Entropy, holography, and p-adic geometry

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Canadian Mathematical Society
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This talk is based on:

- Matthew Heydeman, Matilde Marcolli, Sarthak Parikh, Ingmar Saberi, *Nonarchimedean Holographic Entropy from Networks of Perfect Tensors*, arXiv:1812.04057 to appear in Advances in Theoretical and Mathematical Physics

Other references

- M.Heydeman, M.Marcolli, I.Saberi, B.Stoica, *Tensor networks, p-adic fields, and algebraic curves: arithmetic and the AdS_3/CFT_2 correspondence*, Advances in Theoretical and Mathematical Physics, Vol. 22 (2018) N. 1, 93-176
- S.S.Gubser, M.Heydeman, C.Jepsen, M.Marcolli, S.Parikh, I.Saberi, B.Stoica, B.Trundy, *Edge length dynamics on graphs with applications to p-adic AdS/CFT*, J. High Energy Phys. (2017) no. 6, 157
- M.Marcolli, *Holographic Codes on Bruhat–Tits buildings and Drinfeld Symmetric Spaces*, Pure and Applied Mathematics Quarterly, Vol.16 (2020) N.1, 1-33
- M.Marcolli, *Aspects of p-adic geometry related to entanglement entropy*, in "Integrability, Quantization, and Geometry", AMS 2021.

Manin's *Arithmetical Physics*

- Yuri I. Manin, *New dimensions in geometry*, Arbeitstagung Bonn 1984, pp.59–101, Lecture Notes in Math. 1111, Springer, 1985.
- Yuri I. Manin, *Reflections on Arithmetical Physics*, pp. 293–303, Perspectives in Physics, Academic Press, 1989.

• $\mathrm{Spec}(\mathbb{Z})$ as the arithmetic coordinate of Physics and Geometry

• Observation: Polyakov measure for bosonic string and the Faltings height function at algebraic points of the moduli space of curves ... Is there an adelic Polyakov measure? An arithmetic expression for the string partition function?

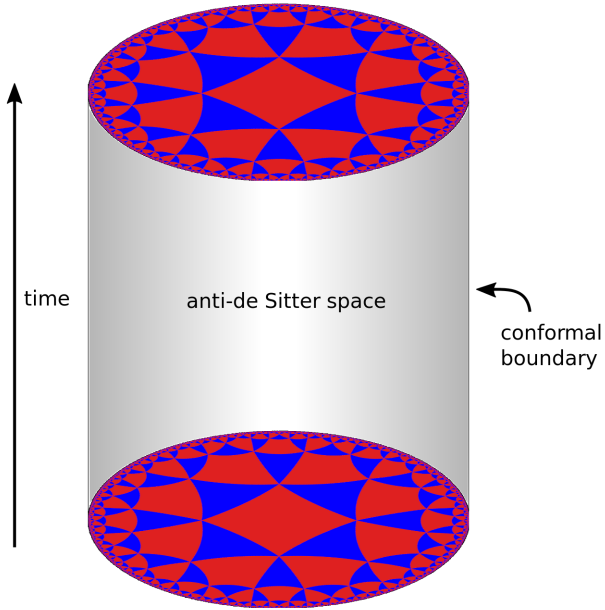
• General Questions: Are the fundamental laws of physics *adelic*? Does physics in the Archimedean setting (partition functions, action functionals, real and complex variables) have *p*-adic shadows? Do these provide convenient "discretized models" of physics powerful enough to recover the Archimedean counterpart?

AdS/CFT Holographic Correspondence

- bulk/boundary spaces
- hyperbolic geometry in the bulk (Lorentzian AdS spaces, Euclidean hyperbolic spaces $\mathbb{H}^{d+1}$)
- conformal boundary at infinity:
  $\partial\mathbb{H}^3 = \mathbb{P}^1(\mathbb{C})$ (AdS_3/CFT_2) or
  $\partial\mathbb{H}^2 = \mathbb{P}^1(\mathbb{R})$ (AdS_2/CFT_1)
- AdS/CFT correspondence: a d-dimensional conformal field theory on the boundary related to a gravitational theory on the $d+1$ dimensional bulk

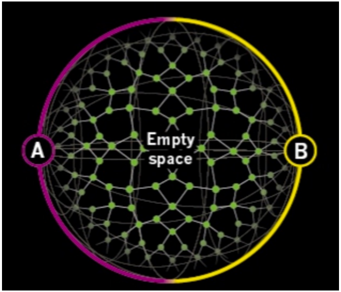
AdS/CFT Holography developed in String Theory since the 1990s

- E. Witten, *Anti-de Sitter space and holography*, arXiv:hep-th/9802150


time
anti-de Sitter space
conformal
boundary

More recent view of AdS/CFT: **Quantum Information**

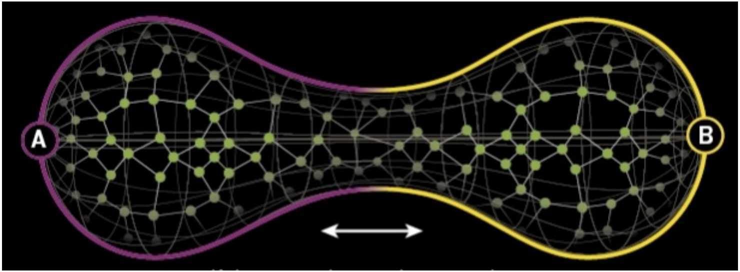
- relation between CFT on the boundary and gravity on the bulk with focus on Information (Entanglement Entropy) of quantum states on the boundary and geometry (gravity) on the bulk.



from R.Cowen, "The quantum source of space-time", Nature 527 (2015) 290–293

Spacetime geometry emerges from quantum entanglement

Entanglement between quantum fields in regions A and B decreases when corresponding regions of bulk space are pulled apart: dynamics of spacetime geometry ($=$ gravity) constructed from quantum entanglement



from R.Cowen, "The quantum source of space-time", Nature 527 (2015) 290–293

Ryu–Takayanagi Formula:

Entanglement Entropy and Bulk Geometry

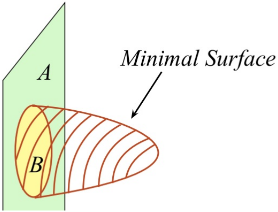
- Entanglement Entropy: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(|\Psi\rangle\langle\Psi|), \quad \mathcal{S}_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

- Entanglement and Geometry: (conjecture)

$$\mathcal{S}_A = \frac{\mathcal{A}(\Sigma_{\min})}{4G}$$

area of minimal surface in the bulk with given boundary $\partial A = \partial B$



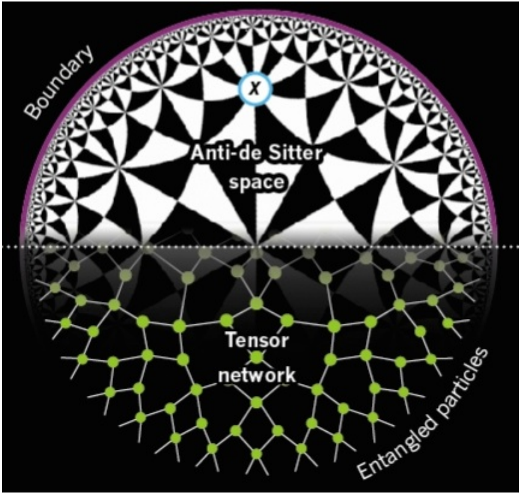
from T.Nishioka,S.Ryu,T.Takayanagi, “Holographic entanglement entropy: an overview”, J.Phys.A 42 (2009) N.50, 504008

Tensor Networks, Quantum Codes, and Geometry from Information

- Fernando Pastawski, Beni Yoshida, Daniel Harlow, John Preskill, *Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence*, JHEP 06 (2015) 149 [HaPPY]

Main Idea: Bulk spacetime geometry is the result of *entanglement* of quantum states in the boundary through a network of quantum error correcting codes

- quantum codes by perfect tensors: maximal entanglement across bipartitions
- network of perfect tensors with contracted legs along a tessellation of hyperbolic space
- uncontracted legs at the boundary (physical spins), and at the center of each tile in the bulk (logical spins)
- holographic state: pure state of boundary spins
- logical inputs on the bulk: encoding by the tensor network (holographic code)



from R.Cowen, "The quantum source of space-time", Nature 527 (2015) 290–293

Pentagon tile holographic code [HaPPY]

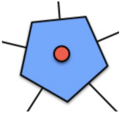
• perfect tensors: $T_{i_1,\ldots,i_n}$ such that, for $\{1,\ldots,n\} = A \cup A^c$ with $\#A \leq \#A^c$, isometry $T : \mathcal{H}_A \to \mathcal{H}_{A^c}$; perfect code (encodes one qbit to $n-1$)

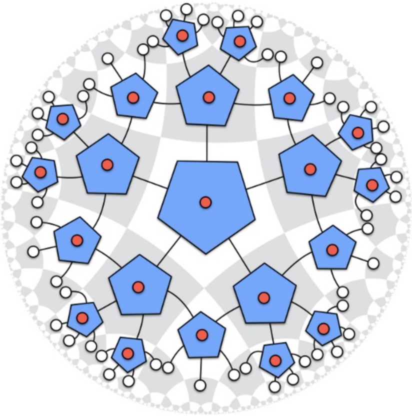
• six legs perfect tensor $T_{i_1\ldots,i_6}$: five qbit perfect code $[[5,1,3]]_2$-quantum code:

$$\mathcal{C} \subset \mathcal{H}^{\otimes 5}, \quad \mathcal{C} = \{\psi \in \mathcal{H}^{\otimes 5} \,:\, S_j\psi = \psi\}$$

$$S_1 = X \otimes Z \otimes Z \otimes X \otimes I$$

X, Y, Z Pauli gates and $S_2, S_3, S_4, S_5 = S_1S_2S_3S_4$ cyclic perms, with $\mathcal{H} = \mathbb{C}^2$ one qbit Hilbert space





from F.Pastawski, B.Yoshida, D.Harlow, J.Preskill, *Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence*, JHEP 06 (2015) 149

Properties of the [HaPPY] code

- quantum error-correcting codes with a tensor network structure as *discretized version* of spacetime
- bulk and boundary degrees of freedom (logical/physical)
- exact prescription for mapping bulk operators to boundary operators
- Ryu–Takayanagi: entanglement entropy in the CFT is computed by the area of a certain minimal surface in the bulk geometry (cutting legs in the tensor network cuts out the bulk region)

Bulk discretization via tensor networks depend on a choice of tessellation and construction of a network of perfect tensors along the tessellation: is there a natural bulk discretization that works?

- Some more recent results on the [HaPPY] code:
  - Elliott Gesteau, Monica Jinwoo Kang, *The infinite-dimensional HaPPY code: entanglement wedge reconstruction and dynamics*, arXiv:2005.05971
  - Elliott Gesteau, Monica Jinwoo Kang, *Thermal states are vital: Entanglement Wedge Reconstruction from Operator-Pushing*, arXiv:2005.07189

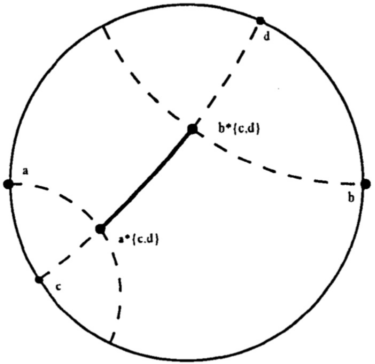
Passing to the limit of an infinite tessellation in the [HaPPY] construction shows some limitations as a model of AdS/CFT holography (lack of long-range entanglement in the boundary and CFT behavior, but good properties of entanglement wedge reconstruction)

Schottky Groups and Holography

- Yuri I. Manin, 3-*dimensional hyperbolic geometry as ∞-adic Arakelov geometry'*, Invent.Math. 104 (1991) N.2, 223–243
- Yuri I. Manin, M. Marcolli, *Holography principle and arithmetic of algebraic curves*, Adv. Theor. Math. Phys. 5 (2001), no. 3, 617–650.

• Holography on Riemann surfaces:

- Conformal boundary: $X(\mathbb{C})$ Riemann surface genus g
  Schottky uniformization $X(\mathbb{C}) = \Omega_\Gamma/\Gamma$ with $\Gamma \sim \mathbb{Z}^{\star g}$
  $\Omega_\Gamma = \mathbb{P}^1(\mathbb{C}) \smallsetminus \Lambda_\Gamma$ domain of discontinuity (Λ_Γ limit set)
- Bulk space: hyperbolic handlebody $\mathfrak{X}_\Gamma = \mathbb{H}^3/\Gamma$ with $X(\mathbb{C}) = \partial \mathfrak{X}_\Gamma$
- Green function on $X(\mathbb{C})$ in terms of geodesic lengths in the bulk space $\mathfrak{X}_\Gamma$ (two-point function of boundary CFT in terms of geometry/gravity in the bulk)



Cross-ratio of four points on the boundary $\mathbb{P}^1(\mathbb{C})$ and geodesic length in the bulk $\mathbb{H}^3$, suitably periodized over Schottky group action for higher genus cases

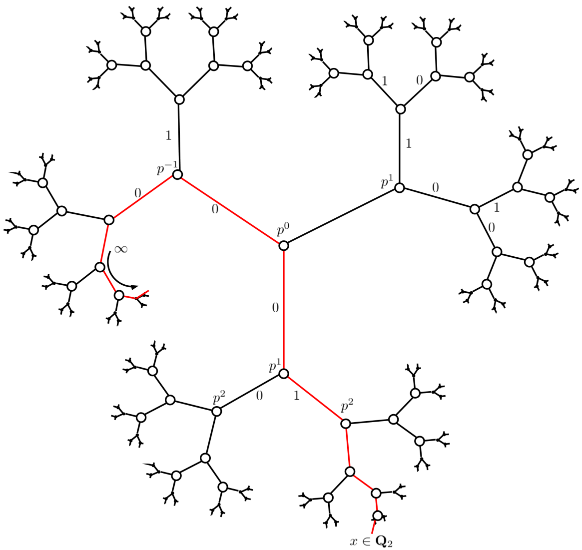
- genus one case, Tate uniformization $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ elliptic curve boundary and hyperbolic solid torus filing $\mathbb{H}^3/q^{\mathbb{Z}}$ as bulk: Bañados–Teitelboim–Zanelli black hole (Euclidean)

p-adic version of AdS/CFT Holography

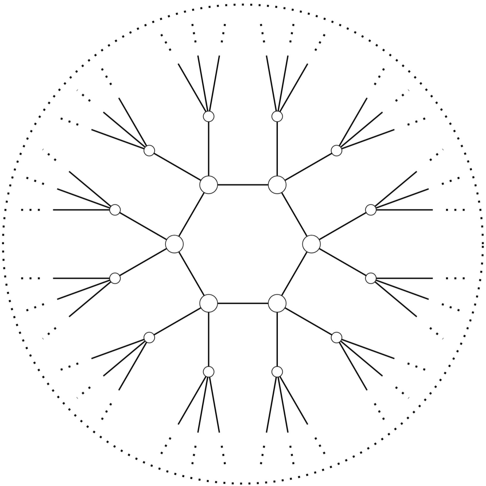
- $\mathbb{K}$ finite extension of $\mathbb{Q}_p$
- bulk space $\mathcal{T}_{\mathbb{K}}$ Bruhat-Tits tree; boundary $\partial \mathcal{T}_{\mathbb{K}} = \mathbb{P}^1(\mathbb{K})$
- p-adic Schottky groups $\Gamma \subset \mathrm{PGL}(2, \mathbb{K})$
- Schottky–Mumford curve boundary $X_\Gamma = \Omega_\Gamma / \Gamma$
- bulk: graph $\mathcal{T}_{\mathbb{K}}/\Gamma$; central finite graph $G = \mathcal{T}_\Gamma / \Gamma$
  $\mathcal{T}_\Gamma \subset \Delta_{\mathbb{K}}$ tree spanned by axes of hyperbolic $\gamma \in \Gamma$
- G dual graph of closed fiber of min model over $\mathcal{O}_{\mathbb{K}}$

Analogous result on geodesics on the bulk and correlation functions on the boundary based on

- Yu.I. Manin, V. Drinfeld, *Periods of p-adic Schottky groups*, J. Reine u. Angew. Math., vol. 262–263 (1973) 239–247



Bruhat–Tits tree of $\mathbb{K} = \mathbb{Q}_2$ and geodesics



p-adic Bañados–Teitelboim–Zanelli black hole with $\mathbb{K} = \mathbb{Q}_3$

p-adic AdS/CFT: holographic codes

- Can simulate the [HaPPY] holographic pentagon code using codes on a uniform tree?
- What kind of codes (classical and quantum) can be naturally built on a Bruhat-Tits tree?
- If replace the discretized bulk space (Bruhat-Tits tree) by the Drinfeld p-adic upper half plane

$$\Omega = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{P}^1(\mathbb{Q}_p)$$

  what kind of holographic codes can be constructed there?

- Example of perfect tensor codes on the tree $\mathcal{T}_{\mathbb{K}}$
  - single 3-ary qubit (qutritt) encodes to three 3-ary qubits

$$\begin{array}{lcl} |0\rangle & \mapsto & |000\rangle + |111\rangle + |222\rangle \\ |1\rangle & \mapsto & |012\rangle + |120\rangle + |201\rangle \\ |2\rangle & \mapsto & |021\rangle + |102\rangle + |210\rangle \end{array}$$

  - polynomial codes $f_a(x) = ax^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0$

$$|a\rangle \mapsto \sum_{b\in\mathbb{F}_q^d} \left(\otimes_{x\in\mathbb{F}_q} |f_a(x)\rangle\right)$$

  - example: $q = 5$

$$|a\rangle \mapsto \sum_{b_0,b_1\in\mathbb{F}_5} |b_0, b_0+b_1+a, b_0+2b_1+4a, b_0+3b_1+4a, b_0+4b_1+a\rangle$$

  - perfect tensors $T_{i_0\ldots i_q}$ with $q+1$-legs for $\mathbb{K}$ finite extension of $\mathbb{Q}_p$ with $\mathbb{F}_q$ residue field

Problem: How to construct a tensor network with the correct properties to obtain a Ryu–Takayanagi formula for the entanglement entropy
if tensor network on the tree, cutting a single leg disconnects the network, while for Ryu–Takayanagi want a "dual network" where cutting legs corresponds to geodesics in the tree

Steps

- method for construction of quantum codes and perfect tensors from classical codes (generalize CRSS)
- construct a "dual graph" to the Bruhat–Tits tree $\mathcal{T}$ using an embedding of the tree as 1-skeleton in the Drinfeld p-adic plane
- each edge cuts a unique edge in the tree; plaquettes at vertices
- use the CRSS quantum codes to assign perfect tensors to a tensor network on this dual graph
- gives correct entanglement entropy computation
- can be generalized to the genus one BTZ black hole case

From classical to quantum codes CRSS algorithm

- $\mathcal{H} = \mathbb{C}^q$ single q-ary qubit, o.n. basis $|a\rangle$ with $a \in \mathbb{F}_q$
- Quantum error correcting codes: subspaces $\mathcal{C} \subset \mathcal{H}_n = \mathcal{H}^{\otimes n}$ error correcting for up to d "q-ary bit flip" and "phase flip" errors $E = E_1 \otimes \cdots \otimes E_n$, $\omega(E) = \#\{i \,:\, E_i \neq I\} < d$

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \lambda_E \, P_{\mathcal{C}}$$

orthogonal projection $P_{\mathcal{C}}$ onto $\mathcal{C}$

- q-ary bit flip and phase flip on $\mathbb{C}^p$: $TR = \xi RT$ with $\xi^p = 1$

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & & & & \\ & \xi & & & \\ & & \xi^2 & & \\ & & & \ddots & \\ & & & & \xi^{p-1} \end{pmatrix}$$

Quantum Error Operators

- $\mathrm{Tr}:\mathbb{F}_q\to\mathbb{F}_p$ trace function $\mathrm{Tr}(a)=\sum_{i=0}^{r-1}a^{p^i}$

$$T_b|a\rangle=|a+b\rangle,\quad R_b|a\rangle=\xi^{\mathrm{Tr}(ab)}|a\rangle,$$

- $b\in\mathbb{F}_q$ as an $\mathbb{F}_p$-vector space, $q=p^r$

$$T_a:=T^{a_1}\otimes\cdots\otimes T^{a_r},\quad R_b:=R^{b_1}\otimes\cdots\otimes R^{b_r}$$

- T_aR_b, $a,b\in\mathbb{F}_q$, o.n. basis $M_{q\times q}(\mathbb{C})$ for $\langle A,B\rangle=\mathrm{Tr}(A^*B)$, generate all possible quantum errors on $\mathcal{H}=\mathbb{C}^q$
- error operators $E_{a,b}$ with $E^p_{a,b}=I$

$$E_{a,b}=T_aR_b=(T_{a_1}\otimes\cdots\otimes T_{a_n})(R_{b_1}\otimes\cdots\otimes R_{b_n})$$

for $a=(a_1,\ldots,a_n)$, $b=(b_1\ldots,b_n)\in\mathbb{F}_q^n$

- commutation and composition rules

$$E_{a,b}E_{a',b'}=\xi^{\langle a,b'\rangle-\langle b,a'\rangle}E_{a',b'}E_{a,b}$$

$$E_{a,b}E_{a',b'}=\xi^{-\langle b,a'\rangle}E_{a+a',b+b'},$$

where $\langle a,b\rangle=\sum_i\langle a_i,b_i\rangle=\sum_{i,j}a_{i,j}b_{i,j}$, with $a_i=(a_{i,j}),b_i=(b_{i,j})\in\mathbb{F}_q$ identified with $\mathbb{F}_p$-vector space

Quantum stabilizer codes

- group $\mathcal{G}_n = \{\xi^i E_{a,b},\ a, b \in \mathbb{F}_q^n,\ 0 \leq i \leq p-1\}$ order pq^{2n}
- quantum stabilizer error-correcting code $\mathcal{C} \subset \mathcal{H}_n$ joint eigenspace of operators $E_{a,b}$ in an abelian subgroup $\mathcal{S} \subset \mathcal{G}_n$
- $\varphi \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_p^r)$ automorphism

$$\langle (a,b),(a',b')\rangle = \langle a, \varphi(b')\rangle - \langle a', \varphi(b)\rangle$$

- $C \subset \mathbb{F}_q^{2n}$ is a classical self-orhogonal code with respect to this pairing $\Rightarrow$ subgroup $\mathcal{S} \subset \mathcal{G}_n$ of $\xi^i E_{a,\varphi(b)}$ with $(a,b) \in C$ is abelian
- CRSS algorithm associates to self-orthogonal classical $[2n, k, d]_q$ code C stabilizer quantum $[[n, n-k, d_Q]]_q$-code

$$d_Q = \min\{h(a,b)\ :\ (a,b) \in C^\perp \smallsetminus C\}$$

$h(a,b) = \#\{i\ :\ a_i \neq 0 \text{ or } b_i \neq 0\}$ and
$C^\perp = \{(v,w) \in \mathbb{F}_q^{2n}\ :\ \langle (a,b),(v,w)\rangle = 0,\ \forall (a,b) \in C\}$

Symplectic vector spaces and Heisenberg groups

- the error operators $E_{ab} = T_a R_b$ give the explicit representation matrices of the Heisenberg group $H(\mathbb{F}_q^{2n})$ with respect to the central character specified by ξ with $\xi^p = 1$
- Symplectic vector space (V, ω) over a finite field $\mathbb{F}_q$ (char$\neq 2$):
  - ω closed: cocycle condition

$$d\omega(u,v,w) = \omega(v,w) - \omega(u+v,w) + \omega(u,v+w) - \omega(u,v) = 0$$

  - ω non-degenerate: given $u \in V$ find v with $\omega(u,v) \neq 0$
- Heisenberg group central extension determined by cocycle ω

$$0 \to \mathbb{F}_q \to \mathrm{Heis}(V,\omega) \to V \to 0$$

$$(v,x) \cdot (w,y) = (v+w, x+y+\frac{1}{2}\omega(v,w))$$

- $H(\mathbb{F}_q^{2n}) = \mathrm{Heis}(\mathbb{F}_q^{2n}, \omega)$ with ω standard Darboux form

Heisenberg groups and quantum codes

- unique irreducible complex representation $\mathcal{H} = \mathcal{H}_\chi(V,\omega)$ of $\mathrm{Heis}(V,\omega)$ with central character $\chi : \mathbb{F}_q \to \mathbb{C}^*$
- functorial geometric quantization over finite fields (Gurevich–Hadani)
- Darboux basis for (V,ω) direct sum of 2-dim symplectic spaces over $\mathbb{F}_q \Rightarrow$ decomposition of $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$ as tensor product of q-ary qbits $\mathbb{C}^q$
- representation matrices $E_{ab} = T_a R_b$ of $\mathrm{Heis}(V,\omega)$ additive basis of $\mathrm{End}(\mathcal{H})$
- isotropic subspace $C \subset V \Rightarrow$ abelian subgroup of $\mathrm{Heis}(V,\omega)$ $\Rightarrow$ mutually diagonalizable, $\mathcal{H}$ sum of $\#C = q^k$ eigenspaces of dimension q^{n-k}
- Each such joint eigenspace of C of dimension q^{n-k} is a quantum code $\mathcal{Q}_C \simeq (\mathbb{C}^q)^{\otimes(n-k)}$ that encodes $n-k$ qbits to n qbits (CRSS quantum code associated to classical code C)

Symplectic vector spaces and perfect tensors

- perfect tensor: $T \in \mathcal{V}^{\otimes m}$ ($\mathcal{V}$ with inner prod to identify with dual, here qbit $\mathcal{V} = \mathbb{C}^q$) such that all splittings (tensor/Hom) for $j \leq m/2$ are *isometries*

$$\mathcal{V}^{\otimes j} \to \mathcal{V}^{\otimes (m-j)}$$

- when isometric injection of the $(n-k)$-qbits code space $\mathcal{Q}_C$ inside the n-qbit space $\mathcal{H}$ is obtained from a partition of the indices of a $(2n-k)$-index tensor into $(n-k)$-qbits (to be encoded), together with n-qbits (encoding space)
- even number of indices of perfect tensor when $\dim C = k$ even
- procedure to produce directly perfect tensors via a version of CRSS algorithm and quantization of symplectic vector spaces over finite fields

Lagrangians

- Symplectic vector space (V, ω) of dim $2n$ over $\mathbb{F}_q$; Lagrangian subspace $L \subset V$ (of dim n)
- Irreducible rep of $\mathrm{Heis}(V, \omega)$ can be realized through a choice of Lagrangian $L \subset V$ (in classical construction of quantum mechanical Hilbert space that identifies position vs momentum repres L, $L^\vee$) $\mathcal{H}_L = \mathcal{H}_\chi(V, L, \omega)$ (invariants under $L^\vee$)
- L chosen in "general position" means that intersection with Darboux decomposition as small as possible, so a basis of $\mathcal{H}_L$ is as far as possible from being a tensor-product basis in the Darboux decomposition

$$(\dim V = 4n) \quad V = \oplus_i V_i \Rightarrow (\mathbb{C}^q)^{\otimes 2n} = \mathcal{H} = \otimes_i \mathcal{H}_i = \otimes_i \mathbb{C}^q$$

$$V = W \oplus W' \text{ splitting of } 2n \text{ indices } k \leq n \text{ and } 2n - k$$

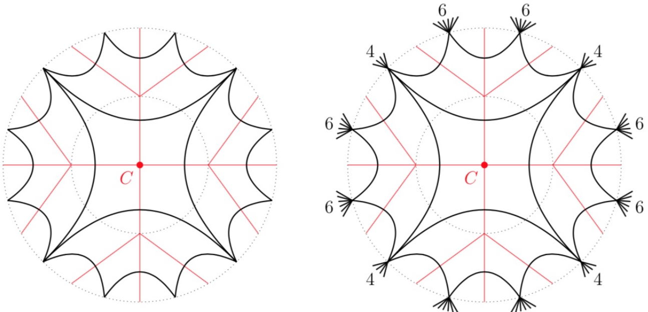
$$\dim(L \cap W) \geq 2(k - n), \quad \dim(L \cap W') \geq 2(n - k)$$

for $k = n$ general position if both zero-dimensional

Lagrangians and perfect tensors

- geometrically maximal rank of perfect tensors with respect to decomposition into groups of qbits corresponds to a "general position" of the Lagrangian (with respect to a given symplectic splitting of V into 2-dim Darboux pieces)
- most non-general position: L' sum of 1-dim Lagrangians in each 2-dim Darboux subspace (maximally decomposable)
- (functorial quantization): symplectomorphism $\psi : W_1 \to W_2 \Rightarrow \mathcal{H}(\psi) : \mathcal{H}(W_1) \to \mathcal{H}(W_2)$
- if Lagrangian L in general position then symplectomorphism $\psi : \bar{W} \to W'$ (with opposite $(\bar{W}, \bar{\omega}) = (W, -\omega)$)
- $\mathcal{H}(\psi) : \mathcal{H}(W)^\vee \to \mathcal{H}(W')$ same as tensor $T \in \mathcal{H}(W) \otimes \mathcal{H}(W') = \mathcal{H}(V)$ is perfect tensor

all nodes of the dual graph tensor network pushed to the boundary, no bulk nodes: finite region example ($q = 3$, perfect tensors $rank = 8$):

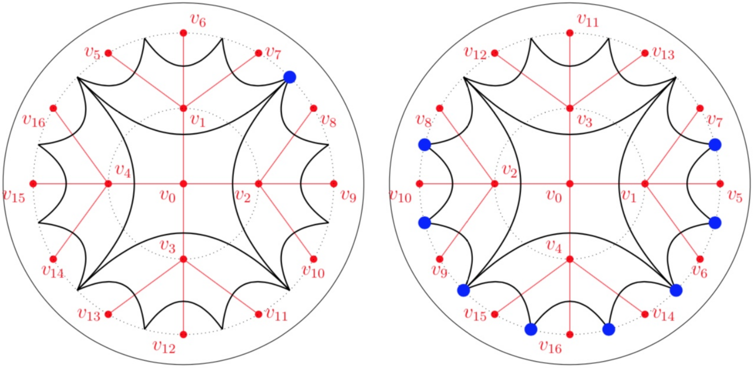


(a) Dual graph to a finite tree

(b) Holographic state with free dangling legs

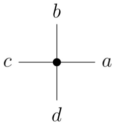
In limit of larger regions with boundary approaching $\mathbb{P}^1(\mathbb{Q}_p)$ length of geodesics and entanglement entropy have a logarithmic UV divergences (rank of perfect tensors also grows with cutoff, so computations at finite levels and limit)

- in real case boundary regions for entanglement entropy defined by intervals, in case of $\mathbb{P}^1(\mathbb{Q}_p)$ this choice of regions depends on the planar embedding dual graph (in terms of p-adic geometry on a section of the projection $\Upsilon : \Omega \to \mathcal{T}_{\mathbb{K}}$ from the Drinfeld plane) but final computation of the von Neumann entropy independent of choice

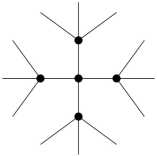


Perfect tensors and density matrices

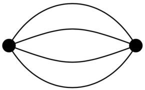
- rank 4 perfect tensor: $|\psi\rangle = T_{abcd}|abcd\rangle$ and $a, b, c, d \in \mathbb{F}_3$

$|\psi\rangle =$ 

- new states obtained by contracting multiple copies of perfect tensor, example:

$|\psi'\rangle =$ 

- density matrix $\quad \rho = \frac{1}{\langle\psi|\psi\rangle}|\psi\rangle\langle\psi|$

$\langle\psi|\psi\rangle =$ 

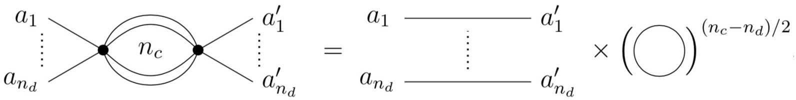
Entanglement entropy computation

- goal: reduced density matrix $\rho_A = \mathrm{Tr}_{A^c}\rho$ for region A (tracing out complementary region) von Neumann entropy

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$

- two rank $r+1$ perfect tensors with n_c contracted indices and n_d free indices $n_c \geq n_d$

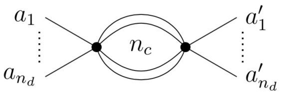
$$T_{a_1 \dots a_{n_d} b_1 \dots b_{n_c}} T_{a'_1 \dots a'_{n_d} b_1 \dots b_{n_c}} = \delta_{a_1 a'_1} \cdots \delta_{a_{n_d} a'_{n_d}} \times r^{(n_c - n_d)/2}$$



- von Neumann entropy for perfect state with $n_c \geq n_d$ contracted

$$S = -\mathrm{Tr}\rho_r \log \rho_r = n_d \log r$$

- when contracted legs $n_c < n_d$ nondiagonal density, in Jordan form

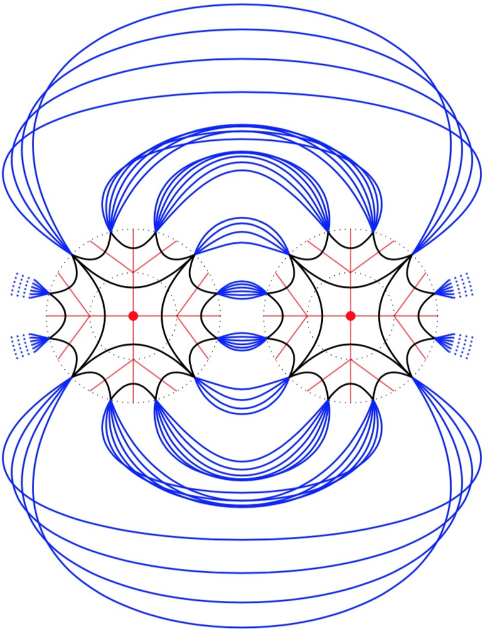


$$= \begin{pmatrix} J_{r^{(r+1)/2-n_c}} & & \\ & \ddots & \\ & & J_{r^{(r+1)/2-n_c}} \end{pmatrix}_{r^{(r+1)/2} \times r^{(r+1)/2}}$$

r^{n_c} nonzero eigenvalues each $\lambda = r^{-n_c}$ with $S = n_c \log r$

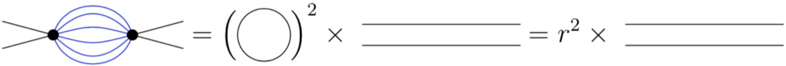
- graphical calculus: diagrammatically computing normalization factor $\langle \psi | \psi \rangle$ (contraction of all free dangling legs)

$$\langle \psi | \psi \rangle = $$



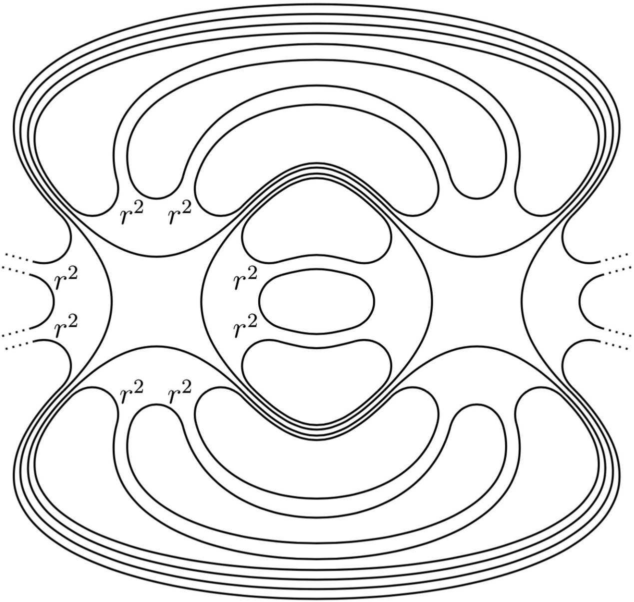
$$.$$

- apply contraction rule at all vertices

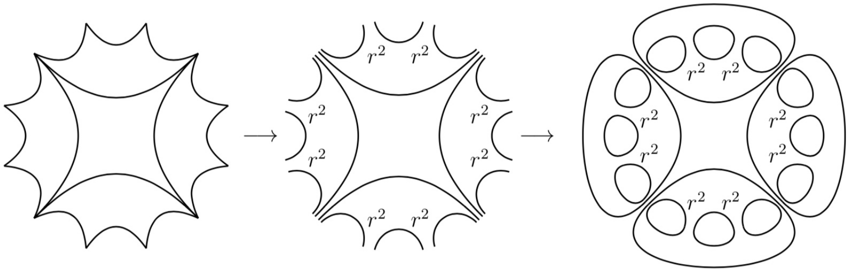


- each new cycle obtained after this operation contributes another factor of r^2 to the computation
- for example above get $\langle\psi|\psi\rangle = r^{32}$ where in this case $r = 7$

$$\langle\psi|\psi\rangle =$$



Splits and cycles operations to compute $\langle\psi|\psi\rangle$



• cutoff size Λ in the network and perfect tensors of $rank = r + 1$ with $(r+1)/2 \geq 2\Lambda$

- vertices of type i and multiplicities $M^{(i)}$

v_c	v_d	multiplicity M
2Λ	$(r+1)-2\Lambda$	$p+1$
$2\Lambda-2$	$(r+1)-(2\Lambda-2)$	$(p+1)(p^1-p^0)$
$2\Lambda-4$	$(r+1)-(2\Lambda-4)$	$(p+1)(p^2-p^1)$
$\vdots$	$\vdots$	$\vdots$
4	$(r+1)-4$	$(p+1)(p^{\Lambda-2}-p^{\Lambda-3})$
2	$(r+1)-2$	$(p+1)(p^{\Lambda-1}-p^{\Lambda-2})$

- total number of vertices $\sum_i M^{(i)} = (p+1)p^{\Lambda-1}$
- splits and cycles

$$N_{splits} = \sum_i M^{(i)} \frac{v_d^{(i)} - v_c^{(i)}}{2} = \frac{p+1}{2(p-1)}(p^\Lambda(r-3) - p^{\Lambda-1}(r+1)+4)$$

$$N_{cycles} = \frac{1}{2}\sum_i M^{(i)} v_c^{(i)} = \frac{p+1}{p-1}(p^\Lambda - 1)$$

- asymptotically for Λ and r large

$$\log\langle\psi|\psi\rangle = (N_{splits} + N_{cycles})\log r \sim \frac{p+1}{2}\, p^\Lambda\, r \log r$$

Entanglement entropy

$$\rho = \frac{1}{\langle\psi|\psi\rangle}|\psi\rangle\langle\psi|, \qquad \rho_A = \mathrm{Tr}_{A^c}\,\rho, \qquad S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

- for $\mathrm{Tr}_{A^c}\,\rho$ two copies of $|\psi\rangle$ gluing vertices along $B = A^c$
- basis of states for which block diagonal form for density matrix with blocks of 1's along diagonal

$$\rho_A = \frac{r^{N_{\text{splits}}+N_{\text{cycles}}}}{\langle\psi|\psi\rangle} \begin{pmatrix} \begin{array}{c} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} r^{\sigma} \\ r^{\sigma} \end{array} & & \\ & \ddots & \\ & & \begin{array}{c} r^{\sigma} \\ r^{\sigma} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \end{array} \end{pmatrix}_{r^{\sigma+C_{AB}} \times r^{\sigma+C_{AB}}}$$

- for this gluing of two copies of $|\psi\rangle$ along $B = A^c$

$$N_{splits} = \sum_{v \in B} \frac{v_d - v_c}{2}, \qquad N_{cycles} = \frac{1}{2}\Big(\sum_{v \in B} v_c - C_{AB}\Big),$$

$C_{AB} =$ number of legs between A and B regions

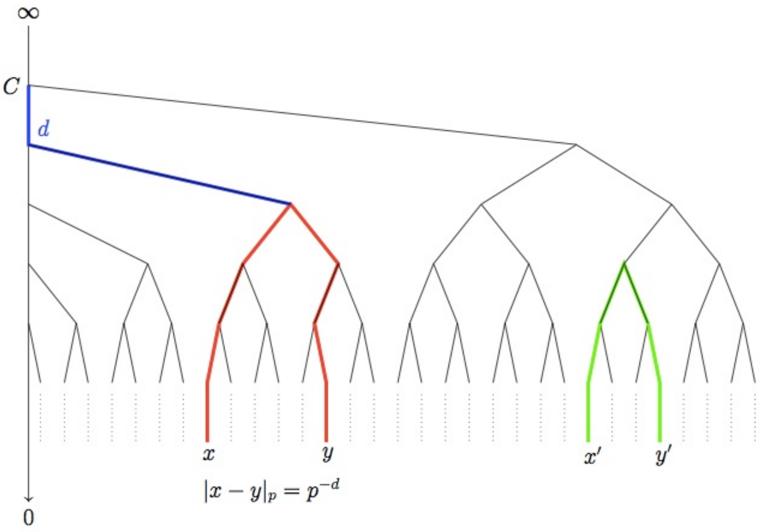
$$\mathrm{Tr}\rho_A = \frac{r^{N_{splits}+N_{cycles}}}{\langle\psi|\psi\rangle} r^{\sigma + C_{AB}}$$

$$S_A = C_{AB} \log r$$

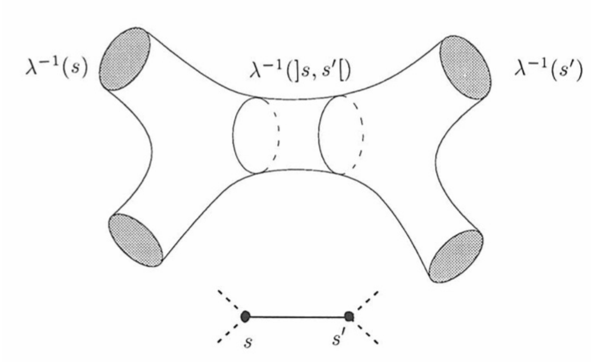
- Ryu-Takayanagi formula: C_{AB} edges on tensor network cut across edges (geodesics) on the the Bruhat–Tits tree separating boundary points x and y into two disconnected parts of the tree

$$C_{AB} = \mathrm{length}(\gamma_{xy}) = 2\Lambda + 2\log_p |x - y|_p$$

assuming edge lengths normalized to 1


∞
C
d
x
y
x'
y'
$|x-y|_p = p^{-d}$
0

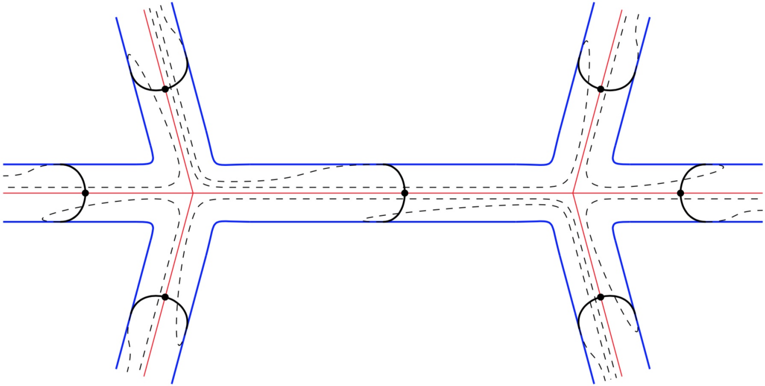
Drinfeld plane and Bruhat–Tits tree



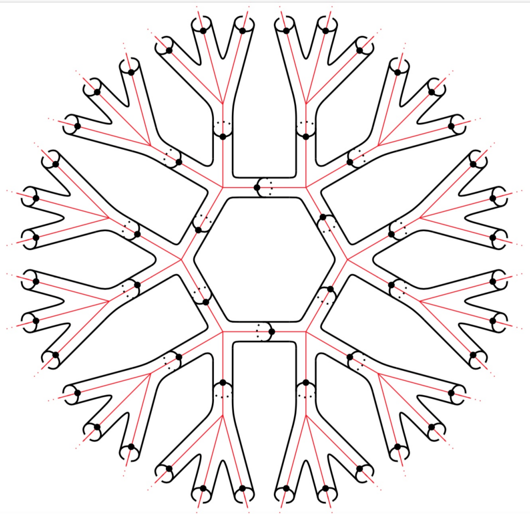
Can embed the Bruhat–Tits tree as a 1-skeleton in the Drinfeld plane via a section of the projection map $\Upsilon : \Omega \to \mathcal{T}_{\mathbb{K}}$

Sketch of the construction of the tensor network

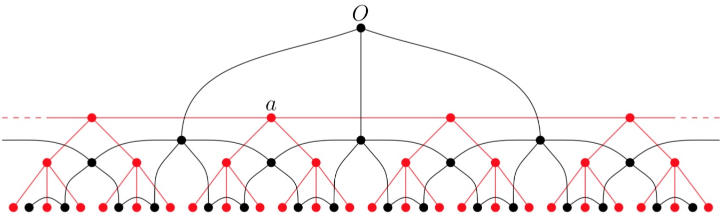
- genus zero case

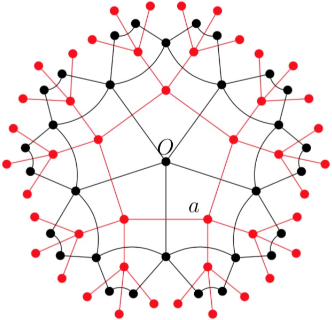


- genus one case

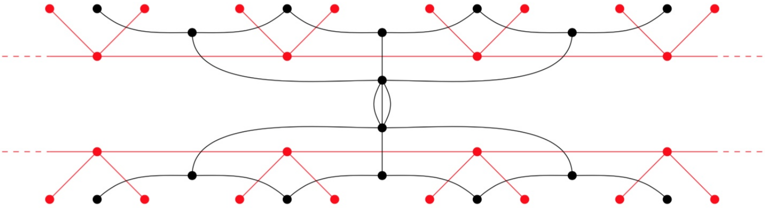


- genus one tensor network





Ryu–Takayanagi with a black hole background



density matrix associated to the thermal state on the boundary of a BTZ black hole tensor network

$$\rho_{BH} = \frac{r^{\frac{(r+1)}{2}-\tau}}{\langle\psi_\tau|\psi_\tau\rangle} \begin{pmatrix} \begin{array}{cc} \begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix} & r^{\sigma} \\ r^{\sigma} & \end{array} & & \\ & \ddots & \\ & & \begin{array}{cc} & r^{\sigma} \\ r^{\sigma} & \begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix} \end{array} \end{pmatrix}_{r^{\sigma+\tau}\times r^{\sigma+\tau}}$$

$$\sigma = \frac{\tau}{2} p^{\Lambda-1}((p-1)r-(p+1))$$

τ edges between the two sides

Black hole entropy

- get Bekenstein–Hawking-type black hole entropy from entanglement entropy computation for boundary holographic state

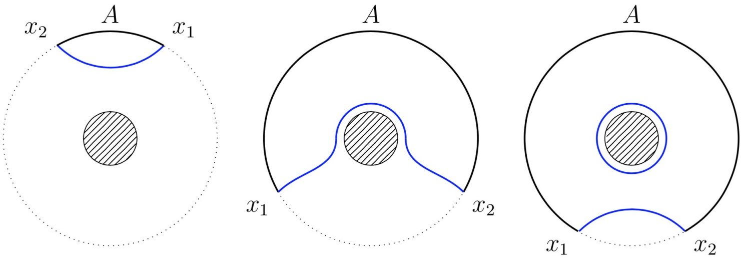
$$S_{BH} = -\mathrm{Tr}\rho_{BH} \log \rho_{BH}$$

- σ dependence cancels
- black hole entropy

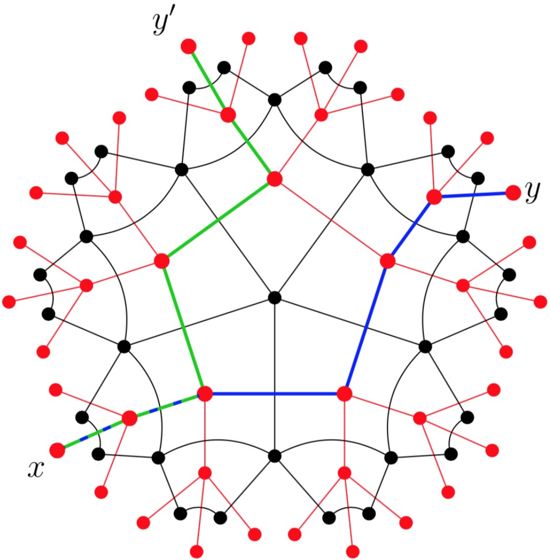
$$S_{BH} = \tau \log r = (\log r) \log_p |q|_p^{-1}$$

- proportional to length of the event horizon with q the Tate uniformization parameter
- boundary elliptic curve $E_q = \mathbb{G}_m / q^{\mathbb{Z}} = \Omega_\Gamma / \Gamma$ (Mumford curve genus one)
- bulk $\mathcal{T} / q^{\mathbb{Z}}$ quotient of Bruhat–Tits tree, $\Omega / q^{\mathbb{Z}}$ quotient of Drinfeld plane
- Krasnov black holes from higher genus Mumford curves

Geometry of geodesics of Ryu–Takayanagi formula



jump of min geodesic to path wrapping other side of the horizon



jump of min geodesic to path wrapping other side of the horizon

Further work (in progress)

- lifting boundary/bulk geometry (gravity and holographic codes) from $\mathbb{P}^1(\mathbb{Q}_p)$ and Bruhat–Tits tree to Berkovich spaces (better geometric setting for holographic duality of boundary/bulk theories)
- holography on higher-rank Bruhat–Tits buildings and hyperbolic groups
- replica argument for p-adic holography (cyclic coverings and Thomae formulae for Mumford curves)
- adelic aspects: recovering holography at archimedean places from its p-adic (disretized) forms