ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XXV, nº 2-3 (2016), p. 517-542.

<http://afst.cedram.org/item?id=AFST_2016_6_25_2-3_517_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXV, n° 2-3, 2016
pp. 517-542

Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies

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To Vadim Schechtman, most cordially

Résumé. — Il est bien connu que l'espace-temps de Bianchi IX avec symétrie du groupe $SO(3)$ montre, dans le voisinage du Big Bang, un comportement chaotique à trajectoires typiques dans le sens inverse du mouvement du temps. Ce comportement (modèle Mixmaster de l'univers) peut être codé par le décalage de fractions continues à deux côtés.
Exactement le même décalage code les suites d'intersections de géodésiques hyperboliques dont l'axe imaginaire pur se situe dans le demi-plan complexe supérieur, c'est-à-dire à flot géodésique dans une surface modulaire appropriée.
Une interprétation physique de cette coincidence a été suggérée dans [MaMar14]: en effet, le chaos Mixmaster est une description approchée du passage d'un univers quantique chaud au moment du Big Bang à l'univers classique refroidissant. Nous discutons et étayons cette suggestion ici, en regardant le modèle Mixmaster pour la deuxième classe d'espaces-temps de Bianchi IX : ceux avec une symétrie $SU(2)$ (métriques d'Einstein autoduales). Nous l'étendons aussi au contexte plus général relié aux équations de Painlevé VI.

Abstract. — It is well known that the so called Bianchi IX spacetimes with $SO(3)$-symmetry in a neighbourhood of the Big Bang exhibit a chaotic behaviour of typical trajectories in the backward movement of time. This behaviour (Mixmaster Model of the Universe) can be encoded by the shift of two-sided continued fractions. Exactly the same shift encodes the sequences of intersections of hyperbolic geodesics with purely imaginary axis in the upper complex half-plane, that is geodesic flow on an appropriate modular surface.
A physical interpretation of this coincidence was suggested in [MaMar14]: namely, that Mixmaster chaos is an approximate description of the passage from a hot quantum Universe at the Big Bang moment to the cooling

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classical Universe. Here we discuss and elaborate this suggestion, looking at the Mixmaster Model from the perspective of the second class of Bianchi IX spacetimes: those with $SU(2)$-symmetry (self-dual Einstein metrics). We also extend it to the more general context related to Painlevé VI equations.

1. Introduction, background and notation

1.1. Plan of the paper

The Mixmaster Model of the early Universe with $SO(3)$-symmetry in a neighbourhood of the Big Bang predicts a chaotic behaviour of "typical" trajectories (in the backward movement of time) encoded by the shift of two-sided continued fractions: cf. [18], [5], [24], and references therein.

The same shift encodes the sequence of intersections with purely imaginary axis of hyperbolic geodesics in the upper complex half-plane, see [28].

This coincidence invites a closer attention, because the accepted mathematical source of the classical Mixmaster chaos involves the behaviour of separatrices on the real boundary of the respective dynamical system (cf. [4]). Geometry of these separatrices and approximate dynamics that it encodes are not visibly related to hyperbolic geodesics.

A physical interpretation of this coincidence was suggested in [23]. Here we discuss and elaborate this suggestion, looking at the Mixmaster model from the perspective of Bianchi IX model with $SU(2)$-symmetry.

More precisely, according to [23], the Mixmaster "classical chaos" should be considered as *an approximation to an unknown quantum description* of the transition from the infinitely hot quantum Universe at the moment of Big Bang to the cooling Universe gradually fitting a classical Einsteinian model. Time axis at the moment of Big Bang is purely imaginary, and it becomes real during the observable history of Universe.

We argued that a mathematical model of such a transition explaining Mixmaster chaos consists in inverse Wick rotation of time axes *mediated by a move of time along random geodesics in the complex hyperbolic half-plane* or rather its appropriate modular quotient. This passage to the modular quotient was critically important for our argument. It was suggested by two initially disjoint evidences. The first one was P. Tod's remark that a conformal version of cosmological time in the Friedman-Robertson-Walker models has a natural structure of the elliptic integral (cf. [23], sec. 4.2). The second evidence was a well known formal coincidence of two encodings: of Kasner's trajectories, on the one hand, and of hyperbolic geodesics with ideal ends, on the other hand.

In this paper we develop and present further details of this picture. Namely, we now look at such a transition from the side of "gravitational instantons" that is, self-dual Einstein spacetimes with $SU(2)$-symmetry. Many such spacetimes have a natural complexification, in particular, time axis can be extended to the complex half-plane, whereas the instantons themselves are defined by restricting time to the imaginary semi-axis.

Following the behaviour of the respective models along oriented geodesics in time connecting imaginary half-axis with real half-axis, we get the new aspect of the Mixmaster picture. This is the main content of this note.

Structure of the paper. In the remaining part of Section 1, we introduce some basic notations and constructions.

Sec. 2 compares (and shows a satisfying agreement) the sequences of Kasner eras in the classical Mixmaster models with sequences of geodesic distances between consecutive intersections of a geodesic with sides of the Farey tessellation. Finally, in sec. 3 and 4 we study an "instanton analogue" of the sequences of Kasner solutions determining chaotic behaviour in the classical Mixmaster model.

Geodesics in upper half-plane and their intersections with boundaries of fundamental domains of modular groups can be also treated as ball trajectories in hyperbolic billiards (see section 2.1). This picture goes back at least to Emil Artin's paper [1]. It was also used in a recent useful survey [19] of physical literature dedicated to Big Bang. In [19], it was suggested to quantize this classical system, see also [15].

1.2. Continued fractions

We denote by $\mathbb{Z}$, resp. $\mathbb{Z}_+$, the set of integers, resp. positive integers; $\mathbb{Q}$, resp. $\mathbb{R}$ is the field of rational, resp. real numbers. For $x \in \mathbb{R}$, we put $[x] := \max\{m \in \mathbb{Z} \,|\, m \le x\}$.

Irrational numbers $x > 1$ admit the canonical infinite continued fraction representation

$$x = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \dots}} =: [k_0, k_1, k_2, \dots], \ k_s \in \mathbb{Z}_+ \tag{1.1}$$

in which $k_0 := [x]$, $k_1 = [1/x]$ etc. Notice that our convention differs from that of [18]: their $[k_1, k_2, \dots]$ means our $[0, k_1, k_2, \dots]$.

1.3. Transformation T

The (partial) map $\widetilde{T} : [0,1]^2 \to [0,1]^2$ is defined by

$$\widetilde{T} : (x,y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y + [1/x]} \right), \tag{1.2}$$

If both coordinates $(x,y) \in [0,1]^2$ are irrational (the complement is a subset of measure zero), we have for uniquely defined $k_s \in \mathbb{Z}_+$:

$$x = [0, k_0, k_1, k_2, \dots], \;\; y = [0, k_{-1}, k_{-2}, \dots].$$

Then

$$\frac{1}{x} - \left[\frac{1}{x} \right] = [0, k_1, k_2, \dots], \quad \frac{1}{y + [1/x]} = \frac{1}{k_0 + y} = [0, k_0, k_{-1}, k_{-2}, \dots].$$

On this subset, $\widetilde{T}$ is bijective and has invariant density

$$\frac{dx\, dy}{\log 2 \cdot (1 + xy)^2}$$

(cf. [24]). Thus we may and will bijectively encode irrational pairs $(x,y) \in [0,1]^2$ by doubly infinite sequences

$$(k) := [\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots], k_i \in \mathbb{Z}_+$$

in such a way that the map $\widetilde{T}$ above becomes the shift of such a sequence denoted T:

$$T(k)_s = k_{s+1}. \tag{1.3}$$

1.4. Continued fractions and chaos in Einsteinian Bianchi IX type models

Bianchi classified metric space-times with a Lie group action transitive on space sections. In particular 4-dim Bianchi IX models of space-time can be of two types: with the symmetry group $SO(3)$ or else $SU(2)$. In the first case, metric has Minkowski's signature, whereas in the second case it is Riemannian. In sec. 1 we survey the now classical results about chaotic behaviour in the $SO(3)$-case (Mixmaster Universe) and prepare ground for the treatment of $SU(2)$-models. Sec. 2 and 3 are dedicated to the $SU(2)$-case.

Consider the real circle defined in $\mathbb{R}^3$ by equations

$$p_a + p_b + p_c = 1, \quad p_a^2 + p_b^2 + p_c^2 = 1. \tag{1.4}$$

Each point of this circle defines a 4-dimensional space-time with metric of Minkowski signature $dt^2 - a(t)dx^2 - b(t)dy^2 - c(t)dz^2$ with *scaling factors* a, b, c:

$$a(t) = t^{p_a},\ b(t) = t^{p_b},\ c(t) = t^{p_c}, t > 0.$$

Such a metric is called the Kasner metric with exponents (p_a, p_b, p_c).

Any point (p_a, p_b, p_c) can be obtained by choosing a unique $u \in [1, \infty]$, putting

$$p_1^{(u)} := -\frac{u}{1+u+u^2} \in [-1/3, 0],\ p_2^{(u)} := \frac{1+u}{1+u+u^2} \in [0, 2/3],$$

$$p_3^{(u)} := \frac{u(1+u)}{1+u+u^2} \in [2/3, 1] \tag{1.5}$$

and then rearranging the exponents $p_1^{(u)} \le p_2^{(u)} \le p_3^{(u)}$ by a bijection $(1, 2, 3) \to (a, b, c)$.

The main result of a series of physical papers dedicated to the Mixmaster Universe can be roughly summarized as follows.

A "typical" solution γ of Einstein equations (vacuum, but also with various energy momentum tensors) with $SO(3)$-symmetry of the Bianchi IX type, followed from an arbitrary (small) value $t_0 > 0$ in the reverse time direction $t \to +0$, oscillates close to a sequence of Kasner type solutions. (See subsection 2.2 below qualifying the use of adjective "typical" in this context).

Somewhat more precisely, introduce the local logarithmic time Ω along this trajectory with inverted orientation. Its differential is $d\Omega := -\dfrac{dt}{abc}$, and the time itself is counted from an arbitrary but fixed moment. Then $\Omega \to +\infty$ approximately as $-\log t$ as $t \to +0$, and we have the following picture.

As $\Omega \cong -\log t \to +\infty$, a "typical" solution γ of the Einstein equations determines a sequence of infinitely increasing moments $\Omega_0 < \Omega_1 < \cdots <$

$\Omega_n < \dots$ and a sequence of irrational real numbers $u_n \in (1, +\infty)$, $n = 0, 1, 2, \dots$.

The time semi-interval $[\Omega_n, \Omega_{n+1})$ is called the *n-th Kasner era* (for the trajectory γ). Within the n-th era, the evolution of a, b, c is approximately described by several consecutive Kasner's formulas. Time intervals where scaling powers (p_i) are (approximately) constant are called *Kasner's cycles.*

The evolution in the n-th era starts at time Ω_n with a certain value $u = u_n > 1$ which determines the sequence of respective scaling powers during the first cycle (1.5):

$$p_1 = -\frac{u}{1+u+u^2},\ p_2 = \frac{1+u}{1+u+u^2},\ p_3 = \frac{u(1+u)}{1+u+u^2}$$

The next cycles inside the same era start with values $u = u_n - 1$, $u_n - 2, \dots$, and scaling powers (1.5) corresponding to these numbers, rearranged corresponding to a bijection $(1, 2, 3) \to (a, b, c)$ which is in turn identical to the previous one, or interchanges b and c (see [22]).

After $k_n := [u_n]$ cycles inside the current era, a jump to the next era comes, with parameter

$$u_{n+1} = \frac{1}{u_n - [u_n]}. \tag{1.6}$$

Moreover, ensuing encoding of γ's and respective sequences (u_i)'s by continued fractions (1.1) of real irrational numbers $x > 1$ is bijective on the set of full measure.

Finally, when we want to include into this picture also the sequence of logarithmic times Ω_n starting new eras, we naturally pass to the two-sided continued fractions and the transformation T. See some details in the next section.

1.5. Doubly infinite sequences and modular geodesics

Let $H := \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$ be the upper complex half-plane with its Poincaré metric $|dz|^2/|\mathrm{Im}\, z|^2$. Denote also by $\overline{H} := H \cup \{\mathbb{Q} \cup \{\infty\}\}$ this half-plane completed with cusps.

The vertical lines $\mathrm{Re}\, z = n, n \in \mathbb{Z}$, and semicircles in $\overline{H}$ connecting pairs of finite cusps $(p/q, p'/q')$ with $pq' - p'q = \pm 1$, cut $\overline{H}$ into the union of geodesic ideal triangles which is called the *Farey tessellation.*

Following [28], consider the set of *oriented* geodesics β's in H with ideal irrational endpoints in $\mathbb{R}$. Let $\beta_{-\infty}$, resp. β_∞ be the initial, resp. the final point of β. Let B be the set of such geodesics with $\beta_{-\infty} \in (-1, 0)$, $\beta_\infty \in (1, \infty)$. Put

$$\beta_{-\infty} = -[0, k_0, k_{-1}, k_{-2}, \dots], \quad \beta_\infty = [k_1, k_2, k_3, \dots], \quad k_i \in \mathbb{Z}_+, \tag{1.7}$$

and encode β by the doubly infinite continued fraction

$$[\dots k_{-2}, k_{-1}, k_0, k_1, k_2, \dots]. \tag{1.8}$$

The geometric meaning of this encoding can be explained as follows. Consider the intersection point $x = x(\beta)$ of β with the imaginary semiaxis in H. Moving along β from x to β_∞, one will intersect an infinite sequence of Farey triangles. Each triangle is entered through a side and left through another side, leaving the ideal intersection point (a cusp) of these sides either to the left, or to the right. Then the infinite word in the alphabet $\{L, R\}$ encoding the consecutive positions of these cusps wrt β will be $L^{k_1}R^{k_2}L^{k_3}R^{k_4}\dots$ Similarly, moving from $\beta_{-\infty}$ to x, we will get the word (infinite to the left) $\dots L^{k_{-1}}R^{k_0}$.

We can enrich the new notation $\dots L^{k_{-1}}R^{k_0}L^{k_1}R^{k_2}L^{k_3}R^{k_4}\dots$ (called *cutting sequence* of our geodesic in [28]) by inserting between the consecutive powers of L, R notations for the respective intersection points of β with the sides of Farey triangles. So $x_0 := x = x(\beta)$ will be put between R^{k_0} and L^{k_1}, and generally we can imagine the word

$$\dots L^{k_{-1}}x_{-1}R^{k_0}x_0L^{k_1}x_1R^{k_2}x_2L^{k_3}x_3R^{k_4}\dots \tag{1.9}$$

We will essentially use this enrichment in the next section.

2. Hyperbolic billiard, geodesic distance, and cosmological time

2.1. Hyperbolic billiard

We will first present a version of the geometric description of geodesic flow: an equivalent dynamical system which is the triangular hyperbolic billiard with infinitely distant corners ("pockets").

Here we use the term "hyperbolic" in order to indicate that sides (boards) of the billiard and trajectories of the ball ("particle") are geodesics with respect to the hyperbolic metric of constant curvature -1 of the billiard table. This is not the standard meaning of the hyperbolicity in this context, where it usually refers to non-vanishing Lyapunov exponents.

Proposition 2.1. — *a) All hyperbolic triangles of the Farey tessellation of $\overline{H}$ are isomorphic as metric spaces.*

b) For any two closed triangles having a common side there exists unique metric isomorphism of them identical along this side. It inverts orientation induced by H. Starting with the basic triangle Δ with vertices $\{0, 1, i\infty\}$ and consecutively using these identifications, one can unambiguously define the map $b : \overline{H} \to \Delta$.

c) Any oriented geodesic on H with irrational end-points in $\mathbb{R}$ is sent by the map b to a billiard ball trajectory on the table Δ never hitting corners.

All this is essentially well known.

It is also worth noticing that although all three sides of Δ are of infinite length, this triangle is *equilateral* in the following sense: there exists a group S_6 of hyperbolic isometries of Δ acting on vertices by arbitrary permutations. This group has a unique fixed point $\rho := \exp(\pi i/3)$ in Δ, *the centroid* of Δ.

In fact, this group is generated by two isometries: $z \mapsto 1 - z^{-1}$ and symmetry with respect to the imaginary axis.

Three finite geodesics connecting the centre ρ with points $i, 1+i, \frac{1+i}{2}$ respectively, subdivide Δ into three geodesic quadrangles, each having one infinite (cusp) corner. We will call these points *centroids* of the respective sides of Δ, and the geodesics (ρ, i) etc. *medians* of Δ.

Each quadrangle is the fundamental domain for $PSL(2, \mathbb{Z})$.

2.2. Billiard encoding of oriented geodesics

Consider the first stretch of the geodesic β encoded by (1.9) that starts at the point x_0 in $(0, i\infty)$. If $k_0 = 1$,the ball along β reaches the opposite side $(1, i\infty)$ and gets reflected to the third side $(0, 1)$. If $k_0 = 2$, it reaches the opposite side, then returns to the initial side $(0, i\infty)$, and only afterwards gets reflected to $(0, 1)$.

More generally, the ball always spends k_0 unobstructed stretches of its trajectory between $(0, i\infty)$ and $(1, i\infty)$, but then is reflected to $(0, 1)$ either from $(1, i\infty)$ (if k_0 is odd), or from $(0, i\infty)$ (if k_0 is even). We can encode this sequence of stretches by the formal word ∞^{k_0} showing exactly how many times the ball is reflected "in the vicinity" of the pocket $i\infty$, that is, does not cross any of the medians.

A contemplation will convince the reader that this allows one to define an alternative encoding of β by the double infinite word in *three letters* , say a, b, c, serving as names of the vertices $\{0, 1, i\infty\}$.

2.3. Kasner's eras in logarithmic time and doubly infinite continued fractions

Now we will explain, how the double infinite continued fractions enter the Mixmaster formalism when we want to mark the consecutive Kasner eras upon the t-axis, or rather upon the Ω-axis, where $\Omega := -\log \int dt/abc$

In the process of construction, these continued fractions will also come with their enrichments, and the first new result of this note will compare this enrichment with the one described by (1.9).

We start with fixing a "typical" space-time γ whose evolution with $t \to +0$ undergoes (approximately) a series of Kasner's eras described by a continued fraction $[k_0, k_1, k_2, \dots]$, where k_s is the number of Kasner's cycles within s-th era $[\Omega_s, \Omega_{s+1})$. We have enriched this encoding by introducing parameters u_s which determine the Kasner exponents within the first cycle of the era number s by (1.5). A further enrichment comes with putting these eras on the Ω-axis. According to [18], [5], [4], if one defines the sequence of numbers δ_s from the relations

$$\Omega_{s+1} = [1 + \delta_s k_s (u_s + 1/\{u_s\})]\Omega_s,$$

then complete information about these numbers can be encoded by the extension to the left of our initial continued fraction:

$$[\dots, k_{-1}, k_0, k_1, k_2, \dots] \tag{2.1}$$

in such a way that

$$\delta_s = x_s^+/(x_s^+ + x_s^-)$$

where

$$x_s^+ = [0, k_s, k_{s+1}, \dots], \quad x_s^- = [0, k_{s-1}, k_{s-2}, \dots]. \tag{2.2}$$

THEOREM 2.2. — *Let a "typical" Bianchi IX Mixmaster Universe be encoded by the double-sided sequence* (2.1)*. Consider also the respective geodesic in H with its enriched encoding* (1.9)*.*

Then we have "asymptotically" as $s \to \infty$, $s \in \mathbb{Z}_+$:

$$\log \Omega_{2s}/\Omega_0 \simeq 2 \sum_{r=0}^{s-1} \text{dist}\,(x_{2r}, x_{2r+1}), \tag{2.3}$$

where dist *denotes the hyperbolic distance between the consecutive intersection points of the geodesic with sides of the Farey tesselation as in* (1.9).

Proof. — According to the formulas (5.1) and (5.5) in [18], and our notation (2.2), we have

$$\log \Omega_{2s}/\Omega_0 \simeq -\sum_{p=1}^{2s} \log(x_p^+ x_p^-)$$
$$= \sum_{p=1}^{2s} \log([k_{p-1}, k_{p-2}, k_{p-3}, \ldots]) \cdot [k_p, k_{p+1}, k_{p+2}, \ldots]). \qquad (2.4)$$

On the other hand, according to the formula (3.2.1) in [28], we have

$$\operatorname{dist}(x_0, x_1) = \frac{1}{2} \log([k_0, k_{-1}, k_{-2}, \ldots]\cdot[k_1, k_2, \ldots]\cdot[k_1, k_0, k_{-1}, \ldots]\cdot[k_2, k_3, \ldots])$$

and hence, more generally,

$$\operatorname{dist}(x_{2r}, x_{2r+1}) = \frac{1}{2} \log([k_{2r}, k_{2r-1}, k_{2r-2}, \ldots]$$
$$\cdot[k_{2r+1}, k_{2r+2}, \ldots] \cdot [k_{2r+1}, k_{2r}, k_{2r-1}, \ldots] \cdot [k_{2r+2}, k_{2r+3}, \ldots]). \qquad (2.5)$$

Inserting (2.5) into the r.h.s. of (2.3), we will see that it agrees with the r.h.s. of (2.4). This completes the proof. □

The formula (2.3) justifies identification of distance measured along a geodesic with (doubly) logarithmic cosmological time in the next section.

During the stretch of *time/geodesic length* which such a geodesic spends in the vicinity of a vertex of Δ, the respective space-time in a certain sense can be approximated by its degenerate version, corresponding to the vertex itself, and this justifies considering the respective segments of geodesics as the "instanton Kasner eras".

3. Mixmaster chaos in complex time and Painlevé VI

3.1. Painlevé VI

Contrary to the separatrix approximation methods, the results about encoding of geodesics β with irrational ends and formulas for the distances between consecutive cutting points are exact, but we did not yet introduce analogs of space-times fibered over geodesics as their time axes. We will do it in this section. The respective space-times are (complexified) versions of Bianchi IX models with $SU(2)$ (rather than $SO(3)$) action, the so called gravitational instantons. An important class of them is described by solutions of the Painlevé VI equation corresponding to a particular point in the space of parameters of these equations: for us, the main references will be [32], [17], and [2].

However, the hyperbolic billiard's picture of Sec. 2 can be lifted to essentially arbitrary Painlevé VI equations, and we will start this section with a brief explanation of the relevant formalism.

Equations of the type Painlevé VI form a four-parametric family. If the parameters $(\alpha, \beta, \gamma, \delta)$ are chosen, the corresponding equation for a function $X(t)$ looks as follows:

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right]. \qquad (3.1)$$

In 1907, R. Fuchs has rewritten (3.1) in the form

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}\right]\int_{\infty}^{(X,Y)} \frac{dx}{\sqrt{x(x-1)(x-t)}} =$$

$$= \alpha Y + \beta\frac{tY}{X^2} + \gamma\frac{(t-1)Y}{(X-1)^2} + (\delta - \frac{1}{2})\frac{t(t-1)Y}{(X-t)^2} \qquad (3.2)$$

Here he enhanced $X := X(t)$ to $(X, Y) := (X(t), Y(t))$ treating the latter pair as a section $P := (X(t), Y(t))$ of the generic elliptic curve $E = E(t)$: $Y^2 = X(X-1)(X-t)$. The section can be local and/or multivalued.

In this form, the left hand side of (3.2) which we denote $\mu(P)$ has a beautiful property: it is a non-linear differential expression (additive differential character) in coordinates of P such that $\mu(P+Q) = \mu(P) + \mu(Q)$ where $P+Q$ means addition of points of the generic elliptic curve E, with infinite section as zero. In particular, $\mu(Q) = 0$ for points of finite order.

To see it, notice that the integral in the l.h.s. of (3.2) is additive modulo periods of our elliptic curve, considered as multivalued functions of t. These periods are annihilated by the Gauss differential operator which is put before the integral sign in (3.2).

The right hand side of (3.2) looks more mysterious. In order to clarify its meaning, notice that $\mu(P)$ is defined up to multiplication by an invertible function of t.

If we choose a differential of the first kind ω on the generic curve and the symbol of the Picard-Fuchs operator of the second order annihilating periods of ω, the character will be defined uniquely. Moreover, it is functorial with respect to base changes (cf. [21], sec. 0.2, 1.2, 1.3). In particular, if we pass to the analytic picture replacing the algebraic family of curves $E(t)$ by the analytic one $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \mapsto \tau \in H$, and denote by z a fixed coordinate on $\mathbb{C}$, then (3.1) and (3.2) can be equivalently written in the form

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau) \tag{3.3}$$

where $(\alpha_0, \dots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1+\tau)$, and

$$\wp(z,\tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau+n)^2} \right). \tag{3.4}$$

Moreover, we have

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau)) \tag{3.5}$$

where

$$e_i(\tau) = \wp(\frac{T_i}{2}, \tau), \tag{3.6}$$

so that $e_1 + e_2 + e_3 = 0$.

The family Painlevé VI was written in this form in [21]. It was considerably generalised by K. Takasaki in [29], in particular, he found its versions for other families of Painlevé equations.

Now, any multivalued solution $z = z(\tau)$ of (3.3) defines a multi-section of the family which is a covering of H. In particular, if we can control its ramification and monodromy, then we may consider its behavior over geodesics with ideal ends in H and study the relevant statistical properties. The most accessible examples are algebraic solutions classified in [3], [20] and other works.

However, here we will return to Bianchi IX models, which according to [17] correspond to the equation with parameters $(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$, solvable in elliptic functions. We will skip the beautiful twistor geometry bridging Painlevé VI and Bianchi IX and simply reproduce the relevant results from [32] and [17], somewhat reworked and simplified in [2].

3.2. $SU(2)$ Bianchi IX metric and scaling factors

Consider the $SU(2)$ Bianchi IX model with metric of the form

$$g = F\left(d\mu^2 + \frac{\sigma_1^2}{W_1^2} + \frac{\sigma_2^2}{W_2^2} + \frac{\sigma_3^2}{W_3^2}\right). \tag{3.7}$$

Here μ is cosmological time, (σ_j) are $SU(2)$-invariant forms along space-sections with $d\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations of $(1,2,3)$, and F is a conformal factor.

By analogy with the $SO(3)$ case and metric $dt^2 - a(t)^2dx^2 - b(t)^2dy^2 - c(t)^2dz^2$, we may and will treat W_i (as well as some natural monomials in W_i and F) as $SU(2)$-*scaling factors.*

However, contrary to the $SO(3)$-case, generic solutions of Einstein equations in the $SU(2)$-case can be written explicitly in terms of elliptic modular functions, whereas their chaotic behaviour along geodesics in the complex half-plane of time is only a reflection of the chaotic behaviour of the respective billiard ball trajectories.

3.3. Theta-functions with characteristics

Explicit formulas in [2] use the following basic function of the complex arguments $i\mu \in H$, $z \in \mathbb{C}$, with parameters (p,q) called theta-characteristics:

$$\vartheta[p,q](z,i\mu) := \sum_{m\in\mathbb{Z}} \exp\{-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\}. \tag{3.8}$$

It can be expressed through the theta-function with vanishing characteristics:

$$\vartheta[p,q](z,i\mu) = \exp\{-\pi p^2\mu + 2\pi ipq\} \cdot \vartheta[0,0](z+pi\mu+q, i\mu). \tag{3.9}$$

All these functions satisfy classical automorphy identities with respect to the action of $PGL(2,\mathbb{Z})$.

THEOREM 3.1. — ([32], [17], [2].) *Put*

$$\vartheta[p,q] := \vartheta[p,q](0,i\mu) \tag{3.10}$$

and

$$\vartheta_2 := \vartheta[1/2,0],\ \ \vartheta_3 := \vartheta[0,0],\ \ \vartheta_4 := \vartheta[0,1/2]. \tag{3.11}$$

(A) Consider the following scaling factors as functions of μ with parameters (p,q):

$$W_1 := \frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p,q+1/2]}{e^{\pi i p}\vartheta[p,q]}, \quad W_2 := \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+1/2,q+1/2]}{e^{\pi i p}\vartheta[p,q]},$$

$$W_3 := -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+1/2,q]}{\vartheta[p,q]}. \tag{3.12}$$

Moreover, define the conformal factor F with non-zero cosmological constant Λ by

$$F := \frac{2}{\pi\Lambda} \frac{W_1 W_2 W_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2} \tag{3.13}$$

The metric (3.7) *with these scaling factors for real $\mu > 0$ is real and satisfies the Einstein equations if either*

$$\Lambda < 0,\ p \in \mathbb{R},\ q \in \frac{1}{2} + i\mathbb{R}, \tag{3.14}$$

or

$$\Lambda > 0,\ q \in \mathbb{R},\ p \in \frac{1}{2} + i\mathbb{R}. \tag{3.15}$$

(B) Consider now a different system of scaling factors

$$W_1' := \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_2, \quad W_2' := \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_3,$$

$$W_3' := \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_4, \tag{3.16}$$

and

$$F' := -C(\mu+q_0)^2\, W_1' W_2' W_3', \tag{3.17}$$

where $q_0, C \in \mathbb{R}$, $C > 0$.

The metric (3.7) *with these scaling factors for real $\mu > 0$ is real and satisfies the Einstein equations with vanishing cosmological constant.*

We will now consider values of $i\mu \in \Delta \subset \overline{H}$ in the vicinity of $i\infty$ but not necessarily lying on the imaginary axis. Since we are interested in the instanton analogs of Kasner's solutions, we will collect basic facts about asymptotics of scaling factors for $i\mu \to i\infty$.

For brevity, we will call a number $r \in \mathbb{R}$ *general*, if $r \notin \mathbb{Z} \cup (1/2 + \mathbb{Z})$.

For such r, denote by $\langle r \rangle \in (-1/2, 0) \cup (0, 1/2)$ such real number that $r + m_0 = \langle r \rangle$ for a certain (unique) $m_0 \in \mathbb{Z}$.

THEOREM 3.2. — *The scaling factors of the Bianchi IX spaces listed in Theorem 3.1 have the following asymptotics near $\mu = +\infty$:*

(i) For $\Lambda = 0$:

$$W_1' \sim -\frac{\pi}{2}, \quad W_2' \sim W_3' \sim \frac{1}{\mu + q_0}. \tag{3.18}$$

(ii) For $\Lambda < 0$ and general p:

$$W_1 \sim -\pi \langle p \rangle \, \exp\{\pi i(\langle p \rangle - p)\}, \quad W_2 \sim \pm W_3,$$

$$W_3 \sim -2\pi i \, \langle p + 1/2 \rangle \cdot \exp\{\pi i \, \mathrm{sgn}\, \langle p \rangle q\} \cdot \exp\{\pi\mu(|\langle p \rangle| - 1/2)\}. \tag{3.19}$$

(iii) For $\Lambda > 0$, real q and $p = 1/2 + ip_0, p_0 \in \mathbb{R}$:

$$W_1 \sim \pi p_0 \, \tan\{\pi(q - p_0\mu)\} - \frac{1}{2}, \quad W_2 \sim -W_3,$$

$$W_3 \sim 2\pi p_0 \cdot (\cos \pi(q - p_0\mu))^{-1}. \tag{3.20}$$

Comments. Theorem 3.2 shows that for general members of all solution families from [2], after eventual sign changes of some W_i's and outside of the pole singularities on the real time axis, we have asymptotically $W_2 = W_3$, $W_1 \neq W_2$.

In the next section, we will show that precisely such a condition allows one to quantize the respective geometric picture in terms Connes-Landi ([8]. This gives additional substance to our vision that chaotic Mixmaster evolution along hyperbolic geodesics reflects a certain "dequantization" of the hot quantum early Universe.

(Sign changes alluded to above are allowed, since Babich and Korotkin get their much simpler formulas by cleverly extracting square roots from expressions given in [17].)

Proof of Theorem 3.2. — Directly from (3.9)-(3.11), we obtain:

$$\vartheta_2 = \sum_{m\in\mathbb{Z}} \exp\{-\pi(m+\frac{1}{2})^2\mu\} \ \sim 2\,\exp\{-\pi\mu/4\}, \tag{3.21}$$

$$\vartheta_3 = \sum_{m\in\mathbb{Z}} \exp\{-\pi m^2\mu\} \ \sim 1 + 2\,\exp\{-\pi\mu\}, \tag{3.22}$$

$$\vartheta_4 = \sum_{m\in\mathbb{Z}} \exp\{-\pi m^2\mu\}(-1)^m \ \sim 1 - 2\,\exp\{-\pi\mu\}. \tag{3.23}$$

Therefore

$$\frac{d}{d\mu}\log\vartheta_2 \sim -\frac{\pi}{4}, \quad \frac{d}{d\mu}\log\vartheta_3 \sim -2\pi\,\exp\{-\pi\mu\}, \quad \frac{d}{d\mu}\log\vartheta_4 \sim 2\pi\,\exp\{-\pi\mu\}.$$

From this and (3.16), (3.17) one gets (3.18) for $\Lambda = 0$.

Now consider the case $\Lambda < 0$, p general.

Then from (3.8), (3.10), and (3.14) one gets

$$\vartheta[p,q] = \sum_{m\in\mathbb{Z}} \exp\{-\pi(m+p)^2\mu + 2\pi i(m+p)q\} \sim$$

$$\sim \exp\{2\pi i\langle p\rangle q\}\cdot\exp\{-\pi\langle p\rangle^2\mu\}, \tag{3.24}$$

because for general p, the leading term of $\vartheta[p,q]$ corresponds to the unique value of m for which $(m+p)^2$ is minimal, that is, equals $\langle p\rangle^2$.

Hence

$$\frac{\partial}{\partial q}\vartheta[p,q] \sim 2\pi i\,\langle p\rangle\exp\{2\pi i\langle p\rangle q\}\cdot\exp\{-\pi\langle p\rangle^2\mu\}. \tag{3.25}$$

Thus, from (3.12), and (3.21)-(3.25) we obtain

$$W_1 = \frac{i}{2}\vartheta_3\vartheta_4\frac{\frac{\partial}{\partial q}\vartheta[p,q+1/2]}{e^{\pi ip}\vartheta[p,q]} \sim \frac{i}{2}\cdot 2\pi i\,\langle p\rangle\exp\{2\pi i\langle p\rangle(q+1/2)\}\cdot\exp\{-\pi\langle p\rangle^2\mu\}\times$$

$$\exp\{-\pi ip\}\cdot\exp\{-2\pi i\langle p\rangle q\}\cdot\exp\{\pi\langle p\rangle^2\mu\} = -\pi\langle p\rangle\,\exp\{\pi i(\langle p\rangle - p)\}.$$

Furthermore,

$$W_2 = \frac{i}{2}\vartheta_2\vartheta_4\frac{\frac{\partial}{\partial q}\vartheta[p+1/2,q+1/2]}{e^{\pi ip}\vartheta[p,q]} \sim$$

$$\sim \frac{i}{2}\cdot 2\,\exp\{-\pi\mu/4\}\cdot 2\pi i\,\langle p+1/2\rangle\cdot\exp\{2\pi i\langle p+1/2\rangle(q+1/2)\}\cdot\exp\{-\pi\langle p+1/2\rangle^2\mu\}$$

$$\times\exp\{-\pi i p\}\cdot\exp\{-2\pi i\langle p\rangle q\}\cdot\exp\{\pi\langle p\rangle^2\mu\}\sim$$

$$-2\pi\,\langle p+1/2\rangle\exp\{\pi i[\langle p+1/2\rangle - p - \mathrm{sgn}\langle p\rangle q]\}\cdot\exp\{\pi\mu(|\langle p\rangle| - 1/2)\}.$$

Notice that exponential terms were rewritten using the identity

$$\langle p+1/2\rangle = \langle p\rangle - \frac{1}{2}\mathrm{sgn}\langle p\rangle.$$

Similarly,

$$W_3 := -\frac{1}{2}\vartheta_2\vartheta_3\frac{\frac{\partial}{\partial q}\vartheta[p+1/2,q]}{\vartheta[p,q]}\sim$$

$$\sim -\frac{1}{2}\cdot 2\,\exp\{-\pi\mu/4\}\cdot 2\pi i\,\langle p+1/2\rangle\cdot\exp\{2\pi i\langle p+1/2\rangle q\}\cdot\exp\{-\pi\langle p+1/2\rangle^2\mu\}\times$$

$$\exp\{-2\pi i\langle p\rangle q\}\cdot\exp\{\pi\langle p\rangle^2\mu\}\sim$$

$$-2\pi i\,\langle p+1/2\rangle\cdot\exp\{\pi i\,\mathrm{sgn}\,\langle p\rangle q\}\cdot\exp\{\pi\mu(|\langle p\rangle| - 1/2)\}.$$

Comparing expressions for W_2 and W_3, one easily sees that $W_2 = \pm W_3$, where the exact sign can be expressed through p and q.

For the conformal factor (3.13) we then get the following asymptotic:

$$F = \frac{2}{\pi\Lambda}\frac{W_1W_2W_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2}\sim$$

$$2i\frac{\langle p+1/2\rangle^2}{\Lambda\langle p\rangle^2}\cdot\exp\{\langle p+1/2\rangle + \langle p\rangle + 2\,\mathrm{sgn}\,\langle p\rangle\, q\}\cdot\exp\{\pi\mu\,(2\,|\langle p\rangle| - 1)\}.$$

Finally, pass to the case $\Lambda > 0$. Put $p = \frac{1}{2} + ip_0,\ p_0\in\mathbb{R}$. We have again to locate first the leading terms as $\mu\to+\infty$ in

$$\vartheta[p,q] = \sum_{m\in\mathbb{Z}}\exp\{-\pi(m+p)^2\mu + 2\pi i(m+p)q\},$$

and also respective terms when p and/or q are shifted by 1/2. Obviously, they correspond to the minimal values of $\mathrm{Re}\,(m+p)^2$, resp. $\mathrm{Re}\,(m+p+1/2)^2$, for $m\in\mathbb{Z}$. Since

$$\mathrm{Re}\,(m+p)^2 = (m+\frac{1}{2})^2 - p_0^2,\quad \mathrm{Re}\,(m+p+\frac{1}{2})^2 = (m+1)^2 - p_0^2,$$

in the first case there are two leading terms, for $m = 0$ and $m = -1$, and in the second case just one, for $m = -1$.

Thus, for $\Lambda > 0$, we have

$$\vartheta[p,q] \sim \exp\{\pi\mu(p_0^2 - 1/4)\} \cdot [\exp\{2\pi i p q - \pi i p_0 \mu\} + \exp\{2\pi i (p-1) q + \pi i p_0 \mu\}].$$

The sum of two terms in square brackets can be rewritten so that in the end we obtain

$$\vartheta[p,q] \sim \exp\{\pi\mu(p_0^2 - 1/4)\} \cdot \exp\{-2\pi p_0 q\} \cdot \cos\pi(q - p_0\mu). \tag{3.26}$$

$$\vartheta[p + 1/2, q] \sim \exp\{\pi\mu p_0^2\} \cdot \exp\{-2\pi p_0 q\}. \tag{3.27}$$

When we have to replace a real q by $q + 1/2$, we may do it formally in the right hand side expressions in (3.26), (3.27).

Therefore, we have from (3.12), (3.22) and (3.27):

$$\frac{W_2}{W_3} = -i \cdot \frac{\vartheta_4}{\vartheta_3} \cdot \frac{\frac{\partial}{\partial q}\vartheta[p + 1/2, q + 1/2]}{e^{\pi i p}\frac{\partial}{\partial q}\vartheta[p + 1/2, q]} \sim$$

$$i \cdot \frac{\exp\{-2\pi p_0(q + 1/2)\}}{\exp\{\pi i(1/2 + i p_0)\} \cdot \exp\{-2\pi p_0 q\}} = -1.$$

Now,

$$W_3 := -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p + 1/2, q]}{\vartheta[p,q]} \sim$$

$$\frac{\exp\{-\pi\mu/4\} \cdot (2\pi p_0) \cdot \exp\{\pi\mu p_0^2\} \cdot \exp\{-2\pi p_0 q\}}{\exp\{\pi\mu(p_0^2 - 1/4)\} \cdot \exp\{-2\pi p_0 q\} \cdot \cos\pi(q - p_0\mu)} \sim$$

$$2\pi p_0 \cdot (\cos\pi(q - p_0\mu))^{-1}.$$

Furthermore,

$$W_1 = \frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p, q + 1/2]}{e^{\pi i p}\vartheta[p,q]} \sim$$

$$\frac{i}{2} \cdot \frac{\exp\{\pi\mu(p_0^2 - 1/4)\} \cdot \frac{\partial}{\partial q}[\exp\{-2\pi p_0(q + 1/2)\} \cdot \cos\pi(q + 1/2 - p_0\mu)]}{\exp\{\pi(i/2 - p_0)\} \cdot \exp\{\pi\mu(p_0^2 - 1/4)\} \cdot \exp\{-2\pi p_0 q\} \cdot \cos\pi(q - p_0\mu)} \sim$$

$$\frac{i}{2} \cdot \frac{\frac{\partial}{\partial q}[\exp\{-2\pi p_0(q + 1/2)\} \cdot \cos\pi(q + 1/2 - p_0\mu)]}{\exp\{\pi(i/2 - p_0)\} \cdot \exp\{-2\pi p_0 q\} \cdot \cos\pi(q - p_0\mu)} \sim$$

$$-\frac{1}{2}\cdot\frac{\frac{\partial}{\partial q}[\exp\{-2\pi p_0(q+1/2)\}\cdot\sin\pi(q-p_0\mu)]}{\exp\{-2\pi p_0(q+1/2)\}\cdot\cos\pi(q-p_0\mu)}\sim$$
$$\pi p_0\,\tan\{\pi(q-p_0\mu)\}-\frac{1}{2}.$$

This completes the proof of Theorem 3.2. □

4. Theta deformations of gravitational instantons

4.1. Theta deformations

In Section 5 of [23] we showed that the gluing of space-times across the singularity using an algebro-geometric blowup can be made compatible with the idea of spacetime coordinates becoming noncommutative in a neighborhood of the initial singularity where quantum gravity effects begin to dominate.

This compatibility is described there in terms of Connes-Landi theta deformations ([8]) and Cirio-Landi-Szabo toric deformations ([7]) of Grassmannians.

Here we consider the same problem in the case of the Bianchi IX models with $SU(2)$-symmetry, namely whether they can be made compatible with the hypothesis of noncommutativity at the Planck scale, using isospectral theta deformations.

The metrics on the S^3 sections, in this case, are only left $SU(2)$-invariant. We show that among all the $SU(2)$ Bianchi IX spacetime, the only ones that admit isospectral theta-deformations of their spatial S^3-sections are those where the metric tensor

$$g=w_1w_2w_3\,d\mu^2+\frac{w_2w_3}{w_1}\,\sigma_1^2+\frac{w_1w_3}{w_2}\,\sigma_2^2+\frac{w_1w_2}{w_3}\,\sigma_3^2 \tag{4.1}$$

is of the special form satisfying $w_1\neq w_2=w_3$ (the two directions σ_2 and σ_3 have equal magnitude). In these metrics, the S^3 sections are Berger spheres. This class includes the general Taub-NUT family ([31], [25]), and the Eguchi-Hanson metrics ([11], [12]). The theta-deformations are obtained, as in the case of the deformations S^3_θ of [8] of the round 3-sphere, by deforming all the tori of the Hopf fibration to noncommutative tori.

Proposition 4.1. — *A Bianchi IX Euclidean spacetime X with $SU(2)$-symmetry admits a noncommutative theta-deformation X_θ, obtained by deforming the tori of the Hopf fibration of each spacial section S^3 to noncommutative tori, if and only if its metric has the $SU(2)\times U(1)$-symmetric* *form*

$$g = w_1 w_3^2 \, d\mu^2 + \frac{w_3^2}{w_1} \, \sigma_1^2 + w_1 \, (\sigma_2^2 + \sigma_3^2). \tag{4.2}$$

Proof. In appropriate local coordinates the $SU(2)$-invariant forms (σ_i) satisfying relations $d\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations (i, j, k) have the explicit form

$$\sigma_1 = x_1 \, dx_2 - x_2 \, dx_1 + x_3 \, dx_0 - x_0 \, dx_3 = \frac{1}{2}(d\psi + \cos\theta \, d\phi),$$

$$\sigma_2 = x_2 \, dx_3 - x_3 dx_2 + x_1 \, dx_0 - x_0 \, dx_1 = \frac{1}{2}(\sin\psi \, d\theta - \sin\theta \cos\psi \, d\phi),$$

$$\sigma_3 = x_3 \, dx_1 - x_1 \, dx_3 + x_2 \, dx_0 - x_0 \, dx_2 = \frac{1}{2}(-\cos\psi \, d\theta - \sin\theta \, \sin\psi \, d\phi),$$

with Euler angles $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ and $0 \leqslant \psi \leqslant 4\pi$ (for the $SU(2)$ case).

The Hopf coordinates (ξ_1, ξ_2, η) are defined by

$$z_1 := x_1 + i x_2 = e^{i(\psi+\phi)} \cos\frac{\theta}{2} = e^{i\xi_1} \cos\eta,$$

$$z_2 := x_3 + i x_0 = e^{i(\psi-\phi)} \sin\frac{\theta}{2} = e^{i\xi_2} \sin\eta.$$

Equivalently, identifying S^3 with unit quaternions, we write $q \in SU(2)$ as

$$q := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{i\xi_1} \cos\eta & e^{i\xi_2} \sin\eta \\ -e^{-i\xi_2} \sin\eta & e^{-i\xi_1} \cos\eta \end{pmatrix},$$

where $|z_1|^2 + |z_2|^2 = 1$ and (ξ_1, ξ_2, η) are the Hopf coordinates as above.

The noncommutative θ-deformations ([8]) of the 3-sphere S^3 are obtained by deforming all the 2-tori of the Hopf fibration to noncommutative 2-tori T^2_θ. Namely, replace q with

$$\begin{pmatrix} U \, \cos\eta & V \, \sin\eta \\ -V^* \, \sin\eta & U^* \, \cos\eta \end{pmatrix},$$

where U, V are the generators of the noncommutative 2-torus T^2_θ.

Then one obtains the algebra generated by $\alpha = U \cos\eta$ and $\beta = V \sin\eta$ with $\alpha\beta = e^{2\pi i\theta}\beta\alpha$, $\alpha^*\beta = e^{-2\pi i\theta}\beta\alpha^*$, $\alpha^*\alpha = \alpha\alpha^*$, $\beta^*\beta = \beta\beta^*$ and $\alpha\alpha^* + \beta\beta^* = 1$. It is shown in [8] that this deformation is *isospectral* with respect to the bi-invariant round metric on S^3, in the sense that the data of the Hilbert space of square integrable spinors $H = L^2(S^3, S)$ and the Dirac operator D for the round metric on S^3 give rise to spectral triples on the deformed algebras S^3_θ.

In fact, the general result of [8] shows that isospectral theta-deformations can be constructed whenever there is an isometric torus action. In particular, in our case the question reduces to whether the action of T^2 that rotates the tori of the Hopf fibration preserves the Bianchi IX metric.

In Hopf coordinates the action of T^2 is given by $(t_1, t_2) : (\xi_1, \xi_2) \mapsto (\xi_1+t_1, \xi_2+t_2)$, or in terms of the Euler angles, $(u, v) : (\phi, \psi) \mapsto (\phi+u, \psi+v)$, with $t_1 = (u + v)/2$ and $t_2 = (v - u)/2$. It is immediate to check that the $U(1)$-action $u : \phi \mapsto \phi + u$ leaves the 1-forms σ_i invariant. This is the $U(1)$-action of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. The form σ_1 is also invariant under the other $U(1)$-action $v : \psi \mapsto \psi + v$, while the other forms σ_2, σ_3 transform as

$$v^*\sigma_2 = \frac{1}{2}(\sin(\psi + \beta)\, d\theta - \cos(\psi + \beta)\, \sin\theta\, d\phi)$$

$$v^*\sigma_3 = \frac{1}{2}(-\cos(\psi + \beta)\, d\theta - \sin(\psi + \beta)\, \sin\theta\, d\phi),$$

hence it is clear that we have $v^*g = g$ for a Bianchi IX metric

$$g = d\mu^2 + a^2\, \sigma_1^2 + b^2\, \sigma_2^2 + c^2\, \sigma_3^2 \tag{4.3}$$

if and only if $b = c$. In the case $b = c$, with

$$g = d\mu^2 + \frac{a^2}{4}(d\psi + \cos\theta\, d\phi)^2 + \frac{c^2}{4}(d\theta^2 + \sin^2\theta\, d\phi^2),$$

the T^2 action is isometric and the resulting theta-deformations are therefore isospectral, with spectral triples (A, H, D), with $A = C^\infty(S^3_\theta)$, and spinors $H = L^2(S^3, S)$ and Dirac operator D with respect to the Bianchi IX metric with $b = c$. $\square$

This is in stark contrast with the situation described in [13], where (Lorentzian and Euclidean) Mixmaster cosmologies of the form

$$\mp dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$$

were considered, with T^3-spatial sections, which always admit isospectral theta-deformations.

We have recalled in the previous section how the self-duality equations for the $SU(2)$ Bianchi IX models can be described in terms of Painlevé

VI equations [32], [17], [26], and how the general solutions (with $w_1 \neq w_2 \neq w_3$) can be written explicitly in terms of theta constants [2], and are obtained from a Darboux-Halphen type system [27], [30]. In the case of the family of Bianchi IX models with $SU(2) \times U(1)$-symmetry, considered in Proposition 4.1, this system has algebraic solutions that give

$$w_2 = w_3 = \frac{1}{\mu - \mu_0}, \quad w_1 = \frac{\mu - \mu_*}{(\mu - \mu_0)^2}, \tag{4.4}$$

with singularities at μ_* (curvature singularity), μ_0 (Taubian infinity) and ∞ (nut). The condition $\mu_* < \mu_0$ avoids naked singularities, by hiding the curvature singularity at μ_* behind the Taubian infinity, see the discussion in Section 5.2 of [27].

Consider the operator

$$D_B = -i \begin{pmatrix} \frac{1}{\lambda} X_1 & X_2 + iX_3 \\ X_2 - iX_3 & -\frac{1}{\lambda} X_1 \end{pmatrix} + \frac{\lambda^2 + 2}{2\lambda}, \tag{4.5}$$

where $\{X_1, X_2, X_3\}$ constitute a basis of the Lie algebra orthogonal for the bi-invariant metric. Assume moreover that the left-invariant metric on S^3 is diagonal in this basis, with eigenvalues $\{w^2/w_1, w_1, w_1\}$, with $w = w_2 = w_3$ and $\lambda = w/w_1$, and where the w_i are as in (4.4). Consider also the operator

$$D = \frac{1}{w_1^{1/2} w} \left(\gamma^0 \left(\frac{\partial}{\partial \mu} + \frac{1}{2} (\frac{\dot{w}}{w} + \frac{1}{2} \frac{\dot{w}_1}{w_1}) \right) + w_1 \; D_B|_{\lambda = \frac{w}{w_1}} \right). \tag{4.6}$$

Proposition 4.2. — *The operators D of* (4.6) *give Dirac operators for isospectral theta deformations of the $SU(2) \times U(1)$-symmetric spacetimes of Proposition 4.1.*

Proof. We consider the frame θ^i with $i \in \{0, 1, 2, 3\}$, given by

$$\theta^0 = uw \, d\mu, \quad \theta^1 = u\lambda \, \sigma_1, \quad \theta^2 = u \, \sigma_2, \quad \theta^3 = u \, \sigma_3,$$

where $u = w_1^{1/2}$ and $\lambda = w/w_1$, for $w = w_2 = w_3$. Since the σ_i satisfy $d\sigma_i = \sigma_j \wedge \sigma_k$ for cyclic permulations $\{i, j, k\}$ of $\{1, 2, 3\}$, we have $d\theta^0 = 0$, and furthermore

$$d\theta^1 = (\dot{u}\lambda + u\dot{\lambda}) \, d\mu \wedge \sigma_1 + u\lambda \, \sigma_2 \wedge \sigma_3 = \frac{1}{uw} (\frac{\dot{u}}{u} + \frac{\dot{\lambda}}{\lambda}) \, \theta^0 \wedge \theta^1 + \frac{1}{u\lambda} \; \lambda^2 \, \theta^2 \wedge \theta^3,$$

$$d\theta^2 = \dot{u} \, d\mu \wedge \sigma_2 + u\sigma_3 \wedge \sigma_1 = \frac{1}{uw} \frac{\dot{u}}{u} \theta^0 \wedge \theta^2 + \frac{1}{u\lambda} \, \theta^3 \wedge \theta^1,$$

$$d\theta^3 = \dot{u} \, d\mu \wedge \sigma_3 + u\sigma_1 \wedge \sigma_2 = \frac{1}{uw} \frac{\dot{u}}{u} \theta^0 \wedge \theta^3 + \frac{1}{u\lambda} \, \theta^1 \wedge \theta^2$$

where dot denotes the time derivative.

Proceeding then as in [6], we use the $d\theta^i$ to write the spin connection and we obtain a Dirac operator of the form

$$D = \gamma^0 \frac{1}{w_1^{1/2} w} \left(\frac{\partial}{\partial \mu} + \frac{1}{2}(\frac{\dot{w}}{w} + \frac{1}{2}\frac{\dot{w}_1}{w_1}) \right) + \frac{w_1^{1/2}}{w} \; D_B|_{\lambda=\frac{w}{w_1}},$$

or equivalently of the form (4.6), where D_B is the Dirac operator on a Berger 3-sphere. The explicit form of Dirac operator on a Berger 3-sphere with metric $\lambda^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2$ was computed in [16], and it is given by the operator (4.5).

As in [13], the Dirac operator of Proposition 4.2 can be seen as involving an anisotropic Hubble parameter H. In the case of the metrics (4.3) of [13] this was of the form

$$H = \frac{1}{3} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right)$$

with a, b, c the scaling factors in (4.3).

In the case of the $SU(2)$ Bianchi IX models, the anisotropic Hubble parameter is again of the form $H = \frac{1}{3}(H_1 + H_2 + H_3)$, where now the H_i correspond to the three directions of the vectors dual to the $SU(2)$-forms σ_i in (4.1). For a metric of the form (4.2), or equivalently

$$g = uw\, d\mu^2 + u^2\lambda^2\, \sigma_1^2 + u^2\sigma_2^2 + u^2\sigma_3^2,$$

with u, λ, w as in Proposition 4.1, we take the anisotropic Hubble parameter to be

$$H = \frac{1}{3} \left(\frac{\dot{u}\lambda + u\dot{\lambda}}{u\lambda} + 2\frac{\dot{u}}{u} \right) = \frac{1}{3} \left(3\frac{\dot{u}}{u} + \frac{\dot{\lambda}}{\lambda} \right),$$

where

$$\frac{\dot{u}}{u} = \frac{1}{2}\frac{\dot{w}_1}{w_1}, \qquad \frac{\dot{\lambda}}{\lambda} = \frac{\dot{w}}{w} - \frac{\dot{w}_1}{w_1},$$

so that

$$H = \frac{1}{3} \left(\frac{\dot{w}}{w} + \frac{1}{2}\frac{\dot{w}_1}{w_1} \right),$$

as in (4.6), so that we can write the 4-dimensional Dirac operator in the form

$$D = \gamma^0 \frac{1}{uw} \left(\frac{\partial}{\partial \mu} + \frac{3}{2} H \right) + D_B,$$

where $D_B = (w_1^{1/2}/w)\; D_B|_{\lambda=\frac{w}{w_1}}$ is the Dirac operator on the spatial sections S^3 with the left $SU(2)$-invariant metric.

Explicit computations of the spectral action for the Bianchi IX metrics have been carried out in [14].

Notice that in the construction above we have considered the same modulus θ for the noncommutative deformation of all the spatial sections S^3 of the Bianchi IX spacetime, but one could also consider a more general situation where the parameter θ of the deformation is itself a function of the cosmological time μ.

This would allow the dependence of the noncommutativity parameter θ on the energy scale (or on the cosmological timeline), with $\theta = 0$ away from the singularity where classical gravity dominates and noncommutativity only appearing near the singularity. Since a non-constant, continuously varying parameter θ crosses rational and irrational values, this would give rise to a Hofstadter butterfly type picture, with both commutativity (up to Morita equivalence, as in the rational noncommutative tori) and true noncommutativity (irrational noncommutative tori).

Another interesting aspect of these noncommutative deformations is the fact that, when we consider a geodesic in the upper half plane encoding Kasner eras in a mixmaster dynamics, the points along the geodesic also determine a family of complex structures on the noncommutative tori T^2_θ of the theta-deformation of the spatial section.

Acknowledgment. — The second author is partially supported by NSF grants DMS-1007207, DMS-1201512, PHY-1205440 and by the Perimeter Institute for Theoretical Physics.

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