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Cuntz–Krieger Algebras and Wavelets on Fractals

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Abstract We consider representations of Cuntz–Krieger algebras on the Hilbert space of square integrable functions on the limit set, identified with a Cantor set in the unit interval. We use these representations and the associated Perron–Frobenius and Ruelle operators to construct families of wavelets on these Cantor sets.

Keywords Cuntz–Krieger · Wavelets · Fractals · Cantor set · Hilbert space · Perron–Frobenius operator · Ruelle operator

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1 Introduction

A class of representations of the Cuntz algebra O_N called permutative representations were studied and classified in [3,4,10]. Besides interest in their own right within the field of operator algebras, Cuntz algebras representations have very interesting applications to wavelets, fractals, and dynamical systems, see [3,4]. Some of these results have been extended to the more general class of Cuntz–Krieger algebras (see [20–22]), where representations of these algebras are related to Perron–Frobenius operators of certain measure space transformations. Similar representations of Cuntz–Krieger algebras were considered in the context of limit sets of Schottky groups and actions on trees in [5–7] for arithmetic applications to Arakelov geometry and p-adic Mumford curves.

In this paper we look at representations of the Cuntz–Krieger algebra having a underlying self-similarity structure. The concept of self-similarity has proved to be fundamental in mathematics as well as in diverse applications, related to the renormalization of structures on nested families of scales. In the theory of wavelets, the scales may be represented in resolutions taking the form of nested systems of linear spaces, while in C^*-algebra theory it gives rise to representations of algebras on generators and relations such as those that define the Cuntz and Cuntz–Krieger algebras.

Cuntz–Krieger algebras arise naturally from semibranching function systems on measure spaces, where the partial inverses σ_i of the coding map σ are not defined everywhere. The resulting algebra is generated by partial isometries S_i associated to the maps in the semibranching function system, and the relations between these generators involve a matrix A with entries equal to zero or one, which describes the decomposition of the domains of the σ_i as a union of ranges of other σ_j in the same family.

Conversely, a Cuntz–Krieger algebra O_A defined by generators and relations in terms of an $N \times N$-matrix A as above determines a semibranching function system on the limit set of infinite sequences in an alphabet on N letters with the admissibility condition that consecutive letters ij can appear in a word if and only if the corresponding entry in the matrix is $A_{ij} = 1$. One can identify this limit set Λ_A as a Cantor set inside the interval $[0, 1]$ by considering points whose N-adic digital expansion satisfies the admissibility condition. On this Cantor set the action of the maps σ_i become simple shifts in the N-adic expansion and the representation of O_A on the Hilbert space $L^2(\Lambda_A, \mu)$, with respect to the Hausdorff measure of the appropriate dimension, has an especially simple form, and so does also the Perron–Frobenius operator for the shift map σ, which is expressed in terms of the generators of the algebra.

The Hausdorff dimension of the limit set Λ_A is computed using the Perron–Frobenius theorem for the non-negative matrix A, which also shows that the components of the Perron–Frobenius eigenvector of A give the measures of the ranges of the maps σ_i in the normalized Hausdorff measure of dimension the Hausdorff dimension of Λ_A, which is the unique probability measure satisfying the self-similarity condition for the fractal set Λ_A.

The Perron–Frobenius eigenvector of the matrix A^t determines a fixed point for the Perron–Frobenius operator for the shift map σ on the limit set Λ_A, which in turn gives a KMS state for an associated time evolution on the algebra O_A at inverse temperature equal to the Hausdorff dimension of Λ_A.

One can construct as in [17] further measures on Λ_A, using operator valued measures and square-integrable functions of unit norm. As in the case of the Cuntz algebras, by analyzing the Fourier transforms of these measures, one sees that one can approximate them with Dirac measures supported at truncations of the N-adic expansions.

Besides the Cantor set $\Lambda_A \subset [0, 1]$, there is another fractal set that one can associate to the same matrix A, namely a Sierpinski fractal $\mathbb{S}_A$ inside the unit cube, given by points (x, y) whose digits in the N-adic expansion satisfy the condition that $A_{x_i y_i} = 1$. The Hausdorff dimension of these sets is simply computed in terms of the number of non-zero entries in A. The shifts in the N-adic expansion determine a semibranching function systems on $\mathbb{S}_A$, where, unlike in the case of Λ_A, the maps are everywhere defined, hence they give rise to an action of a Cuntz algebra or rank depending on the number of non-zero entries in A. There is a natural embedding of Λ_A into $\mathbb{S}_A$ induced by the shift map on Λ_A. The action of the Cuntz algebra determines via this embedding of Λ_A into $\mathbb{S}_A$ an action of a Cuntz–Krieger algebra.

We show how to use the representation of the algebra O_A to construct an orthonormal system of wavelets on $L^2(\Lambda_A, d\mu)$.

We then consider the Ruelle transfer operator for the shift σ on Λ_A, with non-negative valued potential W satisfying the Keane condition that the sum of the values over preimages under σ adds up to one. We show that one can construct from these measures on Λ_{A^t}, for the transpose matrix A^t, in terms of random walks where the probabilities assigned to words of a given length in the alphabet depends upon the values of the potential W. A simple example of a potential satisfying the Keane condition is given in terms of trigonometric functions.

The example of the continued fraction expansion on the Hensley Cantor sets considered in [24,25] is described as an example where the general results of this paper can be applied.

As an application we also show how the technique we described to construct wavelets on the Cantor sets Λ_A can be adapted to construct families of *graph wavelets*, using Cuntz–Krieger algebras associated to finite graphs with no sinks. Graph wavelets are considered a useful tool for spatial network traffic analysis [8].

2 Representations of Cuntz–Krieger Algebras

Let A be an $N \times N$ matrix with entries in $\{0, 1\}$. For consistency with the notation we adopt later in the paper, it is convenient to index the entries $A = (A_{ij})$ with indices $i, j \in \{0, \dots, N-1\}$ instead of $\{1, \dots, N\}$.

Recall that the Cuntz–Krieger algebra O_A associated to such a matrix A is the C^*-algebra generated by N (non-zero) partial isometries $S_0, \dots, S_{N-1}$ satisfying the relations

$$S_i^* S_i = \sum_j A_{ij} S_j S_j^* \tag{2.1}$$

and

$$\sum_{i=0}^{N-1} S_i S_i^* = 1. \tag{2.2}$$

The algebra O_A is uniquely determined by the relations (2.1) and (2.2) and it is linearly spanned by the set of $S_\alpha S_\beta^\star$ with words α and β in $\{0, \dots, N-1\}$ with possibly different lengths $|\alpha|$ and $|\beta|$, see [9].

We are especially interested here in representations of O_A as bounded operators on Hilbert spaces of the form $\mathcal{H} = L^2(X, \mu)$, for (X, μ) a measure space. The representations we are interested in, which include the cases of the arithmetic applications mentioned above, are all constructed in terms of what we refer to as a *semibranching function system*, which will be concretely realized in terms of a shift map on a Cantor-like fractal set and its partial inverses.

Definition 2.1 Consider a measure space (X, μ) and a finite family $\{\sigma_i\}_{i\in I}$, $\#I = N$, of measurable maps $\sigma_i : D_i \to X$, defined on measurable subsets $D_i \subset X$. The family $\{\sigma_i\}$ is a semibranching function system if the following holds.

(1) There exists a corresponding family $\{D_i\}_{i=1}^N$ of measurable subsets of X with the property that

$$\mu(X \smallsetminus \cup_i R_i) = 0, \quad \text{and} \quad \mu(R_i \cap R_j) = 0, \quad \text{for} \quad i \neq j, \tag{2.3}$$

where we denote by R_i the range $R_i = \sigma_i(D_i)$.

(2) There is a Radon–Nikodym derivative

$$\Phi_{\sigma_i} = \frac{d(\mu \circ \sigma_i)}{d\mu}$$

with $\Phi_{\sigma_i} > 0$, μ-almost everywhere on D_i.

A measurable map $\sigma : X \to X$ is called a coding map for the family $\{\sigma_i\}$ if $\sigma \circ \sigma_i(x) = x$ for all $x \in D_i$.

Thus, the maps of the semibranching function system are partial inverses of the coding map σ. Notice that the reverse composition $\sigma_i \circ \sigma$ is only defined when the image of x under σ lands in the domain D_i of σ_i.

Given a semibranching function system $\{\sigma_i\}_{i=0}^{N-1}$ with coding map σ, one can construct an associated family of linear operators $\{T_i\}_{i=0}^{N-1}$ acting on the Hilbert space $L^2(X, \mu)$ by setting

$$(T_i \psi)(x) = \chi_{R_i}(x) \left(\Phi_{\sigma_i}(\sigma(x))\right)^{-1/2} \psi(\sigma(x)), \tag{2.4}$$

with $\psi \in L^2(X, \mu)$, where χ_{R_i} is the characteristic function of $R_i \subset X$.

Lemma 2.2 *The adjoint of the operator T_i of (2.4) is of the form*

$$(T_i^\star \xi)(x) = \chi_{D_i}(x)(\Phi_{\sigma_i}(x))^{1/2}\xi(\sigma_i(x)). \tag{2.5}$$

Proof We have

$$\begin{aligned}\langle T_i\psi, \xi\rangle &= \int_{R_i} \overline{\left(\Phi_{\sigma_i}(\sigma(x))\right)^{-1/2}\psi(\sigma(x))}\xi(x)\, d\mu(x)\\ &= \int_{D_i} \overline{\left(\Phi_{\sigma_i}(u)\right)^{-1/2}\psi(u)}\xi(\sigma_i(u))\,\frac{d\mu\circ\sigma_i}{d\mu}\, d\mu(u)\\ &= \int_{D_i} \overline{\psi(u)}\left(\Phi_{\sigma_i}(u)\right)^{1/2}\xi(\sigma_i(u))\, d\mu(x) = \langle \psi, T_i^*\xi\rangle,\end{aligned}$$

where we used the fact that the Φ_{σ_i} are positive real valued. This gives (2.5). □

We then see easily that the operators T_i and T_i^* satisfy the following relation.

Proposition 2.3 *The operators T_i of (2.4) and their adjoints (2.5) satisfy the relations $T_iT_i^* = P_i$, where P_i is the projection given by multiplication by χ_{R_i}. This gives $\sum_i T_iT_i^* = 1$. Similarly, $T_i^*T_i = Q_i$, where Q_i is the projection given by multiplication by the characteristic function χ_{D_i}.*

Proof We write explicitly the action of the operator $T_iT_i^*$ on elements $\xi \in L^2(X, d\mu)$. We have

$$(T_iT_i^*\xi)(x) = \chi_{R_i}(x)\chi_{D_i}(\sigma(x))\Phi_{\sigma_i}^{-1/2}(\sigma(x))\Phi_{\sigma_i}^{1/2}(\sigma(x))\xi(x) = \chi_{R_i}(x)\xi(x).$$

Equivalently, we can write

$$\begin{aligned}\langle T_i^*\xi, T_i^*\xi\rangle &= \int_{D_i} \Phi_{\sigma_i}(x)\,|\xi(\sigma_i(x))|^2\, d\mu(x)\\ &= \int_{R_i} \Phi_{\sigma_i}(\sigma(u))\,|\xi(u)|^2\,\frac{d\mu\circ\sigma}{d\mu}\, d\mu(u).\end{aligned}$$

Notice then that one has

$$\frac{d\mu\circ\sigma}{d\mu}|_{R_i} = (\Phi_{\sigma_i}\circ\sigma)^{-1}, \tag{2.6}$$

so that we obtain

$$\langle T_i^*\xi, T_i^*\xi\rangle = \int_{R_i} |\xi(u)|^2\, d\mu(u) = \langle P_i\xi, P_i\xi\rangle,$$

which gives $T_i T_i^* = P_i$, the range projection on $L^2(X,\mu)$ realized by the multiplication operator by the characteristic function of the set R_i. By the assumptions (2.3) on the semibranching function system we know that the projections P_i are orthogonal and that $\sum_i P_i = 1$.

We then consider the product $T_i^* T_i$.

We have

$$\langle T_i\xi, T_i\xi\rangle = \int_{R_i} \Phi_{\sigma_i}^{-1}(\sigma(x))\,|\xi(\sigma(x))|^2\,d\mu(x).$$

If $x \in R_i$ then $\sigma(x) \in D_i$ since $\sigma\circ\sigma_i = id$ on D_i. Thus, we write the above as

$$\int_{D_i} \Phi_{\sigma_i}^{-1}(u)\,|\xi(u)|^2\left(\frac{d\mu\circ\sigma}{d\mu}\right)^{-1} d\mu(u) = \int_{D_i} |\xi(u)|^2\,d\mu(u) = \langle \chi_{D_i}\xi, \chi_{D_i}\xi\rangle,$$

where we used again (2.6). This gives $T_i^* T_i = Q_i$, where Q_i is the domain projection given by multiplication by the characteristic function χ_{D_i}. Unlike the range projections P_i, the domain projections Q_i are, in general, not orthogonal. □

When the maps σ_i are defined everywhere on X, one obtains from the operators T_i and T_i^* a representation of the Cuntz algebra O_N in the following way.

Proposition 2.4 *Let $\{\sigma_i\}$ be a semibranching function system on X, where the σ_i are defined on all of X, that is, $D_i = X$ for all $i = 0,\dots,N-1$. Then the operators T_i define a representation of the Cuntz algebra O_N on the Hilbert space $\mathcal{H} = L^2(X,\mu)$. Namely, they satisfy the relations*

$$T_i^* T_i = 1, \quad \sum_i T_i T_i^* = 1. \tag{2.7}$$

Proof Under the assumption that the semibranching function system has $D_i = X$ for all $i \in I$, we obtain from Proposition 2.3 above that the operators T_i and T_i^* of (2.4) and (2.5) satisfy $T_i^* T_i = 1$. Moreover, we know from Proposition 2.3 that $T_i T_i^* = P_i$, the range projections given by multiplication by the characteristic functions χ_{R_i}. Since these range projections are orthogonal and the union of the R_i exhausts X up to sets of measure zero, we obtain that $\sum_i T_i T_i^* = 1$. □

In the case where the maps σ_i are not defined everywhere on X, but only on smaller domains $D_i \subset X$, one can then use the operators T_i and T_i^* of (2.4) and (2.5) to construct representations of Cuntz–Krieger algebras, when the domains D_i have the property that

$$\chi_{D_i} = \sum_j A_{ij}\chi_{R_j}. \tag{2.8}$$

The examples considered in [5–7] are particular cases of this general procedure.

Proposition 2.5 *Let $\{\sigma_i\}$ be a semibranching function system on X, where the σ_i are defined on subsets $D_i \subset X$ satisfying (2.8) (possibly up to sets of measure zero). Also assume that $A_{ii} = 1$ for all $i = 0, \dots, N-1$. Then the operators T_i and T_i^* of (2.4) and (2.5) satisfy the Cuntz–Krieger relations (2.1) and (2.2), namely*

$$\sum_i T_i T_i^* = 1 \quad \text{and} \quad T_i^* T_i = \sum_j A_{ij} T_i T_i^*, \tag{2.9}$$

hence they determine a representation of the Cuntz–Krieger algebra O_A on the Hilbert space $\mathcal{H} = L^2(X, \mu)$.

Proof Using (2.5) and (2.8) we have

$$(T_i^* \xi)(x) = \sum_j A_{ij}\, \chi_{R_j}(x)\, \Phi_{\sigma_i}^{1/2}(x)\, \xi(\sigma_i(x)).$$

We then obtain

$$\begin{aligned}(T_i T_i^* \xi)(x) &= \sum_j A_{ij} \chi_{R_i}(x) \chi_{R_j}(\sigma(x))\, \Phi_{\sigma_i}^{-1/2}(\sigma(x))\, \Phi_{\sigma_i}^{1/2}(\sigma(x))\, \xi(x) \\ &= \sum_j A_{ij} \chi_{R_{ij}}(x) \xi(x) = P_i \xi(x),\end{aligned}$$

since we have from (2.8) that

$$\cup_{j:\, A_{ij}=1} R_{ij} = \{x \in R_i \mid \sigma(x) \in D_i\} = R_i.$$

Since the projections P_i are orthogonal, we then obtain

$$\sum_i T_i T_i^* = 1.$$

This gives (2.2) with $S_i = T_i$. Similarly, we have

$$T_i^* T_i = Q_i$$

from Proposition 2.3, where Q_i is the projection given by multiplication by χ_{D_i}. Using again (2.8) this then gives

$$T_i^* T_i = \sum_j A_{ij} P_j = \sum_j T_i T_i^*,$$

which gives (2.1) with $S_i = T_i$. □

We describe below an important special case of semibranching function system, which gives rise to representations of Cuntz–Krieger algebras of the type described in Proposition 2.5.

2.1 Cantor Sets and Subshifts of Finite Type

Let $\mathfrak{A}$ be an alphabet in N letters, which we can identify with the set $\{0, \dots, N-1\}$. Let Λ_A be the set of all infinite *admissible* words in the alphabet $\mathfrak{A}$, where the admissibility condition is specified by an $N \times N$ matrix A with entries in $\{0, 1\}$. Namely,

$$\Lambda_A := \{w = \{x_n\}_{n=0,1,\dots} \mid x_i \in \mathfrak{A},\ A_{x_i, x_{i+1}} = 1\}. \tag{2.10}$$

We assume further that the matrix A has the property that $A_{ii} = 1$ for each $i = 0, \dots, N-1$, that is, that arbitrarily long strings made of the same letters are allowed in the words of Λ_A.

The set Λ_A can be topologized as a Cantor set, for example by identifying it with the subset of the interval $[0, 1]$ of numbers whose base N expansion satisfies the admissibility condition. However, notice that, when we choose to view Λ_A as a subset of the interval $[0, 1]$, which is convenient in what follows, we identify the rational numbers with infinite periodic sequences rather than with a finite N-adic expansion, so as to be able to act with the shift map σ on all of Λ_A. More precisely, the map between the abstract set Λ_A and its image inside the interval $[0, 1]$ is two-to-one on the periodic sequences. These are of measure zero in the interval, so for our measure theoretic argument we ignore the distinction and use the same notation for both sets.

Let δ_A be the Hausdorff dimension of the set Λ_A, realized as a subset of the interval $[0, 1]$ in this way. We can then consider the Hilbert space $L^2(\Lambda_A, \mu_A)$, where μ_A is the Hausdorff measure in the dimension δ_A.

We consider on Λ_A the self-map given by the one-sided shift

$$\sigma : \Lambda_A \to \Lambda_A, \quad \sigma(x_0 x_1 x_2 \dots x_n \dots) = x_1 x_2 \dots x_n \dots \tag{2.11}$$

Proposition 2.6 *The shift σ is the coding map of the semibranching function system*

$$\sigma_i : D_i \to R_i, \quad \sigma_i(w) = iw, \tag{2.12}$$

where

$$D_i = \{w = \{x_k\} \in \Lambda_A \mid A_{i,x_0} = 1\} \tag{2.13}$$

and

$$R_i = \{w = \{x_k\} \in \Lambda_A \mid x_0 = i\} =: \Lambda_A(i). \tag{2.14}$$

Proof We show that the maps of (2.12) form a semibranching function system. We have

$$\Lambda_A = \cup_i R_i, \quad \text{with} \quad R_i \cap R_j = \emptyset, \quad i \neq j,$$

hence the condition (2.3) of a semibranching function system is satisfied. Moreover, the Radon–Nikodym derivative

$$\Phi_{\sigma_i} = \frac{d\mu_A \circ \sigma_i}{d\mu_A}$$

is well defined and positive, since the map σ_i on $\Lambda_A \subset [0, 1]$ is realized by contractions and translations. In fact, we can write the domain D_i of the map σ_i as

$$D_i = \cup_{j: A_{ij}=1} R_j. \tag{2.15}$$

On each R_j the map σ_i is the restriction of the map of the $I_j \subset [0, 1]$,

$$I_j = \{w \in [0, 1] \,|\, x_0 = j\},$$

where x_0 is the first digit in the N-adic expansion of $w = 0.x_0x_1x_2\ldots$, that maps it to the subset I_{jj} of elements with first and second digit equal to j composed with a translation that maps isometrically $I_{jj} \to I_{ij}$ to the interval of all numbers with first digit i and second digit j. It is then clear that the shift map (2.11) is a coding map for this semibranching function system, since on each D_i we have $\sigma \circ \sigma_i(w) = w$. □

One then sees easily that this gives a representation of the Cuntz–Krieger algebra O_A of the type described in Proposition 2.5 above.

Proposition 2.7 *The operators T_i and T_i^* of (2.4) and (2.5) acting on $\mathcal{H}_A = L^2(\Lambda_A, \mu_A)$ define a representation of O_A with generators $S_i = T_i$.*

Proof The result immediately follows from Proposition 2.5, upon noticing that the condition (2.15) is the needed relation (2.8). We are assuming $A_{ii} = 1$ for all i, so the hypothesis of Proposition 2.5 are satisfied. □

It is well known (see [9]) that the abelian C^*-algebra $C(\Lambda_A)$ sits naturally inside the Cuntz–Krieger algebra O_A as the C^*-subalgebra generated by the range projections

$$S_{x_1} \cdots S_{x_n} S_{x_n}^* \cdots S_{x_1}^*,$$

for arbitrary $x_i \in \mathfrak{A}$ and arbitrary n.

2.2 Perron–Frobenius Operator

Consider the operator $T_\sigma : L^2(X, \mu) \to L^2(X, \mu)$ that composes with the coding map $\sigma : X \to X$,

$$(T_\sigma \psi)(x) = \psi(\sigma(x)). \tag{2.16}$$

It is well known in the theory of dynamical systems that one can associate to a self map $\sigma : X \to X$ of a measure space its Perron–Frobenius operator $\mathcal{P}_\sigma$. This is defined as the adjoint of the composition (2.16) by

$$\int \overline{\psi}\, \mathcal{P}_\sigma(\xi) d\mu = \int \overline{T_\sigma(\psi)}\, \xi \, d\mu. \tag{2.17}$$

Proposition 2.8 *Let $\{\sigma_i\}_{i=1}^N$ be a semibranching function system with coding map $\sigma : X \to X$. Then the Perron–Frobenius operator $\mathcal{P}_\sigma$ is of the form*

$$(\mathcal{P}_\sigma \xi)(x) = \sum_i \chi_{D_i}(x)\, \Phi_{\sigma_i}(x)\, \xi(\sigma_i(x)). \tag{2.18}$$

Proof In the inner product of $\mathcal{H} = L^2(X, \mu)$ we find

$$\begin{aligned} \langle T\psi, \xi \rangle &= \int_X \overline{\psi(\sigma(x))} \xi(x)\, d\mu(x) \\ &= \sum_i \int_{D_i} \overline{\psi(u)} \xi(\sigma_i(u)) \frac{d(\mu \circ \sigma_i)}{d\mu}\, d\mu(u) = \left\langle \psi, \sum_i \chi_{D_i} \Phi_{\sigma_i}\ \xi \circ \sigma_i \right\rangle. \end{aligned}$$

□

Notice the similarity of the Perron–Frobenius operator $\mathcal{P}_\sigma$ to the operators T_i^* of (2.5) above. In fact, using (2.5) and Proposition 2.8, we easily get the following, which was observed already in [20].

Corollary 2.9 *Let $\{\sigma_i\}_{i=1}^N$ be a semibranching function system with coding map $\sigma : X \to X$. Then the Perron–Frobenius operator $\mathcal{P}_\sigma$ is of the form*

$$\mathcal{P}_\sigma = \sum_i \Phi_{\sigma_i}^{1/2}\, T_i^*. \tag{2.19}$$

Notice that, in some particular cases, the functions Φ_{σ_i} may be constant, in which case (2.19) gives just a linear combination of the operators T_i^*. For example, in the cases considered in [6,7] the functions Φ_{σ_i} are locally constant, while they are not in the case considered in [5].

In the case of representations as in Proposition 2.5, we can express the Perron–Frobenius operator in terms of the partial isometries S_i in the following way.

Proposition 2.10 *Let $\{\sigma_i\}$ be a semibranching function system on X, where the σ_i are defined on subsets $D_i \subset X$ satisfying (2.8) (possibly up to sets of measure zero). Then the Perron–Frobenius operator $\mathcal{P}_\sigma$ is a function of the adjoints S_i^* of the generators of the Cuntz–Krieger algebra O_A and the multiplication operators by the functions $\Phi_{\sigma_i}^{1/2}$ by*

$$\mathcal{P}_\sigma = \sum_i \Phi_{\sigma_i}^{1/2}\, S_i^*. \tag{2.20}$$

In the case where the $\Phi_{\sigma_i}^{1/2}$ are constant over D_i, the operator $\mathcal{P}_\sigma$ belongs to the algebra O_A.

Proof The hypothesis are the same as in Proposition 2.5, hence we know that the generators S_i of the Cuntz–Krieger algebra O_A in the representation on $L^2(X,\mu)$ are given by the T_i of (2.4). Then (2.19) gives (2.20). The case where the $\Phi_{\sigma_i}^{1/2}$ are constant over D_i then follows immediately from (2.20), since $\mathcal{P}_\sigma$ is then a linear combination of the S_i^*. □

To avoid having to assume that the Φ_{σ_i} are constant in the result above (although this will in fact be the case in the main example we will be considering later), one can more conveniently work with representations of the Cuntz–Krieger algebras on the Hilbert space of half-densities, analogous to the representations of the Cuntz algebra considered in [16].

Recall that the Hilbert space $\tilde{\mathcal{H}}$ of half densities consists of elements of the form $\psi(d\mu/d\lambda)^{1/2}$, where $\psi \in L^2(X,d\mu)$ and $\mu << \lambda$ with $d\mu/d\lambda$ the Radon–Nikodym derivative, which λ-a.e. positive. Elements are considered modulo λ-a.e. equivalence and the inner product is given by

$$\left\langle \psi \left(\frac{d\mu}{d\lambda}\right)^{1/2}, h\left(\frac{d\nu}{d\lambda}\right)^{1/2} \right\rangle = \int_X \overline{f} \left(\frac{d\mu}{d\lambda}\right)^{1/2} h \left(\frac{d\nu}{d\lambda}\right)^{1/2} d\lambda. \tag{2.21}$$

One often writes elements of $\tilde{\mathcal{H}}$ with the notation $\psi\sqrt{d\mu}$.

Given a semibranching function system on X satisfying (2.8), we can construct representations of the Cuntz–Krieger algebra O_A on the space of half-densities of X, in much the same way as we did in Proposition 2.7 on the space $L^2(X,d\mu)$.

Proposition 2.11 *Let $\{\sigma_i\}$ be a semibranching function system on X, where the σ_i are defined on subsets $D_i \subset X$ satisfying (2.8), possibly up to sets of measure zero. Let $\tilde{\mathcal{H}}$ be the Hilbert space of half-densities on X. Consider the operators*

$$\tilde{S}_i(\psi\sqrt{d\mu}) = \chi_{R_i}\ (\psi\circ\sigma)\ \sqrt{d\mu\circ\sigma}. \tag{2.22}$$

These operators define a representation of the Cuntz–Krieger algebra O_A.

Proof To compute the adjoints S_i^* we check

$$\begin{aligned}
\langle \tilde{S}_i(\psi\sqrt{d\mu}), \xi\sqrt{\nu}\rangle &= \int_{R_i} \overline{\psi(\sigma(x))}\xi(x) \left(\frac{d\mu(\sigma(x))}{d\lambda}\right)^{1/2} \left(\frac{d\nu(x)}{d\lambda}\right)^{1/2} d\lambda(x) \\
&= \int_{D_i} \overline{\psi(u)}\xi(\sigma_i(u)) \left(\frac{d\mu(u)}{d\lambda\circ\sigma_i}\right)^{1/2} \left(\frac{d\nu(\sigma_i(u))}{d\lambda\circ\sigma_i}\right)^{1/2} \frac{d\lambda\circ\sigma_i(u)}{d\lambda}\, d\lambda(u) \\
&= \int_{D_i} \overline{\psi(u)}\xi(\sigma_i(u)) \left(\frac{d\mu(u)}{d\lambda}\right)^{1/2} \left(\frac{d\nu(\sigma_i(u))}{d\lambda}\right)^{1/2} d\lambda(u) \\
&= \langle \psi\sqrt{d\mu}, \chi_{D_i}\, \xi\circ\sigma_i \sqrt{d\nu\circ\sigma_i}\rangle,
\end{aligned}$$

which gives

$$\tilde{S}_i^*(\xi\sqrt{d\nu}) = \chi_{D_i}\ (\xi\circ\sigma_i)\ \sqrt{d\nu\circ\sigma_i}. \tag{2.23}$$

We then check that the operators $\tilde{S}_i$ and $\tilde{S}_i^*$ satisfy the Cuntz–Krieger relations (2.1) and (2.2). We have

$$\begin{aligned}\langle\tilde{S}_i^*(\xi\sqrt{d\nu}),\,\tilde{S}_i(\xi\sqrt{d\nu})\rangle &= \int_{D_i} |\xi(\sigma_i(x))|^2\,\frac{d\nu\circ\sigma_i}{d\lambda}\,d\lambda(x)\\ &= \int_{R_i} |\xi(u)|^2\,\frac{d\nu}{d\lambda\circ\sigma}\,\frac{d\lambda\circ\sigma}{d\lambda}\,d\lambda(u) = \int_{R_i}|\xi|^2\frac{d\nu}{d\lambda}\,d\lambda,\end{aligned}$$

which shows that $\tilde{S}_i\tilde{S}_i^* = \tilde{P}_i$, the range projection given by multiplication by the characteristic function of R_i, so that the relation (2.2) is satisfied by the orthogonality of the projections $\tilde{P}_i$

$$\sum_i \tilde{S}_i\tilde{S}_i^* = 1.$$

We also have

$$\begin{aligned}\langle\tilde{S}_i(\psi\sqrt{d\mu}),\,\tilde{S}_i(\psi\sqrt{d\mu})\rangle &= \int_{R_i} |\psi(\sigma(x))|^2\frac{d\mu\circ\sigma}{d\lambda}\,d\lambda(x)\\ &= \int_{D_i} |\psi(u)|^2\,\frac{d\mu}{d\lambda\circ\sigma_i}\,\frac{d\lambda\circ\sigma_i}{d\lambda}\,d\lambda(u) = \int_{D_i}|\psi(u)|^2\frac{d\mu}{d\lambda}d\lambda(u),\end{aligned}$$

which shows that $\tilde{S}_i^*\tilde{S}_i = \tilde{Q}_i$, where $\tilde{Q}_i$ is the domain projection given by multiplication by the characteristic function of D_i. Using the relation (2.8) this then gives

$$\tilde{S}_i^*\tilde{S}_i = \sum_j A_{ij}\tilde{S}_i\tilde{S}_i^*,$$

which shows that (2.1) is satisfied. □

We then compute explicitly the Perron–Frobenius operator of the coding map $\sigma : X \to X$ acting on the space of half-densities.

Proposition 2.12 *Let $\sigma : X \to X$ be the coding map of a semibranching function system as in Proposition 2.11 above. The Perron–Frobenius operator $\tilde{\mathcal{P}}_\sigma$ on the Hilbert space of half-densities is given by*

$$\tilde{\mathcal{P}}_\sigma = \sum_i \tilde{S}_i^*, \tag{2.24}$$

where $\tilde{S}_i$ are the generators (2.22) of the representation of the Cuntz–Krieger algebra O_A on $\tilde{\mathcal{H}}$.

Proof The translation operator associated to the shift map $\sigma : X \to X$ is acting on the space of half-densities by

$$\tilde{T}_\sigma(\psi\sqrt{d\mu}) = \psi \circ \sigma \sqrt{d\mu \circ \sigma}.$$

The Perron–Frobenius operator $\tilde{\mathcal{P}}_\sigma$ on $\tilde{\mathcal{H}}$ is the adjoint

$$\langle \tilde{T}_\sigma(\psi\sqrt{d\mu}), \xi\sqrt{d\nu}\rangle = \langle \psi\sqrt{d\mu}, \tilde{\mathcal{P}}_\sigma(\xi\sqrt{d\nu})\rangle.$$

This gives

$$\begin{aligned}
&\int_X \overline{\psi(\sigma(x))} \left(\frac{d\mu\circ\sigma}{d\lambda}\right)^{1/2} \xi(x) \left(\frac{d\nu}{d\lambda}\right)^{1/2} d\lambda(x) \\
&= \sum_i \int_{R_i} \overline{\psi(u)} \left(\frac{d\mu}{d\lambda\circ\sigma_i}\right)^{1/2} \xi(\sigma_i(u)) \left(\frac{d\nu\circ\sigma_i}{d\lambda\circ\sigma_i}\right)^{1/2} \frac{d\lambda\circ\sigma_i}{d\lambda} d\lambda(u) \\
&= \sum_i \int_{R_i} \overline{\psi(u)} \left(\frac{d\mu}{d\lambda}\right)^{1/2} \xi(\sigma_i(u)) \left(\frac{d\nu\circ\sigma_i}{d\lambda}\right)^{1/2} d\lambda(u),
\end{aligned}$$

which gives

$$\tilde{\mathcal{P}}_\sigma(\xi\sqrt{d\nu}) = \sum_i \chi_{R_i}\ (\xi\circ\sigma_i)\ \sqrt{d\nu\circ\sigma_i},$$

which is (2.24). □

For example, in the case of the Cuntz–Krieger algebras considered in [5–7], where the representation comes from the action of a Schottky group Γ on its limit set, the generators S_i are associated to a symmetric set of generators $\mathfrak{A} = \{\gamma_1, \dots, \gamma_g, \gamma_1^{-1}, \dots, \gamma_g^{-1}\}$ of a Schottky group of genus g, and the matrix A of the Cuntz–Krieger algebra has $A_{ij} = 1$ for $|i-j| \neq g$ and zero otherwise, corresponding to the admissibility of the infinite sequences $w = a_0 a_1 a_2 \cdots$ of elements of $\mathfrak{A}$ parameterizing points in the limit set Λ_Γ, namely that $a_{i+1} \neq a_i^{-1}$. In this particular class of examples, the Perron–Frobenius operator of Proposition 2.12 has the form

$$\tilde{\mathcal{P}}_\sigma = \tilde{S}^*_{\gamma_1} + \tilde{S}^*_{\gamma_1^{-1}} + \cdots + \tilde{S}^*_{\gamma_g} + \tilde{S}^*_{\gamma_g^{-1}}.$$

This resembles closely a Harper operator for the group Γ, save for the important difference that the operators associated to the symmetric set of generators of Γ here are partial isometries and not unitaries as in the usual Harper operator.

2.3 Projection Valued Measures

We recall how one constructs projection-valued measures using subdivions of compact metric spaces and subdivisions of projections in Hilbert spaces. (We follow the notation and terminology of [18] for the standard material we recall.) We then show how this technique applies to the representations of Cuntz–Krieger algebras described above.

We begin by recalling the notion of partitions and N-adic systems of partitions of a metric space.

Definition 2.13 Let (X, d) be a compact metric space. For subsets $A \subset X$, define the diameter as

$$|A| := \sup\{d(x, y) \,|\, x, y \in A\}. \tag{2.25}$$

A partition $\mathcal{P}$ of X is a family $\{A(i)\}_{i\in I}$, for a (finite) index set I, with the property that

(1) $\bigcup_i A(i) = X$.
(2) $A(i) \cap A(j) = \emptyset$, for $i \neq j$.

For a given $N \geq 2$, an N-adic system of partitions of X is a family (indexed by $k \in \mathbb{N}$) of partitions $\mathcal{P}_k$ of X into Borel subsets $A_k(a)$, indexed by elements of $\mathfrak{A}^k$, where $\mathfrak{A} = \{0, \ldots, N-1\}$ is the given alphabet on N letters, with the properties:

(1) $|A_k(a)| = O(N^{-ck})$, for some $c > 0$.
(2) Every $A_{k+1}(b)$, with $b \in \mathfrak{A}^{k+1}$, is contained in some $A_k(a)$, for some $a \in \mathfrak{A}^k$.

We then recall the equally well known notion of partitions of projections in Hilbert spaces.

Definition 2.14 Let $\mathcal{H}$ be a complex separable Hilbert space. A partition of projections in $\mathcal{H}$ is a collection $\{P(i)\}_{i\in I}$ of projections $P(i) = P(i)^* = P(i)^2$ such that

(1) $P(i)P(j) = 0$, for $i \neq j$.
(2) $\sum_i P(i) = 1$.

An N-adic system of partitions of $\mathcal{H}$ into projections is a family of partitions into projections $\{P_k(a)\}$ indexed by $a \in \mathfrak{A}^k$ such that, for every $P_{k+1}(a)$, there is some $b \in \mathfrak{A}^k$ with $P_k(b)P_{k+1}(a) = P_{k+1}(a)$.

We also recall the notion of operator valued measure.

Definition 2.15 Denote by $\mathcal{B}(X)$ the collection of Borel subsets of a compact metric space X. A positive operator-valued function $E : \mathcal{B}(X) \to \mathcal{L}(\mathcal{H})$ defined on $\mathcal{B}(X)$ with values in bounded linear operators on a Hilbert space $\mathcal{H}$ is called a σ additive measure if, given a sequence $B_1, B_2 \ldots$, in $\mathcal{B}(X)$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$, one has

$$E\left(\bigcup_i B_i\right) = \sum_i E(B_i). \tag{2.26}$$

An orthogonal projection valued measure is a positive operator-valued measure as above satisfying:

(1) $E(B) = E(B)^* = E(B)^2$, for all $B \in \mathcal{B}(X)$.
(2) $E(B_1)E(B_2) = 0$ when $B_1 \cap B_2 = \emptyset$.
(3) $E(X) = 1$, the identity on $\mathcal{H}$.

Note that the values $E(B_i)$ in (2.26) are positive operators, so we take the summation on the right hand side of (2.26) to be convergent in the strong operator topology.

We are interested here in a particular construction of N-adic partitions, for the metric Cantor set Λ_A defined in (2.10) above. As above, we consider the alphabet $\mathfrak{A} = \{0, \dots, N-1\}$. For any $k \in \mathbb{N}$, we denote by $\mathcal{W}_{k,A} \subset \mathfrak{A}^k$ the finite set of all admissible words of length k in the alphabet $\mathfrak{A}$,

$$\mathcal{W}_{k,A} = \{a = (a_1, \dots, a_k) \in \mathfrak{A}^k \mid A_{a_i,a_{i+1}} = 1,\ i = 1, \dots, k\}. \tag{2.27}$$

We also denote by $\Lambda_{k,A}(a)$ the clopen subset of the Cantor set Λ_A given by all words that start with a given $a \in \mathcal{W}_{k,A}$,

$$\Lambda_{k,A}(a) = \{w = (w_1, w_2, \dots, w_n, \dots) \in \Lambda_A \mid (w_1, \dots, w_k) = a\}. \tag{2.28}$$

We then have the following partition and corresponding operator valued measure.

Proposition 2.16 *The subsets $\Lambda_{k,A}(a)$ of (2.28) define an N-adic system of partitions for Λ_A. There is a corresponding N-adic system of projections $P_k(a)$ on the Hilbert space $\mathcal{H} = L^2(\Lambda_A, \mu_A)$ and an orthogonal projection valued measure E on $\mathcal{B}(\Lambda_A)$ satisfying*

$$E(\Lambda_{k,A}(a)) = P_k(a), \tag{2.29}$$

for all $k \in \mathbb{N}$ and for all $a \in \mathcal{W}_{k,A}$.

Proof To see that the $\Lambda_{k,A}(a)$ form an N-adic system of partitions, notice that, when we identify Λ_A with the subset of $[0, 1]$ of numbers with admissible N-adic digital expansion and we measure diameters in the Euclidean distance on $[0, 1]$, we see that the set $\Lambda_{k,A}(a)$, which consists of such numbers with fixed first k digits in the N-adic expansion have

$$|\Lambda_{k,A}(a)| \leq N^{-k}, \tag{2.30}$$

since the sets of all numbers with fixed k digits in the N-adic expansion are intervals of length N^{-k}. Moreover, by construction we have inclusions

$$\Lambda_{k,A}(a_1, \dots, a_k) \subset \Lambda_{k-1,A}(a_1, \dots, a_{k-1}). \tag{2.31}$$

We also have, for fixed k,

$$\Lambda_{k,A}(a) \cap \Lambda_{k,A}(b) = \emptyset, \quad \text{for } a \neq b \in \mathcal{W}_{k,A},$$

and

$$\cup_{a\in\mathcal{W}_{k,A}} \Lambda_{k,A}(a) = \Lambda_A.$$

Thus, we have an N-adic system of partitions.

One knows from [9] that there is an $*$-isomorphism between the C^*-algebra of continuous functions $C(\Lambda_A)$ and the maximal abelian subalgebra of the Cuntz–Krieger algebra O_A generated by all the range projections

$$P_k(a) = S_{a_1}\cdots S_{a_k} S_{a_k}^* \cdots S_{a_1}^*. \tag{2.32}$$

We show that the $P_k(a)$ define an N-adic system of projections on the Hilbert space $\mathcal{H} = L^2(\Lambda_A, \mu_A)$. In the representation of O_A described in Proposition 2.5, the operator $P_k(a)$ acts as the projection given by multiplication by the characteristic function of the set $\Lambda_{k,A}(a)$.

Since the $\Lambda_{k,A}(a)$ form an N-adic system of partitions, in particular, as we have seen above, there are inclusions (2.31). These imply that the corresponding projections satisfy

$$P_{k-1}(a_1,\dots,a_{k-1})\, P_k(a_1,\dots,a_k) = P_k(a_1,\dots,a_k).$$

More precisely, one can see by writing as in (2.32) and using (2.2) that

$$\begin{aligned}\sum_{a_k\in\mathfrak{A}} P_k(a_1,\dots,a_k) &= \sum_{a_k\in\mathfrak{A}} S_{a_1}\cdots S_{a_k} S_{a_k}^* \cdots S_{a_1}^* \\ &= S_{a_1}\cdots S_{a_{k-1}} \left(\sum_{a_k\in\mathfrak{A}} S_{a_k} S_{a_k}^* \right) S_{a_{k-1}}^* \cdots S_{a_1}^* \\ &= S_{a_1}\cdots S_{a_{k-1}} S_{a_{k-1}}^* \cdots S_{a_1}^* = P_{k-1}(a_1,\dots,a_{k-1}).\end{aligned}$$

For every $k \in \mathbf{Z}_+$, let us denote by $\mathcal{U}_k$ the finite dimensional subalgebra of $C(\Lambda_A)$ spanned by the finite linear combinations

$$\sum_{a\in\mathcal{W}_{k,A}} c_a\, \chi_{\Lambda_{k,A}(a)}.$$

The inclusions (2.31) determine embeddings $\mathcal{U}_{k-1} \to \mathcal{U}_k$ and the bound (2.30) on the diameters implies that every function in $C(\Lambda_A)$ can be uniformly approximated with a sequence of functions in $\mathcal{U} = \varinjlim_k \mathcal{U}_k$. Thus, the homomorphism

$$\pi : \sum_{a\in\mathcal{W}_{k,A}} c_a\, \chi_{\Lambda_{k,A}(a)} \mapsto \sum_{a\in\mathcal{W}_{k,A}} c_a\, P_k(a) \tag{2.33}$$

extends, by a standard argument from function theory, from $C(\Lambda_A)$ to all the Baire functions on Λ_A.

It makes sense then to define an operator valued measure by setting

$$E(B) := \pi(\chi_B), \tag{2.34}$$

where we still denote as π the extension above. It follows that $E(\cdot)$ satisfies the properties of Definition 2.15 and is countably additive. It also satisfies $E(\Lambda_{k,A}(a)) = P_k(a)$, for every $k \in \mathbf{Z}_+$ and for all $a \in \mathcal{W}_{k,A}$. □

2.4 Hausdorff Dimension

We consider again the space Λ_A of numbers in the interval $[0, 1]$ whose N-adic expansion is admissible according to the matrix A, that is, $x = 0.a_0a_1 \cdots a_n \cdots$ with $A_{a_i,a_{i+1}} = 1$.

We know that in this case the maps σ_i are defined on domains $D_i \subset \Lambda_A$ satisfying $D_i = \cup_{j:A_{ij}=1} R_j$, where $R_j \subset \Lambda_A$ is the range of σ_j, with $\Lambda_A = \cup_j R_j$ and $R_i \cap R_j = \emptyset$ when $i \neq j$. We then have the following properties.

Theorem 2.17 *Assume that the non-negative matrix A is irreducible, that is, there exists a power A^n for which all entries are positive. Let δ_A be the Hausdorff dimension of Λ_A and $\mu_A = \mu_{\mathcal{H},\delta_A}$ the corresponding Hausdorff measure.*

(1) *On the sets $D_i \subset \Lambda_A$, the Radon–Nikodym derivatives are constant and equal to*

$$\Phi_{\sigma_i} = \frac{d\mu \circ \sigma_i}{d\mu} = N^{-\delta_A}. \tag{2.35}$$

(2) *The Hausdorff measure $\mu = \mu_A$ on Λ_A satisfies*

$$\mu(R_i) = p_i, \tag{2.36}$$

where $p = (p_i)_{i=0,\dots,N-1}$ is the Perron–Frobenius eigenvector of the matrix A,

$$\sum_j A_{ij}\, p_j = r(A)\, p_i, \tag{2.37}$$

with eigenvalue the spectral radius $r(A)$, and normalized to have $\sum_i p_i = 1$.

(3) *The Hausdorff dimension of Λ_A is given by*

$$\delta_A = \dim_H(\Lambda_A) = \frac{\log r(A)}{\log N}, \tag{2.38}$$

with $r(A)$ the spectral radius of the matrix A.

(4) *The measure μ satisfies the self-similarity condition*

$$\mu = N^{-\delta_A} \sum_{k=0}^{n-1} \mu \circ \sigma_k^{-1}, \tag{2.39}$$

where $\mu(\sigma_k^{-1}(E)) = \mu(\{x \in \Lambda_A \mid \sigma_k(x) \in E\})$.

Proof (1) The maps σ_i act as the restrictions to the set D_i of the linear maps

$$\sigma_i(x) = \frac{x+i}{N} \tag{2.40}$$

defined on the interval $[0, 1]$. Thus, we see directly that the Radon–Nikodym derivative of the Hausdorff measure $\mu_{\mathcal{H},s}$ will give

$$\frac{d\mu_{\mathcal{H},s} \circ \sigma_i}{d\mu_{\mathcal{H},s}} = N^{-s}.$$

In particular for $s = \delta_A = \dim_H(\Lambda_A)$ this gives (2.35).

(2) We first show that setting

$$\nu(R_i) = p_i, \tag{2.41}$$

with p the normalized Perron–Frobenius eigenvector of A, defines a probability measure on Λ_A.

The Perron–Frobenius theorem for the matrix A shows that, if $r(A)$ denotes the spectral radius of A, then $r(A)$ is an eigenvalue which has an eigenvector $p = (p_i)$ with non-negative entries. We can normalize it so that $\sum_i p_i = 1$. Setting $\nu(R_i) = p_i$ defines a measure on Λ_A. In fact, it suffices to see that we can define $\nu(\Lambda_{k,A}(a))$ compatibly, for all $a \in \mathcal{W}_{k,A}$. We set

$$\nu(\Lambda_{k,A}(a)) = r(A)^{-k} p_{a_k}, \tag{2.42}$$

where $a = (a_1, \ldots, a_k) \in \mathcal{W}_{k,A}$. To see that (2.42) consistently defines a measure on Λ_A we need to check that

$$\nu(\Lambda_{k,A}(a)) = \sum_{j=0}^{N-1} A_{a_k j}\, \nu(\Lambda_{k+1,A}(aj)). \tag{2.43}$$

We have

$$\sum_j A_{a_k j}\, \nu(\Lambda_{k+1,A}(aj)) = \sum_j A_{a_k j} r(A)^{-k-1} p_j = r(A)^{-k} p_{a_k} = \nu(\Lambda_{k,A}(a)),$$

where we used the Perron–Frobenius relation

$$p_{a_k} = r(A)^{-1} \sum_j A_{a_k j}\, p_j.$$

The measure ν thus satisfies the self-similarity property

$$\nu = r(A)^{-1} \sum_{j=0}^{n-1} \nu \circ \sigma_j^{-1}. \tag{2.44}$$

Indeed, it suffices to check it on sets of the form $E = \Lambda_{k,A}(a)$, for which $\sigma_j^{-1}(\Lambda_{k,A}(a))$ is non-empty for $a_1 = j$, in which case it is $\Lambda_{k-1,A}(\sigma(a))$. Then we have

$$\nu(\Lambda_{k,A}(jb)) = r(A)^{-1} r(A)^{-|b|} p_{b_k} = r(A)^{-1} \nu(\Lambda_{k-1,A}(\sigma(a))),$$

which gives (2.44).

We then compare this with the Hausdorff measure $\mu = \mu_A$. This satisfies

$$\mu(R_i) = N^{-\delta_A} \sum_j A_{ij} \mu(R_j). \tag{2.45}$$

In fact, this follows simply from the fact shown in (1) that the Radon–Nikodym derivatives are constant,

$$\Phi_{\sigma_i} = \frac{d\mu \circ \sigma_i}{d\mu} = N^{-\delta},$$

which gives

$$\mu(R_i) = \int_{D_i} \frac{d\mu \circ \sigma_i}{d\mu} d\mu = N^{-\delta_A} \mu(D_i) = N^{-\delta_A} \sum_j A_{ij} \mu(R_j).$$

Note that it then follows that the measure μ also satisfies

$$\mu(\Lambda_{k,A}(a)) = N^{-k\delta_A} \mu(R_{a_k}), \tag{2.46}$$

for $a = (a_1, \ldots, a_k)$. This follows directly from (2.45) and the fact that

$$\mu(\Lambda_{k,A}(a)) = \sum_j A_{a_k j} \mu(\Lambda_{k+1,A}(aj)).$$

Notice then that (2.45) is saying that the vector $q = (q_i)$ with $q_i = \mu(R_i)$ is also an eigenvector of the matrix A, with eigenvalue $N^{-\delta_A} \leq r(A)$, with the normalization $\sum_i q_i = 1$.

Under the assumption that the non-negative matrix A is irreducible, the Perron–Frobenius theorem for A ensures that the eigenvalue $r(A)$ is simple and that if $q = (q_i)$ is another eigenvector, $Aq = \lambda q$ with $q_i \geq 0$, then $\lambda = r(A)$ and q is a scalar multiple of p. Since both vectors are normalized, this implies that

$$N^{\delta_A} = r(A) \quad \text{and} \quad \nu(R_i) = p_i = q_i = \mu(R_i). \tag{2.47}$$

By (2.46) and (2.42), this implies that the measures μ and ν agree.

(3) then follows immediately from $r(A) = N^{\delta_A}$ and (4) is just the self-similarity (2.44). □

As a particular case, if the matrix A has the property that the value $\alpha = \sum_j A_{ij}$ is the same for all $i = 0, \dots, N-1$, then one has uniform probability for all the R_i, equal to $\mu(R_i) = 1/N$, and the set Λ_A has then Hausdorff dimension $\delta_A = \log(\alpha)/\log(N)$.

We return to consider now in particular the representation of the Cuntz–Krieger algebra O_A on the space $L^2(\Lambda_A, d\mu_A)$ as in Sect. 2.1.

Corollary 2.18 *The Perron–Frobenius operator $\mathcal{P}_\sigma$ on the Hilbert space $L^2(\Lambda_A, d\mu_A)$, with $\mu_A = \mu_{\mathcal{H},\delta_A}$ the Hausdorff measure with $\delta_A = \dim_H(\Lambda_A)$, satisfies*

$$\mathcal{P}_\sigma = N^{-\delta_A/2} \sum_i S_i^*. \tag{2.48}$$

Proof As we have seen in Proposition 2.17, in this case the Φ_{σ_i} are locally constant and equal to $N^{-\delta_A}$, with δ_A the Hausdorff dimension, which in turn is given in terms of the spectral radius of A. Then we have from Proposition 2.10 that the Perron–Frobenius operator $\mathcal{P}_\sigma$ on $L^2(\Lambda_A, d\mu_A)$ is simply given by (2.48), where the S_i generate the representation of the Cuntz–Krieger algebra on $L^2(\Lambda_A, d\mu_A)$. □

We then see that one can use the result of Theorem 2.17 to construct a fixed point for the Perron–Frobenius operator $\mathcal{P}_\sigma$.

Proposition 2.19 *Assume that the matrix A is irreducible, and let ω be the Perron–Frobenius eigenvector for A^t. Then $f = \sum_i \omega_i \chi_{R_i}$ is a fixed point of the Perron–Frobenius operator $\mathcal{P}_\sigma$.*

Proof Let ω be the Perron–Frobenius eigenvector

$$A^t \omega = r(A)\omega.$$

The Perron–Frobenius operator $\mathcal{P}_\sigma$ acting on the function $f = \sum_i \omega_i \chi_{R_i}$ gives

$$\mathcal{P}_\sigma(f) = N^{-\delta_A} \sum_i \chi_{D_i} f \circ \sigma_i,$$

by Corollary 2.18. We have

$$\chi_{R_k} \circ \sigma_i = \delta_{ik} \chi_{D_i},$$

which gives

$$\mathcal{P}_\sigma(f) = N^{-\delta_A} \sum_i \omega_i \chi_{D_i} = N^{-\delta_A} \sum_{ij} \omega_i A_{ij} \chi_{R_j}$$

from (2.8). Using then $A^t\omega = r(A)\omega$ we obtain

$$\mathcal{P}_\sigma(f) = N^{-\delta_A} r(A) \sum_i \omega_i \chi_{R_i} = \sum_i \omega_i \chi_{R_i} = f,$$

where we used the fact that $r(A) = N^{\delta_A}$ as in (2.47). □

There is a well known relation for Cuntz–Krieger algebras between the fixed points of the dual Perron–Frobenius (or Ruelle transfer operator) acting on measures and KMS states with respect to associated time evolutions, see [22]. We discuss the more general case of the Ruelle transfer operators later, but we comment here on the case that follows directly from Theorem 2.17.

Corollary 2.20 *On the Cuntz–Krieger algebra O_A consider the time evolution defined by setting*

$$\sigma_t(S_i) = N^{it} S_i. \tag{2.49}$$

The measure $\mu = \mu_A$ on Λ_A defines a KMS state for the system (O_A, σ_t) at inverse temperature $\beta = \delta_A$.

Proof We define a state φ on O_A associated to the measure μ by setting

$$\varphi(S_a S_b^*) = \begin{cases} 0 & a \neq b, \\ \mu(\Lambda_{k,A}(a)) & a = b \in \mathcal{W}_{k,A}. \end{cases} \tag{2.50}$$

We use here the fact that all elements in O_A can be approximated by linear combinations of elements of the form $S_a S_b^*$. We then need to check that the state φ satisfies the KMS condition at inverse temperature $\beta = \delta_A$ for the time evolution (2.49). Because of the form of the state (2.50), and the fact that the measure μ satisfies (2.46), it suffices to check that

$$\varphi(S_i^* S_i) = N^{\beta} \varphi(S_i S_i^*).$$

This follows since we have

$$\varphi(S_i^* S_i) = \sum_j A_{ij} \varphi(S_j S_j^*) = \sum_j A_{ij} \mu(R_j) = N^{\delta_A} \mu(R_i) = N^{\delta_A} \varphi(S_i S_i^*),$$

using the fact that $p = (p_i)$ with $p_i = \mu(R_i)$ is the Perron–Frobenius eigenvector of the matrix A. □

2.5 Real Valued Measures and Fourier Transforms

Given an element $f \in \mathcal{H}$ with norm $\|f\| = 1$, one can define a real valued measure on $\Lambda_A \subset [0, 1]$ by setting

$$\mu_f(B) := \langle f, E(B) f \rangle, \tag{2.51}$$

with $E(B)$ an operator valued measure as in Sect. 2.3.

Since each such μ_f is a compactly supported measure on the real line, it makes sense to consider its Fourier transform

$$\widehat{\mu_f}(t) := \int e^{itx} \, d\mu_f(x). \tag{2.52}$$

We then have the following result, which is analogous to the case of the Cuntz algebras O_n discussed in [17].

Proposition 2.21 *For every function $f \in \mathcal{H} = L^2(\Lambda_A, d\mu_A)$ with $\|f\| = 1$, the measure $\mu_f(E) = \langle f, P(E) f \rangle$ satisfies*

$$\sum_{k=0}^{N-1} \int_{\Lambda_A} \psi \circ \sigma_k \, d\mu_{S_k^* f} = \int_{\Lambda_A} \psi \, d\mu_f . \tag{2.53}$$

The Fourier transform $\widehat{\mu}_f(t)$ satisfies

$$\widehat{\mu}_f(t) = \sum_{k=0}^{N-1} e^{\frac{itk}{N}} \widehat{\mu_{S_k^* f}} \left(\frac{t}{N} \right). \tag{2.54}$$

Proof We have

$$\sum_k \int_{\Lambda_A} \psi \circ \sigma_k \, d\mu_{S_k^* f} = \sum_k \langle S_k^* f, \pi(\chi_{D_k} \, \psi \circ \sigma_k) S_k^* f \rangle, \tag{2.55}$$

where π denotes the embedding $\pi : C(\Lambda_A) \hookrightarrow O_A$, as in (2.33), which realizes $C(\Lambda_A)$ as an abelian $*$-subalgebra of O_A, with $\pi(\chi_{\Lambda_{k,A}(a)}) = S_a S_a^*$. In the algebra O_A we have the relations

$$\begin{aligned} \pi(f) S_k &= S_k \, \pi(\chi_{D_k} \, f \circ \sigma_k), \\ S_k \pi(f) &= \pi(f \circ \sigma) \, S_k, \\ \pi(f) S_k^* &= S_k^* \, \pi(f \circ \sigma), \\ S_k^* \pi(f) &= \pi(\chi_{D_k} \, f \circ \sigma_k) \, S_k^* . \end{aligned} \tag{2.56}$$

Thus, we have $\pi(\chi_{D_k} \, \psi \circ \sigma_k) S_k^* = S_k^* \pi(\psi)$ and we write (2.55) as

$$\sum_k \langle f, S_k S_k^* \pi(\psi) f \rangle = \sum_k \langle f, \pi(\chi_{R_k} \, \psi) f \rangle = \sum_k \int_{R_k} \psi \, d\mu_f , \tag{2.57}$$

which gives (2.53). We then proceed as in [17], and observe that (2.53), applied to $\psi(x) = e^{itx}$, gives

$$\sum_k \int e^{it \frac{x+k}{N}} \, d\mu_{S_k^* f}(x) = \int e^{itx} \, d\mu_f(x),$$

which gives (2.54). □

We can equivalently see (2.53) as an immediate consequence of (2.39), since we have

$$\begin{aligned}\int \psi d\mu_f &= \langle f, \pi(\psi) f\rangle = \int \psi |f|^2 d\mu \\ &= N^{-\delta} \sum_j \langle \chi_{D_j} f\circ\sigma_j, \pi(\psi\circ\sigma_j)\chi_{D_j} f\circ\sigma_j\rangle \\ &= \sum_j \langle S_j^* f, \pi(\psi\circ\sigma_j) S_j^* f\rangle = \int \psi\circ\sigma_j d\mu_{S_j^* f},\end{aligned}$$

with $S_j^* f = N^{-\delta/2}\chi_{D_j} f\circ\sigma_j$.

Iterating the relation (2.54) one obtains

$$\hat{\mu}_f(t) = \sum_{a\in\mathcal{W}_{k,A}} e^{itx(a)} \hat{\mu}_{S_a^* f}\left(\frac{t}{N^k}\right), \tag{2.58}$$

where for $a = (a_1, \ldots, a_k) \in \mathcal{W}_{k,A}$ we denote by $x(a)$ the expression

$$x(a) = \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots + \frac{a_k}{N^k}. \tag{2.59}$$

As in [17], we then obtain an approximation of the measure μ_f with a family of combinations of Dirac measures in the following way.

Corollary 2.22 *Let $\mu_f^{(k)}$ denote the measure*

$$\mu_f^{(k)}(E) = \sum_{a\in\mathcal{W}_{k,A}} \|S_a^* f\|^2 \delta_a(E), \tag{2.60}$$

where δ_a is the Dirac measure supported at the rational point $x(a)$ in Λ_A whose terminating N-adic expansion is of the form (2.59), for

$$a = (a_1, \ldots, a_k) \in \mathcal{W}_{k,A}.$$

The measures $\mu_f^{(k)}$ weakly converge to μ_f, when considered as functionals on the space of integrable functions ψ on the real line whose Fourier transform satisfies

$$\int |t\hat{\psi}(t)| dt < \infty. \tag{2.61}$$

Proof We show that, for all functions ψ with (2.61), we have

$$\lim_{k\to\infty} \int_{\Lambda_A} \psi\, d\mu_f^{(k)} = \int_{\Lambda_A} \psi\, d\mu_f.$$

Passing to Fourier transforms, we have

$$\int \psi \, d\mu_f^{(k)} - \int \psi \, d\mu_f = \int \hat{\psi}(t)(\hat{\mu}_f^{(k)}(t) - \hat{\mu}_f(t))\frac{dt}{2\pi}.$$

The Fourier transform of $\mu_f^{(k)}$ is clearly of the form

$$\hat{\mu}_f^{(k)}(t) = \sum_{a \in \mathcal{W}_{k,A}} e^{itx(a)} \| S_a^* f \|^2,$$

with $x(a)$ as in (2.59), and one can estimate as in [17]

$$|\hat{\mu}_f(t) - \hat{\mu}_f^{(k)}(t)| \leq | \, t | N^{-k}.$$

This gives

$$\left| \int \psi \, d\mu_f^{(k)} - \int \psi \, d\mu_f \right| \leq \frac{N^{-k}}{2\pi} \int |t\hat{\psi}(t)| \, dt,$$

which gives the weak convergence $\mu_f^{(k)} \to \mu_f$. □

2.6 Sierpinski Fractals

There is another fractal object, besides the limit set Λ_A, that is naturally associated to an $N \times N$-matrix A with entries in $\{0, 1\}$. This is a Sierpinksi fractal constructed in the following way. Consider the square $\mathbb{S} = [0, 1] \times [0, 1]$ and write points $(x, y) \in \mathbb{S}$ in terms of the N-adic expansion

$$(x, y) = \left(\frac{x_1}{N} + \frac{x_2}{N^2} + \cdots + \frac{x_k}{N^k} + \cdots , \frac{y_1}{N} + \frac{y_2}{N^2} + \cdots + \frac{y_k}{N^k} + \cdots \right),$$

with $(x_i, y_i) \in \{0, \ldots, N-1\} \times \{0, \ldots, N-1\} = \mathfrak{A}^2$, for all $i \geq 1$. We then consider the subset $\mathbb{S}_A \subset \mathbb{S}$ given by

$$\mathbb{S}_A = \{(x, y) \in S \mid A_{x_i, y_i} = 1, \ \forall i \geq 1\}. \tag{2.62}$$

This is a Sierpinski fractal whose iterative construction starts by subdividing the unit square $\mathbb{S}$ into the N^2 subsquares of size N^{-2} consisting of points (x, y) with first digits of the N-adic expansion equal to given $(i, j) \in \mathfrak{A}^2$. One then keeps among these only those for which $A_{ij} = 1$. The procedure is then iterated by subdividing each of the remaining squares into N^2 subsquares of size N^{-4} and keeping only those for which the same condition $A_{ij} = 1$ is satisfied, and so on. At each step a square is of size

N^{-2k} is replaced by D squares of size $N^{-2(k+1)}$, where

$$D = \sum_{i=0}^{N-1} d_i, \quad \text{with} \quad d_i = \#\{j \mid A_{ij} = 1\}. \tag{2.63}$$

These satisfy $d_i \leq N$ and $D \leq N^2$. Thus, the Hausdorff dimension of the Sierpinski fractal $\mathbb{S}_A$ is simply

$$\dim_H(\mathbb{S}_A) = \frac{\log D}{2 \log N}. \tag{2.64}$$

One can then consider maps $\tau_{(i,j)} : \mathbb{S}_A \to \mathbb{S}_A$, for (i, j) satisfying $A_{ij} = 1$, given by

$$\tau_{(i,j)}(x, y) = (\tau_i(x), \tau_j(y)) = \left(\frac{x + i}{N}, \frac{x + j}{N} \right). \tag{2.65}$$

Notice how, unlike the σ_i acting on Λ_A that we considered before, here the $\tau_{(i,j)}$ are *everywhere defined* on $\mathbb{S}_A$. Since we are only considering such maps for pairs (i, j) with $A_{ij} = 1$, it is clear that the image $(\tau_i(x), \tau_j(y))$ is still a point in $\mathbb{S}_A$. The corresponding coding map $\tau : \mathbb{S}_A \to \mathbb{S}_A$ is given by

$$\tau(x, y) = (\tau(x), \tau(y)) = (0.x_2 \cdots x_k \cdots , 0.y_2 \cdots y_k \cdots),$$

for $(x, y) = (0.x_1 x_2 \cdots x_k \cdots , 0.y_1 y_2 \cdots y_k \cdots)$.

Lemma 2.23 *The semibranching function system* $\{\tau_{(i,j)}\}$ *for* $(i, j) \in \mathfrak{A}^2$ *with* $A_{ij} = 1$ *determines a representation of the Cuntz algebra* O_D *on the Hilbert space* $L^2(\mathbb{S}_A, \mu)$*, with* μ *the Hausdorff measure of dimension* $\delta = \dim_H(\mathbb{S}_A)$ *as in* (2.64)*.*

Proof Let $\Phi_{(i,j)}$ denote the Radon–Nikodym derivative of the measure μ with respect to composition by $\tau_{(i,j)}$. Since $\tau_{(i,j)}$ is of the form (2.65), we have

$$\Phi_{(i,j)}(x, y) = \frac{d\mu \circ \tau_{(i,j)}}{d\mu} = N^{-2\delta} = \frac{1}{D}. \tag{2.66}$$

We consider the operators $S_{(i,j)}$ and $S^*_{(i,j)}$ defined as in the general case of a semibranching function system in the form

$$S_{(i,j)} f = \chi_{R_{(i,j)}} \cdot (\Phi_{(i,j)} \circ \tau)^{-1/2} \cdot f \circ \tau, \tag{2.67}$$

with $R_{i,j} \subset \mathbb{S}_A$ the range of $\tau_{(i,j)}$. The adjoint $S^*_{(i,j)}$ in the inner product of $L^2(\mathbb{S}_A, \mu)$ is given by

$$\langle S_{(i,j)} f, h \rangle = N^{\delta} \int_{R_{(i,j)}} f \circ \tau \, h \, d\mu = N^{\delta} \int_{\mathbb{S}_A} f \, h \circ \tau_{(i,j)} \, \Phi_{ij} \, d\mu,$$

so that we get

$$S^*_{(i,j)}h = \Phi_{ij}^{1/2}\, h \circ \tau_{(i,j)} = N^{-\delta}\, h \circ \tau_{(i,j)}. \tag{2.68}$$

Thus, one sees that

$$S^*_{(i,j)}S_{(i,j)} = 1, \quad \text{and} \quad \sum_{(i,j):A_{ij}=1} S_{(i,j)}S^*_{(i,j)} = 1, \tag{2.69}$$

since $S_{(i,j)}S^*_{(i,j)}$ is the range projection given by multiplication by $\chi_{R_{(i,j)}}$. Thus, the $S_{(i,j)}$ generate a representation of the Cuntz algebra O_D on $L^2(\mathbb{S}_A, \mu)$. □

In particular, this means that one can apply to the Sierpinski set $\mathbb{S}_A$ all the techniques for constructions of wavelets on fractals from representations of Cuntz algebras developed, for instance, in [3,4,12–14,16–19,26–28], etc.

Notice then that we can embed the limit set Λ_A inside the Sierpinski fractal $\mathbb{S}_A$ in the following way.

Lemma 2.24 *The map*

$$\Xi : \Lambda_A \to \mathbb{S}_A, \quad \Xi(x) = (x, \sigma(x)). \tag{2.70}$$

gives an embedding $\Lambda_A \hookrightarrow \mathbb{S}_A$.

Proof A point $x = (x_1 x_2 \cdots x_n \cdots)$ in Λ_A satisfies $A_{x_i x_{i+1}} = 1$. This means that the point

$$(x, y) = (0.x_1 x_2 \cdots x_n \cdots, 0.x_2 x_3 \cdots x_{n+1} \cdots) = (x, \sigma(x))$$

satisfies $A_{x_i, y_i} = A_{x_i x_{i+1}} = 1$ for all $i \geq 1$, hence it is a point in $\mathbb{S}_A$. The map Ξ is clearly injective since it is the identity on the first coordinate. It is continuous since the preimage of a clopen set $\mathbb{S}_A(i_1 \cdots i_k, j_i \cdots j_k)$ of $\mathbb{S}_A$, given by numbers with fixed first k digits of the N-adic expansion, is either empty, or else, when $j_r = i_{r+1}$ for $r = 1, \ldots, k-1$, it is equal to the clopen set $\Lambda_A(i_1, \ldots, i_k, j_k)$ of Λ_A. □

One can then use this embedding together with the representation of the algebra O_D on $L^2(\mathbb{S}_A, \mu)$ to obtain an induced action of a Cuntz–Krieger algebra.

Proposition 2.25 *The maps $\tau_{i,j}$ restricts to maps defined on domains $D_{i,j} \subset \Xi(\Lambda_A)$. These determine a semibranching function system on $\Xi(\Lambda_A)$ which gives rise to a representation of the algebra $O_{\tilde{A}}$, where the $D \times D$-matrix $\tilde{A}$ is given by*

$$\tilde{A}_{(i,j),(\ell,k)} = \delta_{j,\ell}\, A_{jk}. \tag{2.71}$$

Proof The condition that $\tau_{(i,j)}(x, \sigma(x)) = (\tau_i(x), \tau_j(\sigma(x))$ is in $\Xi(\Lambda_A)$ determines the domain $D_{(i,j)} \subset \Xi(\Lambda_A)$ to be

$$D_{(i,j)} = \{(x, \sigma(x)) \in \Xi(\Lambda_A) \,|\, \sigma_j\sigma(x) = \sigma\sigma_i(x)\} = \Xi(R_j). \tag{2.72}$$

In fact, the condition that $A_{ij} = 1$ implies that $R_j \subset D_i$ in Λ_A, so that $\Xi(D_i \cap R_j) = \Xi(R_j)$. We identify the restriction of continuous functions on $\mathbb{S}_A$ to $\Xi(\Lambda_A)$ with continuous functions on Λ_A and we write equivalently, with a slight abuse of notation, $f(x, \sigma(x))$ or $f(x)$. One then sees that

$$f(\tau_{(i,j)}(x, \sigma(x))) = f(\sigma_i(x))\ \chi_{R_j}(x).$$

This induces an isometry on the Hilbert space $L^2(\Xi(\Lambda_A), \mu_s)$, where μ_s is the Hausdorff measure of dimension $s = \dim_H(\Xi(\Lambda_A))$,

$$\hat{S}^*_{(i,j)} f(x) = N^s \chi_{R_{ij}}(x) f(\sigma(x)),$$

since for a function $f(x, \sigma(x))$ on $\Xi(\Lambda_A)$ we have

$$\chi_{R_{(i,j)}}(x, \sigma(x)) f(\sigma(x), \sigma^2(x)) = \chi_{R_{ij}}(x) f(\sigma(x)).$$

This has adjoint

$$\hat{S}^*_{(i,j)} f(x) = N^{-s} \chi_{R_j}(x) f(\sigma_i(x)).$$

We then obtain

$$\hat{S}_{(i,j)} \hat{S}^*_{(i,j)} f(x) = \chi_{R_{ij}}(x)\ \chi_{R_j}(\sigma(x))\ f(\sigma_i \sigma(x)) = \chi_{R_{ij}}(x)\ f(x)$$

so that we have the relation

$$\sum_{(i,j)} \hat{S}_{(i,j)} \hat{S}^*_{(i,j)} = 1.$$

We also have

$$\hat{S}^*_{(i,j)} \hat{S}_{(i,j)} f(x) = \chi_{R_j}(x)\, \chi_{R_{ij}}(\sigma_i(x)) f(\sigma \sigma_i(x)) = \chi_{R_j}(x)\ f(x).$$

Using the fact that

$$\chi_{R_j} = \sum_k A_{jk} \chi_{R_{jk}},$$

we then obtain the other relation in the form

$$\hat{S}^*_{(i,j)} \hat{S}_{(i,j)} = \sum_k A_{jk} \hat{S}_{(j,k)} \hat{S}^*_{(j,k)}.$$

These correspond to the Cuntz–Krieger relations for the matrix $\tilde{A}$ of (2.71). □

3 Wavelets on Fractals

A general construction of wavelets on self-similar fractals was described in [15], see also [2]. The cases considered there correspond, from the point of view of semibranching function systems, to the case where the σ_i are defined on all of X, as in the case of the Cuntz algebra. To adapt these constructions of wavelets to the main case we are interested in, which is the Cantor sets Λ_A introduced above, one can use the representation of the Cuntz–Krieger algebra O_A on $L^2(\Lambda_A, d\mu_A)$ that we considered in the previous sections, and again the Perron–Frobenius theory for the non-negative matrix A.

We begin by recalling briefly how the construction of [15] works in the case of a semibranching function system on a measure space (X, μ) where the N maps σ_i are defined on all of X. In this case one considers the $(m+1)$-dimensional linear space $\mathfrak{P}^m$ of polynomials on $\mathbb{R}$ of degree $\leq m$, and one denotes by $\mathfrak{S}_0$ the linear subspace of $L^2(X, d\mu)$, generated by the restrictions $P|_{\Lambda_A}$ of polynomials in $\mathfrak{P}^m$. Under the condition that X preserves Markov's inequality (see Sect. 4 of [15]), one knows that one still has $\dim \mathfrak{S}_0 = m+1$. One then considers the linear subspace $\mathfrak{S}_1 \subset L^2(X, d\mu)$ of functions $f \in L^2(X, d\mu)$ that are μ-almost everywhere on $R_i = \sigma_i(X)$ restrictions $P|_{R_i}$ of some polynomial $P \in \mathfrak{P}^m$. Clearly $\mathfrak{S}_0 \subset \mathfrak{S}_1$ and $\dim \mathfrak{S}_1 = N \dim \mathfrak{S}_0 = N(m+1)$, and let ϕ^ℓ, for $\ell = 1, \dots, m+1$ be an orthonormal basis for $\mathfrak{S}_0$. One then considers the orthogonal complement $\mathfrak{S}_1 \ominus \mathfrak{S}_0$, with a fixed choice of an orthonormal basis ψ^ρ, for $\rho = 1, \dots, (N-1)(m+1)$. The functions ϕ^r and ψ^ρ provide the mother wavelets. One then considers the family of linear subspaces $\mathfrak{S}_k$ of $L^2(X, d\mu)$, of functions whose restriction to each subset $\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)$, agrees μ-almost everywhere with the restriction to the same set of a polynomial in $\mathfrak{P}^m$. These satisfy $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \cdots \mathfrak{S}_k \subset \cdots L^2(X, d\mu)$. Moreover, any function in $L^2(X, d\mu)$ can be approximated by elements in

$$\mathfrak{S}_0 \oplus \bigoplus_{k \geq 0} (\mathfrak{S}_{k+1} \ominus \mathfrak{S}_k),$$

since in fact the polynomials of degree zero already suffice, as they give combinations of characteristic functions of the sets $\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)$. The wavelets are then obtained in [15] as

$$\psi_a^\rho = \mu(\sigma_a(X))^{-1/2} \psi^\rho \circ \sigma_a^{-1}. \tag{3.1}$$

for $a = (i_1, \dots, i_k)$ and $\sigma_a = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}$.

We show now how to adapt this construction to the case of the Cantor sets Λ_A. For simplicity, we describe in full only the case where one only considers locally constant functions, that is, where one starts with the 1-dimensional space $\mathfrak{P}^0$. This is the case that is closest to the classical construction based on the Haar wavelets [11].

On the space $\Lambda_A \subset [0,1]$, with the Hausdorff measure $\mu = \mu_A$, let $\mathfrak{S}_k$ denote the linear subspaces of $L^2(\Lambda_A, d\mu_A)$ obtained as above, starting from the 1-dimensional space $\mathfrak{P}^0$. Let

$$\{f^{\ell,k}\}_{k=0,\dots,N-1;\ell=1,\dots,d_k}, \tag{3.2}$$

with

$$d_k = \#\{j \mid A_{kj} = 1\}, \tag{3.3}$$

be a family of locally constant functions on Λ_A such that the support of $f^{\ell,k}$ is contained in R_k and

$$\int_{R_k} \overline{f^{\ell,k}} f^{\ell',k} = \delta_{\ell,\ell'}. \tag{3.4}$$

We also require that

$$\int_{R_k} f^{\ell,k} = 0, \quad \forall \ell = 1, \dots, d_k. \tag{3.5}$$

Lemma 3.1 *A family of functions $f^{\ell,k}$ as in (3.2), satisfying (3.4) and (3.5), can be constructed using linear combinations of characteristic functions $\chi_{R_{kj}}$, where $R_{kj} = \Lambda_{2,A}(kj)$. The resulting $f^{\ell,k}$ give an orthonormal basis of the space $\mathfrak{S}_2 \ominus \mathfrak{S}_1$.*

Proof To see that linear combinations of characteristic functions $\chi_{R_{kj}}$ suffice to construct the functions $f^{\ell,k}$, notice first that the $\chi_{R_{kj}}$ give an orthogonal basis for the space $\mathfrak{S}_2$, which is of dimension $\dim \mathfrak{S}_2 = \sum_k d_k$. We then write the $f^{\ell,k}$ in the form

$$f^{\ell,k} = \sum_j A_{kj} c_j^{\ell,k} \chi_{R_{kj}}, \tag{3.6}$$

where the conditions (3.4) and (3.5) translate into conditions on the coefficients of the form

$$\sum_j A_{kj} \bar{c}_j^{\ell,k} c_j^{\ell',k} p_{kj} = \delta_{\ell,\ell'}, \tag{3.7}$$

where we use the notation

$$p_{kj} = \mu(R_{kj}) = N^{-2\delta_A} p_j, \tag{3.8}$$

according to (2.46), where $p = (p_0, \dots, p_{N-1})$ is the Perron–Frobenius eigenvector $Ap = r(A)p$ for the non-negative matrix A. Similarly, the condition (3.5) becomes

$$\sum_j A_{kj} c_j^{\ell,k} p_{kj} = N^{-2\delta_A} \sum_j A_{kj} c_j^{\ell,k} p_j = 0, \tag{3.9}$$

where we again use (3.8).

Let us introduce the following notation for convenience. Consider on $\mathbb{C}^{d_k} \subset \mathbb{C}^N$ the inner product

$$\langle v, w\rangle_k := \sum_j A_{kj}\bar{v}_j w_j p_j. \tag{3.10}$$

Let $\mathcal{V}_k$ denote the orthogonal complement, in the inner product (3.10) on $\mathbb{C}^{d_k}$ of the vector $u = (1, 1, \dots, 1)$, and let $\{c^{\ell,k} = (c_i^{\ell,k})\}_{\ell=1,\dots,d_k-1}$ be an orthonormal basis of $\mathcal{V}_k$, in the inner product (3.10), namely

$$\langle c^{\ell,k}, u\rangle_k = 0, \quad \text{and} \quad \langle c^{\ell,k} c^{\ell',k}\rangle_k = \delta_{\ell,\ell'}. \tag{3.11}$$

Then for $c^{\ell,k}$ as above, one sees that the functions (3.6) are an orthonormal family satisfying the conditions (3.4) and (3.5).

The space spanned by the $f^{\ell,k}$ is contained in $\mathfrak{S}_2$ by construction. The condition (3.5) ensures that the functions $f^{\ell,k}$ are orthogonal to all the χ_{R_k}, hence they are in $\mathfrak{S}_2 \ominus \mathfrak{S}_1$. They span a space of dimension $\sum_k (d_k - 1) = \sum_k d_k - N = \dim \mathfrak{S}_2 \ominus \mathfrak{S}_1$. □

Theorem 3.2 *Suppose given an orthonormal basis $\{f^{\ell,r}\}$ for $\mathfrak{S}_2 \ominus \mathfrak{S}_1$, constructed as in Lemma 3.1 above. Then the functions of the form*

$$\psi_a^{\ell,r} = S_a\, f^{\ell,r}, \tag{3.12}$$

for $a = (a_1, \dots, a_k) \in \mathcal{W}_{k,A}$, give an orthonormal basis for the space $\mathfrak{S}_{k+1} \ominus \mathfrak{S}_k$ hence, for varying $a \in \mathcal{W}_{k,A}$ and for all $k \geq 0$, they give an orthonormal basis of wavelets for $L^2(\Lambda_A, \mu)$.

Proof We have shown in Lemma 3.1 that the functions $f^{\ell,r}$, for $r = 0, \dots, N-1$ and $\ell = 1, \dots, d_r$, give an orthonormal basis of $\mathfrak{S}_2 \ominus \mathfrak{S}_1$. We then check that the functions $S_a f^{\ell,r}$ give an orthonormal basis for $\mathfrak{S}_{k+1} \ominus \mathfrak{S}_k$. Since in the representation of O_A on $L^2(\Lambda_A, d\mu_A)$ we have constant Radon–Nikodym derivatives $\Phi_{\sigma_i} = N^{-\delta_A}$, this gives

$$S_j f = N^{\delta_A/2}\, \chi_{R_j}\ f \circ \sigma,$$

so that we then have

$$S_a\, f^{\ell,r} = N^{\delta_A k/2}\, \chi_{\Lambda_{k,A}(a)}\ f^{\ell,r} \circ \sigma^k.$$

For $a \in \mathcal{W}_{k,A}$, we have

$$\begin{aligned}\langle S_a f^{\ell,r}, S_{a'} f^{\ell',r'}\rangle &= N^{\delta_A k}\langle \chi_{R_a}\, f^{\ell,r} \circ \sigma^k, \chi_{R_{a'}}\, f^{\ell',r'} \circ \sigma^k\rangle \\ &= N^{\delta_A k}\delta_{a,a'} \int_{R_a} \overline{(f^{\ell,r} \circ \sigma^k)}\, (f^{\ell',r'} \circ \sigma^k)\, d\mu,\end{aligned}$$

where we write $R_a = \Lambda_{k,A}(a)$, for the range of $\sigma_a = \sigma_{a_1} \circ \cdots \circ \sigma_{a_k}$. Notice then that we have, for any function $f \in L^2(\Lambda_A, d\mu)$ and any $a \in \mathcal{W}_{k,A}$,

$$\int_{R_a} f\circ\sigma^k\,d\mu = \int_{D_{a_k}} f\,\frac{d\mu\circ\sigma_a}{d\mu}\,d\mu$$
$$= N^{-\delta_A k}\int_{D_{a_k}} f\,d\mu = N^{-\delta_A k}\sum_j A_{a_k j}\int_{R_j} f\,d\mu. \tag{3.13}$$

Applied to the above this gives

$$\langle S_a f^{\ell,r}, S_{a'} f^{\ell',r'}\rangle = \delta_{a,a'}\delta_{r,r'}A_{a_k r}\int_{R_r}\overline{f^{\ell,r}}\,f^{\ell',r}\,d\mu = \delta_{a,a'}\delta_{r,r'}\delta_{\ell,\ell'}.$$

Thus the $S_a f^{\ell,k}$ form an orthonormal system.

The space spanned by these functions is contained in $\mathfrak{S}_{k+1}$ and a counting of dimensions shows that it has the dimension of $\mathfrak{S}_{k+1}\ominus\mathfrak{S}_k$. To see that the $S_a f^{\ell,k}$ are in fact orthogonal to the elements of $\mathfrak{S}_k$ it suffices to compute

$$\langle S_a f^{\ell,r}, \chi_{\Lambda_{k,A}(b)}\rangle = \delta_{a,b}N^{\delta_A k}\int_{R_a} f^{\ell,r}\circ\sigma^k\,d\mu$$
$$= \delta_{a,b}\sum_j A_{a_k,j}\int_{R_j} f^{\ell,r}\,d\mu = \delta_{a,b}A_{a_k,r}\int_{R_r} f^{\ell,r}\,d\mu = 0,$$

by (3.13) and (3.5). This shows that we obtained an orthonornal basis of $\mathfrak{S}_{k+1}\ominus\mathfrak{S}_k$, hence a wavelet system for $L^2(\Lambda_A, d\mu)$. □

It is useful to remark how the main difference in this case, as opposed to the similar constructions given for instance in [15] that we mentioned above, is that here we need to start from an orthonormal basis of $\mathfrak{S}_2\ominus\mathfrak{S}_1$ instead of $\mathfrak{S}_1\ominus\mathfrak{S}_0$. This reflects the fact that our functions σ_i are not everywhere defined and, while the choice of an orthonormal basis for $\mathfrak{S}_1\ominus\mathfrak{S}_0$ gives the needed information on the ranges R_i, in order to control both the ranges and the domains D_i one needs to go one step further before starting the induction that constructs the wavelets, and consider $\mathfrak{S}_2\ominus\mathfrak{S}_1$. Thus, the wavelet decomposition of a function $f\in L^2(\Lambda_A,\mu)$ will be given by

$$f = \sum_{k=0}^{N-1}\sum_{\ell=1}^{d_k-1}\alpha_{\ell,k}\,f^{\ell,k} + \sum_{j=0}^{\infty}\sum_{a\in\mathcal{W}_{j,A}}\sum_{(\ell,k)}\alpha_{\ell,k,a}\,S_a f^{\ell,k}. \tag{3.14}$$

The more general case where one starts the wavelet construction from the linear space of polynomials $\mathfrak{P}^m$ with $m\geq 1$ can be done along the same lines as Lemma 3.1 and Theorem 3.2. We describe in the next section a different approach to wavelets constructions based on the Ruelle transfer operator for the coding map σ. This is closer to the point of view developed in [12].

4 Ruelle Transfer Operator

A more general version of the Perron–Frobenius operator associated to the coding map $\sigma : \Lambda_A \to \Lambda_A$ is obtained by considering the Ruelle transfer operator. This depends on the choice of a potential function W, defined on Λ_A, and is defined as

$$\mathcal{R}_{\sigma,W} f(x) = \sum_{y\,:\,\sigma(y)=x} W(y)\, f(y). \tag{4.1}$$

Lemma 4.1 *If the function W is real valued, one can describe the operator $\mathcal{R}_{\sigma,W}$ as the adjoint of the operator*

$$T_W f(x) = N^{\delta_A}\, W(x)\, f(\sigma(x)). \tag{4.2}$$

Proof We have

$$\begin{aligned}\langle T_W f, h\rangle &= \int_{\Lambda_A} N^{\delta_A}\, W(x)\, \overline{f(\sigma(x))} h(x)\, d\mu(x) \\ &= \sum_i \int_{D_i} \overline{f(u)}\, W(\sigma_i(u)) h(\sigma_i(u)) d\mu(u),\end{aligned}$$

using the fact that the Radon–Nikodym derivative $d\mu \circ \sigma_i / d\mu = N^{-\delta_A}$. We then write the above as

$$\sum_{i,j} A_{ij} \int_{R_j} \overline{f(u)}\, W(\sigma_i(u)) h(\sigma_i(u)) d\mu(u).$$

We also have

$$\sum_{i,j} A_{ij}\, \chi_{R_j}(x) W(\sigma_i(x)) h(\sigma_i(x)) = \sum_i A_{ix_1} W(\sigma_i(x)) h(\sigma_i(x)).$$

Since the set of preimages of the point x under the coding map is given by

$$\{y \mid \sigma(y) = x\} = \bigcup_{i:A_{ix_1}=1} R_i,$$

we see that the above is in fact

$$\sum_i A_{ix_1} W(\sigma_i(x)) h(\sigma_i(x)) = \sum_{y\,:\,\sigma(y)=x} W(y)\, f(y).$$

This shows that $\langle T_W f, h\rangle = \langle f, \mathcal{R}_{\sigma,W}(h)\rangle$. □

We assume that the potential W of the Ruelle transfer operator satisfies the Keane condition, namely that it has non-negative real values $W : \Lambda_A \to \mathbb{R}_+$, and satisfies

$$\sum_{y:\sigma(y)=x} W(y) = 1. \tag{4.3}$$

Equivalently, this means

$$\sum_i A_{ix_1} W(\sigma_i(x)) = 1. \tag{4.4}$$

4.1 Random Processes

In the same way as described in [12], we relate here harmonic functions for the Ruelle transfer operator, that is, functions satisfying $\mathcal{R}_{\sigma,W} h = h$ to random processes defined by transition probabilities for paths from a given point x to the image under the σ_j and their iterates.

Let A^t be the transpose of the matrix A. Then we have $a^t = (a_k, \dots, a_1) \in \mathcal{W}_{k,A}$ if and only if $a = (a_1, \dots, a_k) \in \mathcal{W}_{k,A^t}$. We construct probability measures on the limit set Λ_{A^t} that are related to fixed points of the Ruelle transfer operator for the coding $\sigma : \Lambda_A \to \Lambda_A$. In the following we denote by R_i and D_i, as before, the ranges and domains of the maps σ_i in Λ_A and by R_i^t and D_i^t the corresponding sets in Λ_{A^t}.

For a given potential W on Λ_A satisfying the Keane condition (4.3), consider a function $x \mapsto P_x^W$, for $x \in D_i \subset \Lambda_A$, where $P_x^W : \mathcal{B}(\Lambda_{A^t} \cap R_i^t) \to \mathbb{R}_+$, is a non-negative function on the Borel subsets of Λ_{A^t} defined by assigning to the $\Lambda_{k,A^t}(a)$ the values

$$P_x^W(\Lambda_{k,A^t}(a)) = A_{a_1 x_1} W(\sigma_{a_1}(x)) W(\sigma_{a_2}\sigma_{a_1}(x)) \cdots W(\sigma_{a_k} \cdots \sigma_{a_1}(x)), \tag{4.5}$$

for $a^t = (a_k, \dots, a_1) \in \mathcal{W}_{k,A}$ and for $x \in D_{a_1} \subset \Lambda_A$.

Lemma 4.2 *The assignment* (4.5)*, for* $x \in D_i \subset \Lambda_A$*, defines a measure on* $R_i^t \subset \Lambda_{A^t}$.

Proof Similarly, to the case of Λ_A seen in (2.43), to check that (4.5) defines a measure one has to check the compatibility condition

$$P_x^W(\Lambda_{k,A^t}(a)) = \sum_j A^t_{a_k j} \, P_x^W(\Lambda_{k+1,A}(aj)), \tag{4.6}$$

for all $x \in D_{a_1} \subset \Lambda_A$. We have

$$P_x^W(\Lambda_{k+1,A}(aj)) = A_{a_1,x_1} W(\sigma_{a_1}(x)) \cdots W(\sigma_{a_k} \cdots \sigma_{a_1}(x)) W(\sigma_j \sigma_{a_k} \cdots \sigma_{a_1}(x)))$$

Moreover, the Keane condition for W on Λ_A gives

$$\sum_j A_{j a_k} W(\sigma_j \sigma_{a_k} \cdots \sigma_{a_1}(x))) = 1,$$

so we obtain (4.6). □

One can think of the values of the potential W as defining a probability of transition, or walk, from x to $\sigma_{a_1}(x)$, so that (4.5) can be regarded as the probability of a random walk from x to $\sigma_{a_k}\cdots\sigma_{a_1}(x)$. We then see that the random process P_x^W is related to the fixed points of the Ruelle transfer operator.

Proposition 4.3 *The random process $x \mapsto P_x^W$ introduced above is related to fixed points of the Ruelle transfer operator in the following ways.*

(1) *Let $E \subset \Lambda_{A^t}$ be a shift invariant set $\sigma^{-1}(E) = E$. Then the function $x \mapsto P_x^W(E)$ is a fixed point of the Ruelle transfer operator with potential W on Λ_A.*

(2) *If the series*

$$h(x) := \sum_{k\geq 1} \sum_{a\in \mathcal{W}_{k,A^t}} A_{a_1 x_1} W(\sigma_{a_1}(x)) \cdots W(\sigma_{a_k}\cdots\sigma_{a_1}(x)) \tag{4.7}$$

converges, then the function $h(x)$ is a fixed point of the Ruelle transfer operator with potential W on Λ_A.

Proof (1) We check that this condition is equivalent to the fixed point condition under the Ruelle transfer operator. For a given set $\Lambda_{k,A^t}(a)$, we have

$$\begin{aligned}\mathcal{R}_{\sigma,W}(P_x^W(\Lambda_{k,A^t}(a))) &= \sum_{y:\sigma(y)=x} W(y)\, P_y^W(\Lambda_{k,A^t}(a)) \\ &= \sum_j A_{jx_1} W(\sigma_j(x))\, P^W_{\sigma_j(x)}(\Lambda_{k,A^t}(a)).\end{aligned}$$

A shift invariant set $\sigma^{-1}(E) = E$ in Λ_{A^t} satisfies

$$\cup_{j,i:A^t_{ji}=1}\sigma_j(E\cap R_i) = E.$$

By construction of the measures P_x^W, we know that $P_x^W(\sigma_j(E\cap R_i^t))$ is non-trivial provides that $x \in D_j$, so that $A_{jx_1} = 1$. Thus, for $\sigma^{-1}(E) = E$, we have

$$\mathcal{R}_{\sigma,W}(P_x^W(E)) = \sum_j A_{ij} P_x^W(\sigma_j(E\cap R_i^t)) = P_x^W(\sigma^{-1}(E)) = P_x^W(E),$$

which shows that $P_x^W(E)$ is a fixed point for $\mathcal{R}_{\sigma,W}$.

(2) Assuming that the series (4.7) converges, we have

$$\begin{aligned}\mathcal{R}_{\sigma,W}h(x) &= \sum_{\sigma(y)=x} W(y)h(y) = \sum_j A_{jx_1} W(\sigma_j(x))h(\sigma_j(x)) \\ &= \sum_j A_{jx_1} W(\sigma_j(x)) \sum_k \sum_a A_{a_1 j} W(\sigma_{a_1}\sigma_j(x))\cdots W(\sigma_{a_k}\cdots\sigma_{a_1}\sigma_j(x)) \\ &= \sum_k \sum_{b=ja\in W_{k+1,A^t}} A_{jx_1} W(\sigma_j(x)) W(\sigma_{a_1}\sigma_j(x))\cdots W(\sigma_{a_k}\cdots\sigma_{a_1}\sigma_j(x)).\end{aligned}$$

This gives $\mathcal{R}_{\sigma,W}h(x) = h(x)$. □

4.2 A Trigonometric Example

We give an example of a potential W satisfying the Keane condition, constructed using trigonometric functions.

Lemma 4.4 *The function*

$$W(x) = \frac{1}{N_1}\left(1 - \cos\left(\frac{2\pi N x}{N_1}\right)\right), \tag{4.8}$$

with $N_1 = \#\{j : A_{jx_1} = 1\}$*, is a potential satisfying the Keane condition* (4.3) *on* Λ_A.

Proof First notice that we have

$$\sum_{j=0}^{N-1} A_{jx_1} \exp\left(\frac{2\pi i N \sigma_j(x)}{N_1}\right) = 0,$$

since $\sigma_j(x) = (x+j)/N$ and the above becomes a sum over all the N_1th roots of unity. It follows directly from this that the real valued trigonometric version also satisfies

$$\sum_{j=0}^{N-1} A_{jx_1} \cos\left(\frac{2\pi N \sigma_j(x)}{N_1}\right) = 0,$$

from which it follows that the potential of (4.8) satisfies

$$\sum_{j=0}^{N-1} A_{jx_1} W(\sigma_j(x)) = 1.$$

Moreover, the function $W(x)$ takes non-negative real values, so it gives a potential with the Keane condition. □

5 Examples and Applications

5.1 Hensley Cantor Sets and Continued Fraction Expansion

In [23] the coding of geodesics on the modular curves $X_\Gamma = \mathbb{H}/\Gamma$, for $\Gamma \subset \mathrm{PGL}_2(\mathbb{Z})$ a finite index subgroup and $\mathbb{H}$ the hyperbolic upper half plane, was related to a generalization of the shift map of the continued fraction expansion $T : [0,1]\times\mathbb{P} \to [0,1]\times\mathbb{P}$,

$$T(x,s) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} s\right), \tag{5.1}$$

where $\mathbb{P} = \mathrm{PGL}_2(\mathbb{Z})/\Gamma$ is the finite coset set. It was then shown in [24,25], that the restriction of this dynamical system to the Hensley Cantor sets, that is, those subsets

$E_N \subset [0,1]$ of points that only contains digits $a_k \leq N$ in the continued fraction expansion, gives rise to a dynamical system

$$\sigma : E_N \times \mathbb{P} \to E_N \times \mathbb{P}, \tag{5.2}$$

which can be identified with the coding map $\sigma : \Lambda_A \to \Lambda_A$ of a semibranching function system $\{\sigma_i\}$ that determines a Cuntz–Krieger algebra O_A. The case where $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$ recovers the Cuntz algebra O_N.

In this setting, one considers the Ruelle transfer operator with potential (without Keane condition)

$$W(x,s) = |T'(x,s)|^\beta$$

so that

$$\begin{aligned}\mathcal{R}_{T,W} f(x,s) &= \sum_{T(y,t)=(x,s)} |T'(y,t)|^\beta f(y,t) \\ &= \sum_{n=1}^{N} \frac{1}{(x+n)^{2\beta}} f\left(\frac{1}{x+n}, \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} s\right).\end{aligned}$$

This can be written in the form

$$\sum_{(n,t)} A_{(n,t),(x_1,s)} W(\sigma_{(n,t)}(x,s)) f(\sigma_{(n,t)}(x,s)),$$

where the matrix A is defined by the condition

$$A_{(n,t),(k,s)} = \begin{cases} 1 & M_n s = t, \\ 0 & \text{otherwise}, \end{cases}$$

where the matrix $M_n \in \mathrm{GL}_2(\mathbb{Z})$, acting on the left on the coset $\mathbb{P}$, is

$$M_n = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}.$$

The shift invariant measure $\mu_{N,\mathbb{P}}$ on $E_N \times \mathbb{P}$ constructed in [24] using the fixed point of the Ruelle transfer operator can then be also seen as in [22] as KMS_β state for the time evolution on the Cuntz–Krieger algebra O_A given by

$$\sigma_t(S_{(k,s)}) = W^{-it} S_{(k,s)},$$

where we identify W^{-it}, for fixed t, with an element in $C(E_N \times \mathbb{P})$. The KMS state is then of the form

$$\varphi_\beta(S_a S_a^*) = \int_{E_N \times \mathbb{P}} f_a(x,s)\, d\mu_{N,\mathbb{P}}(x,s),$$

for $a = ((k_1, s_1), \ldots, (k_r, s_r)) \in \mathcal{W}_{r,A}$ and f_a the element in $C(E_N \times \mathbb{P})$ that corresponds to $S_a S_a^*$. The Ruelle operator can correspondingly be written as

$$\mathcal{R}_{T,W} f = \sum_{(n,t)} S^*_{(n,t)}\, W f\, S_{(n,t)}$$

in term of generators of the Cuntz–Krieger algebra.

5.2 Graph Wavelets from Cuntz–Krieger Algebras

It was recently shown, see for instance [8], that the crucial problem of spatial traffic analysis on networks can be addressed using a form of wavelet analysis which is adapted to the topology of the network graph. These *graph wavelets* are constructed as families of functions $\Psi_\alpha(v)$ on the set of vertices $V(G)$ of a given finite graph G, localized with respect to certain scaling indices α, and with the property that

$$\int_{V(G)} \Psi_\alpha(v) d\mu(v) = 0, \quad \text{and} \quad \int_{V(G)} \bar{\Psi}_\alpha(v) \Psi_{\alpha'}(v) d\mu(v) = \delta_{\alpha,\alpha'}, \tag{5.3}$$

where $\mu(v)$ is a given measure that weights the nodes of the network with assigned probabilities. We show here how to construct families of graph wavelets using the representations of Cuntz–Krieger algebras and the corresponding wavelets on Λ_A constructed in Sect. 3.

Let G be a finite directed graph with no sinks. It is well known that one can associate to such a graph a Cuntz–Krieger algebra in the following way. One considers a collection of projections P_v associated to the vertices $v \in V(G)$ and a collection of partial isometries S_e associated to the oriented edges $e \in E(G)$, with the relations

$$P_v = \sum_{s(e)=v} S_e S_e^* \tag{5.4}$$

for all $v \in V(G)$, and

$$P_{r(e)} = S_e^* S_e, \tag{5.5}$$

for all edges $e \in E(G)$. Assuming that the graph has no sinks, so that all vertices are sources, one has $\sum_v P_v = 1$ so that the isometries S_e satisfy the relation (2.2),

$$\sum_e S_e S_e^* = 1. \tag{5.6}$$

Moreover, for $N = \#E(G)$, one defines the $N \times N$-matrix $A_{ee'}$ by

$$A_{ee'} = \begin{cases} 1 & r(e) = s(e'), \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Then the relation (5.5) reads equivalently as

$$S_e^* S_e = \sum_{e':r(e)=s(e')} S_{e'} S_{e'}^* = \sum_{e'} A_{ee'} S_{e'} S_{e'}^*, \tag{5.8}$$

which gives the other Cuntz–Krieger relation (2.1).

As before, let Λ_A be the limit set associated to the algebra O_A of the graph G. Let $d_e = \#\{e' \,|\, r(e) = s(e')\} = \#\{e' \,|\, A_{ee'} = 1\}$. Consider as in Sect. 3 the orthonormal family of functions $\{f^{\ell,e}\}$ with $e \in E(G)$ and $\ell = 1, \dots, d_e$. As we have seen in Sect. 3 these are the mother wavelets for the orthonormal basis of $L^2(\Lambda_A, \mu_A)$ given by the functions $\{S_a f^{\ell,e}\}$, for varying $a \in \mathcal{W}_{k,A}$ and $k \in \mathbb{N}$. Here an element $a = (e_1, \dots, e_k) \in \mathcal{W}_{k,A}$ is a path in the graph G of length k starting at the vertex $s(e_1)$. Here we use the same mother functions to construct a family of graph wavelets.

Recall from Sect. 3 that the functions $f^{\ell,e}$ are constructed in terms of a family $c^{\ell,e} = (c^{\ell,e}_{e'})$ of vectors satisfying

$$\sum_{e'} A_{e,e'} \bar{c}^{\ell,e}_{e'} c^{\ell',e}_{e'} p_{ee'} = \delta_{e,e'}, \tag{5.9}$$

where $p_{ee'} = \mu(R_{ee'}) = N^{-2\delta_A} p_{e'}$ and

$$\sum_{e'} A_{e,e'} c^{\ell,e}_{e'} p_{e'} = 0. \tag{5.10}$$

Upon rescaling the coefficients $c^{\ell,e}_{e'}$ by a factor N^{δ_A}, we obtain a family satisfying (5.10) and with (5.9) replaced by the similar

$$\sum_{e'} A_{e,e'} \bar{c}^{\ell,e}_{e'} c^{\ell',e}_{e'} p_{e'} = \delta_{e,e'}, \tag{5.11}$$

where we keep the same notation for these rescaled coefficients. The p_e are the components of the Perron–Frobenius eigenvector $Ap = r(A)p$.

After fixing a choice of a base vertex $v_0 \in E(G)$, we define a measure on the set of vertices of the graph by $\mu_{G,v_0}(v_0) = 0$ and

$$\mu_{G,v_0}(v) := p_{e_1} \cdots p_{e_k}, \tag{5.12}$$

where $e_1 \cdots e_k$ is the shortest path in the graph G starting at v_0 and ending at v. This means that we are considering a random walk on the graph starting at v_0, where at the first step one has probability p_e of moving to the nearby vertex $r(e)$ and probability zero of remaining at v_0. The measure (5.12) gives the probability of reaching at time k one of the vertices that are k steps away from v_0.

In addition to fixing the base vertex v_0, we also fix a choice of an edge e_0 with $r(e_0) = v_0$. We then define functions

$$\Psi_\ell(v) = \begin{cases} c_{e'}^{\ell,e_0} & v = r(e'),\ v_0 = r(e_0) = s(e'), \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}$$

These satisfy

$$\int_{V(G)} \Psi_\ell(v) d\mu_{G,v_0}(v) = \sum_{e'} A_{e_0 e'} c_{e'}^{\ell,e_0} p_{e'} = 0 \tag{5.14}$$

and

$$\int_{V(G)} \bar{\Psi}_\ell(v) \Psi_{\ell'}(v) d\mu_{G,v_0}(v) = \sum_{e'} A_{e_0 e'} \bar{c}_{e'}^{\ell,e_0} c_{e'}^{\ell',e_0} p_{e'} = \delta_{\ell,\ell'}. \tag{5.15}$$

We then extend this to a family $\Psi_{\ell_1,\dots,\ell_k}(v)$, where we consider paths $a = (e_1, \dots, e_k) \in \mathcal{W}_{k,A}$ of length k in the graph starting at v_0, with $\ell_i = 1, \dots, d_{e_i}$. We set

$$\Psi_{\ell_1,\dots,\ell_k}(v) = \begin{cases} c_{e_1}^{\ell_1,e_0} c_{e_2}^{\ell_2,e_1} \cdots c_{e_k}^{\ell_k,e_{k-1}} & v = r(e_k),\ v_0 = s(e_1), \\ 0 & \text{otherwise.} \end{cases} \tag{5.16}$$

These again satisfy

$$\begin{aligned} &\int_{V(G)} \Psi_{\ell_1,\dots,\ell_k}(v) d\mu_{G,v_0}(v) \\ &\quad = \sum_{(e_1,\dots,e_k)} A_{e_0 e_1} \cdots A_{e_{k-1} e_k} c_{e_1}^{\ell_1,e_0} c_{e_2}^{\ell_2,e_1} \cdots c_{e_k}^{\ell_k,e_{k-1}} p_{e_1} \cdots p_{e_k} = 0. \end{aligned} \tag{5.17}$$

This vanishes since already $\sum_{e_k} A_{e_{k-1} e_k} c_{e_k}^{\ell_k,e_{k-1}} p_{e_k} = 0$. Moreover, they satisfy

$$\begin{aligned} &\int_{V(G)} \bar{\Psi}_{\ell_1,\dots,\ell_k}(v) \Psi_{\ell'_1,\dots,\ell'_k}(v) d\mu_{G,v_0}(v) \\ &\quad = \sum_{(e_1,\dots,e_k)} A_{e_0 e_1} \cdots A_{e_{k-1} e_k} \bar{c}_{e_1}^{\ell_1,e_0} \cdots \bar{c}_{e_k}^{\ell_k,e_{k-1}} c_{e_1}^{\ell'_1,e_0} \cdots c_{e_k}^{\ell'_k,e_{k-1}} p_{e_1} \cdots p_{e_k} \\ &\quad = \delta_{\ell_1,\ell'_1} \cdots \delta_{\ell_k,\ell'_k}. \end{aligned} \tag{5.18}$$

The functions $\Psi_{\ell_1,\dots,\ell_k}$, for $k \geq 1$, constructed in this way, are supported on concentric regions $\mathcal{U}_k(v_0)$ made of vertices at a distance k from a chosen base vertex v_0. Unlike other types of graph wavelets constructions where the functions are constant on such concentric regions $\mathcal{U}_k(v_0)$ and average to zero over different k, the ones we obtain here are supported on a single $\mathcal{U}_k(v_0)$ with zero average. In terms of traffic analysis on

networks, while one type of graph wavelets may be more suitable in analyzing radial propagation from a vertex, the other may be preferable for directional propagation away from a chosen vertex.

In [6,7] one considered, in the setting of Mumford curves with p-adic Schottky uniformization, the Cuntz–Krieger algebras associated to the finite graphs with no sinks obtained from the action of a p-adic Schottky group on the subtree of the Bruhat–Tits tree spanned by geodesics with boundary points on the limit set in $\mathbb{P}^1(\mathbb{Q}_p)$. In that context it would be interesting to compare the wavelet constructions described in this paper with the p-adic wavelet theory (see for instance [1]).

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