J. reine angew. Math. **429** (1992), 91—106

Journal für die reine und angewandte Mathematik

Density theorems for sampling and interpolation in the Bargmann-Fock space I

By *Kristian Seip* at Trondheim

---

1. Introduction

Our work is based on Beurling's lectures on balayage of Fourier-Stieltjes transforms and interpolation for an interval on $\mathbb{R}^1$ [4]. We observe that Beurling's problems concerning functions of exponential type have natural counterparts for functions of order two, finite type, and find that indeed so have his main results. The most interesting part is however that Beurling's ideas are applicable also in the Hilbert space setting, yielding a complete description of sampling and interpolation in the Bargmann-Fock space. The simplicity of these results is quite remarkable when compared to the situation in the Paley-Wiener space (the corresponding Hilbert space of functions of exponential type) and to the extensive literature on nonharmonic Fourier series and in particular Riesz bases of complex exponentials [24].

This research is motivated by a recent development in signal analysis and applied mathematics, which was initiated by Daubechies, Grossmann and Meyer [7], [6]. Their work inspired us to search for a general characterization of the information needed to represent signals, as functions in the Bargmann-Fock space. Our results can be seen as sharp statements about the Nyquist density and its meaning in this context.

We remark that the results to be presented were announced in [21]. It is the purpose of the present papers to give detailed proofs of those results.

2. Main results

For $\alpha > 0$, let $d\mu_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dx\, dy$, $z = x + iy$, and define the Bargmann-Fock space F_α^2 to be the collection of entire functions $f(z)$ for which

$$\|f\|_2 = \|f\|_{\alpha,2} = \iint_{C} |f(z)|^2 d\mu_\alpha(z) < \infty .$$

F_α^2 is a Hilbert space with reproducing kernel $K(z, \zeta) = e^{\alpha \bar{z} \zeta}$, i.e. for every $f \in F_\alpha^2$ we have

$$f(z) = \langle f, K(z, \cdot) \rangle = \iint_{\mathbb{C}} f(\zeta) \overline{K(z, \zeta)} \, d\mu_\alpha(\zeta) . \tag{1}$$

The normalized reproducing kernels, $k_\zeta(z) = K(\zeta, \zeta)^{-\frac{1}{2}} K(\zeta, z)$, can be viewed as the natural (well-localized) building blocks of F_α^2. They correspond, via the Bargmann transform, to the canonical coherent states of quantum mechanics, and to Gabor wavelets in signal analysis. This relation is the reason for the importance of the Bargmann-Fock space; see [10] for general information, and [7] for more background on the problems treated here.

We say that a discrete set Γ of complex numbers is a *set of sampling* for F_α^2 if there exist positive numbers A and B such that

$$A \|f\|_2^2 \leqq \sum_{z \in \Gamma} e^{-\alpha |z|^2} |f(z)|^2 \leqq B \|f\|_2^2 \tag{2}$$

for all $f \in F_\alpha^2$. If to every l^2-sequence $\{a_j\}$ of complex numbers there exists an $f \in F_\alpha^2$ such that $e^{-\frac{\alpha}{2}|z_j|^2} f(z_j) = a_j$ for all j, $\Gamma = \{z_j\}$ is said to be a *set of interpolation* for F_α^2. A set of sampling corresponds, in the terminology of [9], to a *frame* of coherent states. A set of both sampling and interpolation would correspond to a *Riesz basis* of coherent states; we will prove below that such a set does not exist.

With a view to applications in physic and signal analysis, Daubechies and Grossmann posed the problem of finding those lattices z_{mn}, $m, n \in \mathbb{Z}$ that are sets of sampling [7]. They proved that a lattice could be a set of sampling only if $ab < \pi/\alpha$ and conjectured this condition also to be sufficient. For $ab = \pi/(\alpha N)$, N an integer $\geqq 2$, they found (2) to hold by providing explicit expressions for the optimal constants A, B. Daubechies was later able to show that a lattice is a set of sampling whenever $N^{-1} < 0.996$ [6].

We prove that the density criterion of the Daubechies-Grossmann conjecture applies not only to lattices, but to *arbitrary* discrete sets. We should add here that the conjecture itself was proved independently by Lyubarskii [16], and by Wallstén and the author [23].

For the description to be given of sets of sampling and interpolation, we need Beurling's density concept as generalized by Landau [15]. We consider then *uniformly discrete sets*, i.e. discrete sets $\Gamma = \{z_j\}$ for which $q(\Gamma) = \inf_{j \neq k} |z_j - z_k| > 0$. We fix a compact set I of measure 1 in the complex plane, whose boundary has measure 0. Let $n^-(r)$ and $n^+(r)$ denote respectively the smallest and largest number of points from Γ to be found in a translate of rI. We define the lower and upper uniform densities of Γ to be

$$D^-(\Gamma) = \liminf_{r \to \infty} \frac{n^-(r)}{r^2} \quad \text{and} \quad D^+(\Gamma) = \limsup_{r \to \infty} \frac{n^+(r)}{r^2},$$

respectively. It was proved by Landau that these limits are independent of I.

Our main theorems are the following.

Theorem 2.1. *A discrete set Γ is a set of sampling for F_α^2 if and only if it can be expressed as a finite union of uniformly discrete sets and contains a uniformly discrete subset Γ' for which $D^-(\Gamma') > \alpha/\pi$.*

Theorem 2.2. *A discrete set Γ is a set of interpolation for F_α^2 if and only if it is uniformly discrete and $D^+(\Gamma) < \alpha/\pi$.*

Remark 1. The lattice with $ab = \pi/\alpha$ is called the *von Neumann* lattice since von Neumann claimed (without proof) that it is a set of uniqueness [17]; many proofs have been given later [2], [18], [1], [23]. The (unstable) expansion associated with the von Neumann lattice was suggested for use in electrical engineering by Gabor [11]. See [12] for an attempted repair of the "defect" of the von Neumann lattice that it is neither a set of sampling nor one of interpolation.

Remark 2. Decomposition and interpolation theorems for general discrete sets were obtained in [14], however without any indication of a critical density.

We will need to consider the analogues in our setting of the problems treated in [4]. We introduce then the Banach space F_α^∞, consisting of those entire functions $f(z)$ for which

$$\|f\|_\infty = \|f\|_{\alpha,\infty} = \sup_z e^{-\frac{\alpha}{2}|z|^2} |f(z)| < \infty .$$

Γ is said to be a *set of sampling* for F_α^∞ if there exists a positive number K such that

$$\|f\|_\infty \leqq K \sup_{z\in\Gamma} e^{-\frac{\alpha}{2}|z|^2} |f(z)|$$

for all $f \in F_\alpha^\infty$. If to every bounded sequence $\{a_j\}$ of complex numbers there exists an $f \in F_\alpha^\infty$ such that $e^{-\frac{\alpha}{2}|z_j|^2} f(z_j) = a_j$ for all j, we say that $\Gamma = \{z_j\}$ is a *set of interpolation* for F_α^∞. We have then the following counterparts of Beurling's two density theorems in [4]. Note that we are using the term sampling instead of balayage as in [4], which seems natural since we no longer have the relation to Fourier-Stieltjes transforms.

Theorem 2.3. *A discrete set Γ is a set of sampling for F_α^∞ if and only if it contains a uniformly discrete subset Γ' for which $D^-(\Gamma') > \alpha/\pi$.*

Theorem 2.4. *A discrete set Γ is a set of interpolation for F_α^∞ if and only if it is uniformly discrete and $D^+(\Gamma) < \alpha/\pi$.*

Let us remark, as Beurling did, that the problems and some of the results extend to several variables. We would also like to mention that there are corresponding density theorems for weighted Bergman spaces; see [20], [21].

In this part (Part I) of the paper we prove the necessity parts of the theorems, while Part II [23] deals with the sufficiency. The main problem of Part I consists in showing that we can replace the inequalities in the following statement of [19] by strict inequalities: A uniformly discrete set Γ is a set of sampling for F_α^2 only if $D^-(\Gamma) \geqq \alpha/\pi$ and a set of interpolation for F_α^2 only if $D^+(\Gamma) \leqq \alpha/\pi$.

3. Preparation for proofs

In this section we describe some notational conventions and introduce some tools to be used in the proofs.

We let

$$\|f|\Gamma\|_\infty = \|f|\Gamma\|_{\alpha,\infty} = \sup_{z\in\Gamma} e^{-\frac{\alpha}{2}|z|^2}|f(z)|,$$

and similarly,

$$\|f|\Gamma\|_2 = \|f|\Gamma\|_{\alpha,2} = \left(\sum_{z\in\Gamma} e^{-\alpha|z|^2}|f(z)|^2\right)^{\frac{1}{2}}.$$

For the rest of this section, p is to be taken to be either 2 or ∞ (it could be any $1 \leqq p \leqq \infty$).

For any discrete set Γ, $M_p(\Gamma) = M_p(\Gamma, \alpha)$ will denote the smallest number M_p such that

$$\|f\|_p \leqq M_p \|f|\Gamma\|_p$$

for all $f \in F_\alpha^p$. Γ is consequently a set of sampling for F_α^∞ if and only if $M_\infty(\Gamma) < \infty$, and it is a set of sampling for F_α^2 if and only if $M_2(\Gamma) < \infty$ and $\|f|\Gamma\|_2 < \infty$ for all $f \in F_\alpha^2$.

If $\Gamma = \{z_j\}$ is a set of interpolation for F_α^p, a standard argument based on the closed graph theorem [13], p. 196, shows that the interpolation can be performed in a stable way. This means that there exists a positive number N_p such that for every l^p-sequence $\{a_j\}$ we can find a $f \in F_\alpha^p$ with $f(z_j) = e^{\frac{\alpha}{2}|z_j|^2} a_j$ for all j, and

$$\|f\|_p \leqq N_p \|f|\Gamma\|_p .$$

The smallest such N_p is denoted $N_p(\Gamma) = N_p(\Gamma, \alpha)$, and we put $N_p(\Gamma) = \infty$ if Γ is not a set of interpolation for F_α^p.

The translations

$$(T_\alpha f)(z) = e^{\alpha \bar{a} z - \frac{\alpha}{2}|a|^2} f(z-a)$$

act isometrically in F_α^p. This translation invariance implies immediately that

$$M_p(\Gamma + z) = M_p(\Gamma) \quad \text{and} \quad N_p(\Gamma + z) = N_p(\Gamma),$$

and it will permit us to translate our analysis around an arbitrary point z to 0.

An important feature of F_α^p is the following compactness property: If $\{f_n\}$ is a sequence in the ball

$$\{f \in F_\alpha^p : \|f\|_p \leqq R\},$$

then there is a subsequence $\{f_{n_k}\}$ converging pointwise and uniformly on compact sets to some function in the ball. This is immediate from the definition of F_α^p and a normal family argument.

A sequence Q_j of closed sets converges strongly to Q, denoted $Q_j \to Q$, if $[Q, Q_j] \to 0$; here $[\cdot, \cdot]$ denotes the Fréchet distance between two closed sets. Q_j converges weakly to Q, denoted $Q_j \rightharpoonup Q$ if for every compact set D, $Q_j \cap D \to Q \cap D$.

Following Beurling, for a closed set Γ, we let $W(\Gamma)$ denote the collection of weak limits of translates $\Gamma + z$. The compactness property and the translation invariance of F_α^p make $W(\Gamma)$ a crucial tool in our analysis. Indeed, it turns out that all of Beurling's arguments concerning $W(\Gamma)$ can be carried over to our situation.

Note that by the reproducing formula (1) and the Cauchy-Schwarz inequality, we have $\|f\|_\infty \leqq \|f\|_2$. By translation invariance and subharmonicity of $|f|^2$, we have moreover

$$|f(\zeta)|^2 e^{-\alpha|\zeta|^2} \leqq C(r) \iint_{D(\zeta, r)} |f(z)|^2 \, d\mu_\alpha(z), \tag{3}$$

where (here and in the sequel) $D(\zeta, r)$ denotes the disk of radius r centered at ζ.

Bernstein's theorem plays an essential role in Beurling's analysis. We shall find that the following estimate is a sufficiently good substitute.

Lemma 3.1. *For $f \in F_\alpha^\infty$ we have for $S(w) = e^{-\frac{\alpha}{2}|w|^2} f(w)$,*

$$\big||S(\zeta + z)| - |S(\zeta)|\big| \leqq O(|z|)\|f\|_\infty,$$

where the bound in $O(|z|)$ depends only on α.

Proof. We consider

$$S_\zeta(w) = e^{-\frac{\alpha}{2}|w|^2}(T_{-\zeta} f)(w)$$

and find that

$$\begin{aligned} \big||S(\zeta + z)| - |S(\zeta)|\big| = \big||S_\zeta(z)| - |S_\zeta(0)|\big| &\leqq |(e^{-\frac{\alpha}{2}|z|^2} - 1) f_\zeta(z) + f_\zeta(z) - f_\zeta(0)| \\ &\leqq (e^{\frac{\alpha}{2}|z|^2} - 1)\|f\|_\infty + |z| \sup_{|w| < |z|} |f_\zeta'(w)|. \end{aligned}$$

An application of Cauchy's formula to the last term yields the desired result. □

We describe finally some special properties of the normalized monomials in F_α^2,

$$f_k(z) = \left(\frac{\alpha^k}{k!}\right)^{\frac{1}{2}} z^k,$$

$k = 0, 1, 2, \ldots$, which are the images of the Hermite functions under the Bargmann transform [10], p. 39. They constitute an orthonormal basis for F_α^2. They are in fact orthogonal over any disk centered at 0, i.e.

$$\iint_{D(0,r)} f_k(z)\overline{f_m(z)}\,d\mu_\alpha(z) = \lambda_k(r)\,\delta_{k,m}$$

for any $r > 0$. This implies that the f_k are the eigenfunctions, and the numbers

$$\lambda_k(r) = \frac{1}{k!}\int_0^{\alpha r^2} t^k e^{-t}\,dt$$

are the eigenvalues of the Toeplitz operator with symbol equal to the characteristic function of $D(0, r)$. Such a relation was first described by Bergman [3], pp. 14–18, and this interesting special operator and the behavior of its eigenvalues $\lambda_k(r)$ were studied in [5], see [19] for a remark on the connection to Bergman's work. Such operators were first used in problems on sampling and interpolation by Landau [15].

We note that, by translation invariance, the sequence $\{T_\zeta f_k\}$ also constitutes an orthonormal basis, relating in the same way to the disks $D(\zeta, r)$.

4. Proof of the necessity part of Theorem 2.3

We start by checking that various auxiliary results in Beurling's notes can be carried over to our situation.

Lemma 4.1. *If Γ is a set of sampling for F_α^∞, then Γ contains a uniformly discrete set that is also a set of sampling for F_α^∞.*

Proof. As the proof of Theorem 2 in [4], p. 344, with Lemma 3.1 in place of Bernstein's theorem. □

Lemma 4.2. *$\Gamma_j \rightharpoonup \Gamma$ implies $M_\infty(\Gamma) \leqq \liminf M_\infty(\Gamma_j)$.*

Proof. For any $\varepsilon > 0$, let $f \in F_\alpha^\infty$ be such that $|f\|_\infty = 1$ and $\|f|\Gamma\|_\infty \leqq M^{-1} + \varepsilon$, $M = M_\infty(\Gamma)$ (which may be infinite). We may assume that $|f(0)| \geqq 1 - \varepsilon$. Now consider the function $f(az)$, $a < 1$. We find that

$$e^{-\frac{\alpha}{2}|z|^2}|f(az)| = e^{-\frac{\alpha}{2}|az|^2}|f(az)|e^{-\frac{\alpha}{2}(1-a^2)|z|^2}.$$

In view of Lemma 3.1, we have for $|z - az| < C\varepsilon$ (C depending only on α),

$$|e^{-\frac{\alpha}{2}|az|^2}|f(az)| - e^{-\frac{\alpha}{2}|z|^2}|f(z)|| \leqq \varepsilon .$$

We choose $1 - a = C\varepsilon^3$ so that for $|z - az| \geqq C\varepsilon$,

$$e^{-\frac{\alpha}{2}|z|^2}|f(az)| \leqq e^{-\frac{\alpha}{2\varepsilon}} .$$

Then by the assumption on the sequence, we have for sufficiently small ε and large j, $\|f(a\cdot)|\Gamma_j\|_\infty \leqq M^{-1} + 3\varepsilon$, or $M_\infty(\Gamma_j) \geqq (1-\varepsilon)/(M^{-1} + 3\varepsilon)$. Since ε is arbitrary, the result follows. □

Lemma 4.3. *$M_\infty(\Gamma) < \infty$ if and only if every $\Gamma_0 \in W(\Gamma)$ is a set of uniqueness for F_α^∞.*

Proof. As the proof of Theorem 3 in [4], p. 345. □

The preceding lemma is not used directly in the proof of Theorem 2.3, but it is needed in the proof of the next lemma, and it will also be needed at a later stage.

Lemma 4.4 *If $M_\infty(\Gamma, \alpha) < \infty$, then $M_\infty(\Gamma, \alpha + \varepsilon) < \infty$ for all sufficiently small ε.*

Proof. As the proof of Theorem 4 in [4], p. 345. □

We turn to the proof of the necessity part of Theorem 2.1. In view of Lemma 4.1 and Lemma 4.4, it is clear that we are done if we can prove that $D^-(\Gamma) \geqq \alpha/\pi$ for every uniformly discrete set Γ being a set of sampling. So suppose to the contrary that Γ is a set of sampling and that $D^-(\Gamma) < \alpha/\pi$. Writing

$$D^-(\Gamma) = \frac{\alpha}{(1 + 2\varepsilon)\pi},$$

we can then find arbitrarily large R so that there are $n \leqq \alpha R^2$ points from Γ in some disk of area $\pi(1+\varepsilon)R^2$. By translation invariance, we assume this disk is $D_R = D\big(0, (1+\varepsilon)^{\frac{1}{2}} R\big)$.

Let $p_R(z)$ be the polynomial with $\Gamma \cap D_R$ as its zero-set, normalized so that $\|p_R\|_2 = 1$. Writing

$$p_R(z) = \sum_{k=0}^{n} a_k f_k(z) ,$$

we have then $\sum |a_k|^2 = 1$. We find that

$$\|p_R\|_\infty \geqq \sup_{z \in D_R} e^{-\frac{\alpha}{2}|z|^2}|p_R(z)| \geqq CR^{-1} \tag{4}$$

since

$$\iint_{\Delta_R} |p_R(z)|^2 \, d\mu_\alpha(z) \geqq \lambda_n\big((1+\varepsilon)^{\frac{1}{2}} R\big) \|p_R\|_2^2 \geqq \frac{1}{2} .$$

The last estimate is a consequence of Stirling's formula; see below, and also [5].

On the other hand, for $|z|^2 = (1+t)R^2 \geqq (1+\varepsilon)R^2$, $k \leqq n$,

$$e^{-\alpha|z|^2}|f_k(z)|^2 = \frac{\alpha^k}{k!}(1+t)^k R^{2k} e^{-\alpha(1+t)R^2}$$

$$= \frac{(\alpha R^2)}{k!} e^{-\alpha R^2} e^{-\alpha t R^2 + k\log(1+t)} \leqq C R e^{-\alpha 2c(\alpha,\varepsilon)R^2}$$

by Stirling's formula, with $c(\alpha, \varepsilon) > 0$. Thus by the Cauchy-Schwarz inequality and the fact that $\sum |a_k|^2 = 1$, we have

$$\|p_R|\Gamma\|_\infty \leqq C\sqrt{nR}\, e^{-c(\alpha,\varepsilon)R^2}.$$

Combining this with (4) we see that

$$\frac{\|p_R|\Gamma\|_\infty}{\|p_R\|_\infty} \leqq C R^{\frac{5}{2}} e^{-c(\varepsilon,\alpha)R^2} \to 0$$

as $R \to \infty$. We have reached a contradiction, so $D^-(\Gamma) \geqq \alpha/\pi$, and the proof of Theorem 2.3 is complete.

5. Proof of the necessity part of Theorem 1.4

We follow Beurling's proof of Theorem 1 in [4], p. 351.

Lemma 5.1. *Every set of interpolation for F_α^∞ is uniformly discrete.*

Proof. Let $\{a_j\}$ be a sequence with $|a_j| \leqq 1$, and $|a_k| - |a_m| = 1$ for some arbitrary k and m. Then the inequality

$$1 = \||a_k| - |a_m|\| = \||S(z_k)| - |S(z_m)|\| \leqq N_\infty(\{z_j\})\, O(|z_k - z_m|),$$

deduced from Lemma 3.1, yields the result. □

Lemma 5.2. *$\Gamma_j \rightharpoonup \Gamma$ implies $N_\infty(\Gamma) \leqq \liminf N_\infty(\Gamma_j)$.*

Proof. We may assume that the right-hand side is bounded, and even that $\sup|M_\infty(\Gamma_j)| < \infty$ by picking an appropriate subsequence. The result then follows by the compactness property. □

We introduce now a key notion in the proof. For $z \in \mathbb{C}$, let

$$\varrho_\infty(z, \Gamma) = \sup_f e^{-\frac{\alpha}{2}|z|^2}|f(z)|,$$

where f ranges over those functions f for which $f(\zeta) = 0$, $\zeta \in \Gamma$, and $\|f\|_\infty \leqq 1$.

Lemma 5.3. *$N_\infty(\Gamma) < \infty$ implies $\varrho_\infty(z, \Gamma) > 0$ when $z \notin \Gamma$.*

Proof. As the proof of Lemma 3 in [4], p. 352. □

Lemma 5.4. *For $z_0 \notin \Gamma$, we have*

$$N_\infty(\Gamma \cup \{z_0\}) \leqq \frac{1 + 2N_\infty(\Gamma)}{\varrho_\infty(z_0, \Gamma)}.$$

Proof. We assume, by translation invariance, that $z_0 = 0$, and proceed as in the proof of Lemma 4 [4], p. 353. □

In the next lemma, $d(\cdot, \cdot)$ denotes Euclidean distance.

Lemma 5.5. *Given δ_0, l_0, and α, there exists a positive constant $C = C(\delta_0, l_0, \alpha)$ such that if $N_\infty(\Gamma) \leqq l_0$ and $d(z, \Gamma) \geqq \delta_0$, then*

$$\varrho_\infty(z, \Gamma) \geqq C.$$

Proof. As the proof of Lemma 5 in [4], p. 353. □

The estimate in the following lemma is not sharp, but sufficiently good for our purposes.

Lemma 5.6. *For given l_0 and α, there exists a positive constant $C = C(l_0, \alpha)$ so that if $N_\infty(\Gamma, \alpha) \leqq l_0$, then for every square Q with $|Q| \geqq 1$,*

$$\iint_Q \log \varrho_\infty(z, \Gamma)\, dx\, dy \geqq -C|Q|^2.$$

Proof. By the proof of Lemma 5.1, we can find a point z_0 in Q so that

$$d(z_0, \Gamma) \geqq C(\alpha, l_0).$$

We may assume, by translation invariance, that $z_0 = 0$. The preceding lemma shows that there exists a function f with $|f(0)| \geqq C(\alpha, l_0)$ and $\|f\|_\infty \leqq 1$. Since clearly

$$\varrho_\infty(z, \Gamma) \geqq e^{-\frac{\alpha}{2}|z|^2} |f(z)|,$$

we have by the subharmonicity of $\log|f(z)|$,

$$-C(\alpha, l_0) = \log|f(0)| \leqq \frac{\alpha}{2} r^2 + \frac{1}{2\pi} \int_0^{2\pi} \log \varrho_\infty(re^{i\theta}, \Gamma)\, d\theta.$$

We multiply by r, integrate the inequality with respect to r from 0 to $\sqrt{2|Q|}$, and obtain the desired estimate. □

We may now prove the necessity part of Theorem 2.2. We consider then an arbitrary large square of side length R. We divide it into $[R]\times[R]$ squares, each of side length $s=R/[R]$, $[R]$ denoting the integer part of R. For each such smaller square, say Q_j, we choose a point z_j with $d(z_j,\Gamma)\geqq\delta_0=\delta_0(N_\infty(\Gamma),\alpha)$. We put $\Gamma_j=\Gamma\cup\{z_j\}$, and note that

$$N_\infty(\Gamma_j)\leqq l\,,$$

independent of j by Lemma 5.5. Thus

$$\iint_{Q_j}\log\varrho_\infty(z,\Gamma_j)\,dx\,dy\geqq -C(l,\alpha) \tag{5}$$

by Lemma 5.6. For given $z\in Q_j$, we construct an f vanishing on Γ_j-z, with $\|f\|_\infty\leqq 1$ and $f(0)=\varrho_\infty(0,\Gamma_j-z)=\varrho_\infty(z,\Gamma_j)$. Then by Jensen's formula applied to the disk $|\zeta|<r$,

$$\begin{aligned}\log\varrho_\infty(z,\Gamma_j)&\leqq\frac{1}{2\pi}\int_0^{2\pi}\log|f(z+re^{i\theta})|\,d\theta+\sum_{\zeta\in\Gamma,\,|z-\zeta|<r}\log\frac{|z-\zeta|}{r}+\log\frac{\sqrt{2}\,s}{r}\\&\leqq\frac{\alpha r^2}{2}-\sum_{\zeta\in\Gamma\cap Q^-}\log^+\frac{r}{|z-\zeta|}\log\frac{2}{r}\,,\end{aligned}$$

where Q^- is the square of side length $R-2r$, consisting of those points whose distance to the complement of Q exceeds r. We integrate this inequality with respect to area measure over Q. Using (5) and the fact that ($\zeta=\xi+i\eta$)

$$\iint\log^+\frac{r}{|\zeta|}\,d\xi\,d\eta=\frac{\pi}{2}r^2\,,$$

we then find that

$$n(Q^-)\frac{\pi}{2}r^2\leqq\left(\frac{\alpha}{2}r^2-\log r+C(l,\alpha)\right)R^2\,,$$

$n(Q^-)$ denoting the number of points from Γ contained in Q^-. Hence

$$\frac{n(Q^-)}{(R-2r)^2}\leqq\left(\frac{\alpha}{\pi}-\frac{2}{\pi}\frac{\log r}{r^2}+\frac{C(l,\alpha)}{r^2}\right)\left(1-\frac{2r}{R}\right)^{-2}. \tag{6}$$

By first choosing r so large that

$$\frac{\alpha}{2}-\frac{2}{\pi}\frac{\log r}{r^2}+\frac{C(l,\alpha)}{r^2}<\frac{\alpha}{\pi}\,,$$

and then letting $R\to\infty$, we see from (6) that $D^+(\Gamma)<\alpha/\pi$.

6. A key lemma for the L^2 problem

In this section we prove an auxiliary result that will enable us to employ Beurling's ideas also to the L^2 problem.

We need first the following simple observation.

Lemma 6.1. *Every set of interpolation for F_α^2 is uniformly discrete.*

Proof. Let $\{a_j\}$ be a sequence with $a_k = 1$ for some arbitrary k, and $a_j = 0$ otherwise. Then, since $\|f\|_\infty \leqq \|f\|_2$, we have

$$1 = \| a_k| - |a_m\| = \| S(z_k)| - |S(z_m)\| \leqq N_2(\{z_j\})\sqrt{2}\, O(|z_k - z_m|)$$

by an application of Lemma 3.1. □

Lemma 6.2. *There is no discrete subset of $\mathbb{C}$ that is both a set of sampling and a set of interpolation for F_α^2.*

Proof. Suppose that such a Γ exists. By the preceding lemma it is uniformly discrete, and by the result from [19] quoted in Section 2, we have $D^-(\Gamma) = D^+(\Gamma) = \alpha/\pi$. Pick an arbitrary $\zeta_0 \in \Gamma$, and consider the unique function $g_0 \in F_\alpha^2$ with

$$g_0(z) = \begin{cases} e^{\frac{\alpha}{2}|\zeta_0|^2}, & z = \zeta_0, \\ 0, & z \in \Gamma \setminus \{\zeta_0\}. \end{cases}$$

We put $g(z) = (z - \zeta_0) g_0(z)$, and observe that clearly $g(z)/(z - \zeta) \in F_\alpha^2$ for arbitrary $\zeta \in \Gamma$. Hence we must have

$$\iint \left| \frac{g(z)}{z - \zeta} \right|^2 d\mu_\alpha(z) \leqq C |g'(\zeta)|^2 e^{-\alpha|\zeta|^2}, \tag{7}$$

C independent of $\zeta \in \Gamma$. We note that by the subharmonicity of $|g(z)/(z-\zeta)|^2$, we also have

$$|g'(\zeta)|^2 e^{-\alpha|\zeta|^2} \leqq \frac{C(\varepsilon)}{\varepsilon^2} \iint_{\varepsilon < |z-\zeta| < 2\varepsilon} |g(z)|^2 d\mu_\alpha(z) \leqq \frac{C(\varepsilon)}{\varepsilon^2} \iint_{D(\zeta, 2\varepsilon)} |g(z)|^2 d\mu_\alpha(z). \tag{8}$$

Let $D(w, R+1)$ and $D(w, R)$ be arbitrary concentric disks, and sum (7) over $\Gamma \cap D(w, R)$. By the uniform discreteness and (8) with a sufficiently small ε, this yields

$$\iint \sum_{\zeta \in \Gamma \cap D(w,R)} |\zeta - z|^{-2} |g(z)|^2 d\mu_\alpha(z) \leqq C \iint_{D(w, R+1)} |g(z)|^2 d\mu_\alpha(z),$$

and hence

$$\inf_{z \in D(w, R+1)} \sum_{\zeta \in \Gamma \cap D(w,R)} |\zeta - z|^{-2} \leqq C.$$

But since Γ is uniformly discrete with $D^-(\Gamma) = \alpha/\pi > 0$, we have

$$\inf_{z \in D(w,R+1)} \sum_{\zeta \in \Gamma \cap D(w,R)} |\zeta - z|^{-2} \geqq C \log R$$

for large R, and thus a contradiction. □

By this proof, the difference between the Paley-Wiener and the Bargmann-Fock spaces in our context seems to be that the function $(1 + |z|)^{-1}$ happens to be square integrable on the line but not in the plane.

We state two important consequences of Lemma 6.2.

Lemma 6.3. *If Γ is a set of sampling for F_α^2, then so is $\Gamma \setminus \{\zeta\}$ for any $\zeta \in \Gamma$.*

Proof. The result follows from Lemma 6.2 by the fact that the removal of a vector from a frame leaves either a frame or an incomplete set [9], pp. 360–361; in the latter case it would have to be a Riesz basis. □

Lemma 6.4. *If Γ is a set of interpolation for F_α^2, then so is $\Gamma \cup \{\zeta\}$ for any $\zeta \notin \Gamma$.*

Proof. A set of interpolation that is also a set of uniqueness, is necessarily a set of sampling. Thus if Γ is a set of interpolation for F_α^2, there is a function $g \in F_\alpha^2$ vanishing on Γ, with help of which we may interpolate on $\Gamma \cup \{\zeta\}$. □

7. Proof of Theorem 2.1

The following lemma proves, in conjunction with [23], the sufficiency of the condition in Theorem 2.1.

Lemma 7.1. *There exists a positive constant B such that*

$$\sum_{z \in \Gamma} e^{-\alpha|z|^2} |f(z)|^2 \leqq B \|f\|_2^2$$

for all $f \in F_\alpha^2$ if and only if Γ can be expressed as a finite union of uniformly discrete sets.

Proof. Suppose such a B exists and that there is no bound on the number of points from Γ to be found in translates of, say, a unit square I. We can then find a sequence of points z_j so that the number of points from Γ in $I + z_j$ tends to infinity, implying that

$$\|k_{z_j}|\Gamma\|_2 \to \infty .$$

This is a contradiction, and so there must be a bound on the number of points in $I + z$. Now a simple argument shows that Γ can be expressed as a finite union of uniformly discrete sets.

The opposite implication follows immediately from (3). □

Lemma 7.2. *If Γ is a set of sampling for F_α^2, then Γ contains a uniformly discrete set that is also a set of sampling.*

Proof. For $\varepsilon > 0$, we construct (as we may) a uniformly discrete subset Γ' of Γ such that $d(\zeta, \Gamma') < \varepsilon$ for each $\zeta \in \Gamma$. We have then $\Gamma = \bigcup_{\zeta' \in \Gamma'} (\Gamma \cap D(\zeta', \varepsilon))$. By the preceding lemma, there is a uniform bound, say P, on the number of points in $\Gamma \cap D(\zeta', \varepsilon)$, $\varepsilon \leqq 1$.

We make now an estimate similar to one made in the proof of Theorem 5.1 in [19]. For arbitrary $f \in F_\alpha^2$ we have

$$\iint_{D(\zeta,1)} |f(z)|^2 \, d\mu_\alpha(z) = \lambda_0(1) |f(\zeta)|^2 e^{-\alpha|\zeta|^2} + \sum_{k=1}^{\infty} \lambda_k(1) |\langle f, T_\zeta f_k \rangle|^2 .$$

We sum over $\zeta \in \Gamma$ and find, by the fact that Γ is a finite union of uniformly discrete sets, that there is a positive constant C such that

$$\sum_{\zeta \in \Gamma} \sum_{k=1}^{\infty} \lambda_k(1) |\langle f, T_\zeta f_k \rangle|^2 \leqq C \|f\|_2^2 . \tag{9}$$

For some $\zeta' \in \Gamma'$, let $\zeta \in \Gamma \cap D(\zeta', \varepsilon)$. From the relation

$$f(\zeta) = \frac{K(\zeta', \zeta)}{K(\zeta', \zeta')} f(\zeta') + \sum_{k=1}^{\infty} \langle f, T_{\zeta'} f_k \rangle (T_{\zeta'} f_k)(\zeta)$$

we find that

$$\frac{K(\zeta', \zeta')^{\frac{1}{2}} f(\zeta)}{K(\zeta', \zeta)} - \frac{f(\zeta')}{K(\zeta', \zeta')^{\frac{1}{2}}} = \sum_{k=1}^{\infty} \left(\frac{\alpha^k}{k!} \right)^{\frac{1}{2}} \langle f, T_{\zeta'} f_k \rangle (\zeta - \zeta')^k .$$

We multiply and divide the k-th term in this sum by $(k+1)^{-\frac{1}{2}}$. By the Cauchy-Schwarz inequality we get

$$\left| \frac{K(\zeta', \zeta')^{\frac{1}{2}} f(\zeta)}{K(\zeta', \zeta)} - \frac{f(\zeta')}{K(\zeta', \zeta')^{\frac{1}{2}}} \right|^2 \leqq \sum_{k=1}^{\infty} \frac{\alpha^k}{k!\,(k+1)} |\langle f, T_{\zeta'} f_k \rangle|^2 \sum_{j=1}^{\infty} (j+1)\, \varepsilon^{2j} .$$

We sum this inequality over $\zeta' \in \Gamma'$, use (9) and the estimate

$$\lambda_k(1) \geqq \frac{e^{-\alpha} \alpha^{k+1}}{k!\,(k+1)} ,$$

and obtain thus

$$\sum_{\zeta' \in \Gamma} \left| \frac{K(\zeta', \zeta')^{\frac{1}{2}} f(\zeta)}{K(\zeta', \zeta)} - \frac{f(\zeta')}{K(\zeta', \zeta')^{\frac{1}{2}}} \right|^2 \leqq C \varepsilon^2 \|f\|_2^2 .$$

Using the observation at the beginning of the proof, the triangle inequality and the fact that

$$\frac{K(\zeta',\zeta')^{\frac{1}{2}}}{|K(\zeta',\zeta)|} \geqq \frac{1}{K(\zeta,\zeta)^{\frac{1}{2}}}$$

we find that

$$\|f|\Gamma\|_2 - PC\varepsilon\|f\|_2 \leqq P\|f|\Gamma'\|_2 .$$

The proof is complete since the choice of ε is at our disposal. □

Lemma 7.3. *Suppose* $\inf q(\Gamma_j) > 0$. *Then* $\Gamma_j \rightharpoonup \Gamma$ *implies* $M_2(\Gamma,\alpha) \leqq \liminf M_2(\Gamma_j,\alpha)$.

Proof. For any $\varepsilon > 0$, let $\|f\|_2 = 1$ and $\|f|\Gamma\|_2 \leqq M^{-1} + \varepsilon$, $M = M_2(\Gamma)$ (which may be infinite). Then by the assumption on the sequence, we have for all sufficiently large j, $\|f|\Gamma_j\|_2 \leqq M^{-1} + 2\varepsilon$, or $M_2(\Gamma_j) \geqq (M^{-1} + 2\varepsilon)^{-1}$. This proves the lemma, since ε is arbitrary. □

We may now finish the proof of Theorem 2.1. We consider, by Lemma 7.2, a uniformly discrete set Γ being a set of sampling for F_α^2. Lemma 7.3 and the fact that all sets in $W(\Gamma)$ are uniformly discrete imply that $W(\Gamma)$ consists only of sets of sampling. By Lemma 6.3 we have that every set of sampling for F_α^2 is a set of uniqueness for F_α^∞. For suppose Γ_0 is a set of sampling for F_α^2 and that $g \in F_\alpha^\infty$ vanishes on Γ_0. Then the function

$$f(z) = \frac{g(z)}{(z - z_1)(z - z_2)},$$

$z_1, z_2 \in \Gamma_0$, belongs to F_α^2 and vanishes on $\Gamma_0 \setminus \{z_1, z_2\}$. This contradicts Lemma 6.3.

Thus every set in $W(\Gamma)$ is a set of uniqueness for F_α^∞. It follows from Lemma 4.3 that Γ is also a set of sampling for F_α^α, and thus by Theorem 2.3, $D^-(\Gamma) > \alpha/\pi$.

8. Proof of the necessity part of Theorem 2.2

In this section we will use Beurling's method of proof as in Section 4; this turns out to be possible by Lemma 6.4.

Lemma 8.1. $\Gamma_j \rightharpoonup \Gamma$ *implies* $N_2(\Gamma) \leqq \liminf N_2(\Gamma_j)$.

Proof. As the proof of Lemma 5.2. □

For $z \in \mathbb{C}$, let now

$$\varrho_2(z,\Gamma) = \sup_f e^{-\frac{\alpha}{2}|z|^2}|f(z)|,$$

where f ranges over those functions f for which $f(\zeta) = 0$, $\zeta \in \Gamma$ and $\|f\|_2 \leqq 1$.

Lemma 8.2. *$N_2(\Gamma) < \infty$ implies $\varrho_2(z, \Gamma) > 0$ when $z \notin \Gamma$.*

Proof. As the proof of Lemma 5.3, now with Lemma 6.4 as a crucial ingredient. □

Lemma 8.3. *For $z_0 \notin \Gamma$, we have*

$$N_2(\Gamma \cup \{z_0\}) \leqq \frac{1 + 2N_2(\Gamma)}{\varrho_2(z_0, \Gamma)}.$$

Proof. We assume, by translation invariance, that $z_0 = 0$, and proceed as in the proof of Lemma 4 in [4], p. 353.

Lemma 8.4. *Given δ_0, l_0, and α, there exists a constant $C = C(\delta_0, l_0, \alpha)$ such that if $N_2(\Gamma, \alpha) \leqq l_0$ and $d(z, \Gamma) \geqq \delta_0$, then*

$$\varrho_2(z, \Gamma) \geqq C.$$

Proof. We need only a slight modification of the proof of Lemma 5 in [4], p. 353, but we elaborate the details for the sake of clarity.

Let us assume the lemma is false. Then there exists a sequence Γ_j of sets such that $N_2(\Gamma_j) \leqq l_0$ for all j and points z_j with $d(z_j, \Gamma_j) > \delta_0$ such that

$$\varrho_2(z_j, \Gamma_j) \to 0.$$

By translation invariance we may assume that $z_j = 0$, and also that $\Gamma_j \rightharpoonup \Gamma'$, where Γ' may be empty. By Lemma 8.1 we have $N_2(\Gamma') \leqq l_0$ and $0 \notin \Gamma'$. By the preceding lemma, we can find an $f \in F_\alpha^2$, vanishing on Γ', with $\|f\|_2 \leqq 1$, and $f(0) = r > 0$. Since $\Gamma_j \rightharpoonup \Gamma'$ and f is square-integrable,

$$\varepsilon_j = \|f \,|\, \Gamma_j\|_2 \to 0.$$

Choose $g_j \in F_\alpha^2$ with $g_j(z) = f(z)$ for $z \in \Gamma_j$ and $\|g_j\|_2 \leqq l_0 \varepsilon_j$, and define

$$f_j(z) = \frac{f(z) - g_j(z)}{\|f\|_2 + l_0 \varepsilon_j}.$$

Then $\|f_j\| \leqq 1$ and $f_j(z) = 0$ for $z \in \Gamma_j$. But since $|g_j(0)| \leqq \varepsilon_j \to 0$, we have

$$\varrho_2(0, \Gamma_j) \geqq |f_j(0)| \to \frac{r}{\|f\|_2} > 0,$$

which is a contradiction. □

All we need to say now is that by the fact that $\|f\|_\infty \leqq \|f\|_2$, the rest of the proof is identical to the corresponding part of the proof of Theorem 2.4.

Ackowledgement. This work was done while the author was enjoying a stay at Institut Mittag-Leffler, Djursholm, Sweden, whose warm hospitality is gratefully acknowledged.

References

[1] *H. Bacry, A. Grossmann* and *J. Zak*, Proof of the completeness of lattice states in the kq-representation, Phys. Rev. **B 12** (1975), 1118–1120.
[2] *V. Bargmann, P. Butero, L. Girardello* and *J. R. Klauder*, On the completeness of coherent states, Rep. Mod. Phys. **2** (1971), 221–228.
[3] *S. Bergman*, The Kernel Function and Conformal Mapping, Amer. Math. Soc., Math. Surveys V, New York 1950.
[4] *A. Beurling*, The Collected Works of Arne Beurling, Vol. 2 Harmonic Analysis, Boston 1989.
[5] *I. Daubechies*, Time-frequency localization operators – a geometric phase space approach, IEEE Trans. Inform. Th. **34** (5) (1988), 605–612.
[6] *I. Daubechies*, The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Th. **36** (5) (1990), 961–1005.
[7] *I. Daubechies* and *A. Grossmann*, Frames in the Bargmann space of entire functions, Comm. Pure Appl. Math. **41** (1988), 151–164.
[8] *I. Daubechies, A. Grossmann* and *Y. Meyer*, Painless nonorthogonal expansions, J. Math. Phys. **27** (1986), 1271–1283.
[9] *R. J. Duffin* and *A. C. Schaeffer*, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. **72** (1952), 341–366.
[10] *G. B. Folland*, Harmonic Analysis in Phase Space, Princeton 1989.
[11] *D. Gabor*, Theory of communication, J. Inst. Elect. Eng. (London) **93** (1946), 429–457.
[12] *K. Gröchenig* and *D. Walnut*, A Riesz basis for the Bargmann-Fock space related to sampling and interpolation, Ark. Mat., to appear.
[13] *K. Hoffmann*, Banach Spaces of Analytic Functions, Englewood Cliffs 1962.
[14] *S. Janson, J. Peetre* and *R. Rochberg*, Hankel forms and the Fock space, Revista Mat. Iberoamer. **3** (1987), 61–138.
[15] *H. J. Landau*, Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. **117** (1967), 37–52.
[16] *Y. Lyubarskii*, Frames in the Bargmann space of entire functions, Proceedings of Seminar in Complex Analysis, Lect. Notes Math., to appear.
[17] *J. von Neumann*, Foundations of Quantum Mechanics, Princeton 1955.
[18] *A. M. Perelomov*, On the completeness of a system of coherent states, Theor. Math. Phys. **6** (1971), 156–164.
[19] *K. Seip*, Reproducing formulas and double orthogonality in Bargmann and Bergman spaces, SIAM J. Math. Anal. **22** (3) (1991), 856–876.
[20] *K. Seip*, Regular sets of sampling and interpolation for weighted Bergman spaces, Proc. Amer. Math. Soc., to appear.
[21] *K. Seip*, Density theorems for sampling and interpolation in the Bargmann-Fock space, Bull. Amer. Math. Soc., to appear.
[22] *K. Seip*, Beurling Type Density Theorems in the Unit Disk, Preprint, Trondheim 1991.
[23] *K. Seip* and *R. Wallstén*, Density theorems for sampling and interpolation in the Bargmann-Fock space II, J. reine angew. Math. **429** (1992), 107–113.
[24] *R. M. Young*, An Introduction to Nonharmonic Fourier Series, London–New York 1980.

Division of Mathematical Sciences, University of Trondheim, N-7034 Trondheim, Norway

Eingegangen 18. Juni 1991