Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies

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Topics in Mathematical Physics

Based on:

- Yuri Manin, Matilde Marcolli, *Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies*, arXiv:1504.04005 [gr-qc]

Kasner metrics

• real circle in $\mathbb{R}^3$ defined by equations

$$p_a + p_b + p_c = 1, \qquad p_a^2 + p_b^2 + p_c^2 = 1$$

• each point on this circle defines a metric with Minkowskian (or Euclidean) signature

$$\pm dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$$

with *scaling factors* a, b, c:

$$a(t) = t^{p_a}, \qquad b(t) = t^{p_b}, \qquad c(t) = t^{p_c}, t > 0.$$

Kasner metric with exponents (p_a, p_b, p_c).

u-parameterization

• Points (p_a, p_b, p_c) on the circle parameterized by a coordinate $u \in [1, \infty]$

$$p_1^{(u)} := -\frac{u}{1+u+u^2} \qquad \in [-1/3, 0]$$

$$p_2^{(u)} := \frac{1+u}{1+u+u^2} \qquad \in [0, 2/3]$$

$$p_3^{(u)} := \frac{u(1+u)}{1+u+u^2} \qquad \in [2/3, 1]$$

• Rearrange the exponents $p_1^{(u)} \leq p_2^{(u)} \leq p_3^{(u)}$ by a bijection $(1,2,3) \to (a,b,c)$ (permutation of the 3 space axes)

Mixmaster Universe (1970s)

V. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, *Oscillatory approach to singular point in Relativistic cosmology*. Adv. Phys. 19 (1970), 525–551.

• Anisotropic cosmologies

• Locally described by a Kasner metric

• Sequence of Kasner metrics (Kasner epochs and cycles)

• Within each epoch one direction dominates expansion, the other two oscillate in a series of Kasner cycles

• At the end of each epoch a *bounce* occurs and a possibly different direction becomes responsible for expansion

• Approach: model the dynamics by a discrete dynamical system that determines epochs and cycles
$\Rightarrow$ continued fraction expansion

Continued fraction expansion

• infinite continued fraction expansion
of a number $x \in \mathbb{R} \smallsetminus \mathbb{Q}$, $x > 1$:

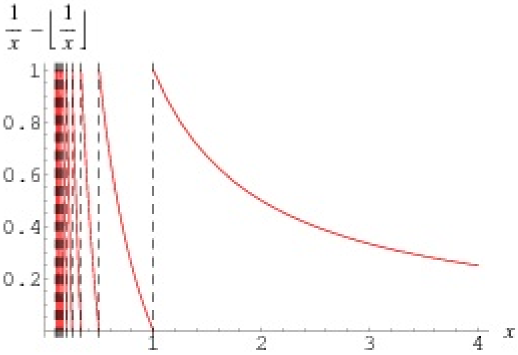
$$x = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \dots}} =: [k_0, k_1, k_2, \dots], \;\; k_s \in \mathbb{N}$$

with $k_0 := [x]$, $k_1 = [1/x]$ etc. digits $k_i \in \mathbb{N}$
for a point in $[0,1]$, expansion $[0, k_1, k_2, \dots, k_n, \dots]$

• Dynamical system: (partial map)
one sided shift of the continued fraction expansion

$$T : [0,1] \to [0,1] \qquad T : x \mapsto \frac{1}{x} - \left[\frac{1}{x}\right]$$

$$[0, k_0, k_1, k_2, \dots, k_n, \dots] \stackrel{T}{\mapsto} [0, k_1, k_2, \dots, k_{n+1}, \dots]$$



One-sided shift of the continued fraction expansion

• Dynamical system: (partial map)
invertible two sided shift

$$\widetilde{T}: [0,1]^2 \to [0,1]^2 \qquad \widetilde{T}: (x,y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y + [1/x]} \right)$$

• On $[0,1]^2 \cap (\mathbb{R}^2 \smallsetminus \mathbb{Q}^2)$ uniquely defined $k_s \in \mathbb{N}$

$$x = [0, k_0, k_1, k_2, \ldots], \qquad y = [0, k_{-1}, k_{-2}, \ldots]$$

$$\frac{1}{x} - \left[\frac{1}{x} \right] = [0, k_1, k_2, \ldots], \quad \frac{1}{y + [1/x]} = \frac{1}{k_0 + y} = [0, k_0, k_{-1}, k_{-2}, \ldots]$$

• On this subset $\widetilde{T}$ bijective with invariant density

$$d\mu(x,y) = \frac{dx\, dy}{\log 2 \cdot (1 + xy)^2}$$

• encode $(x,y) \in [0,1]^2 \cap (\mathbb{R}^2 \smallsetminus \mathbb{Q}^2)$ with doubly infinite sequence $(k) := [\ldots k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots]$, $k_i \in \mathbb{N}$ where $T(k)_s = k_{s+1}$
invertible shift

Continued fractions and Mixmaster Universe

• *typical* solutions of Einstein equations Bianchi IX type with $SO(3)$–symmetry oscillates (near the initial singularity) close to a sequence of Kasner type solutions

• local logarithmic time $d\Omega := -\frac{dt}{abc}$

• for $\Omega \cong -\log t \to +\infty$:

- increasing sequence of times $\Omega_0 < \Omega_1 < \cdots < \Omega_n < \ldots$
- sequence of irrational real numbers $u_n \in (1, +\infty)$, $n = 0, 1, 2, \ldots$

• semi–interval $[\Omega_n, \Omega_{n+1})$ is n–th Kasner epoch

• start at time Ω_n with a value $u = u_n > 1$:

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}$$

• Kasner cycles (within same Kasner epoch)
$u = u_n - 1,\ u_n - 2, \ldots$, with corresponding Kasner metrics

• after $k_n := [u_n]$ cycles inside the same Kasner epoch, a jump to the next epoch with new parameter

$$u_{n+1} = \frac{1}{u_n - [u_n]}$$

• at the end of each epoch a reshuffling of space axes (also determined by the discrete dynamical system)

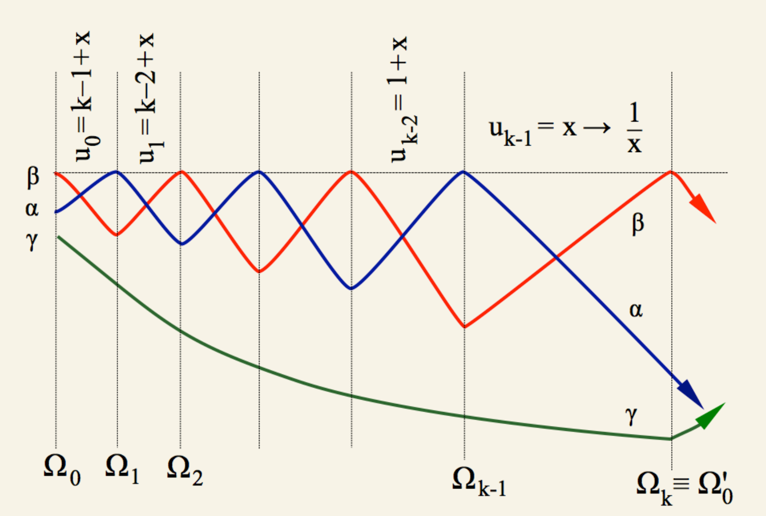
• sequence of logarithmic times Ω_n specified by a sequence δ_n

$$\Omega_{n+1} = [1 + \delta_n k_n (u_n + 1/\{u_n\})]\Omega_n$$

• setting $\eta_n = (1 - \delta_n)/\delta_n$ recursion

$$\eta_{n+1} x_n = \frac{1}{k_n + \eta_n x_{n_1}}$$

with $x_n = u_n - k_n$


$u_0 = k-1+x$
$u_1 = k-2+x$
$u_{k-2} = 1+x$
$u_{k-1} = x \rightarrow \frac{1}{x}$
β
α
γ
β
α
γ
Ω_0
Ω_1
Ω_2
Ω_{k-1}
$\Omega_k \equiv \Omega'_0$

- I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, L. N. Shchur, and Ya. G. Sinai. *On the stochasticity in relativistic cosmology.* Journ. Stat. Phys., Vol. 38, Nos. 1/2 (1985), 97–114

- D.H. Mayer, *Relaxation properties of the Mixmaster Universe*, Phys. Lett. A 121 (1987), no. 8,9, 390–394

Conclusion:

• trajectories of mixmaster universe dynamics are parameterized by pairs $(x, y) \in [0, 1]^2 \cap (\mathbb{R}^2 \smallsetminus \mathbb{Q}^2)$

$$x = [0, k_0, k_1, k_2, \ldots], \qquad y = [0, k_{-1}, k_{-2}, \ldots]$$

x specifies number of Kasner cycles in each Kasner epoch, y specifies the Kasner logarithmic times

• transition from one Kasner epoch to the next is given by the action of the double sided shift of the continued fraction

$$\widetilde{T} : (x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y + [1/x]} \right)$$

Modular curves

• upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \,|\, y = \Im(z) > 0\}$
hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

• geodesics: vertical straight lines orthogonal to x-axis; circular arcs (half-circles) orthogonal to x-axis

• $\mathrm{PSL}_2(\mathbb{R})$ acts transitively and isometrically

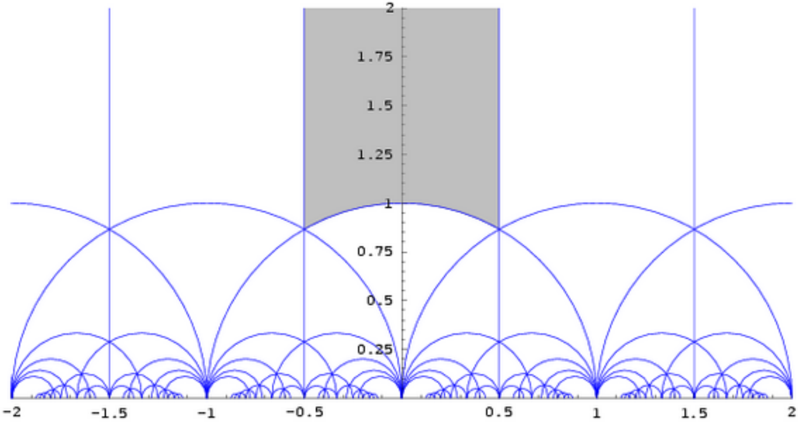
$$z \mapsto \frac{az + b}{cz + d}$$

• action of $\mathrm{PSL}_2(\mathbb{Z})$ and finite index subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$

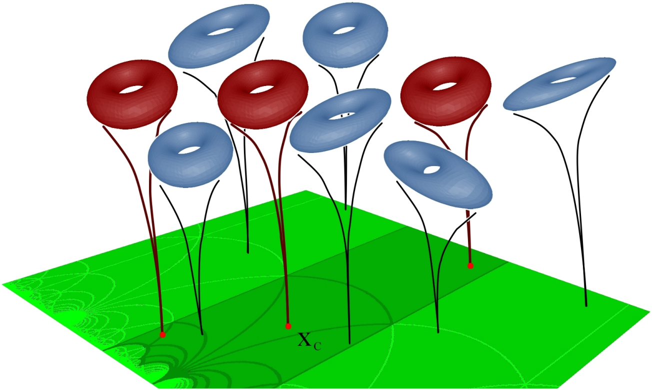
• Elliptic curves $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ modulus $\tau \in \mathbb{H}$;
isomorphic $E_\tau \simeq E_{\tau'}$ when $\tau \sim \tau'$ by $\mathrm{PSL}_2(\mathbb{Z})$ action

• $\mathcal{M} = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ moduli space of elliptic curves (modular curve)

Fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$-action



Elliptic curves and modular curve



(detail from an image by Christian Wuthrich)

Farey tessellation

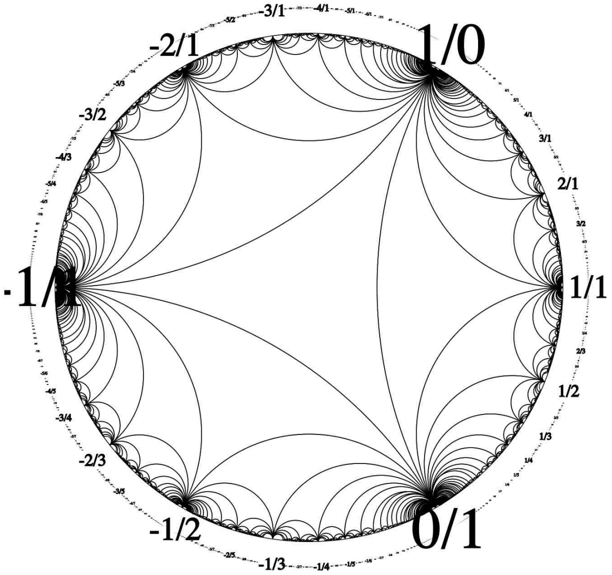
• adding cusps to the upper half plane: $\overline{\mathbb{H}} := \mathbb{H} \cup \{\mathbb{Q} \cup \{\infty\}\}$

• vertical lines $\Re(z) = n, n \in \mathbb{Z}$, and semicircles in $\overline{\mathbb{H}}$ connecting pairs of finite cusps $(p/q, p'/q')$ with $pq' - p'q = \pm 1$

• these cut $\overline{\mathbb{H}}$ into a union of geodesic ideal triangles: Farey tessellation

- C.Series, *The modular surface and continued fractions*, J. London MS, Vol. 2, no. 31 (1985), 69–80

• coding of geodesics on $\mathcal{M} = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ using Farey tessellation and continued fraction



Farey tessellation in the Poincaré disk model of $\overline{\mathbb{H}}$

• $\mathcal{B}$ set of oriented geodesics β in $\mathbb{H}$ with ideal irrational endpoints $\beta_{-\infty}, \beta_\infty$ in $\mathbb{R}$, such that

$$\beta_{-\infty} \in (-1, 0), \qquad \beta_\infty \in (1, \infty)$$

• continued fraction expansion of endpoints

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \ldots], \qquad \beta_\infty = [k_1, k_2, k_3, \ldots], \quad k_i \in \mathbb{N},$$

• β determined by endpoints, by doubly infinite sequence of continued fraction digits

$$[\ldots k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots]$$

• intersection point $x = x(\beta)$ of β with imaginary semiaxis in $\mathbb{H}$

• moving along β: intersect infinite sequence of Farey triangles

• enter each triangle through one side and leave through a different one: the ideal intersection point of these two sides is either to the left or to the right

• infinite sequences in alphabet $\{L, R\}$ (moving in both directions)

$$\ldots L^{k_{-3}} R^{k_{-2}} L^{k_{-1}} R^{k_0} \quad L^{k_1} R^{k_2} L^{k_3} R^{k_4} \ldots$$

Hyperbolic billiard

• all hyperbolic triangles Δ of the Farey tessellation of $\overline{\mathbb{H}}$ are isometric metric spaces

• folding the tessellation: a geodesic $\beta \in \mathcal{B}$ corresponds to a billiard ball trajectory on the table Δ that never hits the corners

• group S_6 of hyperbolic isometries of Δ (permutations on vertices); unique fixed point $\rho := \exp(\pi i/3)$ in Δ

• three geodesic acrs connecting ρ with points $i, 1+i, \frac{1+i}{2}$ subdivide Δ into three geodesic quadrangles, each with one cusp corner

• each quadrangle is a copy of the fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$

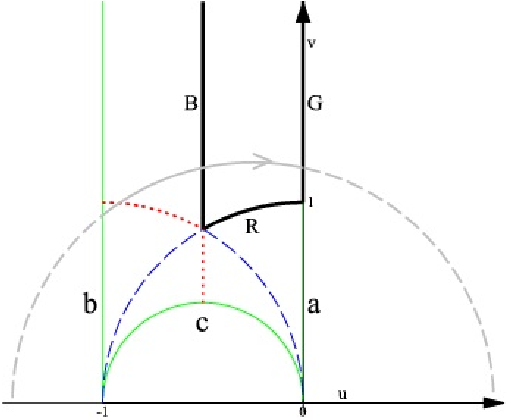
• for a geodesic β with $x(\beta) = x_0$ in $(0, i\infty)$, if $k_0 = 1$, the ball reaches the opposite side $(1, i\infty)$ of Δ and gets reflected to the third side $(0, 1)$

• if $k_0 = 2$, it reaches the opposite side, then returns to the initial side $(0, i\infty)$, and after that gets reflected to $(0, 1)$

• generally the ball always spends k_0 unobstructed stretches of its trajectory between $(0, i\infty)$ and $(1, i\infty)$, then is reflected to $(0, 1)$ either from $(1, i\infty)$ (if k_0 is odd), or from $(0, i\infty)$ (if k_0 is even)

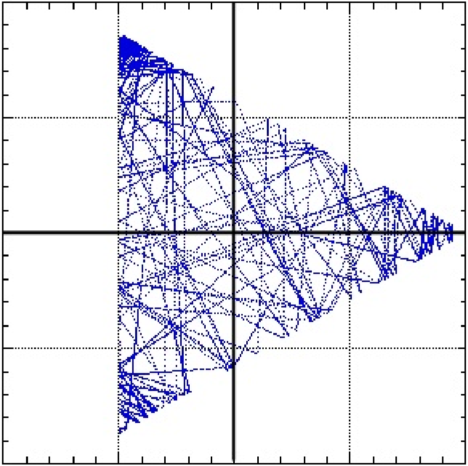
• can equivalently encode the geodesic β in $\mathcal{M} = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ by a doubly infinite sequence in the alphabet $\{a, b, c\}$ labelling the three vertices at infinity $\{0, 1, i\infty\}$ of Δ

Hyperbolic Billiard



(from O.M.Lecian, "Reflections in the hyperbolic plane", arXiv:1303.6343)

Other billiard models for Mixmaster dynamics



(from Beverly Berger, “Numerical Approaches to Spacetime Singularities”)

Enriched encoding of geodesics and Mixmaster trajectories

• hyperbolic billiard as above: insert between consecutive powers of L, R the intersection points of β with sides of Farey triangles:

$$\dots L^{k_{-1}} x_{-1} R^{k_0} x_0 L^{k_1} x_1 R^{k_2} x_2 L^{k_3} x_3 R^{k_4} \dots$$

• Result: when $s \to \infty$, $s \in \mathbb{N}$

$$\log \frac{\Omega_{2s}}{\Omega_0} \simeq 2 \sum_{r=0}^{s-1} \mathrm{dist}\,(x_{2r}, x_{2r+1}),$$

dist = hyperbolic distance between consecutive intersection points of the geodesic with sides of the Farey tesselation

- Sketch of argument: known from mixmaster dynamics that

$$\log \frac{\Omega_{2s}}{\Omega_0} \simeq -\sum_{p=1}^{2s} \log(x_p^+ x_p^-)$$

$$= \sum_{p=1}^{2s} \log([k_{p-1}, k_{p-2}, k_{p-3}, \dots] \cdot [k_p, k_{p+1}, k_{p+2}, \dots])$$

From coding of geodesics also know that

$$\mathrm{dist}(x_0, x_1) = \frac{1}{2} \log([k_0, k_{-1}, k_{-2}, \dots] \cdot [k_1, k_2, \dots] \cdot [k_1, k_0, k_{-1}, \dots] \cdot [k_2, k_3, \dots])$$

and more generally $\mathrm{dist}\,(x_{2r}, x_{2r+1})$ is given by

$$\frac{1}{2} \log([k_{2r}, k_{2r-1}, k_{2r-2}, \dots] \cdot [k_{2r+1}, k_{2r+2}, \dots] \cdot$$

$$\cdot [k_{2r+1}, k_{2r}, k_{2r-1}, \dots] \cdot [k_{2r+2}, k_{2r+3}, \dots])$$

Consequence: identification of distance along geodesic with logarithmic cosmological time

Painlevé VI equations

- *Painlevé transcendents*: solutions of nonlinear second-order ODEs in the plane with *Painlevé property* (the only movable singularities are poles) not solvable in terms of elementary functions; classification in types

- *Painlevé VI*: 4-parameter family $(\alpha, \beta, \gamma, \delta)$

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2$$

$$-\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

Painlevé VI and elliptic curves

- Painlevé VI rewritten as (Fuchs)

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_{\infty}^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}$$

where $(X,Y) := (X(t),Y(t))$ is a section (local and/or multivalued) $P := (X(t),Y(t))$ of the generic elliptic curve $E=E(t):\ \ Y^2=X(X-1)(X-t)$

- left-hand-side $\mu(P)$ satisfies $\mu(P+Q)=\mu(P)+\mu(Q)$ for $P+Q$ addition on the elliptic curve E (in particular $\mu(Q)=0$ for points of finite order)

- analytic description of the elliptic curve $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, with $\tau \in \mathbb{H}$
- then Painlevé VI rewritten as (Manin)

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau)$$

with $(\alpha_0, \ldots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)$, and

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

• also have, for $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

• a multivalued solution $z = z(\tau)$ defines a multi–section of the family, which is a covering of $\mathbb{H}$

• is know ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

- Yu.I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, in "Geometry of Differential Equations", Amer. Math. Soc. Transl. (2) Vol. 186 (1998) 131–151

$SU(2)$-Bianchi IX cosmologies $\mathbb{R} \times S^3$

• another version of Bianchi IX mixmaster cosmologies, with $SU(2)$ symmetry (Euclidean version)

$$g = W_1 W_2 W_3 \, d\mu^2 + \frac{W_2 W_3}{W_1} \, \sigma_1^2 + \frac{W_1 W_3}{W_2} \, \sigma_2^2 + \frac{W_1 W_2}{W_3} \, \sigma_3^2$$

or more generally

$$g = F \, (d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^3})$$

with a conformal factor $F \sim W_1 W_2 W_3$

• $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$

- more explicitly

$$\sigma_1 = x_1\, dx_2 - x_2\, dx_1 + x_3\, dx_0 - x_0\, dx_3 = \frac{1}{2}(d\psi + \cos\theta\, d\phi),$$

$$\sigma_2 = x_2\, dx_3 - x_3 dx_2 + x_1\, dx_0 - x_0\, dx_1 = \frac{1}{2}(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\phi),$$

$$\sigma_3 = x_3\, dx_1 - x_1\, dx_3 + x_2\, dx_0 - x_0\, dx_2 = \frac{1}{2}(-\cos\psi\, d\theta - \sin\theta\, \sin\psi\, d\phi),$$

Euler angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$ ($SU(2)$ case)

- identifying S^3 with unit quaternions $SU(2)$
- The metrics on S^3

$$\frac{W_2 W_3}{W_1}\, \sigma_1^2 + \frac{W_1 W_3}{W_2}\, \sigma_2^2 + \frac{W_1 W_2}{W_3}\, \sigma_3^2$$

are left-invariants under the action of $SU(2)$ but *not* right-invariant (unlike the round metric on S^3)

Bianchi IX gravitational instantons and Painlevé VI

• Euclidean Bianchi IX metrics with $SU(2)$-symmetry that are
- self-dual (Weyl curvature tensor W self-dual)
- Einstein metrics (Ricci tensor proportional to the metric)

• Self-dual equations for a Riemannian 4-manifold are PDEs; with $SU(2)$-symmetry reduce to ODEs

• This ODE is a Painlevé VI equation with

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$$

- N.J. Hitchin. *Twistor spaces, Einstein metrics and isomonodromic deformations*, J.Diff.Geom., Vol. 42, No. 1 (1995), 30–112.
- K.P. Tod. *Self–dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A 190 (1994), 221–224.
- S. Okumura. *The self–dual Einstein–Weyl metric and classical solutions of Painlevé VI*, Lett. in Math. Phys., 46 (1998), 219–232.
- M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337

Theta characteristics

- explicit paremeterization of solutions for coefficients W_i of the Bianchi IX gravitational instantons (from solutions of Painlevé VI)
- theta–characteristics with parameters (p, q):

$$\vartheta[p,q](z,i\mu) := \sum_{m\in\mathbb{Z}} \exp\left(-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\right)$$

- theta-characteristics and theta functions with vanishing characteristics

$$\vartheta[p,q](z,i\mu) = \exp\left(-\pi p^2\mu + 2\pi ipq\right)\cdot\vartheta[0,0](z+pi\mu+q,i\mu)$$

Gravitational instantons and theta characteristics

- use notation $\vartheta[p,q] := \vartheta[p,q](0,i\mu)$, and

$$\vartheta_2 := \vartheta[1/2,0], \qquad \vartheta_3 := \vartheta[0,0], \qquad \vartheta_4 := \vartheta[0,1/2]$$

- self-dual metrics

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^3})$$

with

$$W_1 = -\frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p,q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]}, \qquad W_2 = \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2},q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]},$$

$$W_3 = -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2},q]}{\vartheta[p,q]},$$

- with non-zero cosmological constant Λ:

$$F = \frac{2}{\pi\Lambda}\frac{W_1 W_2 W_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2}$$

- these metrics also satisfy Einstein equation if either
  1. $\Lambda < 0$ with $p \in \mathbb{R}$ and $q \in \frac{1}{2} + i\mathbb{R}$
  2. $\Lambda > 0$ with $q \in \mathbb{R}$ and $p \in \frac{1}{2} + i\mathbb{R}$
- also case with vanishing cosmological constant:

$$W_1' = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu} \log \vartheta_2, \quad W_2' = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu} \log \vartheta_3,$$

$$W_3' = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu} \log \vartheta_4, \quad F' := C(\mu + q_0)^2 \, W_1' W_2' W_3'$$

with $q_0, C \in \mathbb{R}, \; C > 0$.

Asymptotics for $\mu \to \infty$

Result on asymptotics:

1. $\Lambda < 0$: $W_2 \sim \pm W_3$ and

$$W_1 \sim \pi \langle p \rangle \, e^{\pi i \langle p \rangle - p}, \quad W_3 \sim -2\pi i \, \langle p + \frac{1}{2} \rangle \cdot e^{\pi i \, \mathrm{sgn} \, \langle p \rangle q} \cdot e^{\pi \mu (|\langle p \rangle| - \frac{1}{2})}$$

2. $\Lambda > 0$ with $p = \frac{1}{2} + i p_0, p_0 \in \mathbb{R}$: $W_2 \sim -W_3$ and

$$-W_1 \sim \pi p_0 \, \tan(\pi(q - p_0 \mu)) - \frac{1}{2}, \quad W_3 \sim 2\pi p_0 \cdot (\cos(\pi(q - p_0 \mu)))^{-1}$$

3. $\Lambda = 0$: $W_2' \sim W_3'$ and

$$W_1' \sim \frac{\pi}{2}, \quad W_3' \sim \frac{1}{\mu + q_0}$$

Sketch of argument

- $\Lambda = 0$:

$$\vartheta_2 = \sum_{m\in\mathbb{Z}} \exp(-\pi(m+\frac{1}{2})^2\mu) \ \sim 2\,\exp(-\pi\mu/4),$$

$$\vartheta_3 = \sum_{m\in\mathbb{Z}} \exp(-\pi m^2\mu) \ \sim 1 + 2\,\exp(-\pi\mu),$$

$$\vartheta_4 = \sum_{m\in\mathbb{Z}} \exp(-\pi m^2\mu)(-1)^m \ \sim 1 - 2\,\exp(-\pi\mu).$$

So get

$$\frac{d}{d\mu}\log\vartheta_2 \sim -\frac{\pi}{4}, \quad \frac{d}{d\mu}\log\vartheta_3 \sim -2\pi\, e^{-\pi\mu}, \quad \frac{d}{d\mu}\log\vartheta_4 \sim 2\pi\, e^{-\pi\mu}.$$

- $\Lambda < 0$, p general:

$$\vartheta[p,q] = \sum_{m\in\mathbb{Z}} e^{-\pi(m+p)^2\mu + 2\pi i(m+p)q} \sim e^{2\pi i\langle p\rangle q} \cdot e^{-\pi\langle p\rangle^2\mu},$$

leading term of $\vartheta[p,q]$:
unique value of m with $(m+p)^2$ minimal $= \langle p\rangle^2$

- then get

$$\frac{\partial}{\partial q}\vartheta[p,q] \sim 2\pi i\,\langle p\rangle \exp(2\pi i\langle p\rangle q)\cdot\exp(-\pi\langle p\rangle^2\mu)$$

- with identity

$$\langle p+1/2\rangle = \langle p\rangle - \frac{1}{2}\mathrm{sgn}\langle p\rangle$$

$$W_2 = \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+1/2, q+1/2]}{e^{\pi i p}\vartheta[p,q]} \sim$$

$$\sim \frac{i}{2}\cdot 2\,\exp(-\pi\mu/4)\cdot 2\pi i\,\langle p+\frac{1}{2}\rangle \cdot \exp(2\pi i\langle p+\frac{1}{2}\rangle(q+\frac{1}{2}))$$

$$\cdot \exp(-\pi\langle p+\frac{1}{2}\rangle^2\mu)\cdot \exp(-\pi i p)\cdot \exp(-2\pi i\langle p\rangle q)\cdot \exp(\pi\langle p\rangle^2\mu) \sim$$

$$-2\pi\,\langle p+1/2\rangle \exp(\pi i(\langle p+1/2\rangle - p - \operatorname{sgn}\langle p\rangle q))\cdot \exp(\pi\mu(|\langle p\rangle| - 1/2))$$

$$W_3 = -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+1/2, q]}{\vartheta[p,q]} \sim$$

$$\sim -\frac{1}{2}\cdot 2\,\exp(-\pi\mu/4)\cdot 2\pi i\,\langle p+1/2\rangle\cdot\exp(2\pi i\langle p+1/2\rangle q)\cdot\exp(-\pi\langle p+1/2\rangle^2\mu)\cdot$$

$$\exp(-2\pi i\langle p\rangle q)\cdot \exp(\pi\langle p\rangle^2\mu) \sim$$

$$-2\pi i\,\langle p+1/2\rangle \cdot \exp(\pi i\,\operatorname{sgn}\langle p\rangle q)\cdot \exp(\pi\mu(|\langle p\rangle| - 1/2))$$

- conformal factor

$$F = \frac{2}{\pi\Lambda} \frac{W_1 W_2 W_3}{(\frac{\partial}{\partial q} \log \vartheta[p,q])^2} \sim$$

$$2i \frac{\langle p+1/2\rangle^2}{\Lambda \langle p\rangle^2} \cdot \exp(\langle p+1/2\rangle + \langle p\rangle + 2\,\mathrm{sgn}\,\langle p\rangle\; q) \cdot \exp(\pi\mu\,(2\,|\langle p\rangle| - 1))$$

- $\Lambda > 0$ with $p = \frac{1}{2} + ip_0, \;\; p_0 \in \mathbb{R}$

$$\vartheta[p,q] = \sum_{m\in\mathbb{Z}} \exp(-\pi(m+p)^2\mu + 2\pi i(m+p)q),$$

$$\vartheta[p,q] \sim \exp(\pi\mu(p_0^2 - 1/4)) \cdot \exp(-2\pi p_0 q) \cdot \cos\pi(q - p_0\mu)$$

$$\vartheta[p+1/2,q] \sim \exp(\pi\mu p_0^2) \cdot \exp(-2\pi p_0 q)$$

$$\frac{W_2}{W_3} = -i \cdot \frac{\vartheta_4}{\vartheta_3} \cdot \frac{\frac{\partial}{\partial q}\vartheta[p+1/2, q+1/2]}{e^{\pi i p}\frac{\partial}{\partial q}\vartheta[p+1/2,q]} \sim$$

$$i \cdot \frac{\exp(-2\pi p_0(q+1/2))}{\exp(\pi i(1/2 + ip_0)) \cdot \exp(-2\pi p_0 q)} = -1$$

$$W_3 := -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+1/2,q]}{\vartheta[p,q]} \sim$$

$$\frac{\exp(-\pi\mu/4)\cdot(2\pi p_0)\cdot\exp(\pi\mu p_0^2)\cdot\exp(-2\pi p_0 q)}{\exp(\pi\mu(p_0^2-1/4))\cdot\exp(-2\pi p_0 q)\cdot\cos\pi(q-p_0\mu)} \sim$$

$$2\pi p_0\cdot(\cos\pi(q-p_0\mu))^{-1}$$

$$-W_1 = \frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p,q+1/2]}{e^{\pi i p}\vartheta[p,q]} \sim$$

$$\frac{i}{2}\cdot\frac{\exp(\pi\mu(p_0^2-1/4))\cdot\frac{\partial}{\partial q}(\exp(-2\pi p_0(q+1/2))\cdot\cos\pi(q+1/2-p_0\mu))}{\exp(\pi(i/2-p_0))\cdot\exp(\pi\mu(p_0^2-1/4))\cdot\exp(-2\pi p_0 q)\cdot\cos\pi(q-p_0\mu)} \sim$$

$$\frac{i}{2}\cdot\frac{\frac{\partial}{\partial q}(\exp(-2\pi p_0(q+1/2))\cdot\cos\pi(q+1/2-p_0\mu))}{\exp(\pi(i/2-p_0))\cdot\exp(-2\pi p_0 q)\cdot\cos\pi(q-p_0\mu)} \sim$$

$$-\frac{1}{2}\cdot\frac{\frac{\partial}{\partial q}(\exp(-2\pi p_0(q+1/2))\cdot\sin\pi(q-p_0\mu))}{\exp(-2\pi p_0(q+1/2))\cdot\cos\pi(q-p_0\mu)} \sim$$

$$\pi p_0\ \tan(\pi(q-p_0\mu)) - \frac{1}{2}$$

Comments

• Singularities (poles) on the real axis: like Taub-NUT infinity

• Sign changes allowed to get all asymptotics with $W_2 \sim W_3 \neq W_1$ (see Babich, Korotkin)

• instanton analogs of Kasner's solutions with $i\mu \in \Delta \subset \overline{\mathbb{H}}$ in the vicinity of $i\infty$ but not necessarily on the imaginary axis

• behavior $\mu \to \infty$ of these Bianchi IX cosmologies as possible model of (Wick rotated) time at the singularity in algebro-geometric gluing of spacetimes proposed in:

- Yu.I. Manin, M. Marcolli, *Big Bang, blowup, and modular curves: algebraic geometry in cosmology*, SIGMA Symmetry Integrability Geom. Methods Appl. 10 (2014), Paper 073, 20 pp.

Spacetime noncommutativity in the early universe

• noncommutativity hypothesis: near the singularity spacetime coordinates acquire noncommutativity as part of quantum effects

• noncommutative deformation should preserve the metric properties

• Connes–Landi isospectral deformations

• for the 3-sphere S^3 with the round metric: isospectral deformation by making all the tori of the Hopf fibration into noncommutative tori

• do the left-$SU(2)$-invariant Bianchi IX metrics admit similar noncommutative isospectral deformations?

• not always, but yes in the cases that arise as asymptotic behavior at $\mu \to \infty$ of the gravitational instantons

Hopf fibration on S^3

- Hopf coordinates (ξ_1, ξ_2, η)

$$z_1 := x_1 + ix_2 = e^{i(\psi+\phi)} \cos\frac{\theta}{2} = e^{i\xi_1}\cos\eta,$$

$$z_2 := x_3 + ix_0 = e^{i(\psi-\phi)} \sin\frac{\theta}{2} = e^{i\xi_2}\sin\eta.$$

- identifying S^3 with unit quaternions $SU(2)$

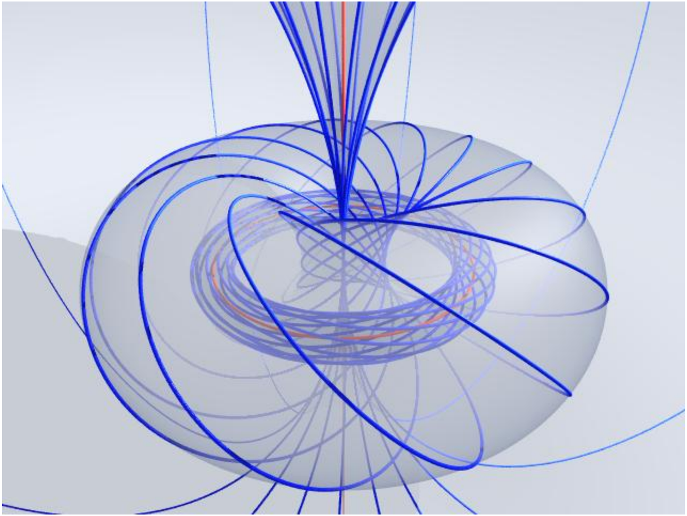
$$q := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{i\xi_1}\cos\eta & e^{i\xi_2}\sin\eta \\ -e^{-i\xi_2}\sin\eta & e^{-i\xi_1}\cos\eta \end{pmatrix}$$

with $|z_1|^2 + |z_2|^2 = 1$ and (ξ_1, ξ_2, η) Hopf coordinates

- Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2$$

The Hopf fibration of S^3



(image by Benoit Kloeckner)

Result: a spacetime with (Euclidean) Bianchi IX metric of the form

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2})$$

admits a Connes–Landi isospectral deformation, obtained by deforming the Hopf fibration, iff $W_2 = W_3$ (or $W_1 = W_2$ up to renaming axes)

- Deformation of the Hopf fibration

$$\begin{pmatrix} U\,\cos\eta & V\,\sin\eta \\ -V^*\,\sin\eta & U^*\,\cos\eta \end{pmatrix}$$

with U, V the generators of the noncommutative 2-torus T^2_θ

$$VU = e^{2\pi i\theta} UV$$

no longer group structure of $SU(2)$

• Deformed C^*-algebra of functions S^3_θ:
generators $\alpha = U \cos\eta$ and $\beta = V \sin\eta$
relations: $\alpha\beta = e^{2\pi i\theta}\beta\alpha$, $\alpha^*\beta = e^{-2\pi i\theta}\beta\alpha^*$, $\alpha^*\alpha = \alpha\alpha^*$, $\beta^*\beta = \beta\beta^*$ and $\alpha\alpha^* + \beta\beta^* = 1$

• Riemannian geometry in noncommutative setting described by spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with $\mathcal{A}$ involutive algebra (smooth functions on NC space), $\mathcal{H}$ Hilbert space with representation of $\mathcal{A}$ (spinors), and Dirac operator $\mathcal{D}$

• *Isospectral* deformation X_θ of a manifold X: $\mathcal{A} = C^\infty(X_\theta)$ noncommutative, with $(\mathcal{H}, \mathcal{D}) = (L^2(X, S), \not{D}_X)$ same as for X

• Connes–Landi: if T^2 acts by isometries on X then $\exists\, X_\theta$

• check when have action of T^2 by isometry on the Bianchi IX, compatible with the Hopf fibration of S^3

• in Hopf coordinates T^2 action on S^3

$$(t_1, t_2) : (\xi_1, \xi_2) \mapsto (\xi_1 + t_1, \xi_2 + t_2)$$

Euler angles $(u, v) : (\phi, \psi) \mapsto (\phi + u, \psi + v)$, with $t_1 = (u + v)/2$ and $t_2 = (v - u)/2$

• $U(1)$-action $u : \phi \mapsto \phi + u$ leaves 1-forms σ_i invariant (rotates circles $S^1 \hookrightarrow S^3$ of Hopf fibration)

• the form σ_1 also invariant under other $U(1)$-action $v : \psi \mapsto \psi + v$

$$v^* \sigma_2 = \frac{1}{2}(\sin(\psi + \beta)\, d\theta - \cos(\psi + \beta)\, \sin\theta\, d\phi)$$

$$v^* \sigma_3 = \frac{1}{2}(-\cos(\psi + \beta)\, d\theta - \sin(\psi + \beta)\, \sin\theta\, d\phi),$$

• then $v^* g = g$ for a Bianchi IX metric

$$g = d\mu^2 + a^2\, \sigma_1^2 + b^2\, \sigma_2^2 + c^2\, \sigma_3^2$$

if and only if $b = c$

- This class of Bianchi IX metrics include

1. Taub-NUT and Eguchi-Hanson gravitational instantons
2. asymptotic form of the general Bianchui IX gravitational instantons

Dirac operator:

- Berger sphere S^3 with $\lambda^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2$

$$D_B = -i \begin{pmatrix} \frac{1}{\lambda} X_1 & X_2 + iX_3 \\ X_2 - iX_3 & -\frac{1}{\lambda} X_1 \end{pmatrix} + \frac{\lambda^2 + 2}{2\lambda},$$

with $\{X_1, X_2, X_3\}$ basis of the Lie algebra

- on the Bianchi IX (Euclidean) spacetime

$$\mathcal{D} = \frac{1}{W_1^{1/2} W} \left(\gamma^0 \left(\frac{\partial}{\partial \mu} + \frac{1}{2} (\frac{\dot{W}}{W} + \frac{1}{2} \frac{\dot{W}_1}{W_1}) \right) + W_1 \ D_B|_{\lambda = \frac{W}{W_1}} \right)$$

with $W = W_2 = W_3$

Conclusion: Bianchi IX gravitational instantons are compatible with spacetime noncommutativity (only at $\mu \to \infty$)