From CFT to CFT?

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Disclaimer: this is a largely speculative talk, meant for an informal discussion session at the workshop "Novel approaches to the finite simple groups" in Banff

Quantum Statistical Mechanics and Class Field Theory

QSM approach to Class Field Theory: number field $\mathbb{K}$

• QSM system $(\mathcal{A}, \sigma_t)$
• partition function $Z(\beta)$ is Dedekind zeta function $\zeta_{\mathbb{K}}(\beta)$.
• phase transition spontaneous symmetry breaking at pole $\beta = 1$
• unique equilibrium state above critical temperature
• quotient $C_{\mathbb{K}}/D_{\mathbb{K}}$ (idèle class group by connected component) acts by symmetries
• subalgebra $\mathcal{A}_0$ of $\mathcal{A}$: values of extremal ground states on $\mathcal{A}_0$ are algebraic numbers and generate K^{ab}
• Galois action by C_K/D_K via CFT isom
$\theta : C_K/D_K \to \mathrm{Gal}(K^{ab}/K)$

Quantum Statistical Mechanics

$\mathcal{A}$ = algebra of observables (C^*-algebra)
State: $\varphi : \mathcal{A} \to \mathbb{C}$ linear

$$\varphi(1) = 1, \quad \varphi(a^* a) \geq 0$$

Time evolution $\sigma_t \in \mathrm{Aut}(\mathcal{A})$
rep $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on Hilbert space $\mathcal{H}$

$$\text{Hamiltonian} \quad H = \frac{d}{dt} \sigma_t |_{t=0}$$

$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

Symmetries

- Automorphisms: $G \subset \mathrm{Aut}(\mathcal{A})$, $g\sigma_t = \sigma_t g$; inner: $a \mapsto uau^*$ with u = unitary, $\sigma_t(u) = u$,
- Endomorphisms: $\rho\sigma_t = \sigma_t\rho \quad e = \rho(1)$ (need $\varphi(e) \neq 0$)

$$\rho^*(\varphi) = \frac{1}{\varphi(e)} \varphi \circ \rho$$

inner: u = isometry with $\sigma_t(u) = \lambda^{it} u$

Equilibrium states (inverse temperature $\beta = 1/kT$)

$$\frac{1}{Z(\beta)} \mathrm{Tr}\left(a\, e^{-\beta H}\right) \qquad Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right)$$

More general: KMS states $\varphi \in \mathrm{KMS}_\beta$ $(0 < \beta < \infty)$
$\forall a, b \in \mathcal{A}$ $\exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$

At $T > 0$ simplex KMS_β $\rightsquigarrow$ extremal $\mathcal{E}_\beta$ (points of NC space)
At $T = 0$: KMS_∞ = weak limits of KMS_β

$$\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a)$$

The Bost–Connes system

Algebra $\mathcal{A}_{\mathbb{Q},BC} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ generators and relations

$$\begin{aligned} &\mu_n \mu_m = \mu_{nm} \\ &\mu_n \mu_m^* = \mu_m^* \mu_n \quad \text{when } (n,m)=1 \\ &\mu_n^* \mu_n = 1 \end{aligned}$$

$$e(r+s) = e(r)e(s), \quad e(0) = 1$$

$$\rho_n(e(r)) = \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ with time evolution

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it} \mu_n$$

Rep on $\ell^2(\mathbb{N})$, partition function $\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$

- J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (1995)

KMS states of the BC system

- Representations π_ρ on $\ell^2(N)$:

$$\mu_n \epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m \epsilon_m$$

$\zeta_r = \rho(e(r))$ root of 1, for $\rho \in \hat{\mathbb{Z}}^*$

- Low temperature extremal KMS ($\beta > 1$) Gibbs states

$$\varphi_{\beta,\rho}(a) = \frac{\mathrm{Tr}(\pi_\rho(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \quad \rho \in \hat{\mathbb{Z}}^*$$

- phase transition at $\beta = 1$; high temperature: unique KMS state
- Zero temperature: evaluations $\varphi_{\infty,\rho}(e(r)) = \zeta_r$

$$\varphi_{\infty,\rho}(a) = \langle \epsilon_1, \pi_\rho(a) \epsilon_1 \rangle$$

Intertwining: $a \in \mathcal{A}_{\mathbb{Q},BC}$, $\gamma \in \hat{\mathbb{Z}}^*$

$$\varphi_{\infty,\rho}(\gamma a) = \theta_\gamma(\varphi_{\infty,\rho}(a))$$

$$\theta : \hat{\mathbb{Z}}^* \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

Class field theory isomorphism

Gibbs states near the phase transition

• Gibbs states of BC system are polylogs at roots of unity

$$\varphi_{\beta,\rho}(e(r)) = \zeta(\beta)^{-1} \sum_{n\geq 1} \frac{\zeta_r^n}{n^\beta} = \zeta(\beta)^{-1} \mathrm{Li}_\beta(\zeta_r)$$

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

• The zeta function $\zeta(\beta)$ has a pole at $\beta = 1$

• Behavior of these KMS states as $\beta \to 1$: near criticality behavior of expectation values

Useful polylogs identities

- Fourier sums: $\zeta(s,a) =$ Hurwitz zeta function

$$\mathrm{Li}_s(e^{2\pi i m/p}) = p^{-s} \sum_{k=1}^{p} e^{2\pi i m k/p} \zeta(s, \frac{k}{p})$$

- multiplication formula:

$$\sum_{m=0}^{p-1} \mathrm{Li}_s(z e^{2\pi i m/p}) = p^{1-s}\, \mathrm{Li}_s(z^p)$$

- Fermi–Dirac distribution $-\mathrm{Li}_{s+1}(e^{-\mu})$
- $-\log(-\mu)$ limit as $s \to 1$:

$$\mathrm{Li}_s(e^{\mu}) = \Gamma(1-s)(-\mu)^{s-1} + \sum_{k=0}^{\infty} \frac{\zeta(s-k)}{k!} \mu^k$$

$$\lim_{s \to k+1} \left(\frac{\zeta(s-k)}{k!} \mu^k + \Gamma(1-s)(-\mu)^{s-1} \right) = \frac{\mu^k}{k!} (H_k - \log(-\mu))$$

$H_n = \sum_{k=1}^{n} 1/k$ harmonic numbers

QSM systems and complex multiplication

(Connes–Marcolli–Ramachandran)

$\mathbb{K} = \mathbb{Q}(\sqrt{-D})$ imaginary quadratic field

- 1-dimensional $\mathbb{K}$-lattice (Λ, ϕ): fin gen $\mathcal{O}$-submod $\Lambda \subset \mathbb{C}$ with $\Lambda \otimes_{\mathcal{O}} \mathbb{K} \cong \mathbb{K}$ and $\mathcal{O}$-mod morphism $\phi : \mathbb{K}/\mathcal{O} \to \mathbb{K}\Lambda/\Lambda$ (invertible is ϕ isom)
- (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if $\mathbb{K}\Lambda_1 = \mathbb{K}\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$
- space of 1-dim $\mathbb{K}$-lattices $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*)$: adelic description of lattices (Λ, ϕ) as (ρ, s), $\rho \in \hat{\mathcal{O}}$ and $s \in \mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*$, mod $(\rho, s) \mapsto (x^{-1}\rho, xs)$, $x \in \hat{\mathcal{O}}^*$
- commensurability classes: $\mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^*$ with $\mathbb{A}^{\cdot}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{C}^*$ (nontrivial archimedean component)

• Groupoid algebra of commensurability of 1-dim $\mathbb{K}$-lattices $C_0(\mathbb{A}^{\cdot}_{\mathbb{K}}) \rtimes \mathbb{K}^*$, up to scaling: $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*)$ mod $\mathbb{C}^*$ is $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*)$ and commensurability by $(\rho, s) \mapsto (s_J\rho, s_J^{-1}s)$, $J \subset \mathcal{O}$ ideal adelically $J = s_J\hat{\mathcal{O}} \cap \mathbb{K}$

$$C(\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*)) \rtimes J^+_{\mathbb{K}}$$

• Class number from $\mathbb{A}^*_{\mathbb{K},f}/(\mathbb{K}^* \times \hat{\mathcal{O}}^*) = \mathrm{Cl}(\mathcal{O})$

• $\hat{\mathcal{O}}^*$ acts by automorphisms, semigroup $\hat{\mathcal{O}} \cap \mathbb{A}^*_{\mathbb{K},f}$ by endomorphisms, $\mathcal{O}^\times$ by inner

• time evolution

$$\sigma_t(f)((\Lambda,\phi),(\Lambda',\phi')) = \left|\frac{Covol(\Lambda')}{Covol(\Lambda)}\right|^{it} f((\Lambda,\phi),(\Lambda',\phi'))$$

by $\mathfrak{n}(J)^{it}$ on commens class of invertible (positive energy)

• Arithmetic subalgebra from modular functions evaluated at CM points in $\mathbb{H}$ (using restriction from GL_2-system of Connes–Marcolli)

• Partition function $\zeta_{\mathbb{K}}(\beta)$ Dedekind zeta function

• Gibbs states low temperature; phase transition at $\beta = 1$; unique KMS above

• At zero temperature intertwining of symmetries and Galois action (Class Field Theory)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{K}^* & \longrightarrow & \mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f}) & \xrightarrow{\simeq} & \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & \mathbb{Q}^* & \longrightarrow & \mathrm{GL}_2(\mathbb{A}_f) & \xrightarrow{\simeq} & \mathrm{Aut}(F) & \longrightarrow & 1. \end{array}$$

using Shimura reciprocity and GL_2-system

General systems for number fields (Ha–Paugam)

$$\mathcal{A}_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}}, \quad \text{with} \quad X_{\mathbb{K}} := G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}},$$

$\hat{\mathcal{O}}_{\mathbb{K}}$ = ring of finite integral adeles, $J^+_{\mathbb{K}}$ = is the semigroup of ideals, acting on $X_{\mathbb{K}}$ by Artin reciprocity

- Crossed product algebra $\mathcal{A}_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}}$, generators and relations: $f \in C(X_{\mathbb{K}})$ and $\mu_{\mathfrak{n}}$, $\mathfrak{n} \in J^+_{\mathbb{K}}$

$$\mu_{\mathfrak{n}} \mu_{\mathfrak{n}}^* = e_{\mathfrak{n}}; \;\; \mu_{\mathfrak{n}}^* \mu_{\mathfrak{n}} = 1; \;\; \rho_{\mathfrak{n}}(f) = \mu_{\mathfrak{n}} f \mu_{\mathfrak{n}}^*;$$

$$\sigma_{\mathfrak{n}}(f) e_{\mathfrak{n}} = \mu_{\mathfrak{n}}^* f \mu_{\mathfrak{n}}; \;\; \sigma_{\mathfrak{n}}(\rho_{\mathfrak{n}}(f)) = f; \;\; \rho_{\mathfrak{n}}(\sigma_{\mathfrak{n}}(f)) = f e_{\mathfrak{n}}$$

- Artin reciprocity map $\vartheta_{\mathbb{K}} : \mathbb{A}_{\mathbb{K}}^* \to G_{\mathbb{K}}^{ab}$, write $\vartheta_{\mathbb{K}}(\mathfrak{n})$ for ideal $\mathfrak{n}$ seen as idele by non-canonical section s of

$$\mathbb{A}_{\mathbb{K},f}^* \twoheadrightarrow J_{\mathbb{K}} \ (\text{with section } s : J_{\mathbb{K}} \to \mathbb{A}_{\mathbb{K},f}^*) \qquad : \qquad (x_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$$

- semigroup action: $\mathfrak{n} \in J_{\mathbb{K}}^+$ acting on $f \in C(X_{\mathbb{K}})$ as

$$\rho_{\mathfrak{n}}(f)(\gamma, \rho) = f(\vartheta_{\mathbb{K}}(\mathfrak{n})\gamma, s(\mathfrak{n})^{-1}\rho) e_{\mathfrak{n}},$$

$e_{\mathfrak{n}} = \mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^*$ projector onto $[(\gamma, \rho)]$ with $s(\mathfrak{n})^{-1}\rho \in \hat{\mathcal{O}}_{\mathbb{K}}$

- partial inverse of semigroup action:

$$\sigma_{\mathfrak{n}}(f)(x) = f(\mathfrak{n} * x) \quad \text{with} \quad \mathfrak{n} * [(\gamma, \rho)] = [(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}\gamma, \mathfrak{n}\rho)]$$

- time evolution on $J_{\mathbb{K}}^+$ as phase factor $N(\mathfrak{n})^{it}$

$$\sigma_{\mathbb{K},t}(f) = f \quad \text{and} \quad \sigma_{\mathbb{K},t}(\mu_{\mathfrak{n}}) = N(\mathfrak{n})^{it}\, \mu_{\mathfrak{n}}$$

for $f \in C(G_{\mathbb{K}}^{ab} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}})$ and for $\mathfrak{n} \in J_{\mathbb{K}}^+$

Properties of QSM systems for number fields

• Partition function Dedekind $\zeta_{\mathbb{K}}(\beta)$; symmetry action of $G_{\mathbb{K}}^{ab}$

• Complete classification of KMS states (Laca–Larsen–Neshveyev): low temperature Gibbs states; phase transition at $\beta = 1$; unique high temperature state

• Reconstruction (Cornelissen–Marcolli): isomorphism of QSM systems $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (\mathcal{A}_{\mathbb{K}'}, \sigma_{\mathbb{K}'})$ preserving suitable algebraic subalgebra determines field isomorphism $\mathbb{K} \simeq \mathbb{K}'$

• Arithmetic subalgebra for Class Field Theory via endomotives (Yalkinoglu)

Gibbs states near the phase transition:

- Partition function = Dedekind zeta function has pole at $\beta = 1$
- Residue given by class number formula

$$\lim_{s\to 1}(s-1)\zeta_{\mathbb{K}}(s) = \frac{2^{r_1}(2\pi)^{r_2} h_{\mathbb{K}} \mathrm{Reg}_{\mathbb{K}}}{w_{\mathbb{K}}\sqrt{|D_{\mathbb{K}}|}}$$

with $[\mathbb{K}:\mathbb{Q}] = n = r_1 + 2r_2$ (r_1 real and r_2 pairs of complex embeddings), $h_{\mathbb{K}} = \#\mathrm{Cl}(\mathcal{O})$ class number; $\mathrm{Reg}_{\mathbb{K}}$ regulator; $w_{\mathbb{K}}$ numer of roots of unity in $\mathbb{K}$; $D_{\mathbb{K}}$ discriminant

- For a quadratic field $\zeta_{\mathbb{K}}(s) = \zeta(s)L(\chi,s)$ with $L(\chi,s)$ Dirichlet L-series with character $\chi(n) = (\frac{D_{\mathbb{K}}}{n})$ Legendre symbol

$$(\frac{a}{p}) = \begin{cases} 1 & a \text{ square mod } p \ (a \neq 0 \text{ mod } p) \\ -1 & a \text{ not square mod } p \\ 0 & a = 0 \text{ mod } p \end{cases}$$

• KMS states at low temperature

$$\varphi_{\gamma,\beta}(f) = \frac{1}{\zeta_{\mathbb{K}}(\beta)} \sum_{\mathfrak{n} \in J_{\mathbb{K}}^{+}} \frac{f(\mathfrak{n} * \gamma)}{N_{\mathbb{K}}(\mathfrak{n})^{\beta}}$$

• Also know (Cornelissen–Marcolli) L-functions with Grossencharacters related to values of KMS states: character $\chi \in \widehat{G}_{\mathbb{K}}^{ab}$ gives function

$$f_{\chi}(\gamma,\rho) := \begin{cases} \chi^{-1}(\gamma \vartheta_{\mathbb{K}}(\rho')) & \text{if } \forall v \mid \mathfrak{f}_{\chi}, \rho_v \in \hat{\mathcal{O}}_{\mathbb{K},v}^{*} \\ 0 & \text{otherwise,} \end{cases}$$

with $\rho' \in \hat{\mathcal{O}}_{\mathbb{K}}^{*}$ such that $\rho'_v = \rho_v$ for all $v \mid \mathfrak{f}_{\chi}$

$$\varphi_{\beta,\gamma}(f_{\chi}) = \frac{1}{\zeta_{\mathbb{K}}(\beta)\chi(\gamma)} \cdot L_{\mathbb{K}}(\chi,\beta)$$

• Case of CM field: values at $s = 1$ of L-functions with Grossencharacter (see Birch and Swinnerton–Dyer conjecture for CM elliptic curves)

From Class Field Theory to Conformal Field Theory

- Can one construct a (Rational) Conformal Field Theory from the data of the behavior near the phase transition of the KMS states of the QSM system for an imaginary quadratic field?

- There are RCFTs associated to imaginary quadratic fields (Gukov–Vafa)

- Near critical behavior of statistical systems often determines a conformal field theory (Ising model)

- Natural question: are the RCFT of imaginary quadratic fields related to the QSM systems of imaginary quadratic fields near criticality?

- All information about $\mathbb{K}$ is encoded in $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and in its low temperature KMS states, and the RCFTs are determined by data of $\mathbb{K}$

The Gukov–Vafa RCFTs

$$Z(q,\bar{q}) = \frac{1}{\eta^2\bar{\eta}^2} \sum_{(p,\bar{p})\in\Gamma^{(2,2)}} q^{p^2/2}\bar{q}^{\bar{p}^2/2}$$

sum over momentum lattice

$$\Gamma^{(2,2)} = \frac{i}{\sqrt{2\tau_2\rho_2}}\mathbb{Z}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \bar{\rho} \\ \rho \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \tau \\ \tau \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \bar{\rho}\tau \\ \rho\tau \end{pmatrix}$$

$\tau = \tau_1 + i\tau_2 \in \mathbb{H}$, $\rho = \rho_1 + i\rho_2 \in \mathbb{H}$ (complex and Kähler parameter), $E = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and Dedekind η-function

$$\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Properties of these CFTs

- Rational: $\Gamma_0 \subset \Gamma_L$ finite index sublattice

$$\Gamma_0 = \{p | \begin{pmatrix} p \\ 0 \end{pmatrix} \in \Gamma^{(2,2)}\} \quad \Gamma_L = \{p | \begin{pmatrix} p \\ \star \end{pmatrix} \in \Gamma^{(2,2)}\}$$

- Rational $\Leftrightarrow \tau, \rho \in \mathbb{Q}(\sqrt{D})$ $(D < 0)$ imaginary quadratic field (both the elliptic curve E_τ and its mirror are CM same field)

- diagonal form of partition function

$$Z(q, \bar{q}) = \sum_\omega \chi_\omega(q) \bar{\chi}_\omega(\bar{q})$$

when τ solution of $a\tau^2 + b\tau + c = 0$ discriminant $D = b^2 - 4ac$ and $\rho = \mathfrak{f} a\tau$ ($\rho \in \mathcal{O}_\mathfrak{f}$ order with conductor $\mathfrak{f}$)

- dim of chiral algebra $\mathfrak{f}^2 D$, where $\mathfrak{f}$ coeff of intersection form of Γ_0

$$\chi_\omega(q) = \frac{1}{\eta^2} \sum_{v \in \Gamma_0} q^{\frac{1}{2}(v+\omega)^2}$$

ω = reps of chiral algebra ($\omega \in \Gamma_L / \Gamma_0$)

Data that classify diagonal $c = 2$ RCFTs:

• $D < 0$ square free discriminant: $\mathbb{K} = \mathbb{Q}(\sqrt{D})$

• $\mathfrak{f} > 0$ conductor of order $\mathcal{O}_f$
$\mathfrak{f} = \{\alpha \in \mathcal{O} \,|\, \alpha\mathcal{O} \subset \mathcal{O}_\mathfrak{f}\}$: for imaginary quadratic gen over $\mathbb{Z}$ by multiple of discriminant ($\mathfrak{f}^2 D$ discriminant of order)

• an element in the class group $\mathrm{Cl}(\mathcal{O})$ (equiv class of integral binary quadratic form discriminant D); $\mathrm{Cl}(\mathcal{O}) = \mathrm{Gal}(\mathbb{K}(j(\tau))/\mathbb{K})$ acts transitively on the $j(\tau)$

$\Rightarrow$ extract these data from low temperature KMS states of QSM system for imaginary quadratic field

Gukov–Vafa RCFTs and the QSM system for CM fields

- Elements of $\mathrm{Cl}(\mathcal{O})$ from action by symmetries

$$1 \to \hat{\mathcal{O}}^*/\mathcal{O}^* \to \mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^* \to \mathrm{Cl}(\mathcal{O}) \to 1$$

where $\hat{\mathcal{O}}^*/\mathcal{O}^*$ automorphisms

$$\theta_s(f)((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} f((\Lambda, s^{-1}\phi),(\Lambda', s^{-1}\phi')) & \text{both divisible by } J \\ 0 & \text{otherwise} \end{cases}$$

for $s \in \hat{\mathcal{O}} \cap \mathbb{A}^*_{\mathbb{K},f}$ and $J = s\hat{\mathcal{O}} \cap \mathbb{K}$; divisibility by J: ϕ well defined modulo $J\Lambda$

- Also Λ (as $\mathcal{O}$-module) determines a class in $K_0(\mathcal{O})$; for invertible $\mathbb{K}$-lattice invariant of commensurability; $K_0(\mathcal{O}) = \mathbb{Z} + \mathrm{Cl}(\mathcal{O})$ (here rank $\in \mathbb{Z}$ is one so just $\mathrm{Cl}(\mathcal{O})$)

- low temperature KMS states $\Leftrightarrow$ invertible $\mathbb{K}$-lattices

Orders

• $\mathcal{O}_\mathfrak{f} \subset \mathcal{O}$ subring and $\mathbb{Z}$-module rank $[\mathbb{K} : \mathbb{Q}]$

• an order $\mathcal{O}_\mathfrak{f}$ in an imaginary quadratic field $\mathbb{K}$ determines an abelian extension $\mathbb{L}$ of $\mathbb{K}$, unramified over the conductor $\mathfrak{f}$ with $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = \mathrm{Cl}(\mathcal{O}_\mathfrak{f})$ (ring class field of $\mathcal{O}_\mathfrak{f}$)

• see abelian extensions and ramification from KMS states

• can describe ramification through ranges of projectors $e_{\mathbb{K},\mathfrak{n}} = \mu_\mathfrak{n}\mu_\mathfrak{n}^*$ and abelian extensions through the L-functions