Cuntz–Krieger algebras and wavelets on fractals

Matilde Marcolli

UC Riverside, June 2012

joint work with Anna Maria Paolucci

• Matilde Marcolli, Anna Maria Paolucci *Cuntz–Krieger algebras and wavelets on fractals*, Complex Analysis and Operator Theory, Vol.5 (2011) N.1, 41–81.

Cuntz–Krieger algebras

- $A = (A_{ij})_{i,j=0,\dots,N-1}$ entries in $\{0,1\}$
- Algebra O_A: generators partial isometries $S_0, \dots, S_{N-1}$; relations

$$S_i^* S_i = \sum_j A_{ij} S_j S_j^*$$

$$\sum_{i=0}^{N-1} S_i S_i^* = 1$$

- O_A universal C^*-algebra determined by this presentation
- linearly spanned by monomials $S_\alpha S_\beta^\star$: words α and β in $\{0, \dots, N-1\}$ with possibly different lengths $|\alpha|$ and $|\beta|$
- O_A arises in dynamical systems (topological Markov chains)
- interesting class of *noncommutative spaces*

semi-branching function system

- measure space (X, μ)
- finite family $\{\sigma_i\}_{i\in I}$, $\#I = N$, measurable maps $\sigma_i : D_i \to X$, defined on measurable subsets $D_i \subset X$
- ranges $R_i = \sigma_i(D_i)$

$$\mu(X \smallsetminus \cup_i R_i) = 0, \quad \text{and} \quad \mu(R_i \cap R_j) = 0, \quad \text{for} \quad i \neq j$$

- Radon–Nikodym derivative

$$\Phi_{\sigma_i} = \frac{d(\mu \circ \sigma_i)}{d\mu}$$

with $\Phi_{\sigma_i} > 0$, μ-ae on D_i.

- $\sigma : X \to X$ *coding map* for $\{\sigma_i\}$ if $\sigma \circ \sigma_i(x) = x$, for all $x \in D_i$ (partial inverses of the coding map)

Operators from semi-branching function systems

- Given $\{\sigma_i\}_{i=0}^{N-1}$ with coding map $\sigma \Rightarrow$ operators on $L^2(X,\mu)$

$$(T_i\psi)(x) = \chi_{R_i}(x)\,(\Phi_{\sigma_i}(\sigma(x)))^{-1/2}$$

$$(T_i^\star\xi)(x) = \chi_{D_i}(x)(\Phi_{\sigma_i}(x))^{1/2}$$

- relations $T_iT_i^* = P_i$ (projection by χ_{R_i}) with $\sum_i T_iT_i^* = 1$ and $T_i^*T_i = Q_i$ (projection by χ_{D_i})
- If σ_i defined on all of $X \Rightarrow$ Cuntz algebra O_N, relations $T_i^*T_i = 1$ and $\sum_i T_iT_i^* = 1$
- in general case $D_i \subset X \Rightarrow$ Cuntz–Krieger algebra O_A if D_i satisfy

$$\chi_{D_i} = \sum_j A_{ij}\chi_{R_j}$$

- construct representations of O_A on Hilbert spaces $L^2(X,\mu)$

Fractal sets

• $\mathfrak{A} = \{0, \ldots, N-1\}$ alphabet; $\{0,1\}$-matrix $A = (A_{ij})_{i,j=0,\ldots,N-1}$

• Λ_A set of all infinite *admissible* words

$$\Lambda_A := \{ w = \{x_n\}_{n=0,1,\ldots} \,|\, x_i \in \mathfrak{A},\ A_{x_i,x_{i+1}} = 1 \}$$

• Cantor set topology on Λ_A (by cylinder sets)

• embeds in $[0,1]$: numbers whose base N expansion satisfies admissibility condition

• $\delta_A = \dim_H(\Lambda_A)$ and μ_A: δ_A-Hausdorff measure on Λ_A

subshift of finite type

- one-sided shift on Λ_A

$$\sigma : \Lambda_A \to \Lambda_A, \quad \sigma(x_0 x_1 x_2 \dots x_n \dots) = x_1 x_2 \dots x_n \dots$$

- coding map for semi-branching function system

$$\sigma_i : D_i \to R_i, \quad \sigma_i(w) = iw$$

$$D_i = \{w = \{x_k\} \in \Lambda_A \,|\, A_{i,x_0} = 1\} = \cup_{j:\, A_{ij}=1} R_j$$

$$R_i = \{w = \{x_k\} \in \Lambda_A \,|\, x_0 = i\} =: \Lambda_A(i)$$

- representation of O_A on $L^2(\Lambda_A, \mu_A)$ by T_i and T_i^*

Perron–Frobenius operator

- composition with coding map $T_\sigma : L^2(X,\mu) \to L^2(X,\mu)$

$$(T_\sigma \psi)(x) = \psi(\sigma(x))$$

- Perron–Frobenius operator $\mathcal{P}_\sigma$ adjoint

$$\int \overline{\psi}\, \mathcal{P}_\sigma(\xi) d\mu = \int \overline{T_\sigma(\psi)}\, \xi \, d\mu$$

- semi-branching function system $\{\sigma_i\}_{i=1}^N$ with coding map $\sigma : X \to X$

$$\mathcal{P}_\sigma = \sum_i \Phi_{\sigma_i}^{1/2}\, T_i^*$$

- if the Φ_{σ_i} constant over D_i, then $\mathcal{P}_\sigma$ belongs to the algebra O_A

• when non-constant: Hilbert space of half-densities $\psi(d\mu/d\lambda)^{1/2}$ for $\psi \in L^2(X, d\mu)$: representation of O_A with

$$\tilde{S}_i(\psi\sqrt{d\mu}) = \chi_{R_i}\ (\psi \circ \sigma)$$

• Perron–Frobenius operator (half-densities case)

$$\tilde{\mathcal{P}}_\sigma = \sum_i \tilde{S}_i^*$$

• Example: Schottky group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ free group rk g; symmetric set of generators $\mathfrak{A} = \{\gamma_1, \ldots, \gamma_g, \gamma_1^{-1}, \ldots, \gamma_g^{-1}\}$; CK algebra with $A_{ij} = 1$ for $|i - j| \neq g$;

$$\tilde{\mathcal{P}}_\sigma = \tilde{S}^*_{\gamma_1} + \tilde{S}^*_{\gamma_1^{-1}} + \cdots + \tilde{S}^*_{\gamma_g} + \tilde{S}^*_{\gamma_g^{-1}}$$

Perron–Frobenius is Harper operator of the group Γ

projection valued measures and space partitions

• (X, d) compact metric space

• partition $\mathcal{P}$ of X: (finite) family of subsets $\{A(i)\}_{i\in I}$: $\bigcup_i A(i) = X$; and $A(i) \cap A(j) = \emptyset$, for $i \neq j$

• N-adic system of partitions of X: partitions $\mathcal{P}_k$ for $k \in \mathbb{N}$ with subsets $A_k(a)$ indexed by elements of $\mathfrak{A}^k$ with alphabet $\mathfrak{A} = \{0, \ldots, N-1\}$ such that $|A_k(a)| = O(N^{-ck})$, some $c > 0$ and every $A_{k+1}(b)$, $b \in \mathfrak{A}^{k+1}$, contained in some $A_k(a)$, $a \in \mathfrak{A}^k$

• $\mathcal{H}$ be a complex separable Hilbert space: partition collection $\{P(i)\}_{i\in I}$ of projections $P(i) = P(i)^* = P(i)^2$ with $P(i)P(j) = 0$, for $i \neq j$ and $\sum_i P(i) = 1$

• N-adic system of partitions of $\mathcal{H}$: family $\{P_k(a)\}$ indexed by $a \in \mathfrak{A}^k$ such that for all $P_{k+1}(a)$, there is some $b \in \mathfrak{A}^k$ with $P_k(b)P_{k+1}(a) = P_{k+1}(a)$

• $\mathcal{B}(X)$ Borel subsets of X

• positive operator-valued function $E : \mathcal{B}(X) \to \mathcal{L}(\mathcal{H})$ is σ additive measure if given $B_1, B_2 \dots,$ in $\mathcal{B}(X)$ with $B_i \cap B_j = \emptyset$ for $i \neq j$ (convergence in the strong operator topology)

$$E\left(\bigcup_i B_i\right) = \sum_i E(B_i)$$

• orthogonal projection valued measure if also $E(B) = E(B)^* = E(B)^2$, for all $B \in \mathcal{B}(X)$ and $E(B_1)E(B_2) = 0$ when $B_1 \cap B_2 = \emptyset$, with $E(X) = 1$ identity on $\mathcal{H}$

N-adic system of partitions for the fractal Λ_A

$$\mathcal{W}_{k,A} = \{a = (a_1, \ldots, a_k) \in \mathfrak{A}^k \,|\, A_{a_i, a_{i+1}} = 1,\ i = 1, \ldots, k\}$$

$$\Lambda_{k,A}(a) = \{w = (w_1, w_2, \ldots, w_n, \ldots) \in \Lambda_A \,|\, (w_1, \ldots, w_k) = a\}$$

- subsets $\Lambda_{k,A}(a)$ define an N-adic system of partitions of Λ_A
- projections $P_k(a)$ by characteristic function $\chi_{\Lambda_{k,A}(a)}$

$$P_{k-1}(a_1, \ldots, a_{k-1}) P_k(a_1, \ldots, a_k) = P_k(a_1, \ldots, a_k)$$

- operator valued measure $E(B) := \pi(\chi_B)$

$$\pi : \sum_{a \in \mathcal{W}_{k,A}} c_a \, \chi_{\Lambda_{k,A}(a)} \mapsto \sum_{a \in \mathcal{W}_{k,A}} c_a \, P_k(a)$$

Hausdorff dimension and Hausdorff measure

- assume matrix A irreducible ($\exists\ A^n$ with all entries positive)
- then Radon-Nikodym derivatives constant on $D_i \subset \Lambda_A$

$$\Phi_{\sigma_i} = \frac{d\mu \circ \sigma_i}{d\mu} = N^{-\delta_A}$$

- Hausdorff measure $\mu = \mu_A$ on Λ_A satisfies $\mu(R_i) = p_i$, where

$p = (p_i)_{i=0,\dots,N-1}$ Perron–Frobenius eigenvector of A

$$\sum_j A_{ij}\, p_j = r(A)\, p_i$$

- Hausdorff dimension of Λ_A

$$\delta_A = \dim_H(\Lambda_A) = \frac{\log r(A)}{\log N}$$

with $r(A)$ spectral radius (Perron–Frobenius eigenvalue)

- Self–similarity, for $\mu(\sigma_k^{-1}(E)) = \mu(\{x \in \Lambda_A \,|\, \sigma_k(x) \in E\})$

$$\mu = N^{-\delta_A} \sum_{k=0}^{n-1} \mu \circ \sigma_k^{-1}$$

Perron–Frobenius operator on Λ_A

- $\mathcal{P}_\sigma$ on $L^2(\Lambda_A, d\mu_A)$ satisfies

$$\mathcal{P}_\sigma = N^{-\delta_A/2} \sum_i S_i^*$$

with S_i generators of O_A representation

- A irreducible and ω Perron–Frobenius eigenvector:

$$f = \sum_i \omega_i \chi_{R_i}$$

fixed point of the Perron–Frobenius operator: $\mathcal{P}_\sigma f = f$

time evolution on O_A and KMS state

• time evolution on a C^*-algebra: $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ one-parameter group of automorphisms

• on O_A time evolution

$$\sigma_t(S_k) = N^{it} S_k$$

• KMS equilibrium states for time evolution: $\varphi \in \mathrm{KMS}_\beta$ $(0 < \beta < \infty)$: $\forall a, b \in \mathcal{A}$ $\exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$

• measure $\mu = \mu_A$ on Λ_A gives KMS state for (O_A, σ_t) at inverse temperature $\beta = \delta_A$

$$\varphi(S_a S_b^*) = \begin{cases} 0 & a \neq b \\ \mu(\Lambda_{k,A}(a)) & a = b \in \mathcal{W}_{k,A}. \end{cases}$$

real valued measures and Fourier transforms

• $f \in \mathcal{H}$ of norm $\|f\| = 1 \Rightarrow$ real valued measure $\mu_f(B) := \langle f, E(B)f \rangle$ from operator valued measure

• Fourier transform $\widehat{\mu_f}(t) := \int e^{itx}\, d\mu_f(x)$

• $f \in \mathcal{H} = L^2(\Lambda_A, d\mu_A)$ with $\|f\| = 1$ gives $\mu_f(E) = \langle f, P(E)f \rangle$ with

$$\sum_{k=0}^{N-1} \int_{\Lambda_A} \psi \circ \sigma_k \, d\mu_{S_k^* f} = \int_{\Lambda_A} \psi d\mu_f$$

and Fourier transform

$$\widehat{\mu}_f(t) = \sum_{k=0}^{N-1} e^{\frac{itk}{N}} \widehat{\mu_{S_k^* f}}(\frac{t}{N})$$

- iteration:

$$\hat{\mu}_f(t) = \sum_{a \in \mathcal{W}_{k,A}} e^{itx(a)} \hat{\mu}_{S_a^* f}\left(\frac{t}{N^k}\right)$$

with

$$x(a) = \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots + \frac{a_k}{N^k}$$

- good approximation by Dirac measures (weak convergence)

$$\mu_f^{(k)}(E) = \sum_{a \in \mathcal{W}_{k,A}} \|S_a^* f\|^2 \delta_a(E)$$

$\delta_a =$ Dirac measure supported at the rational point $x(a)$ in Λ_A

other fractal objects associated to O_A: Sierpinski fractals

• square $\mathbb{S} = [0,1] \times [0,1]$ points $(x,y) \in \mathbb{S}$ with N-adic expansion

$$(x,y) = \left(\frac{x_1}{N} + \frac{x_2}{N^2} + \cdots + \frac{x_k}{N^k} + \cdots, \frac{y_1}{N} + \frac{y_2}{N^2} + \cdots + \frac{y_k}{N^k} + \cdots\right)$$

$(x_i, y_i) \in \{0, \ldots, N-1\} \times \{0, \ldots, N-1\} = \mathfrak{A}^2$

• subset $\mathbb{S}_A \subset \mathbb{S}$ given by

$$\mathbb{S}_A = \{(x,y) \in S \mid A_{x_i, y_i} = 1, \ \forall i \geq 1\}$$

• iterative construction: k-th step square of size N^{-2k} replaced by D squares of size $N^{-2(k+1)}$

$$D = \sum_{i=0}^{N-1} d_i, \quad \text{with} \quad d_i = \#\{j \mid A_{ij} = 1\}$$

• Hausdorff dimension $\dim_H(\mathbb{S}_A) = \frac{\log D}{2 \log N}$

semi-branching function system on $\mathbb{S}_A$

- maps $\tau_{(i,j)} : \mathbb{S}_A \to \mathbb{S}_A$, for (i,j) with $A_{ij} = 1$

$$\tau_{(i,j)}(x,y) = (\tau_i(x), \tau_j(y)) = (\frac{x+i}{N}, \frac{x+j}{N})$$

the $\tau_{(i,j)}$ are *everywhere defined* on $\mathbb{S}_A$

- Radon–Nikodym derivatives

$$\Phi_{(i,j)}(x,y) = \frac{d\mu \circ \tau_{(i,j)}}{d\mu} = N^{-2\delta} = \frac{1}{D}$$

- representation of Cuntz algebra O_D on $L^2(\mathbb{S}_A, \mu)$, Hausdorff measure of dimension $\delta = \dim_H(\mathbb{S}_A)$

$$S_{(i,j)} f = \chi_{R_{(i,j)}} \cdot (\Phi_{(i,j)} \circ \tau)^{-1/2} \cdot f \circ \tau$$

$$S^*_{(i,j)} S_{(i,j)} = 1, \quad \text{and} \quad \sum_{(i,j): A_{ij}=1} S_{(i,j)} S^*_{(i,j)} = 1$$

Λ_A and $\mathbb{S}_A$

- embedding $\Xi : \Lambda_A \hookrightarrow \mathbb{S}_A$ with $\Xi(x) = (x, \sigma(x))$
- maps $\tau_{i,j}$ restricts to maps defined on $D_{i,j} \subset \Xi(\Lambda_A)$

$$D_{(i,j)} = \{(x, \sigma(x)) \in \Xi(\Lambda_A) \,|\, \sigma_j \sigma(x) = \sigma \sigma_i(x)\} = \Xi(R_j)$$

- semi-branching function system that gives representation of CK algebra $O_{\tilde{A}}$, with $D \times D$-matrix $\tilde{A}$

$$\tilde{A}_{(i,j),(\ell,k)} = \delta_{j,\ell} \, A_{jk}$$

- representation on $L^2(\Xi(\Lambda_A), \mu_s)$ with $s = \dim_H(\Xi(\Lambda_A))$

$$\hat{S}^*_{(i,j)} f(x) = N^s \chi_{R_{ij}}(x) f(\sigma(x))$$

wavelets on fractals (general construction: Jonsson)

- $\{\sigma_i\}$ semi-branching function system on (X,μ) with $X\subset\mathbb{R}$; $\mathfrak{P}^m$ polynomials on $\mathbb{R}$ of degree $\leq m$
- $\mathfrak{S}_0$ lin subspace of $L^2(X,d\mu)$ gen by restrictions $P|_{\Lambda_A}$ of polynomials in $\mathfrak{P}^m$; $\mathfrak{S}_1\subset L^2(X,d\mu)$ functions $f\in L^2(X,d\mu)$ restrictions $P|_{R_i}$ of $P\in\mathfrak{P}^m$, μ-ae on $R_i=\sigma_i(X)$: $\mathfrak{S}_0\subset\mathfrak{S}_1$ and $\dim\mathfrak{S}_1=N\dim\mathfrak{S}_0=N(m+1)$
- ϕ^ℓ, for $\ell=1,\dots,m+1$, o.n. basis for $\mathfrak{S}_0$; ψ^ρ, for $\rho=1,\dots,(N-1)(m+1)$, o.n. basis of $\mathfrak{S}_1\ominus\mathfrak{S}_0$
- lin subspaces $\mathfrak{S}_k\subset\mathfrak{S}_{k+1}$ of $L^2(X,d\mu)$ functions whose restriction to $\sigma_{i_1}\circ\cdots\circ\sigma_{i_k}(X)$ are μ-ae restrictions of polynomials in $\mathfrak{P}^m$
- functions ϕ^r and ψ^ρ provide the mother wavelets and

$$\psi_a^\rho=\mu(\sigma_a(X))^{-1/2}\psi^\rho\circ\sigma_a^{-1}$$

for $a=(i_1,\dots,i_k)$ and $\sigma_a=\sigma_{i_1}\circ\cdots\circ\sigma_{i_k}$

wavelets on Λ_A

- in $L^2(\Lambda_A, d\mu_A)$ locally constant functions

$$\{f^{\ell,k}\}_{k=0,\dots,N-1;\ell=1,\dots,d_k}$$

with $d_k = \#\{j \mid A_{kj} = 1\}$ and support of $f^{\ell,k}$ in R_k

$$\int_{R_k} \overline{f^{\ell,k}} f^{\ell',k} = \delta_{\ell,\ell'} \quad \text{and} \quad \int_{R_k} f^{\ell,k} = 0, \quad \forall \ell = 1,\dots,d_k$$

- $f^{\ell,k}$ lin combinations of characteristic functions of $R_{kj} = \Lambda_{2,A}(kj)$

$$f^{\ell,k} = \sum_j A_{kj} c_j^{\ell,k} \chi_{R_{kj}}$$

- $p_{kj} = \mu(R_{kj}) = N^{-2\delta_A} p_j$ with $p = (p_0, \dots, p_{N-1})$

Perron–Frobenius eigenvector $Ap = r(A)p$: conditions on $f^{\ell,k}$ become

$$\sum_j A_{kj} \bar{c}_j^{\ell,k} c_j^{\ell',k} p_{kj} = \delta_{\ell,\ell'}$$

$$\sum_j A_{kj} c_j^{\ell,k} p_{kj} = N^{-2\delta_A} \sum_j A_{kj} c_j^{\ell,k} p_j = 0$$

- $\mathbb{C}^{d_k} \subset \mathbb{C}^N$ with inner product

$$\langle v, w \rangle_k := \sum_j A_{kj} \bar{v}_j w_j p_j$$

with p Perron–Frobenius eigenvector of A

- $\mathcal{V}_k = \{u = (1, 1, \ldots, 1)\}^\perp \subset \mathbb{C}^{d_k}$ and $\{c^{\ell,k} = (c_i^{\ell,k})\}_{\ell=1,\ldots,d_k-1}$ o.n. basis
- then $f^{\ell,k} = \sum_j A_{kj} c_j^{\ell,k} \chi_{R_{kj}}$ gives right family:

$$\psi_a^{\ell,r} = S_a f^{\ell,r} \quad \text{for} \quad a = (a_1, \ldots, a_k) \in \mathcal{W}_{k,A}$$

varying $k \Rightarrow$ orthonormal basis of wavelets for $L^2(\Lambda_A, \mu_A)$

- wavelet decomposition

$$f = \sum_{k=0}^{N-1} \sum_{\ell=1}^{d_k-1} \alpha_{\ell,k} f^{\ell,k} + \sum_{j=0}^{\infty} \sum_{a \in \mathcal{W}_{j,A}} \sum_{(\ell,k)} \alpha_{\ell,k,a} S_a f^{\ell,k}$$

Ruelle transfer operator

- generalizes Perron–Frobenius operator: coding map $\sigma : \Lambda_A \to \Lambda_A$
- potential function W

$$\mathcal{R}_{\sigma,W} f(x) = \sum_{y\,:\,\sigma(y)=x} W(y)\, f(y)$$

if W real valued: adjoint of

$$T_W f(x) = N^{\delta_A}\, W(x)\, f(\sigma(x))$$

- Keane condition: $W : \Lambda_A \to \mathbb{R}_+$ with

$$\sum_{y:\sigma(y)=x} W(y) = 1 \quad \text{that is} \quad \sum_i A_{ix_1} W(\sigma_i(x)) = 1$$

- Example of potential satisfying the Keane condition

$$W(x) = \frac{1}{N_1}\left(1 - \cos\left(\frac{2\pi N x}{N_1}\right)\right)$$

with $N_1 = \#\{j \,:\, A_{jx_1} = 1\}$

Random processes

• harmonic functions for Ruelle transfer operator $\mathcal{R}_{\sigma,W} h = h$
$\Leftrightarrow$ random processes along paths under the σ_j iterates

• transpose A^t, potential W with Keane condition $\Rightarrow$ measure on Λ_{A^t} for fixed initial point x

$$P_x^W(\Lambda_{k,A^t}(a)) = A_{a_1 x_1} W(\sigma_{a_1}(x)) W(\sigma_{a_2}\sigma_{a_1}(x)) \cdots W(\sigma_{a_k} \cdots \sigma_{a_1}(x))$$

random walk from x to $\sigma_{a_k} \cdots \sigma_{a_1}(x)$

• $E \subset \Lambda_{A^t}$ shift invariant $\sigma^{-1}(E) = E$ then $x \mapsto P_x^W(E)$ fixed point of Ruelle transfer operator and if

$$h(x) := \sum_{k \geq 1} \sum_{a \in \mathcal{W}_{k,A^t}} A_{a_1 x_1} W(\sigma_{a_1}(x)) \cdots W(\sigma_{a_k} \cdots \sigma_{a_1}(x))$$

converges then $h(x)$ fixed point of Ruelle transfer operator

conclusions

These results are part of a broader program aimed at showing that:

• Techniques from operator algebra and noncommutative geometry can be applied to the study of the geometry of fractals

• fractals are good examples of “noncommutative spaces” (although they exist as ordinary spaces)

• dynamical systems methods (semi-branching fuction systems, subshifts of finite type, Perron–Frobenius and Ruelle operators) can be used to study noncommutative geometries

• noncommutative geometry can be useful in addressing questions in signal analysis, coding theory, and information