Arithmetic Structures in Spectral Models of Gravity

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MPIM Bonn, December 23, 2016

References:

• Farzad Fathizadeh, Matilde Marcolli, *Periods and motives in the spectral action of Robertson-Walker spacetimes*, arXiv:1611.01815

• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Spectral Action for Bianchi type-IX cosmological models*, arXiv:1506.06779, J. High Energy Phys. (2015) 85, 28 pp.

• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Modular forms in the spectral action of Bianchi IX gravitational instantons*, arXiv:1511.05321

Spectral action models of gravity (modified gravity)

• Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$

1. unital associative algebra $\mathcal{A}$
2. represented as bounded operators on a Hilbert space $\mathcal{H}$
3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$

• prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$

• extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)

Action functional

- Suppose *finitely summable* $ST = (\mathcal{A}, \mathcal{H}, D)$

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) < \infty, \quad \Re(s) >> 0$$

- Spectral action (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

f = smooth approximation to (even) cutoff

Asymptotic expansion (Chamseddine–Connes) for (almost) commutative geometries:

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma^+_{ST}} f_\beta \, \Lambda^\beta \, ⨍ |D|^{-\beta} \; + \; f(0)\, \zeta_D(0)$$

- Residues
$$⨍ |D|^{-\beta} = \frac{1}{2} \mathrm{Res}_{s=\beta} \, \zeta_D(s)$$

- Momenta $f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$

- Dimension Spectrum Σ_{ST} poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$

- positive dimension spectrum $\Sigma^+_{ST} = \Sigma_{ST} \cap \mathbb{R}^*_+$

Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha} \zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

Pseudo-differential Calculus: (manifold case)
to obtain *full* asymptotic expansion of the Spectral Action

- Dirac operator D and pseudodifferential symbol of D^2

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

each p_k homogeneous of order k in ξ

- Cauchy integral formula

$$e^{-tD^2} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} \, (D^2 - \lambda)^{-1} \, d\lambda$$

- Seeley de-Witt coefficients ($m = \dim M$)

$$\mathrm{Tr}(e^{-tD^2}) \sim_{t\to 0+} \quad t^{-m/2} \sum_{n=0}^{\infty} a_{2n}(D^2) t^n$$

Parametrix Method

- D^2 order 2 elliptic differential operator: exists a parametrix R_λ with

$$\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x,\xi,\lambda)$$

- $r_j(x,\xi,\lambda)$ pseudodifferential symbol order $-2-j$

$$r_j(x,t\xi,t^2\lambda) = t^{-2-j} r_j(x,\xi,\lambda)$$

- R_λ approximates $(D^2-\lambda)^{-1}$ with $\sigma((D^2-\lambda)R_\lambda) \sim 1$
- recursive equation:

$$\sigma((D^2-\lambda)R_\lambda) \sim ((p_2(x,\xi)-\lambda)+p_1(x,\xi)+p_0(x,\xi))\circ\left(\sum_{j=0}^{\infty} r_j(x,\xi,\lambda)\right) \sim 1$$

- solution for R_λ constructed recursively:

$$r_0(x,\xi,\lambda) = (p_2(x,\xi) - \lambda)^{-1}$$

$$r_n(x,\xi,\lambda) = -\sum \frac{1}{\alpha!}\partial_\xi^\alpha r_j(x,\xi,\lambda)\, D_x^\alpha p_k(x,\xi)\, r_0(x,\xi,\lambda),$$

summation over all $\alpha \in \mathbb{Z}^4_{\geq 0}, j \in \{0,1,\ldots,n-1\}, k \in \{0,1,2\}$, with $|\alpha| + j + 2 - k = n$

Seeley-deWitt coefficients and Parametrix Method

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i}\int\int_\gamma e^{-\lambda}\,\mathrm{tr}\,(r_{2n}(x,\xi,\lambda))\; d\lambda\, d^m\xi$$

- odd j coefficients vanish: $r_j(x,\xi,\lambda)$ odd function of ξ

A different method: Wodzicki residue

• Wodzicki residue: unique trace functional on algebra of pseudodifferential operators on smooth sections of vector bundle over smooth manifold

• classical pseudodifferential operator P_σ of order $d \in \mathbb{Z}$ local symbol

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi) \qquad (\xi \to \infty),$$

σ_{d-j} positively homogeneous order $d-j$ in ξ

• Residue:

$$\mathrm{Res}(P_\sigma) = \int_{S^*M} \mathrm{Tr}\left(\sigma_{-m}(x,\xi)\right) \, d^{m-1}\xi \, d^m x,$$

$S^*M = \{(x,\xi) \in T^*M; ||\xi||_g = 1\}$ cosphere bundle

• spectral formulation of residue: pseudodifferential operator P_σ, Laplacian Δ

$$P_\sigma \mapsto \mathrm{Res}_{s=0}\mathrm{Tr}(P_\sigma \Delta^{-s})$$

same up to a constant $c_m = 2^{m+1}\pi^m$

• Mellin transform (for simplicity $\mathrm{Ker}(\Delta) = 0$):

$$\mathrm{Tr}(\Delta^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^s \,\frac{dt}{t}$$

• heat kernel expansion

$$\mathrm{Tr}\left(e^{-t\Delta}\right) = t^{-m/2} \sum_{n=0}^{N} a_{2n} t^n + O(t^{-m/2+N+1})$$

- find for any non-negative integer $n \leq m/2 - 1$:

$$\mathrm{Res}_{s=m/2-n} \mathrm{Tr}(\Delta^{-s}) = \frac{a_{2n}(\Delta)}{\Gamma(m/2-n)},$$

- in particular

$$\mathrm{Res}_{s=1} \mathrm{Tr}(\Delta^{-s}) = a_{m-2}(\Delta)$$

- in terms of Wodzicki residue:

$$a_{m-2}(\Delta) = \frac{1}{c_m} \mathrm{Res}(\Delta^{-1}) = \frac{1}{2^{m+1}\pi^m} \mathrm{Res}(\Delta^{-1})$$

applied to $\Delta = D^2$

• coefficient $a_2(D^2)$

$$a_2(D^2) = \frac{1}{c_4}\mathrm{Res}(D^{-2}) = \frac{1}{32\pi^4}\int_{S^*M} \mathrm{Tr}\left(\sigma_{-4}(D^{-2})\right)\, d^3\xi\, d^4x$$

• for other coefficients, introduce an auxiliary product space for correct counting of dimensions: use flat r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$

$$\Delta = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

$\Delta_{\mathbb{T}^r}$ flat Laplacian on $\mathbb{T}^r$

$$a_{2+r}(D^2) = \frac{1}{2^5\,\pi^{4+r/2}}\mathrm{Res}(\Delta^{-1})$$

because Künneth formula gives

$$a_{2+r}((x,x'),\Delta) = a_{2+r}(x,D^2)a_0(x',\mathbb{T}^r) = 2^{-r}\pi^{-r/2}a_{2+r}(x,D^2)$$

with volume term only non-zero heat coefficient for flat metric

- obtain for all coefficients

$$a_{2+r}(D^2) = \frac{1}{2^5 \pi^{4+r/2}} \int \mathrm{Tr}\left(\sigma_{-4-r}(\Delta^{-1})\right) d^{3+r}\xi \, d^4x.$$

- writing $\sigma(\Delta^{-1}) \sim \sum_{j=-2}^{-\infty} \sigma_j(x,\xi)$ inductively

$$\begin{aligned}
\sigma_{-2}(x,\xi) &= p_2'(x,\xi)^{-1}, \\
\sigma_{-2-n}(x,\xi) &= -\sum \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_j(x,\xi)\, D_x^\alpha p_k(x,\xi)\, \sigma_{-2}(x,\xi) \qquad (n>0),
\end{aligned}$$

summation over all multi-indices non-negative integers α, $-2-n < j \leq -2, 0 \leq k \leq 2$, with $|\alpha| - j - k = n$

Robertson–Walker spacetime

- Topologically $S^3 \times \mathbb{R}$
- Metric (Euclidean)

$$ds^2 = dt^2 + a(t)^2 d\sigma^2$$

scaling factor $a(t)$, round metric $d\sigma^2$ on S^3

- Hopf coordinates on S^3

$$x = (t, \eta, \phi_1, \phi_2) \mapsto (t, \sin\eta\cos\phi_1, \sin\eta\sin\phi_2, \cos\eta\cos\phi_1, \cos\eta\sin\phi_2),$$

$$0 < \eta < \frac{\pi}{2}, \qquad 0 < \phi_1 < 2\pi, \qquad 0 < \phi_2 < 2\pi.$$

- Robertson-Walker metric in Hopf coordinates

$$ds^2 = dt^2 + a(t)^2 \left(d\eta^2 + \sin^2(\eta)\, d\phi_1^2 + \cos^2(\eta)\, d\phi_2^2 \right)$$

Dirac operator

• orthonormal coframe $\{\theta^a\}$

$$D = \sum_a \theta^a \nabla^S_{\theta_a}$$

• spin connection ∇^S with matrix of 1-forms $\omega = (\omega^a_b)$ with

$$\nabla \theta^a = \sum_b \omega^a_b \otimes \theta^b$$

• metric-compatibility and torsion-freeness (Levi–Civita connection)

$$\omega^a_b = -\omega^b_a, \qquad d\theta^a = \sum_b \omega^a_b \wedge \theta^b$$

• Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega_{ac}^b \gamma^a \gamma^b$$

with $\omega_a^b = \sum_c \omega_{ac}^b \theta^c$

• matrices γ^a Clifford action of θ^a on spin bundle:
$(\gamma^a)^2 = -I$
$\gamma^a \gamma^b + \gamma^b \gamma^a = 0$ for $a \neq b$

Pseudodifferential Symbol $\sigma_D(x,\xi)$ of Dirac operator D sum $q_1(x,\xi)+q_0(x,\xi)$ with $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)\in T_x^*M\simeq\mathbb{R}^4$ cotangent fiber at $x=(t,\eta,\phi_1,\phi_2)$

$$q_1(x,\xi) = \begin{pmatrix} 0 & 0 & \frac{i\sec(\eta)\xi_4}{a(t)}-\xi_1 & \frac{i\xi_2}{a(t)}+\frac{\csc(\eta)\xi_3}{a(t)} \\ 0 & 0 & \frac{i\xi_2}{a(t)}-\frac{\csc(\eta)\xi_3}{a(t)} & -\xi_1-\frac{i\sec(\eta)\xi_4}{a(t)} \\ -\xi_1-\frac{i\sec(\eta)\xi_4}{a(t)} & -\frac{i\xi_2}{a(t)}-\frac{\csc(\eta)\xi_3}{a(t)} & 0 & 0 \\ \frac{\csc(\eta)\xi_3}{a(t)}-\frac{i\xi_2}{a(t)} & \frac{i\sec(\eta)\xi_4}{a(t)}-\xi_1 & 0 & 0 \end{pmatrix},$$

$$q_0(\xi) = \begin{pmatrix} 0 & 0 & \frac{3ia'(t)}{2a(t)} & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} \\ 0 & 0 & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} \\ \frac{3ia'(t)}{2a(t)} & \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & 0 & 0 \\ \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} & 0 & 0 \end{pmatrix}. \tag{1}$$

Pseudodifferential symbol of square D^2 of Dirac operator:

$$\sigma_{D^2}(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

$$p_2(x,\xi) = q_1(x,\xi)\, q_1(x,\xi) = \left(\sum g^{\mu\nu}\xi_\mu\xi_\nu\right) I_{4\times 4}$$

$$= \left(\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\csc^2(\eta)\xi_3^2}{a(t)^2} + \frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right) I_{4\times 4},$$

$$p_1(x,\xi) = q_0(x,\xi)\, q_1(x,\xi) + q_1(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_1}{\partial x_j}(x,\xi),$$

$$p_0(x,\xi) = q_0(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_0}{\partial x_j}(x,\xi).$$

The a_2 term

• 1-density (unit cotangent sphere bundle integral)

$$\mathrm{wres}_x P_\sigma = \left(\int_{|\xi|=1} \mathrm{tr}\,(\sigma_{-m}(x,\xi))\, |\sigma_{\xi,\, m-1}| \right) |dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{m-1}|$$

• Wodzicki residue of ΨDO P_σ

$$\mathrm{Res}\,(P_\sigma) = \int_M \mathrm{wres}_x P_\sigma$$

• $a_2(D^2)$ coefficient, with $(D^2)^{-1}$ parametrix

$$a_2 = \frac{1}{2^5\,\pi^4}\,\mathrm{Res}\left((D^2)^{-1}\right),$$

• dimension of manifold is 4: need term $\sigma_{-4}(x,\xi)$ homogeneous order -4 in expansion of symbol of $(D^2)^{-1}$

• computer calculation of $\mathrm{tr}(\sigma_{-4}(x,\xi))$ takes a couple of pages to write out (sum of fractions involving trigonometric functions and powers of ξ_i, scaling factor $a(t)$ and derivative)

• important properties of resulting expression:

- each term with an odd power of ξ_j in numerator will integrate to 0 in integration of 1-density
- numerical coefficients of all terms in integrand are *rational numbers*
- treat scaling factor $a(t)$ and derivative $a'(t)$, $a''(t)$ as affine variables $\alpha, \alpha_1, \alpha_2$ (integration without performing time integration)
- there is a natural change of coordinates replacing trigonometric functions by polynomials: rational function

change of coordinates

$$u_0 = \sin^2(\eta), \qquad u_1 = \xi_1, \qquad u_2 = \xi_2,$$
$$u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4,$$

Then have

$$\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\xi_3^2 \csc^2(\eta)}{a(t)^2} + \frac{\xi_4^2 \sec^2(\eta)}{a(t)^2} = u_1^2 + \frac{1}{a(t)^2}(u_2^2 + u_3^2 + u_4^2),$$

$$\cot^2(\eta) = \frac{1-u_0}{u_0},$$

$$\csc^2(\eta) = \frac{1}{u_0},$$

$$\sec^2(\eta) = \frac{1}{1-u_0},$$

$$\cot(\eta)\cot(2\eta) = \frac{\cot^2(\eta)}{2} - \frac{1}{2},$$

$$\csc^2(2\eta) = \frac{1}{4}\csc^2(\eta)\sec^2(\eta),$$

$$\tan^2(\eta) = \sec^2(\eta) - 1,$$

$$\tan(\eta)\cot(2\eta) = \frac{1}{2} - \frac{\tan^2(\eta)}{2},$$

$$\cot^2(2\eta) = \frac{\tan^2(\eta)}{8} + \frac{\cot^2(\eta)}{8} + \frac{1}{8}\csc^2(\eta)\sec^2(\eta) - \frac{3}{4}.$$

Also exponents of the variables ξ_j are even positive integers

a_2-term as a period integral $C \cdot \int_{A_4} \Omega^{\alpha}_{(\alpha_1,\alpha_2)}$ with $C \in \mathbb{Q}[(2\pi i)^{-1}]$

- Algebraic differential form

$$\Omega = f\,\widetilde{\sigma}_3,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2) \in \mathbb{A}^2$

- functions $f(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = f_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2))^\ell}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = P_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2]$ with r, k, m and ℓ non-negative integers

• algebraic differential form $\widetilde{\sigma}_3 = \widetilde{\sigma}_3(u_0, u_1, u_2, u_3, u_4)$

$$\frac{1}{2}\Big(u_1\, du_0\, du_2\, du_3\, du_4 - u_2\, du_0\, du_1\, du_3\, du_4 + u_3\, du_0\, du_1\, du_2\, du_4 - u_4\, du_0\, du_1\, du_2\, du_3\Big)$$

• forms $\Omega^\alpha = \Omega^\alpha_{(\alpha_1,\alpha_2)}$ restricting to fixed value of $\alpha \in \mathbb{A}^1 \smallsetminus \{0\}$: two parameter family

• defined on the complement in $\mathbb{A}^5$ of union of two affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and hypersurface $\widehat{CZ}_\alpha$ defined by vanishing of the quadratic form

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2)$$

• $\mathbb{Q}$-semialgebraic set: subset S of some $\mathbb{R}^n$

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, P(x_1, \dots, x_n) \geq 0\},$$

for some polynomial $P \in \mathbb{Q}[x_1, \dots, x_n]$, and complements, intersections, unions

• domain of integration $\mathbb{Q}$-semialgebraic set

$$A_4 = \left\{ (u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5(\mathbb{R}) \,:\, \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0) u_4^2 = 1, \\ 0 < u_i < 1, \text{ for } i = 0, 1, 2 \end{array} \right\}$$

a_4-term and Wodzicki Residue

$$a_4 = \frac{1}{2^5 \pi^5} \mathrm{Res}(\Delta_4^{-1})$$

need $\mathrm{tr}(\sigma_{-6}(\Delta_4^{-1}))$ of order -6 in expansion of symbol of Δ_4^{-1}

- general recursive procedure with auxiliary flat tori T^r

$$\Delta_{r+2} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r}$$

$$\sigma_{-2}(\Delta_{r+2}^{-1}) = \left(p_2(x, \xi_1, \xi_2, \xi_3, \xi_4) + (\xi_5^2 + \cdots + \xi_{4+r}^2) I_{4\times 4}\right)^{-1}$$

then recursively $\sigma_{-2-n}(\Delta_{r+2}^{-1})$ given by

$$-\left(\sum_{\substack{0\leq j<n,\, 0\leq k\leq 2 \\ \alpha\in\mathbb{Z}^4_{\geq 0} \\ -2-j-|\alpha|+k=-n}} \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \sigma_{-2-j}(\Delta_{r+2}^{-1})\right) \left(\partial_x^\alpha p_k\right) \right) \sigma_{-2}(\Delta_{r+2}^{-1}).$$

a_4-term as a period integral $C \cdot \int_{A_6} \Omega^{\alpha}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

- algebraic differential form

$$\Omega = f\, \widetilde{\sigma}_5,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{A}^7$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{A}^4$

- functions $f_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)}{\alpha^{2r} u_0^k (1 - u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2)^\ell}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, u_5, u_6\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$ where r, k, m and ℓ non-negative integers

• algebraic form $\widetilde{\sigma}_5 = \widetilde{\sigma}_5(u_0, u_1, u_2, u_3, u_4, u_5, u_6)$

$$\begin{aligned}\widetilde{\sigma}_5 \quad = \quad & \frac{1}{2}\Big(u_1\, du_0\, du_2\, du_3\, du_4\, du_5\, du_6 - u_2\, du_0\, du_1\, du_3\, du_4\, du_5\, du_6 \\ & + u_3\, du_0\, du_1\, du_2\, du_4\, du_5\, du_6 - u_4\, du_0\, du_1\, du_2\, du_3\, du_5\, du_6 \\ & + u_5\, du_0\, du_1\, du_2\, du_3\, du_4\, du_6 - u_6\, du_0\, du_1\, du_2\, du_3\, du_4\, du_5 \Big).\end{aligned}$$

• forms $\Omega^\alpha = \Omega^\alpha_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ restricting to a fixed $\alpha \in \mathbb{A}^1 \smallsetminus \{0\}$: four-parameter family

• domain of definition complement in $\mathbb{A}^7$ of the union of the affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and the hypersurface $\widehat{CZ}_\alpha$ defined by the vanishing of the quadratic form

$$Q_{\alpha,4} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2$$

- domain of integration $\mathbb{Q}$-semialgebraic set

$$A_6 = \left\{ (u_0, \ldots, u_6) \in \mathbb{A}^7(\mathbb{R}) \, : \, \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1 - u_0) u_4^2 + u_5^2 + u_6^2 = 1 \\ 0 < u_i < 1, \quad i = 0, 1, 2, 5, 6 \end{array} \right\}$$

- the change of variables used here

$$u_0 = \sin^2(\eta), \qquad u_1 = \xi_1, \qquad u_2 = \xi_2$$

$$u_3 = \csc(\eta)\, \xi_3, \qquad u_4 = \sec(\eta)\, \xi_4, \qquad u_5 = \xi_5, \qquad u_6 = \xi_6$$

higher order terms a_{2n}

$$a_{2n} = \frac{1}{2^5 \, \pi^{3+n}} \mathrm{Res}(\Delta_{2n}^{-1})$$

using auxiliary flat tori T^{2n-2}

$$\Delta_{2n} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^{2n-2}}$$

need term σ_{-2n-2} homogeneous of order $-2n-2$ in expansion of pseudodifferential symbol of parametrix Δ_{2n}^{-1}

- recursive argument for structure of term σ_{-2n-2}

- term $\mathrm{tr}(\sigma_{-2n-2})$ given by

$$\sum_{j=1}^{M_n} c_{j,2n}\, u_0^{\beta_{0,1,j}/2}\,(1-u_0)^{\beta_{0,2,j}/2}\,\frac{u_1^{\beta_{1,j}}\, u_2^{\beta_{2,j}}\,\cdots\, u_{2n+2}^{\beta_{2n+2,j}}}{Q_{\alpha,2n}^{\rho_{j,2n}}}\,\alpha^{k_{0,j}}\,\alpha_1^{k_{1,j}}\,\cdots\,\alpha_{2n}^{k_{2n,j}},$$

where

$$\alpha = a(t), \qquad \alpha_1 = a'(t), \qquad \alpha_2 = a''(t), \qquad \ldots \qquad \alpha_{2n} = a^{2n}(t),$$

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2+u_3^2+u_4^2) + u_5^2 + \cdots + u_{2n+2}^2,$$

$$c_{j,2n}\in\mathbb{Q}, \quad \beta_{0,1,j},\beta_{0,2,j},k_{0,j}\in\mathbb{Z}, \quad \beta_{1,j},\ldots,\beta_{2n+2,j},\rho_{j,2n},k_{1,j},\ldots,k_{2n,j}\in\mathbb{Z}_{\geq 0}.$$

- using change of coordinates

$$u_0 = \sin^2(\eta), \qquad u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4$$

$$u_j = \xi_j, \qquad j = 1,2,5,6,\ldots,2n+2$$

a_{2n}-terms as period integrals $C \cdot \int_{A_{2n}} \Omega^{\alpha}_{\alpha_1,\ldots,\alpha_{2n}}$

- algebraic differential form

$$\Omega^{\alpha}_{\alpha_1,\ldots,\alpha_{2n}}(u_0, u_1, \ldots, u_{2n+2})$$

- domain of definition complement

$$\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)$$

with hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and $\widehat{CZ}_{\alpha,2n}$ the hypersurface defined by the vanishing of the quadric

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2$$

- $\mathbb{Q}$-semialgebraic set A_{2n+2}

$$A_{2n+2} = \left\{ (u_0, \ldots, u_{2n+2}) \in \mathbb{A}^{2n+3}(\mathbb{R}) \ : \ \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1 - u_0) u_4^2 + \sum_{i=5}^{2n+2} u_i^2 = 1 \\ 0 < u_i < 1, \quad i = 0, 1, 2, 5, 6, \ldots, 2n+2 \end{array} \right\}$$

Mixed Motives associated to these periods

$$\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

divisor Σ containing boundary of domain of integration A_{2n}

- motives of quadrics (Rost, Vishik)
  - hyperbolic form $\mathbb{H} := \langle 1, -1 \rangle$
  - $Q = d \cdot \mathbb{H}$ of dimension $2d$

$$\mathfrak{m}(Z_{d\mathbb{H}}) = \mathbb{Z}(d-1)[2d-2] \oplus \mathbb{Z}(d-1)[2d-2] \oplus \bigoplus_{i=0,\ldots,d-2,d,\ldots,2d-2} \mathbb{Z}(i)[2i]$$

  - $Q = d \cdot \mathbb{H} \perp \langle 1 \rangle$ in dimension $2d+1$

$$\mathfrak{m}(Z_{d\mathbb{H}\perp\langle 1\rangle}) = \bigoplus_{i=0,\ldots,2d-1} \mathbb{Z}(i)[2i]$$

  - if $\exists$ quadratic field extension $\mathbb{K}$ where Q hyperbolic

$$\mathfrak{m}(Z_Q) = \begin{cases} \mathfrak{m}_1 \oplus \mathfrak{m}_1(1)[2] & m = 2 \mod 4 \\ \mathfrak{m}_1 \oplus \mathcal{R}_{Q,\mathbb{K}} \oplus \mathfrak{m}_1(1)[2] & m = 0 \mod 4 \end{cases}$$

involving *forms of Tate motives*

- quadratic field extension $\mathbb{Q}(\sqrt{-1})$, assuming $\alpha \in \mathbb{Q}^*$

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2)$$

change of variables

$$X = u_1 + \frac{i}{\alpha} u_2, \quad Y = u_1 - \frac{i}{\alpha} u_2, \quad Z = \frac{i}{\alpha}(u_3 + iu_4), \quad W = \frac{i}{\alpha}(u_3 - iu_4)$$

identification of Z_α with the Segre quadric

$$\{XY - ZW = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

- similar for a_{2n}-term case

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2 + \cdots + u_{2n+1}^2 + u_{2n+2}^2$$

inductively: change of coordinates

$$X = u_{2n+1} + iu_{2n+2}, \qquad Y = u_{2n+1} - iu_{2n+2}$$

puts $Q_{\alpha,2n}$ in the form

$$Q_{\alpha,2n} = Q_{\alpha,2n-2}(u_1, \ldots, u_{2n}) + XY.$$

classes in the Grothendieck ring

- $Z_{\alpha,2n}$ quadric in $\mathbb{P}^{2n+1}$ determined by $Q_{\alpha,2n}$

$$[\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n+1} - \mathbb{L}^n$$

$$[\mathbb{A}^{2n+3} \smallsetminus \widehat{CZ}_{\alpha,2n}] = \mathbb{L}^{2n+3} - \mathbb{L}^{2n+2} - \mathbb{L}^{n+2} + \mathbb{L}^{n+1}$$

$$[\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)] = \mathbb{L}^{2n+3} - 3\mathbb{L}^{2n+2} + 2\mathbb{L}^{2n+1} - \mathbb{L}^{n+2} + 3\mathbb{L}^{n+1} - 2\mathbb{L}^n$$

- based on an inductive argument using identities

1. $[\mathbb{A}^N \smallsetminus \hat{Z}] = (\mathbb{L}-1)[\mathbb{P}^{N-1} \smallsetminus Z]$
2. $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = (\mathbb{L}-1)[\mathbb{P}^N \smallsetminus CZ]$
3. $[CZ] = \mathbb{L}\,[Z] + 1$
4. $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = \mathbb{L}^{N+1} - \mathbb{L}(\mathbb{L}-1)[Z] - \mathbb{L}$
5. $[\mathbb{A}^{N+1} \smallsetminus (\widehat{CZ} \cup H \cup H')] = \mathbb{L}^{N+1} - 2\mathbb{L}^N - (\mathbb{L}-2)(\mathbb{L}-1)[Z] - (\mathbb{L}-2)$.

with $Z \subset \mathbb{P}^{N-1}$, $\hat{Z} \subset \mathbb{A}^N$ affine cone, CZ projective cone in $\mathbb{P}^N$, H and H' affine hyperplanes with $H \cap H' = \emptyset$, intersections $\widehat{CZ} \cap H$ and $\widehat{CZ} \cap H'$ sections $\hat{Z}$ of cone

Mixed Tate

- mixed motive (over field $\mathbb{Q}(\sqrt{-1})$)

$$\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

is mixed Tate

- over $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ quadratic form

$$Q_{\alpha,2n}|_{\mathbb{Q}(\sqrt{-1})} = (n+1) \cdot \mathbb{H},$$

so motive

$$\mathfrak{m}(Z_{\alpha,2n}|_{\mathbb{K}}) = \mathbb{Z}(n)[2n] \oplus \mathbb{Z}(n)[2n] \oplus \bigoplus_{i=0,\ldots,n-1,n+1,\ldots 2n} \mathbb{Z}(i)[2i]$$

- rest of the argument shown in example of a_2 for simplicity

• $\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)$ is mixed Tate

$$\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha) \to \mathfrak{m}(\mathbb{P}^3) \to \mathfrak{m}(Z_\alpha)(1)[2] \to \mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)[1]$$

Gysin distinguished triangle of the closed codim one embedding $Z_\alpha \hookrightarrow \mathbb{P}^3$

• projective cone CZ_α in $\mathbb{P}^4$: homotopy invariance for $\mathbb{A}^1$-fibration $\mathbb{P}^4 \smallsetminus CZ_\alpha \to \mathbb{P}^3 \smallsetminus Z_\alpha$

$$\mathfrak{m}_c^j(\mathbb{P}^4 \smallsetminus CZ_\alpha) = \mathfrak{m}_c^{j-2}(\mathbb{P}^3 \smallsetminus Z_\alpha)(-1)$$

motive $\mathfrak{m}(\mathbb{P}^4 \smallsetminus CZ_\alpha)$ also mixed Tate

• motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ mixed Tate: $\mathbb{P}^1$-bundle $\mathcal{P}$ compactification of $\mathbb{G}_m$-bundle

$$\mathcal{T} = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha \to X = \mathbb{P}^4 \smallsetminus CZ_\alpha$$

and Gysin distinguished triangle

$$\mathfrak{m}(\mathcal{T}) \to \mathfrak{m}(\mathcal{P}) \to \mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})^*(1)[2] \to \mathfrak{m}(\mathcal{T})[1]$$

$\mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})$ mixed Tate since $\mathcal{P} \smallsetminus \mathcal{T}$ two copies of X, so $\mathfrak{m}(\mathcal{T})$ mixed Tate

• union $\widehat{CZ}_\alpha \cup H_0 \cup H_1$ is mixed Tate: motives $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ and $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ and motive of intersection $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ are mixed Tate

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(U \cup V) \to \mathfrak{m}(U \cap V)[1]$$

Mayer-Vietoris distinguished triangle with $U = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha$ and $V = \mathbb{A}^5 \smallsetminus (H_0 \cup H_1)$

- $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ mixed Tate by previous
- $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ also mixed Tate since $\mathfrak{m}(H_0 \cup H_1)$ is
- $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ mixed Tate because intersection $\widehat{CZ}_\alpha \cap (H_0 \cup H_1)$ two sections of the cone and $\mathfrak{m}(\hat{Z}_\alpha)$ Tate
- then also $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ mixed Tate

• divisor Σ in $\mathbb{A}^5$ is a union of coordinate hyperplanes and their translates: mixed Tate
• $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1), \Sigma)$ also mixed Tate: distinguished triangle with $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ and $\mathfrak{m}(\Sigma)$

$SU(2)$-Bianchi IX cosmologies (Euclidean, compactified)

• another version of Bianchi IX mixmaster cosmologies, with $SU(2)$ symmetry (Euclidean version)

$$g = w_1 w_2 w_3 \, dt^2 + \frac{w_2 w_3}{w_1}\,\sigma_1^2 + \frac{w_1 w_3}{w_2}\,\sigma_2^2 + \frac{w_1 w_2}{w_3}\,\sigma_3^2$$

with $w_i = w_i(t)$, or more generally

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with a conformal factor $F \sim w_1 w_2 w_3$

• $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i,j,k) of $(1,2,3)$

- more explicitly ds^2 is

$$w_1 w_2 w_3\, dt\, dt + \frac{w_1 w_2 \cos(\eta)}{w_3} d\phi\, d\psi + \frac{w_1 w_2 \cos(\eta)}{w_3} d\psi\, d\phi$$
$$+ \left(\frac{w_2 w_3 \sin^2(\eta) \cos^2(\psi)}{w_1} + w_1 \left(\frac{w_3 \sin^2(\eta) \sin^2(\psi)}{w_2} + \frac{w_2 \cos^2(\eta)}{w_3} \right) \right) d\phi\, d\phi$$
$$+ \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta) \sin(\psi) \cos(\psi)}{w_1 w_2} d\eta\, d\phi + \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta) \sin(\psi) \cos(\psi)}{w_1 w_2} d\phi\, d\eta$$
$$+ \left(\frac{w_2 w_3 \sin^2(\psi)}{w_1} + \frac{w_1 w_3 \cos^2(\psi)}{w_2} \right) d\eta\, d\eta + \frac{w_1 w_2}{w_3} d\psi\, d\psi$$

- identifying S^3 with unit quaternions $SU(2)$
- The metrics on S^3

$$\frac{w_2 w_3}{w_1}\, \sigma_1^2 + \frac{w_1 w_3}{w_2}\, \sigma_2^2 + \frac{w_1 w_2}{w_3}\, \sigma_3^2$$

are left-invariants under the action of $SU(2)$ but *not* right-invariant (unlike the round metric on S^3)

Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b$$

with $\omega^b_a = \sum_c \omega^b_{ac} \theta^c$

- matrices γ^a Clifford action of θ^a on spin bundle:

$(\gamma^a)^2 = -I$
$\gamma^a\gamma^b + \gamma^b\gamma^a = 0$ for $a \neq b$

Dirac operator on Bianchi IX metrics

- local coordinates $(x^\mu) = (t, \eta, \phi, \psi)$ with $\mathbb{S}^3$ parametrized by

$$(\eta, \phi, \psi) \mapsto \left(\cos(\eta/2) e^{i(\phi+\psi)/2}, \sin(\eta/2) e^{i(\phi-\psi)/2} \right)$$

with $0 \le \eta \le \pi, 0 \le \phi < 2\pi, 0 \le \psi < 4\pi$.

- orthonormal frame

$$\begin{aligned}
\theta^0 &= \sqrt{w_1 w_2 w_3}\, dt, \\
\theta^1 &= \sin(\eta) \cos(\psi) \sqrt{\frac{w_2 w_3}{w_1}}\, d\phi - \sin(\psi) \sqrt{\frac{w_2 w_3}{w_1}}\, d\eta, \\
\theta^2 &= \sin(\eta) \sin(\psi) \sqrt{\frac{w_1 w_3}{w_2}}\, d\phi + \cos(\psi) \sqrt{\frac{w_1 w_3}{w_2}}\, d\eta, \\
\theta^3 &= \cos(\eta) \sqrt{\frac{w_1 w_2}{w_3}}\, d\phi + \sqrt{\frac{w_1 w_2}{w_3}}\, d\psi.
\end{aligned}$$

• non-vanishing ω_{ac}^{b}

$$\omega_{11}^{0} = -\frac{w_2\left(w_1 w_3' - w_3 w_1'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega_{22}^{0} = -\frac{w_2\left(w_3 w_1' + w_1 w_3'\right) - w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}},$$

$$\omega_{33}^{0} = -\frac{w_2\left(w_3 w_1' - w_1 w_3'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega_{23}^{1} = -\frac{w_1^2 w_2^2 - w_3^2\left(w_1^2 + w_2^2\right)}{2(w_1 w_2 w_3)^{3/2}},$$

$$\omega_{32}^{1} = -\frac{w_1^2\left(w_2^2 - w_3^2\right) + w_2^2 w_3^2}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega_{31}^{2} = -\frac{w_2^2 w_3^2 - w_1^2\left(w_2^2 + w_3^2\right)}{2(w_1 w_2 w_3)^{3/2}}.$$

pseudo-differential symbol of Dirac

$$
\begin{aligned}
\sigma(D)(x,\xi) &= \sum_{a,\mu} i\gamma^a e_a^\mu \xi_{\mu+1} + \frac{1}{4\sqrt{w_1 w_2 w_3}}\left(\frac{w_1'}{w_1}+\frac{w_2'}{w_2}+\frac{w_3'}{w_3}\right)\gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4 \\
&= -\frac{i\gamma^2\sqrt{w_1}\,(\csc(\eta)\cos(\psi)\,(\xi_4\cos(\eta)-\xi_3)+\xi_2\sin(\psi))}{\sqrt{w_2}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^3\sqrt{w_2}\,(\sin(\psi)\,(\xi_3\csc(\eta)-\xi_4\cot(\eta))+\xi_2\cos(\psi))}{\sqrt{w_1}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^1\xi_1}{\sqrt{w_1}\sqrt{w_2}\sqrt{w_3}} + \frac{i\gamma^4\xi_4\sqrt{w_3}}{\sqrt{w_1}\sqrt{w_2}} \\
&\quad + \frac{1}{4\sqrt{w_1 w_2 w_3}}\left(\frac{w_1'}{w_1}+\frac{w_2'}{w_2}+\frac{w_3'}{w_3}\right)\gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4
\end{aligned}
$$

- with non-vanishing e_a^μ:

$$e_0^0 = \frac{1}{\sqrt{w_1 w_2 w_3}}, \qquad e_1^1 = -\frac{\sqrt{w_1}\sin(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^1 = \frac{\sqrt{w_2}\cos(\psi)}{\sqrt{w_1 w_3}}, \qquad e_1^2 = \frac{\sqrt{w_1}\csc(\eta)\cos(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^2 = \frac{\sqrt{w_2}\csc(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, \qquad e_1^3 = -\frac{\sqrt{w_1}\cot(\eta)\cos(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^3 = -\frac{\sqrt{w_2}\cot(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, \qquad e_3^3 = \frac{\sqrt{w_3}}{\sqrt{w_1 w_2}}$$

- get from the symbol the homogeneous components $p_k(x,\xi)$ with

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

- Example: for $p_0(x,\xi)$ get

$$\begin{gathered}
\Big(-\frac{w_1'}{8w_1w_2^2} - \frac{w_1'}{8w_1w_3^2} + \frac{3w_1'}{8w_1^3} - \frac{w_2'}{8w_1^2w_2} - \frac{w_3'}{8w_1^2w_3} - \frac{w_2'}{8w_2w_3^2} \\
+\frac{3w_2'}{8w_2^3} - \frac{w_3'}{8w_2^2w_3} + \frac{3w_3'}{8w_3^3}\Big)\gamma^1\gamma^2\gamma^3\gamma^4 + \\
\Big(-\frac{w_1''}{4w_1^2w_2w_3} + \frac{w_1'w_2'}{8w_1^2w_2^2w_3} + \frac{w_1'w_3'}{8w_1^2w_2w_3^2} + \frac{5w_1'^2}{16w_1^3w_2w_3} - \frac{w_2''}{4w_1w_2^2w_3} \\
+\frac{w_2'w_3'}{8w_1w_2^2w_3^2} + \frac{5w_2'^2}{16w_1w_2^3w_3} - \frac{w_3''}{4w_1w_2w_3^2} + \frac{5w_3'^2}{16w_1w_2w_3^3} + \frac{w_2w_3}{16w_1^3} \\
+\frac{w_3}{8w_1w_2} + \frac{w_1w_3}{16w_2^3} + \frac{w_2}{8w_1w_3} + \frac{w_1}{8w_2w_3} + \frac{w_1w_2}{16w_3^3}\Big)I.
\end{gathered}$$

- also manageable expression for $p_2(x,\xi)$, longer one for $p_1(x,\xi)$

Applying Parametrix Method to this Dirac operator

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i} \int \int_\gamma e^{-\lambda}\, \mathrm{tr}\,(r_{2n}(x, \xi, \lambda))\; d\lambda\, d^m\xi$$

- Find a_0, a_2, a_4 explicitly

$$a_0(D^2) = 4 w_1 w_2 w_3$$

$$\begin{aligned} a_2(D^2) \quad = \quad & -\frac{w_1^2}{3} - \frac{w_2^2}{3} - \frac{w_3^2}{3} + \frac{w_1^2 w_2^2}{6 w_3^2} + \frac{w_1^2 w_3^2}{6 w_2^2} + \frac{w_2^2 w_3^2}{6 w_1^2} - \frac{(w_1')^{\,2}}{6 w_1^2} - \frac{(w_2')^{\,2}}{6 w_2^2} \\ & - \frac{(w_3')^{\,2}}{6 w_3^2} - \frac{w_1' w_2'}{3 w_1 w_2} - \frac{w_1' w_3'}{3 w_1 w_3} - \frac{w_2' w_3'}{3 w_2 w_3} + \frac{w_1''}{3 w_1} + \frac{w_2''}{3 w_2} + \frac{w_3''}{3 w_3}. \end{aligned}$$

and a much longer and more complicated expression for $a_4(D^2)$

Observation: all coefficients in these expressions (also for a_4) are rational numbers ... what about other terms in expansion?

Wodzicki Residue Method for $SU(2)$-Bianchi IX metrics

• setting $\zeta_{\mu+1} = \sum_\nu e^\nu_\mu \xi_{\nu+1}$ find inductively for $n \geq 2$

$$\sigma_{-2-n}(x,\xi)|_{S^*(M\times\mathbb{T}^{n-2})} = \sigma_{-2-n}(x,\xi(\zeta))|_{\zeta\in\mathbb{S}^{n+1}} = (w_1 w_2 w_3)^{-\frac{3}{2}n} P_n(\zeta)$$

polynomials $P_n(\zeta)$ coefficients functions of w_i and derivatives

• these explicitly give

$$a_{2n}(D^2) = (w_1 w_2 w_3)^{1-3n} Q_{2n}\left(w_1, w_2, w_3, w_1', w_2', w_3', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}\right)$$

with Q_{2n} polynomials with rational coefficients

$$Q_{2n} = \frac{1}{2\pi^{n+1}} \int_{\mathbb{S}^{2n+1}} \mathrm{Tr}(P_{2n}(\zeta)(\Delta^{-1}))\, d^{2n+1}\zeta$$

Question: is this rationality a sign of an arithmetic structure of Bianchi IX gravitational instantons that persists in the Spectral Action?

Bianchi IX gravitational instantons and Painlevé VI

• Euclidean Bianchi IX metrics with $SU(2)$-symmetry that are
- self-dual (Weyl curvature tensor W self-dual)
- Einstein metrics (Ricci tensor proportional to the metric)

• Self-dual equations for a Riemannian 4-manifold are PDEs; with $SU(2)$-symmetry reduce to ODEs

• This ODE is a Painlevé VI equation with

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$$

- N.J. Hitchin. *Twistor spaces, Einstein metrics and isomonodromic deformations*, J.Diff.Geom., Vol. 42, No. 1 (1995), 30–112.
- K.P. Tod. *Self–dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A 190 (1994), 221–224.
- S. Okumura. *The self–dual Einstein–Weyl metric and classical solutions of Painlevé VI*, Lett. in Math. Phys., 46 (1998), 219–232.
- M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337

Painlevé VI equations

• *Painlevé transcendents*: solutions of nonlinear second-order ODEs in the plane with *Painlevé property* (the only movable singularities are poles) not solvable in terms of elementary functions; classification in types

• *Painlevé VI*: 4-parameter family $(\alpha, \beta, \gamma, \delta)$

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2$$

$$-\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

Painlevé VI and elliptic curves

• Painlevé VI rewritten as (Fuchs)

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_\infty^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}$$

where $(X,Y):=(X(t),Y(t))$ is a section
(local and/or multivalued) $P:=(X(t),Y(t))$
of the generic elliptic curve $E=E(t):\ Y^2=X(X-1)(X-t)$

• left-hand-side $\mu(P)$ satisfies $\mu(P+Q)=\mu(P)+\mu(Q)$ for $P+Q$ addition on the elliptic curve E (in particular $\mu(Q)=0$ for points of finite order)

- analytic description of the elliptic curve $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, with $\tau \in \mathbb{H}$
- then Painlevé VI rewritten as (Manin)

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z\left(z + \frac{T_j}{2}, \tau\right)$$

with $(\alpha_0, \ldots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)$, and

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

• also have, for $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

• a multivalued solution $z = z(\tau)$ defines a multi–section of the family, which is a covering of $\mathbb{H}$

• is know ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

- Yu.I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, in "Geometry of Differential Equations", Amer. Math. Soc. Transl. (2) Vol. 186 (1998) 131–151

Theta characteristics

• explicit paremeterization of solutions for coefficients W_i of the Bianchi IX gravitational instantons (from solutions of Painlevé VI)

• theta–characteristics with parameters (p,q):

$$\vartheta[p,q](z,i\mu) := \sum_{m\in\mathbb{Z}} \exp\left(-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\right)$$

• theta-characteristics and theta functions with vanishing characteristics

$$\vartheta[p,q](z,i\mu) = \exp\left(-\pi p^2\mu + 2\pi i pq\right)\cdot\vartheta[0,0](z+pi\mu+q,i\mu)$$

Gravitational instantons and theta characteristics

- use notation $\vartheta[p,q] := \vartheta[p,q](0, i\mu)$, and

$$\vartheta_2 := \vartheta[1/2, 0], \qquad \vartheta_3 := \vartheta[0,0], \qquad \vartheta_4 := \vartheta[0, 1/2]$$

- self-dual metrics

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with

$$w_1 = -\frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]}, \qquad w_2 = \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]},$$

$$w_3 = -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q]}{\vartheta[p,q]},$$

- with non-zero cosmological constant Λ:

$$F = \frac{2}{\pi\Lambda} \frac{w_1 w_2 w_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2}$$

- these metrics also satisfy Einstein equation if either
  1. $\Lambda < 0$ with $p \in \mathbb{R}$ and $q \in \frac{1}{2} + i\mathbb{R}$
  2. $\Lambda > 0$ with $q \in \mathbb{R}$ and $p \in \frac{1}{2} + i\mathbb{R}$

- also case with vanishing cosmological constant:

$$w_1 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_2, \quad w_2 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_3,$$

$$w_3 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_4, \qquad F = C(\mu + q_0)^2\, w_1 w_2 w_3$$

with $q_0, C \in \mathbb{R}$, $C > 0$.

- M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337
- Yuri Manin, Matilde Marcolli, *Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies*, arXiv:1504.04005 [gr-qc]

Bianchi IX: time-dependent conformal perturbations

- original triaxial Bianchi IX:

$$ds^2 = w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1}\sigma_1^2 + \frac{w_3 w_1}{w_2}\sigma_2^2 + \frac{w_1 w_2}{w_3}\sigma_3^2$$

$w_i = w_i(\mu)$ cosmic time μ

- time-dependent conformal perturbation:

$$d\tilde{s}^2 = F ds^2 = F \left(w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1}\sigma_1^2 + \frac{w_3 w_1}{w_2}\sigma_2^2 + \frac{w_1 w_2}{w_3}\sigma_3^2 \right)$$

with $F = F(\mu)$

- effect on Dirac operator:

$$\tilde{D} = \frac{1}{\sqrt{F}} D + \frac{3F'}{4F^{\frac{3}{2}} w_1 w_2 w_3}\gamma^0$$

D Dirac operator of unperturbed Bianchi IX

- spectral action expansion for $\tilde{D}$ from heat kernel

$$\mathrm{Tr}\left(\exp(-t\tilde{D}^2)\right) \sim t^{-2} \sum_{n=0}^{\infty} \tilde{a}_{2n} t^n, \qquad t \to 0^+$$

- rationality result for coefficients of the spectral action

$$\tilde{a}_{2n} = \frac{\tilde{Q}_{2n}\left(w_1, w_2, w_3, F, w_1', w_2', w_3', F', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}, F^{(2n)}\right)}{F^{2n}(w_1 w_2 w_3)^{3n-1}}$$

$\tilde{Q}_{2n}$ polynomial with rational coefficients

• zeroth coefficient: volume form (cosmological term)

$$\tilde{a}_0 = 4F^2 w_1 w_2 w_3$$

• second coefficient $\tilde{a}_2$: Einstein-Hilbert action

$$-\frac{F}{3}\left(w_1^2+w_2^2+w_3^2\right)+\frac{F}{6}\left(\frac{w_1^2 w_2^2 - w_3'^2}{w_3^2}+\frac{w_1^2 w_3^2 - w_2'^2}{w_2^2}+\frac{w_2^2 w_3^2 - w_1'^2}{w_1^2}\right)$$

$$-\frac{F}{3}\left(\frac{w_1' w_2'}{w_1 w_2}+\frac{w_1' w_3'}{w_1 w_3}+\frac{w_2' w_3'}{w_2 w_3}\right)+\frac{F}{3}\left(\frac{w_1''}{w_1}+\frac{w_2''}{w_2}+\frac{w_3''}{w_3}\right)-\frac{F'^2}{2F}+F''$$

• much longer and more complicated explicit formula for $\tilde{a}_4$ (Weyl conformal gravity and Gauss-Bonnet gravity)

Gravitational Instantons

- now assuming conformally perturbed Bianchi IX is self-dual Einstin metric and use parameterization by theta functions
- two-parameter family with non-vanishing cosmological constant:

$$w_1[p,q](i\mu) = -\frac{i}{2}\vartheta_3(i\mu)\vartheta_4(i\mu)\frac{\partial_q\vartheta[p,q+\frac{1}{2}](i\mu)}{e^{\pi i p}\vartheta[p,q](i\mu)}$$

$$w_2[p,q](i\mu) = \frac{i}{2}\vartheta_2(i\mu)\vartheta_4(i\mu)\frac{\partial_q\vartheta[p+\frac{1}{2},q+\frac{1}{2}](i\mu)}{e^{\pi i p}\vartheta[p,q](i\mu)}$$

$$w_3[p,q](i\mu) = -\frac{1}{2}\vartheta_2(i\mu)\vartheta_3(i\mu)\frac{\partial_q\vartheta[p+\frac{1}{2},q](i\mu)}{\vartheta[p,q](i\mu)}$$

$$F[p,q](i\mu) = \frac{2}{\pi\Lambda}\frac{1}{(\partial_q \ln\vartheta[p,q](i\mu))^2} = \frac{2}{\pi\Lambda}\left(\frac{\vartheta[p,q](i\mu)}{\partial_q\vartheta[p,q](i\mu)}\right)^2$$

- one-parameter family with vanishing cosmological constant:

$$\begin{aligned} w_1[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_2(i\mu), \\ w_2[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_3(i\mu), \\ w_3[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_4(i\mu), \\ F[q_0](i\mu) &= C(\mu+q_0)^2, \end{aligned}$$

C arbitrary positive constant

Modularity

• generators of the modular group $\mathrm{PSL}_2(\mathbb{Z})$

$$T_1(\tau) = \tau + 1, \qquad S(\tau) = \frac{-1}{\tau}, \qquad \tau \in \mathbb{H}$$

• using behavior of theta functions and derivatives under modular transformations (two-parameter family):

$$\begin{aligned}
w_1[p,q](i\mu+1) &= w_1[p,q+p+\frac{1}{2}](i\mu), & w_1^{(n)}[p,q](i\mu+1) &= w_1^{(n)}[p,q+p+\frac{1}{2}](i\mu),\\
w_2[p,q](i\mu+1) &= w_3[p,q+p+\frac{1}{2}](i\mu), & w_2^{(n)}[p,q](i\mu+1) &= w_3^{(n)}[p,q+p+\frac{1}{2}](i\mu),\\
w_3[p,q](i\mu+1) &= w_2[p,q+p+\frac{1}{2}](i\mu), & w_3^{(n)}[p,q](i\mu+1) &= w_2^{(n)}[p,q+p+\frac{1}{2}](i\mu).
\end{aligned}$$

- for μ with $\Re(\mu) > 0$:

$$
\begin{aligned}
w_3[p,q](\frac{i}{\mu}) &= -\mu^2 w_1[-q,p](i\mu), \\
w_3^{'}[p,q](\frac{i}{\mu}) &= \mu^4 w_1^{'}[-q,p](i\mu) + 2\mu^3 w_1[-q,p](i\mu), \\
w_3^{''}[p,q](\frac{i}{\mu}) &= -\mu^6 w_1^{''}[-q,p](i\mu) - 6\mu^5 w_1^{'}[-q,p](i\mu) - 6\mu^4 w_1[-q,p](i\mu), \\
w_3^{(3)}[p,q](\frac{i}{\mu}) &= \mu^8 w_1^{(3)}[-q,p](i\mu) + 12\mu^7 w_1^{''}[-q,p](i\mu) + 36\mu^6 w_1^{'}[-q,p](i\mu) \\
&\quad +24\mu^5 w_1[-q,p](i\mu), \\
w_3^{(4)}[p,q](\frac{i}{\mu}) &= -\mu^{10} w_1^{(4)}[-q,p](i\mu) - 20\mu^9 w_1^{(3)}[-q,p](i\mu) - 120\mu^8 w_1^{''}[-q,p](i\mu) \\
&\quad -240\mu^7 w_1^{'}[-q,p](i\mu) - 120\mu^6 w_1[-q,p](i\mu).
\end{aligned}
$$

- similar results for w_2 and w_3 under modular generator S

• conformal factor:

$$\begin{aligned} F[p,q](i\mu+1) &= F[p,q+p+\frac{1}{2}](i\mu), \\ F^{(n)}[p,q](i\mu+1) &= F^{(n)}[p,q+p+\frac{1}{2}](i\mu). \end{aligned}$$

$$\begin{aligned} F[p,q](\frac{i}{\mu}) &= -\mu^{-2}F[-q,p](i\mu), \\ F^{'}[p,q](\frac{i}{\mu}) &= F^{'}[-q,p](i\mu) - 2\mu^{-1}F[-q,p](i\mu), \\ F^{''}[p,q](\frac{i}{\mu}) &= -\mu^{2}F^{''}[-q,p](i\mu) + 2\mu F^{'}[-q,p](i\mu) - 2F[-q,p](i\mu), \\ F^{(3)}[p,q](\frac{i}{\mu}) &= \mu^{4}F^{(3)}[-q,p](i\mu), \\ F^{(4)}[p,q](\frac{i}{\mu}) &= -\mu^{6}F^{(4)}[-q,p](i\mu) - 4\mu^{5}F^{(3)}[-q,p](i\mu). \end{aligned}$$

• similar results for the case of the one-parameter family with vanishing cosmological constant

• modularity of spectral action coefficients:

$$\tilde{a}_0[p,q](i\mu+1) = \tilde{a}_0[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_2[p,q](i\mu+1) = a_2[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_4[p,q](i\mu+1) = \tilde{a}_4[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_0[p,q](\frac{i}{\mu}) = -\mu^2\tilde{a}_0[-q,p](i\mu)$$

$$\tilde{a}_2[p,q](\frac{i}{\mu}) = -\mu^2\tilde{a}_2[-q,p](i\mu)$$

$$\tilde{a}_4[p,q](\frac{i}{\mu}) = -\mu^2\tilde{a}_4[-q,p](i\mu)$$

Modularity of remaining coefficients $\tilde{a}_{2n}$

• Dirac operators $\tilde{D}^2[p,q]$, $\tilde{D}^2[p,q+p+\frac{1}{2}]$ and $\tilde{D}^2[-q,p]$ are isospectral

• heat kernel $K_t[p,q]$ of $\exp\left(-t\tilde{D}^2[p,q]\right)$ in terms of eigenvalues and eigenspinors $\Rightarrow$ modularity

$$\begin{aligned} K_t[p,q](i\mu_1+1, i\mu_2+1) &= K_t[p,q+p+\frac{1}{2}](i\mu_1, i\mu_2), \\ K_t[p,q](-\frac{1}{i\mu_1}, -\frac{1}{i\mu_2}) &= (i\mu_2)^2 K_t[-q,p](i\mu_1, i\mu_2). \end{aligned}$$

• then modularity of coefficients $\tilde{a}_{2n}$:

$$\begin{aligned} \tilde{a}_{2n}[p,q](i\mu+1) &= \tilde{a}_{2n}[p,q+p+\frac{1}{2}](i\mu), \\ \tilde{a}_{2n}[p,q](\frac{i}{\mu}) &= (i\mu)^2 \tilde{a}_{2n}[-q,p](i\mu). \end{aligned}$$

Vector valued modular forms

• coefficients satisfy:

$$\tilde{a}_{2n}[p+1,q] = \tilde{a}_{2n}[p,q+1] = \tilde{a}_{2n}[p,q],$$

• $\mathrm{PSL}_2(\mathbb{Z})$ action on $(p,q) \in \mathbb{R}/\mathbb{Z}^2$:

$$\begin{aligned} \tilde{S}(p,q) &= (-q,p) \\ \tilde{T}_1(p,q) &= (p, q+p+\frac{1}{2}) \end{aligned}$$

finite orbits $\mathcal{O}_{(p,q)}$ on rationals

• $\tilde{a}_{2n}[p',q'](i\mu)$, with $(p',q') \in \mathcal{O}_{(p,q)}$, vector-valued modular form of weight 2 for the modular group $\mathrm{PSL}_2(\mathbb{Z})$

- summing over orbits:

$$\tilde{a}_{2n}(i\mu;\mathcal{O}_{(p,q)}) = \sum_{(p',q')\in\mathcal{O}_{(p,q)}} \tilde{a}_{2n}[p',q'](i\mu)$$

is an ordinary modular form of weight 2 for $\mathrm{PSL}_2(\mathbb{Z})$

- Question: which modular form is it?
- analyze zeros and poles structure to find out
  - Example: for all n, modular form $\tilde{a}_{2n}(i\mu;\mathcal{O}_{(0,\frac{1}{3})})$ in one-dimensional space spanned by

$$\frac{G_{14}(i\mu)}{\Delta(i\mu)},$$

    with Δ modular discriminant (cusp form weight 12) and G_{14} is Eisenstein series weight 14

  - Example: for all n, modular form $\tilde{a}_{2n}(i\mu;\mathcal{O}_{(\frac{1}{6},\frac{5}{6})})$ in one-dimensional space spanned by

$$\frac{\Delta(i\mu)G_6(i\mu)}{G_4(i\mu)^4}$$

Further Developments (ongoing)

- better description of the modular forms arising in the spectral action of Bianchi IX gravitational instantons
- an analysis of the spectral action coefficients for Bianchi IX gravitational instantons in terms of motives of quadrics and periods as in the Robertson–Walker case
- possible existence of other classes of spacetimes for which the spectral action has similar arithmetic properties

That's all for now...