanics, but pairs of projections typically do not commute $P_1 P_2 \neq P_2 P_1$
- *Gleason theorem*: any probability measure $\mu : \mathcal{B}(\mathcal{H}) \to [0,1]$ that satisfies $\mu(\oplus_i P_i) = \sum_i \mu(P_i)$ on mutually orthogonal projections is of the form $\mu(P) = \mathrm{Tr}(\rho P)$ for some density matrix
- states on a finite dimensional C^*-algebra are of the form $\varphi(A) = \mathrm{Tr}(\rho A)$ for some density matrix

Qbit: Bloch sphere

- single particle of spin 1/2: spin up or spin down
- state space $\mathcal{H} = \mathbb{C}^2$ spanned by $|\uparrow\rangle$, $|\downarrow\rangle$
- single qbit space
- pure states $\mathbb{P}^1(\mathbb{C}) \simeq S^2$, Bloch sphere
- mixed states: 3-dim ball B with $\partial B = S^2$ (convex combinations of points of S^2)



- Pauli matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- 2×2 hermitian density matrix can always be written as

$$\rho = \begin{pmatrix} \frac{1}{2} + z & x - iy \\ z + iy & \frac{1}{2} - z \end{pmatrix} = \frac{1}{2}\mathrm{id} + \tau \cdot \underline{\sigma},$$

$\tau = (x, y, z)$ (Bloch vector) and $\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

- positivity $\rho \geq 0$ (iff nonnegative eigenvalues) iff

$$x^2 + y^2 + z^2 \leq \frac{1}{4}$$

Bloch ball coordinates

Fubini–Study metric

- $\mathbb{C}^{n+1}$ with standard hermitian metric (flat Euclidean metric on $\mathbb{R}^{2n+2}$)

$$ds^2 = \sum_{i=0}^{n} dz_i \otimes d\bar{z}_i$$

- not $\mathbb{C}^*$-invariant but $U(1)$-invariant
- restriction of ds^2 to the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ induced the *round metric* $ds^2_{S^{2n+1}}$
- realize $\mathbb{P}^n(\mathbb{C})$ as quotient $\mathbb{P}^n(\mathbb{C}) = S^{2n+1}/S^1$: Hopf fibration

$$S^1 \to S^{2n+1} \to \mathbb{P}^n(\mathbb{C})$$

- by $U(1)$-invariance $ds^2_{S^{2n+1}}$ descends to a metric on $\mathbb{P}^n(\mathbb{C})$ (Fubini–Study metric)

- projective coordinates $(Z_0 : \cdots : Z_n)$ in $\mathbb{P}^n(\mathbb{C})$, affine chart $\mathbb{C}^n$ with affine coordinates $(1, z_1, \ldots, z_n)$

$$ds^2_{FS} = \frac{(1 + z_i \bar{z}^i) dz_j d\bar{z}^j - \bar{z}^j z_i dz_j d\bar{z}^i}{(1 + z_i \bar{z}^i)^2}$$

(sum over repeated indices)

- Kähler potential $K = \log(1 + z_i \bar{z}^i)$

$$ds^2_{FS} = g_{i\bar{j}} dz^i d\bar{z}^j, \qquad g_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} K$$

- projective coordinates $(Z_0 : \cdots : Z_n)$

$$ds^2_{FS} = \frac{2 Z_{[\alpha} dZ_{\beta]} \bar{Z}^{[\alpha} d\bar{Z}^{\beta]}}{(Z_\alpha \bar{Z}^\alpha)^2}$$

with $Z_{[\alpha} W_{\beta]} := \frac{1}{2}(Z_\alpha W_\beta - Z_\beta W_\alpha)$ skew part of tensor

- for $Z_\alpha = (Z_0 : \cdots : Z_n)$ and $W_\alpha = (W_0 : \ldots : W_n)$ points in $\mathbb{P}^n(\mathbb{C})$ representing pure states $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ geodesic distance in FS metric

$$\mathrm{dist}_{FS}(\psi,\phi) = \arccos \sqrt{\frac{\langle\psi|\phi\rangle\,\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle\,\langle\phi|\phi\rangle}} = \arccos \sqrt{\frac{Z_\alpha \bar{W}^\alpha \;\; W_\beta \bar{Z}^\beta}{Z_\alpha \bar{Z}^\alpha \;\; W_\beta \bar{W}^\beta}}$$

- on $\mathbb{P}^1(\mathbb{C}) = S^3/S^1 = S^2$ Fubini–Study metric is round metric of radius $1/2$ (Bloch sphere)

$$ds^2_{FS} = \frac{dx^2 + dy^2}{(1+r^2)^2} = \frac{1}{4}(\sin^2\theta d\phi^2 + d\theta^2)$$

affine chart coordinates $z = x + iy \in \mathbb{C}$ and $x = r\cos\theta$, $y = r\sin\theta$ with (ϕ,θ) coordinates on S^2 related via stereographic projection

- cell decomposition

  $$\mathbb{P}^n(\mathbb{C}) = \mathbb{A}^n(\mathbb{C}) \cup \mathbb{A}^{n-1}(\mathbb{C}) \cup \cdots \cup \mathbb{A}^1(\mathbb{C}) \cup \mathbb{A}^0(\mathbb{C}) = \mathbb{A}^n(\mathbb{C}) \cup \mathbb{P}^{n-1}(\mathbb{C})$$

  case of $\mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C}) \cup \mathbb{A}^0(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \simeq S^2$ one point compactification
- linear subspaces $\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ systems of linear equations $\sum_\alpha P_\alpha Z_\alpha = 0$ in the projective coordinates $Z_\alpha = (Z_0 : \cdots : Z_n)$
- general subvarieties (or schemes): systems of homogeneous polynomial equations in the Z_α
- space parameterizing linear subspaces of dimension k in $\mathbb{P}^n(\mathbb{C})$: Grassmannian $Gr(k, n)$
- lilnes and hyperplanes duality $Gr(n-1, n) \simeq Gr(1, n)$ and more generally projective duality $Gr(k, V) \simeq Gr(n-k, V^*)$

- projective group: $\mathrm{GL}_n(\mathbb{C})$ acts as linear transformations on $\mathbb{C}^n$ so

$$\mathrm{SL}_n(\mathbb{C})/(\mathbb{Z}/n\mathbb{Z})$$

acts on $\mathbb{P}^{n-1}$

- case of $\mathbb{P}^1(\mathbb{C})$ projective group

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/(\mathbb{Z}/2\mathbb{Z})$$

- action by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} = \begin{pmatrix} aZ_0 + bZ_1 \\ cZ_0 + dZ_1 \end{pmatrix}$$

in an affine chart $(z, 1)$ with $z = Z_0/Z_1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

Segre embeddings

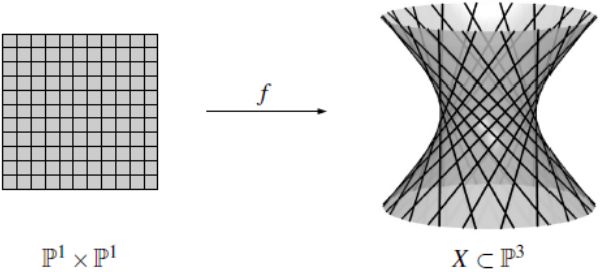
- tensor product $V \otimes W = \mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1} \simeq \mathbb{C}^{(n+1)(m+1)}$ with $(v \otimes w)_{ij} = v_i w_j$
- product of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$ is not a projective space but it embeds via Segre embedding

$$\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

  induced by the tensor product of vector spaces $Z_\alpha = Z_{\mu\mu'} = X_\mu Y_{\mu'}$
- image is the subvariety of $\mathbb{P}^{(n+1)(m+1)-1}$ defined by the equations

$$Z_{\mu\mu'} Z_{\nu\nu'} = Z_{\mu\nu'} Z_{\nu\mu'}$$

- Example: Segre quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ equation $Z_0 Z_3 - Z_1 Z_2 = 0$ (where $Z_0 = z_0 w_0$, $Z_1 = z_0 w_1$, $Z_2 = z_1 w_0$, $Z_3 = z_1 w_1$)



- ruled surface: two rulings

$$\begin{cases} Z_0 = \lambda Z_1 \\ Z_2 = \lambda Z_3 \end{cases} \quad \text{and} \quad \begin{cases} Z_2 = \lambda Z_0 \\ Z_3 = \lambda Z_1 \end{cases}$$

each pair of linear equations a line $\mathbb{P}^1$ in $\mathbb{P}^3$, each $\mathbb{P}^1$ factor in $\mathbb{P}^1 \times \mathbb{P}^1$ goes to a family of lines

pure states and projective spaces

- pure states in $\mathbb{P}^1(\mathbb{C})$: single qbit
- pure states in $\mathbb{P}^n(\mathbb{C})$ in terms of qbits?
- vector $\psi = (Z_0, \ldots, Z_n) \in \mathbb{C}^{n+1} \Rightarrow$ polynomial

$$P_\psi(t) = Z_0 t^n + Z_1 t^{n-1} \cdots + Z_{n-1} t + Z_n$$

- vector up to scaling by $\mathbb{C}^*$ (affine chart where $Z_0 \neq 0$) $\Rightarrow$ monic polynomial

$$P_\psi(t) = t^n + z_1 t^{n-1} \cdots + z_{n-1} t + z_n$$

  can be identified uniquely with (unordered) set of roots
- points in $\mathbb{P}^n(\mathbb{C}) \Leftrightarrow$ unordered sets of n points in $\mathbb{P}^1(\mathbb{C})$
- identification as symmetric product

$$S^n(X) = \underbrace{X \times \cdots \times X}_{n-\text{times}} / S_n \qquad \mathbb{P}^n(\mathbb{C}) = S^n(\mathbb{P}^1(\mathbb{C}))$$

- in general symmetric products of a smooth variety are singular, but not for complex curves (Riemann surfaces): $S^n(\Sigma_g)$ are smooth

- Wigner's theorem: all isometries of $\mathbb{P}^n(\mathbb{C})$ arise from unitary or anti-unitary transformations of $\mathbb{C}^{n+1}$

$$SU(n{+}1)/(\mathbb{Z}/(n{+}1)\mathbb{Z}), \quad \text{for } n=1\text{:} \;\; SU(2)/(\mathbb{Z}/2\mathbb{Z}) = SO(3)$$

- infinitesimal isometries generators of $\mathrm{Lie}(SU(n+1))$ hermitian matrices H
- corresponding flow $i\dot{Z}^\alpha = H^\alpha_\beta Z^\beta$ (Schrödinger)
- pure and mixed states: $\mathbb{P}^n(\mathbb{C}) \hookrightarrow$ Hermitian $(n+1)\times(n+1)$-matrices

$$ds^2 = \frac{1}{2}\mathrm{Tr}(d\rho\; d\rho)$$

Space of density matrices

- density matrices

$$\mathcal{M}^{(N)} = \{\rho \in M_{N\times N}(\mathbb{C}) \,|\, \rho^* = \rho, \ \rho \geq 0, \ \mathrm{Tr}(\rho) = 1\}$$

- positivity $\langle \psi, \rho\psi \rangle \geq 0$, all $\psi \in \mathbb{C}^N$; $\rho = a^* a$; spectrum $\mathrm{Spec}(\rho) \subset \mathbb{R}_+$
- pure states are one-dimensional *projections* $\rho = |\psi\rangle\langle\psi|$ hence idempotent $\rho^2 = \rho$
- seen that pure states form a $\mathbb{P}^{N-1}(\mathbb{C})$ embedded as set of extremal points of $\mathcal{M}^{(N)}$
- Hilbert–Schmidt inner product $\langle A, B \rangle = \mathrm{Tr}(A^* B)$

- $\mathrm{Herm}(N)$ *real* vector space of hermitian matrices N^2-dimensional

$$\mathrm{Herm}(N) \simeq \mathrm{Lie}(U(N))$$

$$A = \tau_0 \, \mathrm{id} + \sum_{i=1}^{N^2-1} \tau_i \, \sigma_i$$

$\sigma_i =$ basis of $\mathrm{Lie}(SU(N))$

$$\tau_0 = \frac{\mathrm{Tr}(A)}{N}, \qquad \tau_i = \frac{1}{2}\mathrm{Tr}(\sigma_i A)$$

- $\rho_{(N)} = \frac{1}{N}\mathrm{id}$ maximally mixed state, like uniform probability in classical case: tracial state $\varphi(A) = \mathrm{Tr}(\rho_{(N)} A) = \frac{1}{N}\mathrm{Tr}(A)$
- subspace $\mathrm{Lie}(SU(N))$ of matrices with $\mathrm{Tr}(A) = 0$
- $\mathcal{P} \subset \mathrm{Herm}(N)$ positive cone $\rho \geq 0$

- can write density matrices in the form

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})} \quad \text{with} \quad H = \sum_{i=1}^{N^2-1} x_i \sigma_i$$

$\sigma_i =$ basis of $\mathrm{Lie}(SU(N))$ and x_i "exponential coordinates", with β inverse temperature

- one-parameter unitary group $U = e^{itH}$
- time evolution $\dot{\rho} = i[\rho, H]$ infinitesimal of

$$\sigma_t(\rho) = e^{itH} \rho e^{-itH}$$

- $\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$ equilibrium state

- Kadison theorem: $\phi : \mathcal{M}^{(N)} \to \mathcal{M}^{(N)}$ bijection preserving convex structure

$$\phi(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda\phi(\rho_1) + (1-\lambda)\phi(\rho_2)$$

is given by $\rho \mapsto U\rho U^*$ with U unitary or anti-unitary

- preserving convex structure: affine and sends extremal points to extremal points, hence symmetry of $\mathbb{P}^{N-1}(\mathbb{C})$ so from Wigner theorem implemented by unitary/antiunitary
- adjoint action of unitaries is adjoint action of $SU(N)$ on its Lie algebra

$$\rho' = U\rho U^* = \frac{1}{N}\mathrm{id} + \sum_{i=1}^{N^2-1} \tau_i U\sigma_i U^* = \frac{1}{N}\mathrm{id} + \sum_{i=1}^{N^2-1} \tau_i' \sigma_i$$

- rotated Bloch vector

$$\tau_i' = \frac{1}{2}\mathrm{Tr}(\rho'\sigma_i) = \frac{1}{2}\sum_j \mathrm{Tr}(\sigma_i U \sigma_j U^*)\tau_j$$

- the entries of an orthogonal matrix $O = (O_{ij})$ since $(OO^t)_{ij} = \delta_{ij}$

$$O_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_i U \sigma_j U^*)$$

- this realizes embedding

$$SU(N)/(\mathbb{Z}/N\mathbb{Z}) \hookrightarrow SO(N^2 - 1)$$

- case of $N = 2$ have $SU(2)/(\mathbb{Z}/2\mathbb{Z}) = SO(3)$

structure of $\mathcal{M}^{(N)}$ as a convex set

- $\rho = \rho^*$ diagonalizable: eigenvalues $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ are a classical probability distribution
- boundary strata of $\mathcal{M}^{(N)}$, where at least one of the eigenvalues is equal to zero
- copies of $\mathcal{M}^{(k)}$ with $k < N$ in the boundary
- extremal points (pure states) where all but one are zero (one-dimensional projections)
- *fix a basis*: those $\rho \in \mathcal{M}^{(N)}$ that are diagonal in that fixed basis form an $(N-1)$-dimensional simplex $\Delta_{N-1} \subset \mathcal{M}^{(N)}$ (eigenvalue simplex)
- one such eigenvalue simplex for each choice of basis; each ρ is in an eigenvalue simplex (for basis that diagonalizes it)

- structure organized by orbits of the unitary group
- diagonalization: $\rho = U\Lambda U^*$ with U unitary and Λ diagonal
- consider a Λ and the $U(N)$-orbit $\Lambda \mapsto U\Lambda U^*$
- if B is diagonal and unitary then $[\Lambda, B] = 0$ so $U\Lambda U^* = UB\Lambda B^* U^*$
- if diagonal entries of Λ are all distinct this is the only ambiguity
- if k entries agree then a further $U(k)$ that commutes with Λ
- densities ρ with nondegenerate spectrum have orbit the flag manifold

$$U(N)/U(1) \times \cdots \times U(1) = \mathrm{Flag}^{(N)}_{1,2,\ldots,N-1}$$

- if degeneracies k_i with $\sum_{i=1}^m k_i = N$ in the spectrum then orbit of ρ flag manifold

$$U(N)/U(k_1) \times \cdots \times U(k_m) = \mathrm{Flag}^{(N)}_{k_1,k_1+k_2,\ldots,\sum_i k_i}$$

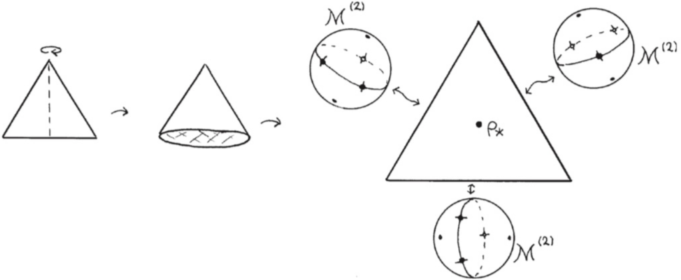


Figure 8.5 An attempt to visualise $\mathcal{M}^{(3)}$. We rotate the eigenvalue simplex to obtain a cone, then we rotate it in another dimension to turn the base of the cone into a Bloch ball rather than a disc; that is a maximal face of $\mathcal{M}^{(3)}$. On the right, we imagine that we have done this to all the three edges of the simplex. In each maximal face we have placed three equidistant points – it happens that when these points are placed correctly on all the three spheres, they form a regular simplex inscribed in $\mathcal{M}^{(3)}$.

from I.Bengtsson, K.Zyczkowski, "Geometry of quantum states", Cambridge University Press, 2017

- resulting structure of $\mathcal{M}^{(N)}$ subdivided into products of simplices and flag manifolds
- Λ diagonal densities is a classical simplex Δ_{N-1}
- first divide into $N!$ pieces (different orderings of eigenvalues)
- one of these pieces $\tilde{\Delta}_{N-1}$ Weyl chamber: $(N-1)$-dimensional space of $U(N)$ orbits
- subdivide the Weyl chamber $\tilde{\Delta}_{N-1}$ into pieces $K_{k_1,\ldots,k_m}$ with $k_1 + \cdots + k_m = N$, according to degeneracies of eigenvalues
- structure of $\mathcal{M}^{(N)}$

$$\mathcal{M}^{(N)} = \bigcup_{k_1+\cdots k_m=N} \mathrm{Flag}^{(N)}_{k_1,k_1+k_2,\ldots,\sum_i k_i} \times K_{k_1,\ldots,k_m}$$

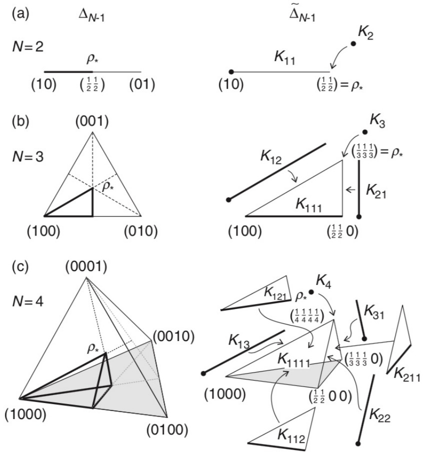


Figure 8.6 The eigenvalue simplex and the Weyl chamber for $N = 2, 3$ and 4. The Weyl chamber $\tilde{\Delta}_{N-1}$, enlarged on the right-hand side, can be decomposed according to the degeneracy into 2^{N-1} parts.

from I.Bengtsson, K.Zyczkowski, "Geometry of quantum states", Cambridge University Press, 2017

Entropy for quantum information

- analog of Shannon entropy $S(P) = -\sum_i p_i \log p_i$
- von Neumann entropy for density matrices $\rho \in \mathcal{M}^{(N)}$

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$

where use spectral theorem to define $\log \rho$

- if ρ diagonal

$$\rho = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \Rightarrow S(\rho) = -\sum_i \lambda_i \log \lambda_i$$

- zero for pure states; $\log N$ for maximally mixed $\rho_{(N)} = N^{-1}\mathrm{id}$

- disjoint states ρ_i have orthogonal ranges (nontrivial eigenvectors span orthogonal subspaces)
- extensivity property: ρ_i disjoint and $\rho = \sum_i p_i \rho_i$ with probabilities $P = (p_i)$

$$S(\rho) = S(P) + \sum_{i=1}^{N} p_i S(\rho_i)$$

follows from Shannon entropy via diagonalization

- concavity: $\rho = \lambda \rho_1 + (1-\lambda)\rho_2$

$$S(\rho) \geq \lambda S(\rho_1) + (1-\lambda) S(\rho_2)$$

- subadditivity: ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ with marginals (partial traces) $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}(\rho)$ and $\rho_2 = \mathrm{Tr}_{\mathcal{H}_1}(\rho)$

$$S(\rho) \leq S(\rho_1) + S(\rho_2)$$

- equality if $\rho = \rho_1 \otimes \rho_2$ independent subsystems

relative entropy in quantum information

- analog of Kullback–Leibler divergence

$$S(\rho|\sigma) = \mathrm{Tr}(\rho(\log \rho - \log \sigma))$$

- can be ∞ (if σ has zero eigevalue) and in general $S(\rho|\sigma) \neq S(\sigma|\rho)$
- for diagonal matrices Kullback–Leibler divergence
- unitary invariance $S(U\rho U^*|U\sigma U^*) = S(\rho|\sigma)$
- positivity: $S(\rho|\sigma) \geq 0$ and zero for $\rho = \sigma$
- joint convexity

$$S(\lambda\rho_a + (1-\lambda)\rho_b|\lambda\rho_c + (1-\lambda)\rho_d) \leq \lambda S(\rho_a|\rho_c) + (1-\lambda)S(\rho_b|\rho_d)$$

- monotonicity under partial trace: ρ, σ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ with marginals (partial traces) $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}(\rho)$ and $\rho_2 = \mathrm{Tr}_{\mathcal{H}_1}(\rho)$ same for σ_1, σ_2

$$S(\rho_i|\sigma_i) \leq S(\rho|\sigma)$$

- for maximally mixed state $S(\rho|\rho_{(N)}) = \log N - S(\rho)$ von Neumann

Kullback–Leibler divergence revisited

- classical case corresponds to requiring $[\rho, h] = 0$ (so diagonal in same basis)

$$S(\rho + h|\rho) = \langle h, \frac{1}{2}\rho^{-1} h\rangle + O(h^3)$$

with $\langle h, \frac{1}{2}\rho^{-1} h\rangle$ Fisher-Rao metric

$$\begin{aligned} S(\rho + h|\rho) &= \mathrm{Tr}((\rho + h)\log(\rho + h)) - \mathrm{Tr}((\rho + h)\log\rho) \\ &= \mathrm{Tr}(\rho\log(\rho(I + \rho^{-1}h))) + \mathrm{Tr}(h\log(\rho(I + \rho^{-1}h))) - \mathrm{Tr}(\rho\log\rho) \\ &= \mathrm{Tr}(\rho\log(I + \rho^{-1}h)) + \mathrm{Tr}(h\log(I + \rho^{-1}h)) \end{aligned}$$

$$S(\rho{+}h|\rho) = \mathrm{Tr}(\rho\,\rho^{-1}h) - \frac{1}{2}\mathrm{Tr}(\rho\,\rho^{-1}h\rho^{-1}h) + \mathrm{Tr}(h\rho^{-1}h) + O(h^3) = \frac{1}{2}\mathrm{Tr}(h\rho^{-1}h)$$

using $\log(I + \rho^{-1}h) = \rho^{-1}h - \frac{1}{2}\rho^{-1}h\rho^{-1}h + O(h^3)$ and $\mathrm{Tr}(h) = 0$ and $h = h^*$

$$\frac{1}{2}\mathrm{Tr}(h\rho^{-1}h) = \langle h, \frac{1}{2}\rho^{-1}h\rangle$$

Baker–Campbell–Hausdorff formula

- quantum case $[\rho, h] \neq 0$ need to replace $\log(\rho(I+\rho^{-1}h)) = \log(\rho) + \log(I+\rho^{-1}h))$ with BCH formula
- Baker–Campbell–Hausdorff formula:

$$\log(e^X e^Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{i=1}^{n} \sum_{a_i+b_i>0} \kappa(a,b)[X^{a_1}Y^{b_1}X^{a_2}Y^{b_2}\cdots X^{a_n}Y^{b_n}]$$

- $\kappa(a,b)$ combinatorial coefficients

$$\kappa(a,b) = \frac{(\sum_i (a_i+b_i))^{-1}}{a_1! b_1! \cdots a_n! b_n!}$$

- $[X^{a_1}Y^{b_1}X^{a_2}Y^{b_2}\cdots X^{a_n}Y^{b_n}]$ iterated commutators starting with a_1 commutators with X, followed by b_1 commutators with Y, etc

- more explicitly

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]])$$
$$-\frac{1}{24}[Y,[X,[X,Y]]] - \frac{1}{720}([[[[X,Y],Y],Y],Y]$$
$$+[[[[Y,X],X],X],X])$$
$$+\frac{1}{360}([[[[X,Y],Y],Y],X] + [[[[Y,X],X],X],Y]) + \cdots$$

- BCH up to second order terms in Y:

$$\log(e^X e^Y) = X + \frac{\mathrm{ad}_X e^{\mathrm{ad}_X}}{e^{\mathrm{ad}_X} - 1} Y + O(Y^2)$$

where $\mathrm{ad}_X Y := [X,Y]$

Second order term in the relative entropy

- quantum case $[\rho, h] \neq 0$

$$S(\rho + h|\rho) = \langle h, (F(\rho) - \frac{1}{2})\rho^{-1} h\rangle + O(h^3)$$

- $F(\rho)$ is given by

$$F(\rho) = \frac{\mathrm{ad}_{\log \rho}\; e^{\mathrm{ad}_{\log \rho}}}{e^{\mathrm{ad}_{\log \rho}} - 1}$$

$$\begin{aligned} S(\rho+h|\rho) &= \mathrm{Tr}((\rho+h)\log(\rho+h)) - \mathrm{Tr}((\rho+h)\log\rho) \\ &= \mathrm{Tr}(\rho\log(\rho(I+\rho^{-1}h))) + \mathrm{Tr}(h\log(\rho(I+\rho^{-1}h))) - \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(h\log\rho) \\ &= \mathrm{Tr}(\rho\log\rho) + \mathrm{Tr}(\rho\log(I+\rho^{-1}h)) + \frac{1}{2}\mathrm{Tr}(\rho[\log\rho, \log(I+\rho^{-1}h)]) + \cdots \\ &+ \mathrm{Tr}(h\log\rho) + \mathrm{Tr}(h(F(\rho)\log(I+\rho^{-1}h) + O(h^2))) - \mathrm{Tr}(\rho\log\rho) - \mathrm{Tr}(h\log\rho) \end{aligned}$$

up to second order $\log(I+\rho^{-1}h) = \rho^{-1}h - \frac{1}{2}\rho^{-1}h\rho^{-1}h + O(h^3)$ so get

$$\begin{aligned} S(\rho+h|\rho) &= \mathrm{Tr}(h) - \frac{1}{2}\mathrm{Tr}(h\rho^{-1}h) \\ &+\frac{1}{2}\mathrm{Tr}(\rho[\log\rho, \rho^{-1}h]) - \frac{1}{4}\mathrm{Tr}(\rho[\log\rho, \rho^{-1}h\rho^{-1}h]) + \cdots \\ &+\mathrm{Tr}(hF(\rho)\rho^{-1}h) + \cdots \end{aligned}$$

have $\mathrm{Tr}(h) = 0$ and up to second order in h iterated commutators contain at most one $Y = -\frac{1}{2}\rho^{-1}h\rho^{-1}h$ and all other equal to $X = \log\rho$ or at most two $Y = \rho^{-1}h$ with all the other $X = \log\rho$; commute ρ with the $X = \log\rho$ variables, obtain trace of a commutator (involving variables X and $Y' = -\frac{1}{2}h\rho^{-1}h$) and trace vanishes on commutators:

$$S(\rho+h|\rho) = -\frac{1}{2}\mathrm{Tr}(h\rho^{-1}h) + \mathrm{Tr}(h(F(\rho)\log(I+\rho^{-1}h)) + O(h^3).$$

- leading term in Taylor series expansion of $\langle h, (F(\rho) - \frac{1}{2})\rho^{-1} h\rangle$ recovers classical Fisher metric $\langle h, \frac{1}{2}\rho^{-1} h\rangle$

$$(F(\rho) - \frac{1}{2} I)\rho^{-1} = \left(\frac{\mathrm{ad}_{\log \rho} \; e^{\mathrm{ad}_{\log \rho}}}{e^{\mathrm{ad}_{\log \rho}} - 1} - \frac{1}{2} I \right) \rho^{-1} = (\frac{1}{2} I + \frac{1}{2} \mathrm{ad}_{\log \rho} + \cdots)\rho^{-1}$$

- quadratic form $\langle h, (F(\rho) - \frac{1}{2})\rho^{-1} h\rangle$ contains the quantum corrections to the classical Fisher metric
- quadratic form $\langle h, (F(\rho) - \frac{1}{2})\rho^{-1} h\rangle$ is positive definite

positivity

- basis in which ρ is diagonal $\rho = (\lambda_i)_{i=1}^N$, but h is not
- commutator $[\log \rho, h]$ is given by
$[\log \rho, h]_{ij} = (\log \lambda_i - \log \lambda_j) h_{ij}$ and
$(\mathrm{ad}^k_{\log \rho} h)_{ij} = (\log \lambda_i - \log \lambda_j)^k h_{ij}$
- using $h^* = h$

$$\langle h, \mathrm{ad}^k_{\log \rho} \, \rho^{-1} h \rangle = \sum_{i,j} h^2_{i,j} \frac{(\log \lambda_i - \log \lambda_j)^k}{\lambda_j} = \sum_{i<j} h^2_{i,j} \Lambda^{(k)}_{i,j}$$

- coefficients $\Lambda^{(k)}_{i,j} \geq 0$

$$\Lambda^{(k)}_{ij} := \begin{cases} \dfrac{(\lambda_i - \lambda_j)(\log \lambda_i - \log \lambda_j)^k}{\lambda_i \lambda_j} & \text{if } k = 2\ell + 1 \\ \dfrac{(\lambda_i + \lambda_j)(\log \lambda_i - \log \lambda_j)^k}{\lambda_i \lambda_j} & \text{if } k = 2\ell. \end{cases}$$

- expression $F(\rho) - 1/2$ can be expanded as

$$\frac{1}{2} + \frac{1}{2}\mathrm{ad}_{\log\rho} + \frac{1}{12}\mathrm{ad}_{\log\rho}^2 - \frac{1}{720}\mathrm{ad}_{\log\rho}^4 + \frac{1}{30240}\mathrm{ad}_{\log\rho}^6 - \frac{1}{1209600}\mathrm{ad}_{\log\rho}^8 + \cdots$$

- consider function $G(t)$ even, with $G(t) \sim t^2/4$ and $G'(t) > 0$ for $t \to 0$

$$G(t) = \frac{te^t}{e^t - 1} - \frac{1}{2} - \frac{1}{2}(1+t) = \frac{e^{t/2}(\frac{t}{2} - 1) + e^{-t/2}(\frac{t}{2} + 1)}{e^{t/2} - e^{-t/2}}$$

- after first order term $\frac{1}{2}\mathrm{ad}_{\log\rho}$ only even powers appear in Taylor series expansion of $F(\rho) - 1/2$ of form:

$$\frac{\lambda_i + \lambda_j}{\lambda_i\lambda_j}(\frac{1}{2} + G(\log(\lambda_i) - \log(\lambda_j)))$$

double expansion of relative entropy

- classical case $[\rho, h] = [\rho, \ell] = [h, \ell] = 0$

$$S(\rho + h|\rho + \ell) \sim \langle (h - \ell), \frac{1}{2}\rho^{-1}(h - \ell)\rangle$$

  same Fisher metric term

- quantum case with nontrivial commutation of ρ, h, ℓ

$$S(\rho+h|\rho+\ell) \sim \langle (h-\ell), \frac{1}{2}\rho^{-1}(h-\ell)\rangle + \langle h, (F(\rho) - I)\rho^{-1}(h-\ell)\rangle$$

- first term still Fisher metric $\langle (h - \ell), \frac{1}{2}\rho^{-1}(h - \ell)\rangle \geq 0$ but remaining term $\langle h, (F(\rho) - I)\rho^{-1}(h - \ell)\rangle$ no longer necessarily non-negative

Completely positive maps

- evolution of a quantum system:
  1. in isolation: $\rho \mapsto U\rho U^*$ unitary evolution
  2. non-isolated $\Rightarrow$ non-unitary processes $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathcal{H}_1$ system and $\mathcal{H}_2$ environment (ancillary)

$$\rho \mapsto \rho' = \mathrm{Tr}_{\mathcal{H}_2}(U(\rho \otimes \sigma)U^*)$$

  U unitary on $\mathcal{H}$

- case where $\sigma = |\psi\rangle\langle\psi|$ pure state and $|\phi\rangle$ o.n. basis of $\mathcal{H}_2$

$$A_\phi = \langle\phi|U|\psi\rangle \in \mathcal{B}(\mathcal{H}_1)$$

$$\rho' = \mathrm{Tr}_{\mathcal{H}_2}(U(\rho \otimes \sigma)U^*) = \mathrm{Tr}_{\mathcal{H}_2}(U\rho \otimes |\psi\rangle\langle\psi|U^*)$$

$$= \sum_\phi \langle\phi|U|\psi\rangle\rho\langle\psi|U^*|\phi\rangle = \sum_\phi A_\phi \rho A_\phi^*$$

$$\sum_\phi A_\phi^* A_\phi = \sum_\phi \langle\psi|U^*|\phi\rangle\langle\phi|U|\psi\rangle = \langle\psi|U^*U|\psi\rangle = \mathrm{id}_{\mathcal{H}_1}$$

operator sum representation of completely positive maps

- family of operators $\{A_i\}$ in $\mathcal{B}(\mathcal{H}_1)$, one for each ϕ_i o.n. basis of $\mathcal{H}_2$

$$\sum_i A_i^* A_i = 1, \qquad \rho' = \sum_i A_i \rho A_i^*$$

- measurement postulate: space of all possible measurement outcomes

$$\{A_i\} \qquad \sum_i A_i^* A_i = 1$$

(completeness relation)

- quantum measurement performed on ρ produces new state

$$\rho \mapsto \rho_i = \frac{A_i \rho A_i^*}{\mathrm{Tr}(A_i \rho A_i^*)}$$

with probability $p_i = \mathrm{Tr}(A_i \rho A_i^*)$ where $\sum_i p_i = 1$ by completeness

- projective measurement: case where $A_i = A_i^* = P_i$ projectors $P_i^2 = P_i = P_i^*$ and orthogonal $P_i P_j = \delta_{ij} P_i$

$$\rho \mapsto \sum_{i=1}^{N} P_i \rho P_i$$

- outcome of projective measurement

$$\rho_i = \frac{P_i \rho P_i}{\mathrm{Tr}(P_i \rho P_i)} \quad \text{with probability} \quad p_i = \mathrm{Tr}(P_i \rho P_i) = \mathrm{Tr}(P_i \rho)$$

- positive operator valued measures (not necessarily projections)

$$\mathrm{id} = \sum_{i=1}^{k} E_i, \qquad E_i = E_i^*, \qquad E_i \geq 0$$

$$p_i = \mathrm{Tr}(E_i \rho) = \mathrm{Tr}(A_i \rho A_i^*) \quad \text{with} \quad E_i = A_i^* A_I$$

- any positive operator valued measure $\{E_i\}_{i=1}^k$ defines an affine map from $\mathcal{M}^{(N)}$ to Δ_{k-1}

$$\rho \mapsto P = (p_i)_{i=1}^k, \quad p_i = \mathrm{Tr}(E_i \rho)$$

- $\{E_i\}$ statistically complete if image $P = (p_i)$ determines ρ uniquely (need N^2 elements)
- $\{E_i\}$ pure if each $E_i = |\phi_i\rangle\langle\phi_i|$ has rank one
- can always purify by passing to spectral projections

Quantum channels

- what are all possible "good" physical operations $\Phi : \mathcal{M}^{(N)} \to \mathcal{M}^{(N)}$ on the set of quantum states?
- seen case of $\rho \mapsto \sum_i A_i \rho A_i^*$ with $\sum_i A_i^* A_i = 1$
- is this the most general case? analog of stochastic matrices for classical probabilities
- completely positive maps: positive maps $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ send positive elements to positive elements; *completely* positive maps if on all extensions $\mathcal{H} \otimes \mathcal{H}'$ the map $\Phi \otimes \mathrm{id}_{\mathcal{H}'}$ is a positive map
- quantum channels: trace preserving completely positive maps

- quantum channels Φ can always be written (non-uniquely) in Kraus form as

$$\Phi(\rho) = \sum_i A_i \rho A_i^*, \quad \text{with} \quad \sum_i A_i^* A_i = 1$$

- can also represent completely positive trace preserving maps with $\Phi(\rho_X) = \rho_Y$ through associated stochastic Choi matrix S_Φ with

$$(\rho_Y)_{ij} = \sum_{a,b} (S_\Phi)_{\substack{ab \\ ij}} (\rho_X)_{ab}$$

- Kraus representations from factorizations $S_\Phi = AA^*$

Entropy and channel capacity

- entropy of an operation $\Phi :: \mathcal{M}^{(N)} \to \mathcal{M}^{(N)}$ using Choi matrix

$$S(\Phi) := S(\frac{1}{N} S_\Phi)$$

- larger entropy when more terms in Kraus decomposition of Φ, so when farther away from unitary (more decoherence)
- can check this way how much entropy a quantum channel introduces when acting on an initial pure state
- entropy exchange of Φ

$$C_\Phi(\rho) := \max_{\mathcal{E}_\rho} \sum_i S(\Phi\sigma_i | \Phi\rho)$$

over set of all representations of ρ as a mixed state

$$\mathcal{E}_\rho = \{\sigma_i, p_i \,|\, \rho = \sum_i p_i \sigma_i\}$$

- channel capacity:

$$C(\Phi) := \max_\rho C_\Phi(\rho)$$

Quantum information and categories

- probabilistic category $\mathcal{PC}$ associated to a category $\mathcal{C}$ with zero object and sum: wreath product of $\mathcal{C}$ and the category $\mathcal{FP}$ of finite classical probabilities
- similar idea for quantum probabilities $Q\mathcal{C}$
- category of quantum probabilities: finite set $X \Rightarrow$ Hilbert space $\mathcal{H}_X = \oplus_{x\in X}\mathbb{C}_x$ with $\mathbb{C}_x$ one-dimensional space at site $x \in X$
- can also replace $\mathbb{C}_x$ with Hilbert space $\mathcal{V}$ of fixed dimension: the internal degrees of freedom at site $x \in X$
- category $\mathcal{FQ}$ of finite quantum probabilities
    - objects: pairs (X, ρ_X) finite set X and density matrix ρ_X on $\mathcal{H}_X$
    - morphisms: $\mathrm{Mor}_{\mathcal{FQ}}((X, \rho_X), (Y, \rho_Y))$ are given by quantum channels Φ, completely positive trace preserving maps with $\Phi(\rho_X) = \rho_Y$

quantum probabilistic categories $\mathcal{QC}$

- a category $\mathcal{C}$ with zero object and categorical sum
- quantum probabilistic version $\mathcal{QC}$
  - objects: $\rho C = ((C_a, C_b), \rho_{ab})_{ab}$, with (C_a, C_b) finite collection of pairs of objects in $\mathcal{C}$ and $\rho = (\rho_{ab})$ density matrix
  - morphisms: for $\rho C = ((C_a, C_b), \rho_{ab})$ and $\rho' C' = ((C'_i, C'_j), \rho'_{ij})$, morphisms $\Xi \in \mathrm{Mor}_{\mathcal{QC}}(\rho C, \rho' C')$ given by finite collection

$$\Xi = \{(\phi_{ai,r}, \psi_{bj,r})\}, (S_{\Phi_r})_{\substack{ab \\ ij}}\}$$

    where $\sum_r S_{\Phi_r} = S_\Phi$ Choi matrix of quantum channel Φ with $\Phi(\rho) = \rho'$
- composition of morphisms $\Xi' \circ \Xi$ given by collection

$$\Xi' \circ \Xi = \{(\phi_{ua,r'} \circ \phi_{ai,r}, \psi_{vb,r'} \circ \psi_{bj,r}), (S_{\Phi_r})_{\substack{ab \\ ij}} (S_{\Phi'_{r'}})_{\substack{ij \\ uv}}\}_{r,r'}$$

  which satisfies

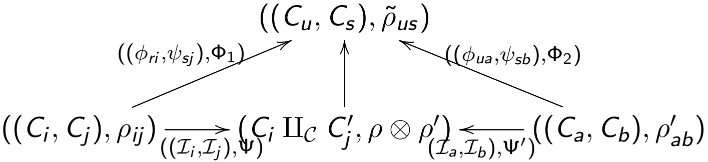
$$\sum_{r,r',i,j} (S_{\Phi_r})_{\substack{ab \\ ij}} (S_{\Phi'_{r'}})_{\substack{ij \\ uv}} = \sum_{i,j} (S_\Phi)_{\substack{ab \\ ij}} (S_{\Phi'})_{\substack{ij \\ uv}} = (S_{\Phi' \circ \Phi})_{\substack{ab \\ uv}}$$

- objects $\rho C = ((C_a, C_b), \rho_{ab})_{ab}$, for $a, b = 1, \dots, N$ include case $N = 1$ just objects $C \in \mathrm{Obj}(\mathcal{C})$ with weight $\rho = 1$ and morphisms in $\mathcal{C}$: embedding of category $\mathcal{C}$ into its quantum probability version $\mathcal{QC}$
- off-diagonal terms ρ_{ij} of density matrix ρ describe interference between amplitudes of the i-th and j-th state = a measure of coherence of the mixed state
- objects ρC of category $\mathcal{QC}$ have an assigned amount of coherence of pairs of objects C_i, C_j in $\mathcal{C}$, described by the coefficients ρ_{ij} of density matrix

- if category $\mathcal{C}$ has zero object 0 and categorical sum $\amalg$ then $Q\mathcal{C}$ also does
- zero object: pair $(0,1)$ with 0 the zero object of $\mathcal{C}$ with $\rho = 1$
- coproduct is of the form

$$\rho C \amalg \rho' C' = (C_i \amalg_{\mathcal{C}} C'_j, \rho \otimes \rho')$$

- satisfies universal property of coproduct



- $\mathcal{I}_i : C_i \to C_i \amalg_{\mathcal{C}} C'_j$ from universal property of coproduct in $\mathcal{C}$
- maps $\Psi\rho = \rho \otimes \rho'$ and $\Psi'\rho' = \rho \otimes \rho'$ given by

$$\Psi_{(i'j'),(ab)}^{\quad ij} = \delta_{ii'}\delta_{jj'}\rho'_{ab} \quad \text{and} \quad \Psi'^{\quad ab}_{(ij),(a'b')} = \delta_{aa'}\delta_{bb'}\rho_{ij}$$

- map $\rho C \amalg \rho' C' \to \tilde{\rho}\tilde{C}$ that makes the diagram commute

$$\tilde{\Phi}_{(ij),(ab)}^{\quad us} = \tilde{\rho}_{us}^{-1}(\Phi_1)_{ij}^{us}(\Phi_2)_{ab}^{us}$$

when the entry $\tilde{\rho}_{us} \neq 0$ and

$$\tilde{\Phi}_{(ij),(ab)}^{\quad us} = (\Phi_1)_{ij}^{us}\,\delta_{ab} + (\Phi_2)_{ab}^{us}\,\delta_{ij}$$

when matrix entry $\tilde{\rho}_{us} = 0$

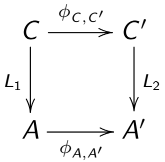
- coproduct induced on $\mathcal{FQ}$: product of independent systems $\rho \amalg_{\mathcal{FQ}} \rho' = \rho \otimes \rho'$

decoherence subcategory

- decoherence subcategory $\mathbb{P}\mathcal{C}$ of $\mathcal{QC}$ (case of mixed states with diagonal density matrices – in a fixed basis)
  - objects given by pairs $(C,z) = ((C_1,\ldots,C_n),(z_1:\cdots:z_n))$ with $C_i \in \mathrm{Obj}(\mathcal{C})$ and $z=(z_1:\cdots:z_n) \in \mathbb{P}^{n-1}(\mathbb{C})$
  - morphisms given by a morphism $\Phi : \mathbb{P}^{n-1} \to \mathbb{P}^{m-1}$ induced by a linear map $\tilde{\Phi} : \mathbb{C}^n \to \mathbb{C}^m$ up to scalars with $\Phi z = z'$ and a collection $\{(\tilde{\phi}_{ji,r} : C_i \to C'_j, \tilde{\Phi}_r)\}$ with $\sum_r \tilde{\Phi}_r = \tilde{\Phi}$
- coproduct $(C,z) \amalg (C',z') = ((C_i \amalg C_j)_{ij}, \alpha_{n,m}(z,z'))$ where $\alpha_{n,m} : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^{nm-1}$ is the Segre embedding

variant: categories of arrows

- variant of construction of categories $\mathcal{QC}$ by working with arrows of $\mathcal{C}$ instead of pairs of objects in $\mathcal{C}$
- category $\mathcal{C}$ with zero object and sum: category of arrows $\mathcal{AC}$
  - objects: $\phi_{C,C'}$ given by elements of $\mathrm{Mor}_{\mathcal{C}}(C, C')$ for arbitrary $C, C' \in \mathrm{Obj}(\mathcal{C})$
  - morphisms: $L \in \mathrm{Mor}_{\mathcal{AC}}(\phi_{C,C'}, \phi_{A,A'})$ pairs $L = (L_1, L_2)$ with $L_1 \in \mathrm{Mor}_{\mathcal{C}}(C, A)$ and $L_2 \in \mathrm{Mor}_{\mathcal{C}}(C', A')$ such diagram commutes:



- category $\mathcal{AC}$ also has zero object and sum
  - zero object of $\mathcal{AC}$ is identity morphism 1_0 of zero object of $\mathcal{C}$
  - coproduct $\phi_{C,C'} \amalg_{\mathcal{AC}} \phi_{A,A'}$ given by unique morphism $\phi_{C \amalg A, C' \amalg A'} : C \amalg_{\mathcal{C}} A \to C' \amalg_{\mathcal{C}} A'$ determined by the morphisms $\phi_{C,C'}$ and $\phi_{A,A'}$ via universal property of coproduct of $\mathcal{C}$

- associate to category of arrows $\mathcal{AC}$ the category $\mathcal{QAC}$, wreath product with finite quantum probabilities $\mathcal{FQ}$
  - objects: $\rho\phi = \{\phi_{ij}, \rho_{ij}\}$ given by collections of morphisms $\phi_{ij} : C_i \to C_j$ in $\mathcal{C}$, for $i, j = 1, \ldots, N$, some $N \in \mathbb{N}$, together with an $N \times N$ density matrix $\rho = (\rho_{ij})$
  - morphisms: $\mathrm{Mor}_{\mathcal{QAC}}(\rho\phi, \rho'\phi')$, with $\phi = (\phi_{ij})$ and $\phi' = (\phi'_{ab})$ are pairs (L, Φ) of a quantum channel $\Phi(\rho) = \rho'$, with Choi matrix $(S_\Phi)_{\substack{ij \\ ab}}$ and a finite collection

$$L = \{(L_{\substack{ij \\ ab}, r}, (S_{\Phi_r})_{\substack{ij \\ ab}})\}$$

of morphisms

$$L_{\substack{ij \\ ab}, r} : \phi_{ij} \to \phi'_{ab}$$

in $\mathcal{AC}$ with associated S_{Φ_r} satisfying

$$\sum_r S_{\Phi_r} = S_\Phi$$