Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies

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Topics in Mathematical Physics
Based on:

Kasner metrics

- real circle in $\mathbb{R}^3$ defined by equations
  
  \[ p_a + p_b + p_c = 1, \quad p_a^2 + p_b^2 + p_c^2 = 1 \]

- each point on this circle defines a metric with Minkowskian (or Euclidean) signature
  
  \[ \pm dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2 \]

with scaling factors $a, b, c$:

\[ a(t) = t^{p_a}, \quad b(t) = t^{p_b}, \quad c(t) = t^{p_c}, \quad t > 0. \]

Kasner metric with exponents $(p_a, p_b, p_c)$. 
u-parameterization

- Points \((p_a, p_b, p_c)\) on the circle parameterized by a coordinate \(u \in [1, \infty]\)

\[
p_1^{(u)} := -\frac{u}{1 + u + u^2} \quad \in \left[-\frac{1}{3}, 0\right]
\]

\[
p_2^{(u)} := \frac{1 + u}{1 + u + u^2} \quad \in \left[0, \frac{2}{3}\right]
\]

\[
p_3^{(u)} := \frac{u(1 + u)}{1 + u + u^2} \quad \in \left[\frac{2}{3}, 1\right]
\]

- Rearrange the exponents \(p_1^{(u)} \leq p_2^{(u)} \leq p_3^{(u)}\) by a bijection \((1, 2, 3) \rightarrow (a, b, c)\) (permutation of the 3 space axes)
Mixmaster Universe (1970s)


- Anisotropic cosmologies
- Locally described by a Kasner metric
- Sequence of Kasner metrics (Kasner epochs and cycles)
- Within each epoch one direction dominates expansion, the other two oscillate in a series of Kasner cycles
- At the end of each epoch a bounce occurs and a possibly different direction becomes responsible for expansion
- Approach: model the dynamics by a discrete dynamical system that determines epochs and cycles
  $\Rightarrow$ continued fraction expansion
Continued fraction expansion

- infinite continued fraction expansion

of a number \( x \in \mathbb{R} \setminus \mathbb{Q}, \ x > 1: \)

\[
x = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \ldots}} =: [k_0, k_1, k_2, \ldots], \ k_s \in \mathbb{N}
\]

with \( k_0 := [x], \ k_1 = [1/x] \) etc. digits \( k_i \in \mathbb{N} \)

for a point in \([0, 1], \) expansion \([0, k_1, k_2, \ldots, k_n, \ldots]\)

- Dynamical system: (partial map)

one sided shift of the continued fraction expansion

\[
T : [0, 1] \to [0, 1] \quad T : x \mapsto \frac{1}{x} - \left[ \frac{1}{x} \right]
\]

\([0, k_0, k_1, k_2, \ldots, k_n, \ldots] \xrightarrow{T} [0, k_1, k_2, \ldots, k_{n+1}, \ldots]\)
One-sided shift of the continued fraction expansion
• Dynamical system: (partial map)
  invertible two sided shift

  \[ \tilde{T} : [0, 1]^2 \to [0, 1]^2 \quad \tilde{T} : (x, y) \mapsto \left( \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{y + \lfloor 1/x \rfloor} \right) \]

• On \([0, 1]^2 \cap (\mathbb{R}^2 \setminus \mathbb{Q}^2)\) uniquely defined \(k_s \in \mathbb{N}\)

  \[ x = [0, k_0, k_1, k_2, \ldots], \quad y = [0, k_{-1}, k_{-2}, \ldots] \]

  \[ \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = [0, k_1, k_2, \ldots], \quad \frac{1}{y + \lfloor 1/x \rfloor} = \frac{1}{k_0 + y} = [0, k_0, k_{-1}, k_{-2}, \ldots] \]

• On this subset \(\tilde{T}\) bijective with invariant density

  \[ d\mu(x, y) = \frac{dx \, dy}{\log 2 \cdot (1 + xy)^2} \]

• encode \((x, y) \in [0, 1]^2 \cap (\mathbb{R}^2 \setminus \mathbb{Q}^2)\) with doubly infinite sequence
  \( (k) := [\ldots k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots], \quad k_i \in \mathbb{N} \) where \(T(k)_s = k_{s+1}\)

invertible shift
Continued fractions and Mixmaster Universe

• *typical* solutions of Einstein equations Bianchi IX type with $SO(3)$–symmetry oscillates (near the initial singularity) close to a sequence of Kasner type solutions

• local logarithmic time $d\Omega := -\frac{dt}{abc}$

• for $\Omega \cong -\log t \to +\infty$:
  - increasing sequence of times $\Omega_0 < \Omega_1 < \cdots < \Omega_n < \ldots$
  - sequence of irrational real numbers $u_n \in (1, +\infty)$, $n = 0, 1, 2, \ldots$

• semi–interval $[\Omega_n, \Omega_{n+1})$ is $n$–th Kasner epoch

• start at time $\Omega_n$ with a value $u = u_n > 1$:

  $$p_1 = -\frac{u}{1 + u + u^2}, \quad p_2 = \frac{1 + u}{1 + u + u^2}, \quad p_3 = \frac{u(1 + u)}{1 + u + u^2}$$

• Kasner cycles (within same Kasner epoch)
  $u = u_n - 1, u_n - 2, \ldots$, with corresponding Kasner metrics
• after $k_n := [u_n]$ cycles inside the same Kasner epoch, a jump to the next epoch with new parameter

$$u_{n+1} = \frac{1}{u_n - [u_n]}$$

• at the end of each epoch a reshuffling of space axes (also determined by the discrete dynamical system)
• sequence of logarithmic times $\Omega_n$ specified by a sequence $\delta_n$

$$\Omega_{n+1} = [1 + \delta_n k_n(u_n + 1/\{u_n\})]\Omega_n$$

• setting $\eta_n = (1 - \delta_n)/\delta_n$ recursion

$$\eta_{n+1}x_n = \frac{1}{k_n + \eta_n x_n_1}$$

with $x_n = u_n - k_n$


**Conclusion:**

- trajectories of mixmaster universe dynamics are parameterized by pairs \((x, y) \in [0, 1]^2 \cap (\mathbb{R}^2 \setminus \mathbb{Q}^2)\)

\[
x = [0, k_0, k_1, k_2, \ldots], \quad y = [0, k_{-1}, k_{-2}, \ldots]
\]

\(x\) specifies number of Kasner cycles in each Kasner epoch, \(y\) specifies the Kasner logarithmic times

- transition from one Kasner epoch to the next is given by the action of the double sided shift of the continued fraction

\[
\tilde{T} : (x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y + [1/x]}\right)
\]
Modular curves

- upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y = \Im(z) > 0\}$
- hyperbolic metric
  $$ds^2 = \frac{dx^2 + dy^2}{y^2}$$
- geodesics: vertical straight lines orthogonal to $x$-axis; circular arcs (half-circles) orthogonal to $x$-axis
- $\text{PSL}_2(\mathbb{R})$ acts transitively and isometrically
  $$z \mapsto \frac{az + b}{cz + d}$$
- action of $\text{PSL}_2(\mathbb{Z})$ and finite index subgroups $\Gamma \subset \text{PSL}_2(\mathbb{Z})$
- Elliptic curves $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ modulus $\tau \in \mathbb{H}$; isomorphic $E_\tau \simeq E_{\tau'}$ when $\tau \sim \tau'$ by $\text{PSL}_2(\mathbb{Z})$ action
- $\mathcal{M} = \mathbb{H}/\text{PSL}_2(\mathbb{Z})$ moduli space of elliptic curves (modular curve)
Fundamental domain of $\text{PSL}_2(\mathbb{Z})$-action
Elliptic curves and modular curve

(detail from an image by Christian Wuthrich)
Farey tessellation

• adding cusps to the upper half plane: \( \overline{\mathbb{H}} := \mathbb{H} \cup \{\mathbb{Q} \cup \{\infty\}\} \)

• vertical lines \( \Re(z) = n, n \in \mathbb{Z} \), and semicircles in \( \overline{\mathbb{H}} \) connecting pairs of finite cusps \((p/q, p'/q')\) with \( pq' - p'q = \pm 1 \)

• these cut \( \overline{\mathbb{H}} \) into a union of geodesic ideal triangles: Farey tessellation


• **coding of geodesics** on \( \mathcal{M} = \mathbb{H}/\text{PSL}_2(\mathbb{Z}) \)

  using Farey tessellation and continued fraction
Farey tessellation in the Poincaré disk model of $\mathbb{H}$
• $\mathcal{B}$ set of oriented geodesics $\beta$ in $\mathbb{H}$ with ideal irrational endpoints $\beta_{-\infty}, \beta_\infty$ in $\mathbb{R}$, such that

$$\beta_{-\infty} \in (-1, 0), \quad \beta_\infty \in (1, \infty)$$

• continued fraction expansion of endpoints

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \ldots], \quad \beta_\infty = [k_1, k_2, k_3, \ldots], \quad k_i \in \mathbb{N},$$

• $\beta$ determined by endpoints, by doubly infinite sequence of continued fraction digits

$$[\ldots k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots]$$

• intersection point $x = x(\beta)$ of $\beta$ with imaginary semiaxis in $\mathbb{H}$

• moving along $\beta$: intersect infinite sequence of Farey triangles

• enter each triangle through one side and leave through a different one: the ideal intersection point of these two sides is either to the left or to the right

• infinite sequences in alphabet $\{L, R\}$ (moving in both directions)

$$\ldots L^{k_{-3}} R^{k_{-2}} L^{k_{-1}} R^{k_0} L^{k_1} R^{k_2} L^{k_3} R^{k_4} \ldots$$
Hyperbolic billiard

- all hyperbolic triangles $\Delta$ of the Farey tessellation of $\mathbb{H}$ are isometric metric spaces

- folding the tessellation: a geodesic $\beta \in B$ corresponds to a billiard ball trajectory on the table $\Delta$ that never hits the corners

- group $S_6$ of hyperbolic isometries of $\Delta$ (permutations on vertices); unique fixed point $\rho := \exp(\pi i/3)$ in $\Delta$

- three geodesic arcs connecting $\rho$ with points $i, 1 + i, \frac{1+i}{2}$ subdivide $\Delta$ into three geodesic quadrangles, each with one cusp corner

- each quadrangle is a copy of the fundamental domain for $\text{PSL}_2(\mathbb{Z})$
• for a geodesic $\beta$ with $x(\beta) = x_0$ in $(0, i\infty)$, if $k_0 = 1$, the ball reaches the opposite side $(1, i\infty)$ of $\Delta$ and gets reflected to the third side $(0, 1)$$$
abla$

• if $k_0 = 2$, it reaches the opposite side, then returns to the initial side $(0, i\infty)$, and after that gets reflected to $(0, 1)$$$
abla$

• generally the ball always spends $k_0$ unobstructed stretches of its trajectory between $(0, i\infty)$ and $(1, i\infty)$, then is reflected to $(0, 1)$ either from $(1, i\infty)$ (if $k_0$ is odd), or from $(0, i\infty)$ (if $k_0$ is even)$$$
abla$

• can equivalently encode the geodesic $\beta$ in $\mathcal{M} = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ by a doubly infinite sequence in the alphabet $\{a, b, c\}$ labelling the three vertices at infinity $\{0, 1, i\infty\}$ of $\Delta$
Hyperbolic Billiard

Other billiard models for Mixmaster dynamics

(from Beverly Berger, “Numerical Approaches to Spacetime Singularities”)
Enriched encoding of geodesics and Mixmaster trajectories

• hyperbolic billiard as above: insert between consecutive powers of $L, R$ the intersection points of $\beta$ with sides of Farey triangles:

$$\ldots L^{k-1}x_{-1}R^{k_0}x_0L^{k_1}x_1R^{k_2}x_2L^{k_3}x_3R^{k_4}\ldots$$

• Result: when $s \to \infty$, $s \in \mathbb{N}$

$$\log \frac{\Omega_{2s}}{\Omega_0} \simeq 2 \sum_{r=0}^{s-1} \text{dist}(x_{2r}, x_{2r+1}),$$

$\text{dist} =$ hyperbolic distance between consecutive intersection points of the geodesic with sides of the Farey tessellation
Sketch of argument: known from mixmaster dynamics that

\[ \log \frac{\Omega_{2s}}{\Omega_0} \simeq -\sum_{p=1}^{2s} \log(x_p^+ x_p^-) \]

\[ = \sum_{p=1}^{2s} \log([k_{p-1}, k_{p-2}, k_{p-3}, \ldots] \cdot [k_p, k_{p+1}, k_{p+2}, \ldots]) \]

From coding of geodesics also know that

\[ \text{dist}(x_0, x_1) = \frac{1}{2} \log([k_0, k_{-1}, k_{-2}, \ldots] \cdot [k_1, k_2, \ldots] \cdot [k_1, k_0, k_{-1}, \ldots] \cdot [k_2, k_3, \ldots]) \]

and more generally \( \text{dist}(x_{2r}, x_{2r+1}) \) is given by

\[ \frac{1}{2} \log([k_{2r}, k_{2r-1}, k_{2r-2}, \ldots] \cdot [k_{2r+1}, k_{2r+2}, \ldots] \cdot [k_{2r+1}, k_{2r}, k_{2r-1}, \ldots] \cdot [k_{2r+2}, k_{2r+3}, \ldots]) \]

Consequence: identification of distance along geodesic with logarithmic cosmological time
Painlevé VI equations

- *Painlevé transcendents*: solutions of nonlinear second-order ODEs in the plane with *Painlevé property* (the only movable singularities are poles) not solvable in terms of elementary functions; classification in types

- *Painlevé VI*: 4-parameter family \((\alpha, \beta, \gamma, \delta)\)

\[
\frac{d^2X}{dt^2} = \frac{1}{2} \left( \frac{1}{X} + \frac{1}{X - 1} + \frac{1}{X - t} \right) \left( \frac{dX}{dt} \right)^2
- \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{X - t} \right) \frac{dX}{dt} + 
\frac{X(X - 1)(X - t)}{t^2(t - 1)^2} \left( \alpha + \beta \frac{t}{X^2} + \gamma \frac{t - 1}{(X - 1)^2} + \delta \frac{t(t - 1)}{(X - t)^2} \right).
\]
Painlevé VI and elliptic curves

- Painlevé VI rewritten as (Fuchs)

\[
t(1-t) \left[ t(1-t) \frac{d^2}{dt^2} + (1-2t) \frac{d}{dt} - \frac{1}{4} \right] \int_\infty^{(X,Y)} \frac{dx}{\sqrt{x(x-1)(x-t)}} = \\
= \alpha Y + \beta \frac{tY}{X^2} + \gamma \frac{(t-1)Y}{(X-1)^2} + \left( \delta - \frac{1}{2} \right) \frac{t(t-1)Y}{(X-t)^2}
\]

where \((X, Y) := (X(t), Y(t))\) is a section (local and/or multivalued) \(P := (X(t), Y(t))\) of the generic elliptic curve \(E = E(t) : Y^2 = X(X-1)(X-t)\)

- left-hand-side \(\mu(P)\) satisfies \(\mu(P + Q) = \mu(P) + \mu(Q)\) for \(P + Q\) addition on the elliptic curve \(E\) (in particular \(\mu(Q) = 0\) for points of finite order)
analytic description of the elliptic curve \( E_\tau = \mathbb{C}/\Lambda \) with \( \Lambda = \mathbb{Z} + \tau\mathbb{Z} \), with \( \tau \in \mathbb{H} \)

then Painlevé VI rewritten as (Manin)

\[
\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp(z + \frac{T_j}{2}, \tau)
\]

with \((\alpha_0, \ldots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)\) and \((T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)\), and

\[
\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left( \frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right)
\]
• also have, for $e_i(\tau) = \wp\left(\frac{T_i}{2}, \tau\right)$

$$\wp_z(z, \tau)^2 = 4(\wp(z, \tau) - e_1(\tau))(\wp(z, \tau) - e_2(\tau))(\wp(z, \tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

• a multivalued solution $z = z(\tau)$ defines a multi–section of the family, which is a covering of $\mathbb{H}$

• is know ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

\textit{SU}(2)-Bianchi IX cosmologies $\mathbb{R} \times S^3$

- another version of Bianchi IX mixmaster cosmologies, with $SU(2)$ symmetry (Euclidean version)

$$g = W_1 W_2 W_3 \, d\mu^2 + \frac{W_2 W_3}{W_1} \sigma_1^2 + \frac{W_1 W_3}{W_2} \sigma_2^2 + \frac{W_1 W_2}{W_3} \sigma_3^2$$

or more generally

$$g = F \left( d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2} \right)$$

with a conformal factor $F \sim W_1 W_2 W_3$

- $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations $(i, j, k)$ of $(1, 2, 3)$
• more explicitly

\[ \sigma_1 = x_1 \, dx_2 - x_2 \, dx_1 + x_3 \, dx_0 - x_0 \, dx_3 = \frac{1}{2}(d\psi + \cos \theta \, d\phi), \]

\[ \sigma_2 = x_2 \, dx_3 - x_3 \, dx_2 + x_1 \, dx_0 - x_0 \, dx_1 = \frac{1}{2}(\sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi), \]

\[ \sigma_3 = x_3 \, dx_1 - x_1 \, dx_3 + x_2 \, dx_0 - x_0 \, dx_2 = \frac{1}{2}(-\cos \psi \, d\theta - \sin \theta \sin \psi \, d\phi), \]

Euler angles \( 0 \leq \theta \leq \pi, \, 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 4\pi \) (SU(2) case)

• identifying \( S^3 \) with unit quaternions SU(2)

• The metrics on \( S^3 \)

\[ \frac{W_2}{W_1} \sigma_1^2 + \frac{W_1}{W_2} \sigma_2^2 + \frac{W_1}{W_3} \sigma_3^2 \]

are left-invariants under the action of SU(2) but not right-invariant (unlike the round metric on \( S^3 \))
Blanchi IX gravitational instantons and Painlevé VI

- Euclidean Blanchi IX metrics with $SU(2)$-symmetry that are
  - self-dual (Weyl curvature tensor $W$ self-dual)
  - Einstein metrics (Ricci tensor proportional to the metric)

- Self-dual equations for a Riemannian 4-manifold are PDEs; with $SU(2)$-symmetry reduce to ODEs

- This ODE is a Painlevé VI equation with

$$ (\alpha, \beta, \gamma, \delta) = \left( \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8} \right) $$


Theta characteristics

• explicit parameterization of solutions for coefficients $W_i$ of the Bianchi IX gravitational instantons (from solutions of Painlevé VI)
• theta–characteristics with parameters $(p, q)$:

$$\vartheta[p, q](z, i \mu) := \sum_{m \in \mathbb{Z}} \exp \left( -\pi (m + p)^2 \mu + 2\pi i (m + p)(z + q) \right)$$

• theta-characteristics and theta functions with vanishing characteristics

$$\vartheta[p, q](z, i \mu) = \exp \left( -\pi p^2 \mu + 2\pi ipq \right) \cdot \vartheta[0, 0](z + pi \mu + q, i \mu)$$
Gravitational instantons and theta characteristics

- use notation $\vartheta[p, q] := \vartheta[p, q](0, i\mu)$, and
  \[ \vartheta_2 := \vartheta[1/2, 0], \quad \vartheta_3 := \vartheta[0, 0], \quad \vartheta_4 := \vartheta[0, 1/2] \]

- self-dual metrics
  \[ g = F \left( d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2} \right) \]

with
  \[ W_1 = -\frac{i}{2} \vartheta_3 \vartheta_4 \frac{\partial}{\partial q} \vartheta[p, q + \frac{1}{2}] e^{\pi i p \vartheta[p, q]}, \quad W_2 = \frac{i}{2} \vartheta_2 \vartheta_4 \frac{\partial}{\partial q} \vartheta[p + \frac{1}{2}, q + \frac{1}{2}] e^{\pi i p \vartheta[p, q]}, \]
  \[ W_3 = -\frac{1}{2} \vartheta_2 \vartheta_3 \frac{\partial}{\partial q} \vartheta[p + \frac{1}{2}, q] \vartheta[p, q], \]

- with non-zero cosmological constant $\Lambda$:
  \[ F = \frac{2}{\pi \Lambda} \left( \frac{W_1 W_2 W_3}{(\frac{\partial}{\partial q} \log \vartheta[p, q])^2} \right) \]
• these metrics also satisfy Einstein equation if either
  1. $\Lambda < 0$ with $p \in \mathbb{R}$ and $q \in \frac{1}{2} + i\mathbb{R}$
  2. $\Lambda > 0$ with $q \in \mathbb{R}$ and $p \in \frac{1}{2} + i\mathbb{R}$

• also case with vanishing cosmological constant:

$$W_1' = \frac{1}{\mu + q_0} + 2 \frac{d}{d\mu} \log \vartheta_2, \quad W_2' = \frac{1}{\mu + q_0} + 2 \frac{d}{d\mu} \log \vartheta_3,$$
$$W_3' = \frac{1}{\mu + q_0} + 2 \frac{d}{d\mu} \log \vartheta_4, \quad F' := C(\mu + q_0)^2 W_1' W_2' W_3'$$

with $q_0, C \in \mathbb{R}, C > 0$. 
Asymptotics for $\mu \to \infty$

Result on asymptotics:

1. $\Lambda < 0$: $W_2 \sim \pm W_3$ and

$$W_1 \sim \pi \langle p \rangle e^{\pi i \langle p \rangle - p}, \quad W_3 \sim -2\pi i \langle p + \frac{1}{2} \rangle \cdot e^{\pi i \text{sgn} \langle p \rangle q} \cdot e^{\pi \mu (|\langle p \rangle| - \frac{1}{2})}$$

2. $\Lambda > 0$ with $p = \frac{1}{2} + ip_0, p_0 \in \mathbb{R}$: $W_2 \sim -W_3$ and

$$-W_1 \sim \pi p_0 \tan(\pi(q-p_0\mu)) - \frac{1}{2}, \quad W_3 \sim 2\pi p_0 \cdot (\cos(\pi(q-p_0\mu)))^{-1}$$

3. $\Lambda = 0$: $W'_2 \sim W'_3$ and

$$W'_1 \sim \frac{\pi}{2}, \quad W'_3 \sim \frac{1}{\mu + q_0}$$
Sketch of argument

- \( \Lambda = 0 \):

\[
\vartheta_2 = \sum_{m \in \mathbb{Z}} \exp(-\pi (m + \frac{1}{2})^2 \mu) \sim 2 \exp(-\pi \mu / 4),
\]

\[
\vartheta_3 = \sum_{m \in \mathbb{Z}} \exp(-\pi m^2 \mu) \sim 1 + 2 \exp(-\pi \mu),
\]

\[
\vartheta_4 = \sum_{m \in \mathbb{Z}} \exp(-\pi m^2 \mu)(-1)^m \sim 1 - 2 \exp(-\pi \mu).
\]

So get

\[
\frac{d}{d\mu} \log \vartheta_2 \sim -\frac{\pi}{4}, \quad \frac{d}{d\mu} \log \vartheta_3 \sim -2\pi e^{-\pi \mu}, \quad \frac{d}{d\mu} \log \vartheta_4 \sim 2\pi e^{-\pi \mu}.
\]
• $\Lambda < 0$, $p$ general:

$$\vartheta[p, q] = \sum_{m \in \mathbb{Z}} e^{-\pi (m+p)^2 \mu} + 2\pi i (m+p)q \sim e^{2\pi i \langle p \rangle q} \cdot e^{-\pi \langle p \rangle^2 \mu},$$

leading term of $\vartheta[p, q]$:
unique value of $m$ with $(m + p)^2$ minimal $= \langle p \rangle^2$
• then get

$$\frac{\partial}{\partial q} \vartheta[p, q] \sim 2\pi i \langle p \rangle \exp(2\pi i \langle p \rangle q) \cdot \exp(-\pi \langle p \rangle^2 \mu)$$

• with identity

$$\langle p + 1/2 \rangle = \langle p \rangle - \frac{1}{2} \text{sgn} \langle p \rangle$$
\[ W_2 = \frac{i}{2} \vartheta_2 \vartheta_4 \frac{\partial}{\partial q} \vartheta[p + 1/2, q + 1/2] \]
\[ e^{\pi i p} \vartheta[p, q] \sim \]
\[ \sim \frac{i}{2} \cdot 2 \exp(-\pi \mu/4) \cdot 2\pi i \left\langle p + \frac{1}{2} \right\rangle \cdot \exp(2\pi i \left\langle p + \frac{1}{2} \right\rangle (q + \frac{1}{2})) \]
\[ \cdot \exp(-\pi \left\langle p + \frac{1}{2} \right\rangle^2 \mu) \cdot \exp(-\pi i p) \cdot \exp(-2\pi i \left\langle p \right\rangle q) \cdot \exp(\pi \left\langle p \right\rangle^2 \mu) \sim \]
\[ -2\pi \left\langle p + 1/2 \right\rangle \exp(\pi i (\left\langle p + 1/2 \right\rangle - p - \text{sgn} \left\langle p \right\rangle q)) \cdot \exp(\pi \mu (|\left\langle p \right\rangle| - 1/2)) \]

\[ W_3 = -\frac{1}{2} \vartheta_2 \vartheta_3 \frac{\partial}{\partial q} \vartheta[p + 1/2, q] \]
\[ \sim -\frac{1}{2} \cdot 2 \exp(-\pi \mu/4) \cdot 2\pi i \left\langle p + 1/2 \right\rangle \cdot \exp(2\pi i \left\langle p + 1/2 \right\rangle q) \cdot \exp(-\pi \left\langle p + 1/2 \right\rangle^2 \mu) \cdot \exp(-2\pi i \left\langle p \right\rangle q) \cdot \exp(\pi \left\langle p \right\rangle^2 \mu) \sim \]
\[ -2\pi i \left\langle p + 1/2 \right\rangle \cdot \exp(\pi i \text{sgn} \left\langle p \right\rangle q) \cdot \exp(\pi \mu (|\left\langle p \right\rangle| - 1/2)) \]
• conformal factor

\[
F = \frac{2}{\pi \Lambda} \frac{W_1 W_2 W_3}{(\frac{\partial}{\partial q} \log \vartheta[p, q])^2} \sim \\
2i \left< \frac{p + 1/2}{\Lambda \langle p \rangle^2} \right> \cdot \exp\left(\left< p + 1/2 \right> + \left< p \right> + 2 \text{sgn} \left< p \right> q \right) \cdot \exp\left(\pi \mu \left(2 \left| < p > \right| - 1 \right)\right)
\]

• \( \Lambda > 0 \) with \( p = \frac{1}{2} + ip_0, \ p_0 \in \mathbb{R} \)

\[
\vartheta[p, q] = \sum_{m \in \mathbb{Z}} \exp\left(-\pi (m + p)^2 \mu + 2\pi i (m + p)q\right),
\]

\[
\vartheta[p, q] \sim \exp(\pi \mu (p_0^2 - 1/4)) \cdot \exp(-2\pi p_0 q) \cdot \cos \pi (q - p_0 \mu)
\]

\[
\vartheta[p + 1/2, q] \sim \exp(\pi \mu p_0^2) \cdot \exp(-2\pi p_0 q)
\]

\[
\frac{W_2}{W_3} = -i \cdot \frac{\vartheta_4}{\vartheta_3} \cdot \frac{\partial}{\partial q} \frac{\vartheta[p + 1/2, q + 1/2]}{e^{\pi i p} \frac{\partial}{\partial q} \vartheta[p + 1/2, q]} \sim \\
i \cdot \frac{\exp(-2\pi p_0 (q + 1/2))}{\exp(\pi i (1/2 + ip_0)) \cdot \exp(-2\pi p_0 q)} = -1
\]
\[ W_3 := -\frac{1}{2} \vartheta_2 \vartheta_3 \frac{\partial}{\partial q} \vartheta[p + 1/2, q] \sim \exp(-\pi \mu/4) \cdot (2\pi p_0) \cdot \exp(\pi \mu p_0^2) \cdot \exp(-2\pi p_0 q) \] 
\[ \frac{\exp(\pi \mu(p_0^2 - 1/4)) \cdot \exp(-2\pi p_0 q) \cdot \cos \pi(q - p_0 \mu)}{\exp(2\pi p_0 q) \cdot \cos \pi(q - p_0 \mu)} \sim 2\pi p_0 \cdot (\cos \pi(q - p_0 \mu))^{-1} \]

\[ -W_1 = \frac{i}{2} \vartheta_3 \vartheta_4 \frac{\partial}{\partial q} \vartheta[p, q + 1/2] \sim \frac{i}{2} \exp(\pi \mu(p_0^2 - 1/4)) \cdot \frac{\partial}{\partial q} \left( \exp(-2\pi p_0(q + 1/2)) \cdot \cos \pi(q + 1/2 - p_0 \mu) \right) \] 
\[ \frac{2}{\exp(\pi(i/2 - p_0))} \cdot \exp(\pi \mu(p_0^2 - 1/4)) \cdot \exp(-2\pi p_0 q) \cdot \cos \pi(q - p_0 \mu) \sim \frac{i}{2} \frac{\partial}{\partial q} \left( \exp(-2\pi p_0(q + 1/2)) \cdot \cos \pi(q + 1/2 - p_0 \mu) \right) \] 
\[ \frac{1}{2} \frac{\partial}{\partial q} \left( \exp(-2\pi p_0(q + 1/2)) \cdot \sin \pi(q - p_0 \mu) \right) \sim \frac{1}{2} \frac{\partial}{\partial q} \left( \exp(-2\pi p_0(q + 1/2)) \cdot \cos \pi(q - p_0 \mu) \right) \]

\[ \pi p_0 \tan(\pi(q - p_0 \mu)) - \frac{1}{2} \]
Comments

- Singularities (poles) on the real axis: like Taub-NUT infinity
- Sign changes allowed to get all asymptotics with $W_2 \sim W_3 \neq W_1$ (see Babich, Korotkin)
- Instanton analogs of Kasner’s solutions with $i\mu \in \Delta \subset \mathbb{H}$ in the vicinity of $i\infty$ but not necessarily on the imaginary axis
- Behavior $\mu \to \infty$ of these Bianchi IX cosmologies as possible model of (Wick rotated) time at the singularity in algebro-geometric gluing of spacetimes proposed in:
Spacetime noncommutativity in the early universe

- noncommutativity hypothesis: near the singularity spacetime coordinates acquire noncommutativity as part of quantum effects
- noncommutative deformation should preserve the metric properties
- Connes–Landi isospectral deformations
- for the 3-sphere $S^3$ with the round metric: isospectral deformation by making all the tori of the Hopf fibration into noncommutative tori
- do the left-$SU(2)$-invariant Bianchi IX metrics admit similar noncommutative isospectral deformations?
- not always, but yes in the cases that arise as asymptotic behavior at $\mu \to \infty$ of the gravitational instantons
Hopf fibration on $S^3$

- Hopf coordinates $(\xi_1, \xi_2, \eta)$

\[ z_1 := x_1 + ix_2 = e^{i(\psi + \phi)} \cos \frac{\theta}{2} = e^{i\xi_1} \cos \eta, \]

\[ z_2 := x_3 + ix_0 = e^{i(\psi - \phi)} \sin \frac{\theta}{2} = e^{i\xi_2} \sin \eta. \]

- Identifying $S^3$ with unit quaternions $SU(2)$

\[ q := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{i\xi_1} \cos \eta & e^{i\xi_2} \sin \eta \\ -e^{-i\xi_2} \sin \eta & e^{-i\xi_1} \cos \eta \end{pmatrix} \]

with $|z_1|^2 + |z_2|^2 = 1$ and $(\xi_1, \xi_2, \eta)$ Hopf coordinates

- Hopf fibration

\[ S^1 \hookrightarrow S^3 \twoheadrightarrow S^2 \]
The Hopf fibration of $S^3$

(image by Benoit Kloeckner)
Result: a spacetime with (Euclidean) Bianchi IX metric of the form

$$g = F \left( d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2} \right)$$

admits a Connes–Landi isospectral deformation, obtained by deforming the Hopf fibration, iff $W_2 = W_3$ (or $W_1 = W_2$ up to renaming axes)

- Deformation of the Hopf fibration

$$\left( \begin{array}{cc} U \cos \eta & V \sin \eta \\ -V^* \sin \eta & U^* \cos \eta \end{array} \right)$$

with $U, V$ the generators of the noncommutative 2-torus $T^2_\theta$

$$VU = e^{2\pi i \theta} UV$$

no longer group structure of $SU(2)$
• Deformed $C^*$-algebra of functions $S^3_\theta$:
  generators $\alpha = U \cos \eta$ and $\beta = V \sin \eta$
  relations: $\alpha \beta = e^{2\pi i \theta} \beta \alpha$, $\alpha^* \beta = e^{-2\pi i \theta} \alpha^* \beta$, $\alpha^* \alpha = \alpha \alpha^*$, $\beta^* \beta = \beta \beta^*$ and $\alpha \alpha^* + \beta \beta^* = 1$

• Riemannian geometry in noncommutative setting described by spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with $\mathcal{A}$ involutive algebra (smooth functions on NC space), $\mathcal{H}$ Hilbert space with representation of $\mathcal{A}$ (spinors), and Dirac operator $\mathcal{D}$

• Isospectral deformation $X_\theta$ of a manifold $X$: $\mathcal{A} = C^\infty(X_\theta)$ noncommutative, with $(\mathcal{H}, \mathcal{D}) = (L^2(X, S), \mathcal{D}_X)$ same as for $X$

• Connes–Landi: if $T^2$ acts by isometries on $X$ then $\exists X_\theta$

• check when have action of $T^2$ by isometry on the Bianchi IX, compatible with the Hopf fibration of $S^3$
in Hopf coordinates $T^2$ action on $S^3$

$$(t_1, t_2) : (\xi_1, \xi_2) \mapsto (\xi_1 + t_1, \xi_2 + t_2)$$

Euler angles $(u, v) : (\phi, \psi) \mapsto (\phi + u, \psi + v)$, with $t_1 = (u + v)/2$ and $t_2 = (v - u)/2$

$U(1)$-action $u : \phi \mapsto \phi + u$ leaves 1-forms $\sigma_i$ invariant (rotates circles $S^1 \hookrightarrow S^3$ of Hopf fibration)

the form $\sigma_1$ also invariant under other $U(1)$-action $v : \psi \mapsto \psi + v$

$$v^* \sigma_2 = \frac{1}{2} (\sin(\psi + \beta) \, d\theta - \cos(\psi + \beta) \, \sin \theta \, d\phi)$$

$$v^* \sigma_3 = \frac{1}{2} (-\cos(\psi + \beta) \, d\theta - \sin(\psi + \beta) \, \sin \theta \, d\phi),$$

then $v^* g = g$ for a Bianchi IX metric

$$g = d\mu^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2$$

if and only if $b = c$
This class of Bianchi IX metrics include

1. Taub-NUT and Eguchi-Hanson gravitational instantons
2. asymptotic form of the general Bianchi IX gravitational instantons

Dirac operator:

- Berger sphere $S^3$ with $\lambda^2 \sigma_1^2 + \sigma_2^2 + \sigma_3^2$

$$D_B = -i \left( \frac{1}{\lambda} X_1 X_2 + iX_3 \right) + \frac{\lambda^2 + 2}{2\lambda},$$

with $\{X_1, X_2, X_3\}$ basis of the Lie algebra

- on the Bianchi IX (Euclidean) spacetime

$$D = \frac{1}{W_{1/2}^1 W} \left( \gamma^0 \left( \frac{\partial}{\partial \mu} + \frac{1}{2} \left( \frac{\dot{W}}{W} + \frac{1}{2} \frac{\dot{W}_1}{W_1} \right) \right) + W_1 D_B |_{\lambda = \frac{W}{W_1}} \right)$$

with $W = W_2 = W_3$

**Conclusion:** Bianchi IX gravitational instantons are compatible with spacetime noncommutativity (only at $\mu \rightarrow \infty$).