Bost-Connes, $\mathbb{F}_1$, 3-manifolds

Matilde Marcolli

2008

The Bost–Connes system

Algebra $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ generators and relations

$$\begin{array}{l} \mu_n\mu_m = \mu_{nm} \\ \mu_n\mu_m^* = \mu_m^*\mu_n \quad \text{when } (n,m)=1 \\ \mu_n^*\mu_n = 1 \end{array}$$

$$e(r+s) = e(r)e(s), \qquad e(0) = 1$$

$$\rho_n(e(r)) = \mu_n e(r) \mu_n^* = \frac{1}{n}\sum_{ns=r} e(s)$$

C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

Time evolution

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it}\mu_n$$

Hamiltonian $\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$

Complete classification of extremal KMS states (phase transition)

Observations

- Representations π_ρ on $\ell^2(N)$:

$$\mu_n \epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m \epsilon_m$$

$\zeta_r = \rho(e(r))$ root of 1, for $\rho \in \hat{\mathbb{Z}}$

- Low temperature KMS states ($\beta > 1$)

$$\varphi_{\beta,\rho}(a) = \frac{\mathrm{Tr}(\pi_\rho(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \quad \rho \in \hat{\mathbb{Z}}^*$$

- Zero temperature: evaluations

$$\varphi_{\infty,\rho}(a) = \langle \epsilon_1, \pi_\rho(a)\epsilon_1 \rangle$$

$\varphi_{\infty,\rho}(e(r)) = \zeta_r$

- $\mu_n \mu_n^* = e_n$ idempotent

$$e_n = \frac{1}{n} \sum_{ns=0} e(s)$$

$e_n \epsilon_m$ projection of $\ell^2(\mathbb{N})$ on $n|m$ (range of multiplication by n)

Bost–Connes endomotive (Connes–Consani–M.)

$A = \varinjlim_n A_n$ with $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$
abelian semigroup action $S = \mathbb{N}$ on $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

Endomotives (A, S) from self maps of algebraic varieties $s : Y \to Y$, $s(y_0) = y_0$ unbranched, $X_s = s^{-1}(y_0)$, $X = \varprojlim X_s = \mathrm{Spec}(A)$

$$\xi_{s,s'} : X_{s'} \to X_s, \quad \xi_{s,s'}(y) = r(y), \quad s' = rs \in S$$

Bost–Connes endomotive:
$\mathbb{G}_m$ with self maps $u \mapsto u^k$

$s_k : P(t, t^{-1}) \mapsto P(t^k, t^{-k}), \;\; k \in \mathbb{N}, \;\; P \in \mathbb{Q}[t, t^{-1}]$

$X_k = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}]/(t^k - 1)) = s_k^{-1}(1)$ and $X = \varprojlim_k X_k$

Integer model of the Bost–Connes algebra

(Connes–Consani–M.)

$\mathcal{A}_{\mathbb{Z},BC}$ generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and μ_n^*, $\tilde{\mu}_n$

$$\begin{array}{l} \tilde{\mu}_n \tilde{\mu}_m = \tilde{\mu}_{nm} \\ \mu_n^* \mu_m^* = \mu_{nm}^* \\ \mu_n^* \tilde{\mu}_n = n \\ \tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n \quad (n,m) = 1. \end{array}$$

$$\mu_n^* x = \sigma_n(x) \mu_n^* \qquad \text{and} \qquad x \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(x)$$

where $\sigma_n(e(r)) = e(nr)$ for $r \in \mathbb{Q}/\mathbb{Z}$

Note: $\rho_n(x) = \mu_n x \mu_n^*$ ring homomorphism but not $\tilde{\rho}_n(x) = \tilde{\mu}_n x \mu_n^*$ (correspondences "crossed product" $\mathcal{A}_{\mathbb{Z},BC} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$)

Gadgets over $\mathbb{F}_1$ (Soulé)

$$(X, \mathcal{A}_X, e_{x,\sigma})$$

• $X : \mathcal{R} \to Sets$ covariant functor,
$\mathcal{R}$ finitely generated flat rings

• $\mathcal{A}_X$ complex algebra

• evaluation maps: for all $x \in X(R)$, $\sigma : R \to \mathbb{C}$
$\Rightarrow e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}$ algebra homomorphism

$$e_{f(y),\sigma} = e_{y,\sigma\circ f}$$

for $f : R' \to R$ ring homomorphism

Affine varieties $V_{\mathbb{Z}} \Rightarrow$ gadget $X = G(V_{\mathbb{Z}})$ with $X(R) = \mathrm{Hom}(O(V), R)$ and $\mathcal{A}_X = O(V) \otimes \mathbb{C}$

Affine variety over $\mathbb{F}_1$ (Soulé)
Gadget with $X(R)$ finite; variety $X_{\mathbb{Z}}$ and morphism of gadgets

$$X \to G(X_{\mathbb{Z}})$$

such that all $X \to G(V_{\mathbb{Z}})$ come from $X_{\mathbb{Z}} \to V_{\mathbb{Z}}$

Bost–Connes and $\mathbb{F}_1$ (Connes–Consani–M.)

Affine varieties $\mu^{(n)}$ over $\mathbb{F}_1$ defined by gadgets $G(\text{Spec}(\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]))$; projective system

Endomorphisms σ_n (of varieties over $\mathbb{Z}$, of gadgets, of $\mathbb{F}_1$-varieties)

Extensions $\mathbb{F}_{1^n}$: free actions of roots of 1 (Kapranov–Smirnov)

$\zeta \mapsto \zeta^n, \; n \in \mathbb{N}$ and $\zeta \mapsto \zeta^\alpha \leftrightarrow e(\alpha(r)), \; \alpha \in \hat{\mathbb{Z}}$

Frobenius action on $\mathbb{F}_{1^\infty}$

In reductions mod p of integral Bost–Connes endomotive $\Rightarrow$ Frobenius

Bost–Connes = extensions $\mathbb{F}_{1^n}$ plus Frobenius

The Habiro ring

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q)_n)$$

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

$\mathbb{Z}[q]/((q)_n) \twoheadrightarrow \mathbb{Z}[q]/((q)_k)$ for $k \le n$ since $(q)_k | (q)_n$

<u>Evaluation maps</u> at roots of 1

$$ev_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta]$$

surjective ring homomorphisms
give an *injective* homomorphism:

$$ev : \widehat{\mathbb{Z}[q]} \to \prod_{\zeta \in \mathcal{Z}} \mathbb{Z}[\zeta]$$

<u>Taylor series expansions</u>

$$\tau_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]]$$

injective ring homomorphism

$$P(q) = a_0 + a_1(\zeta + (q-\zeta)) + \cdots + a_\ell(\zeta + (q-\zeta))^\ell =$$

$$\sum_{j=0}^{\ell} a_j \sum_{k=0}^{j} \binom{j}{k} \zeta^{j-k}(q-\zeta)^k =$$

$$\sum_{k \ge 0} \frac{1}{k!} P^{(k)}(\zeta)(q-\zeta)^k$$

Endomorphisms of the Habiro ring

$$\sigma_n(f)(q) = f(q^n)$$

lifts $P(\zeta) \mapsto P(\zeta^n)$ in $\mathbb{Z}[\zeta]$ through ev_ζ

$\sigma_n : \mathbb{Z}[q]/((q)_m) \to \mathbb{Z}[q]/(\sigma_n(q)_m) \to \mathbb{Z}[q]/((q)_m)$

where $(q)_m | \sigma_n(q)_m \Rightarrow \sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$

$$ev_\zeta(\sigma_n(f)) = ev_{\zeta^n}(f)$$

but on Taylor series $\tau_\zeta(\sigma_n(f)) \neq \tau_{\zeta^n}(f)$

$$\tau_\zeta(\sigma_n(f)) = \sum_{k \geq 0} \frac{1}{k!} ev_\zeta((f \circ \sigma_n)^{(k)})(q-\zeta)^k$$

Representations of the Habiro ring and the Bost–Connes endomotive

$f \in \widehat{\mathbb{Z}[q]}$ and $\zeta \in \mathcal{Z} \Rightarrow$ bounded operator on $\ell^2(\mathbb{N})$

$$E_{\zeta,f}\, \epsilon_n = ev_{\zeta^n}(f)\, \epsilon_n$$

$\Rightarrow C^*$-algebra $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$

$\rho \in \hat{\mathbb{Z}}^* \Rightarrow$ isomorphism $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]}) \cong C^*(\mathbb{Q}/\mathbb{Z})$

$C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$ and μ_n and $\mu_n^* \Rightarrow$ Bost–Connes algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$

With Taylor expansion: operators

$$T_{\zeta,f}\, \epsilon_{n,m} = \sum_{k \geq 0} \frac{1}{k!} (f \circ \sigma_n)^{(k)}(\zeta)\, \epsilon_{n,m+k}$$

on $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N} \cup \{0\})$

$\Rightarrow C^*$-algebra $C^*_{\tau,\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$

Integral model from Habiro ring

Ring $\mathcal{A}_{\mathbb{Z},q}$ generated by $\widehat{\mathbb{Z}[q]}$ and μ_n and μ_n^* with relations

$$\mu_n^* f = \sigma_n(f)\, \mu_n^* \quad \text{and} \quad f\, \mu_n = \mu_n \sigma_n(f)$$

Semigroup crossed product

$$\mathcal{A}_{\mathbb{Z},q} = \widehat{\mathbb{Z}[q]}_\infty \rtimes_\rho \mathbb{N}$$

where $A_N = \{\mu_N\, f\, \mu_N^* \,|\, f \in \widehat{\mathbb{Z}[q]}\}$

$$A_N \cdot A_M \subset A_{NM/(N,M)}$$

$A_\infty = \widehat{\mathbb{Z}[q]}_\infty = \cup_N A_N$
Endomorphisms $\rho_n : A_\infty \to A_\infty$

$$\rho_n(a) = \mu_n\, a\, \mu_n^*, \quad \rho_n : A_N \to A_{nN}$$

$$\sigma_n(a) = \mu_n^* a \mu_n, \quad \sigma_n : A_N \to A_{N/n}, \quad \text{if } n|N$$

$$e_n = \mu_n \mu_n^* : A_N \to \mu_{nN/(n,N)}\, \hat{R}_{n/(n,N)}\, \mu^*_{nN/(n,N)} \subset A_{nN/(n,N)}$$

$a_N \mapsto e_n a_N e_n$ with $\hat{R}_n =$ range of σ_n on $\widehat{\mathbb{Z}[q]}$

Subring $\tilde{\mathcal{A}}_{\mathbb{Z},q}$ generated by $\widehat{\mathbb{Z}[q]}$ and $\tilde{\mu}_n$ and μ_n^* maps to $\mathcal{A}_{\mathbb{Z},BC}$ via evaluations at $\zeta \in \mathcal{Z}$

Time evolution and KMS states

$0 < \hbar < 1$ with $\log r + x \log \hbar = 0$ for $r \in \mathbb{Q}_+^*$ and $x \in \mathbb{Z} \Rightarrow (r, x) = (1, 0)$

$$\sigma_t(T_{\zeta,f})\epsilon_{n,m} = \sum_{k\geq 0} \frac{1}{k!}(P\circ\sigma_n)^{(k)}(\zeta)\,\hbar^{-ikt}\,\delta_k\epsilon_{n,m}$$

$$\sigma_t(\delta_k) = \hbar^{-ikt}\delta_k, \qquad \sigma_t(\mu_n) = n^{it}\,\mu_n$$

$\delta_k\epsilon_{n,m} = \epsilon_{n,m+k}$
Hamiltonian $H\,\epsilon_{n,m} = (\log(n) + m\log(\hbar))\,\epsilon_{n,m}$
Partition function

$$Z_\hbar(\beta) = \frac{\zeta(\beta)}{1-\hbar^\beta}$$

Low temperature KMS states

$$\varphi_\beta(\delta_\ell^* T_{\zeta,f}) = \hbar^\ell(1-\hbar^\beta)\zeta(\beta)^{-1}\sum_n \tau_{\zeta,\ell}(f\circ\sigma_n)n^{-\beta}$$

Bost–Connes case included

$$\varphi_\beta(T_{\zeta,f}) = (1-\hbar^\beta)\zeta(\beta)^{-1}\sum_n ev_\zeta(f\circ\sigma_n)\;n^{-\beta}$$

Zero temperature

$$\varphi_\infty(T_{\zeta,f}) = ev_\zeta(f)$$

$\lim_{\beta\to\infty}\varphi_\beta(\delta_\ell^* T_{\zeta,f})/\hbar^\beta$ recover all terms in the Taylor series

Multivariable Habiro rings (Manin)

$$\widehat{\mathbb{Z}[q_1, \ldots, q_n]} = \varprojlim_N \mathbb{Z}[q_1, \ldots, q_n]/I_{n,N}$$

where $I_{n,N}$ is the ideal

$$\big((q_1-1)(q_1^2-1)\cdots(q_1^N-1), \ldots, (q_n-1)(q_n^2-1)\cdots(q_n^N-1)\big)$$

<u>Evaluations at roots of 1</u>

$$ev_{(\zeta_1,\ldots,\zeta_n)} : \widehat{\mathbb{Z}[q_1, \ldots, q_n]} \to \mathbb{Z}[\zeta_1, \ldots, \zeta_n]$$

<u>Taylor expansions</u>

$$\tau_Z : \widehat{\mathbb{Z}[q_1, \ldots, q_n]} \to \mathbb{Z}[\zeta_1, \ldots, \zeta_n][[q_1-\zeta_1, \ldots, q_n-\zeta_n]]$$

$Z = (\zeta_1, \ldots, \zeta_n)$ in $\mathcal{Z}^n$

Endomorphisms of tori and Habiro rings

Torus $\mathbb{T}^n = (\mathbb{G}_m)^n$, algebra $\mathbb{Q}[t_i, t_i^{-1}]$

$$t^\alpha = (t_i^\alpha)_{i=1,\dots,n} \quad \text{with} \quad t_i^\alpha = \prod_j t_j^{\alpha_{ij}}$$

$\alpha \in M_n(\mathbb{Z})^+$ homomorphisms semigroup; automorphisms $\mathrm{SL}_n(\mathbb{Z})$

<u>Action on Habiro rings</u>

$$\mathbb{Z}[\widehat{q_1, \dots}, q_n] = \varprojlim_N \mathbb{Z}[q_1, \dots, q_n, q_1^{-1}, \dots, q_n^{-1}]/\mathcal{J}_{n,N}$$

$\mathcal{J}_{n,N}$ ideal generated by the $(q_i - 1)\cdots(q_i^N - 1)$, for $i = 1, \dots, n$ and the $(q_i^{-1} - 1)\cdots(q_i^{-N} - 1)$

Semigroup action $\alpha \in M_n(\mathbb{Z})^+$:

$$q \mapsto \sigma_\alpha(q) = \sigma_\alpha(q_1, \dots, q_n) =$$

$$(q_1^{\alpha_{11}} q_2^{\alpha_{12}} \cdots q_n^{\alpha_{1n}}, \dots, q_1^{\alpha_{n1}} q_2^{\alpha_{n2}} \cdots q_n^{\alpha_{nn}}) = q^\alpha$$

Multivariable Bost–Connes endomotives

Variety $\mathbb{T}^n = (\mathbb{G}_m)^n$ endomorphisms $\alpha \in M_n(\mathbb{Z})^+$

$$X_\alpha = \{t = (t_1, \ldots, t_n) \in \mathbb{T}^n \mid s_\alpha(t) = t_0\}$$

$$\xi_{\alpha,\beta} : X_\beta \to X_\alpha, \quad t \mapsto t^\gamma, \quad \alpha = \beta\gamma \in M_n(\mathbb{Z})^+$$

$X = \varprojlim_\alpha X_\alpha$ with semigroup action

$C(X(\bar{\mathbb{Q}})) \cong \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ generators $e(r_1) \otimes \cdots \otimes e(r_n)$

$$\mathcal{A}_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+$$

generated by $e(\underline{r})$ and μ_α, μ_α^*

$$\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r}) \mu_\alpha^* = \frac{1}{\det \alpha} \sum_{\alpha(\underline{s}) = \underline{r}} e(\underline{s})$$

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha = e(\alpha(\underline{r}))$$

From representations of multivariable Habiro rings on $\ell^2(M_n(\mathbb{Z})^+)$

$$ev_{(\zeta_1,\ldots,\zeta_n)}(f)\epsilon_\alpha = f(\zeta^{\tilde{\alpha}})\epsilon_\alpha, \quad \tilde{\alpha} = \det(\alpha)\alpha^{-1}$$

ring $\widehat{\mathbb{Z}[q_1, \ldots, q_n]}_\infty \rtimes_\rho M_n(\mathbb{Z})^+$

Λ-rings, endomotives, and $\mathbb{F}_1$

Grothendieck: characteristic classes, Riemann–Roch

Torsion free R with action of semigroup $\mathbb{N}$ by endomorphisms lifting Frobenius

$$s_p(x) - x^p \in pR, \quad \forall x \in R$$

Morphisms: $f \circ s_k = s_k \circ f$

$\mathbb{Q}$-algebra $A \Rightarrow \Lambda$-ring
iff action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}$ on $\mathcal{X} = \mathrm{Hom}(A, \bar{\mathbb{Q}})$ factors through an action of $\hat{\mathbb{Z}}$

The Bost–Connes endomotives is a direct limit of Λ-rings

$$R_n = \mathbb{Z}[t, t^{-1}]/(t^n - 1) \quad s_k(P)(t, t^{-1}) = P(t^k, t^{-k})$$

Action of $\hat{\mathbb{Z}}$: Frobenius over $\mathbb{F}_{1^\infty}$

Every torsion free finite rank Λ-ring embeds in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ with action of $\mathbb{N}$ compatible with $S_{n,diag} \subset M_n(\mathbb{Z})^+$ (Bogner–Smit)

$\Rightarrow$ multivariable Bost–Connes endomotives as universal Λ-rings

Time evolution and Hamiltonian

$$\sigma_t(e(\underline{r})) = e(\underline{r}), \quad \sigma_t(\mu_\alpha) = \det(\alpha)^{it}\,\mu_\alpha$$

Hamiltonian $H\epsilon_\beta = \log\det(\beta)\,\epsilon_\beta$
Infinite multiplicities: $\mathrm{SL}_n(\mathbb{Z})$ symmetry

As in the case of 2-dim $\mathbb{Q}$-lattices (Connes-M.):

$$\mathcal{U}_\Gamma = \{(\alpha,\rho)\in \Gamma\backslash \mathrm{GL}_n(\mathbb{Q})^+ \times_\Gamma \hat{\mathbb{Z}}^n \,|\, \alpha(\rho)\in\hat{\mathbb{Z}}^n\}$$

Quotient $\mathcal{U}_\Gamma$ by $\mathrm{SL}_n(\mathbb{Z})\times\mathrm{SL}_n(\mathbb{Z})$

$$(\gamma_1,\gamma_2) : (\alpha,\rho)\mapsto(\gamma_1\alpha\gamma_2^{-1},\gamma_2(\rho))$$

Convolution algebra

$$(f_1\star f_2)(\alpha,\rho) = \sum_{(\alpha,\rho)=(\alpha_1,\rho_1)\circ(\alpha_2,\rho_2)\in\mathcal{U}_\Gamma} f_1(\alpha_1,\rho_1)f_2(\alpha_2,\rho_2)$$

$f^*(\alpha,\rho)=\overline{f(\alpha^{-1},\alpha(\rho))}$ and $\sigma_t(f)(\alpha,\rho)=\det(\alpha)^{it}f(\alpha,\rho)$

$$(\pi_\rho(f)\xi)(\alpha) = \sum_{\beta\in\Gamma\backslash\mathrm{GL}_n(\mathbb{Q})^+\,:\,\beta\rho\in\hat{\mathbb{Z}}^*} f(\alpha\beta^{-1},\beta(\rho))\xi(\beta)$$

On $\ell^2(\Gamma\backslash G_\rho)$. If $\rho\in(\hat{\mathbb{Z}}^*)^n$:

$$G_\rho = \{\alpha\in\mathrm{GL}_n(\mathbb{Q})^+\,|\,\alpha(\rho)\in\hat{\mathbb{Z}}^n\} = M_n(\mathbb{Z})^+$$

$$Z(\beta) = \sum_{m\in\Gamma\backslash M_n(\mathbb{Z})^+} \det(m)^{-\beta}$$

Quantum channels and states

States given by "density matrices" ϱ

$$\varphi(a) = \frac{\mathrm{Tr}(a\varrho)}{\mathrm{Tr}(\varrho)}$$

Gibbs states when $\varrho = e^{-\beta H}$

Transforming density matrices by endomorphisms (quantum channels)

$$s: a \mapsto s(a) = \mu_s a \mu_s^*$$

semigroup S acting on algebra $\mathcal{A}$ by endomorphisms realized by isometries μ_s, $s \in S$

$$\varphi_s(a) = (s^*\varphi)(a) = \frac{\mathrm{Tr}(a\varrho_s)}{\mathrm{Tr}(\varrho_s)}, \quad \text{with } \varrho_s = \mu_s^* \varrho \mu_s$$

Action on density matrices $\varrho \mapsto \varrho_s = \mu_s^* \varrho \mu_s$

MZVs of cones (Terasoma)

$$\zeta_{\mathcal{C}}(\ell_1,\ldots,\ell_k,\chi) = \sum_{v\in\mathcal{C}^0\cap\mathbb{Z}^n} \frac{\chi(v)}{\ell_1(v)\cdots\ell_k(v)}$$

ℓ_i = $\mathbb{Q}$-linear forms on $\mathbb{Q}^n$ positive on interior $\mathcal{C}^0$ of $\mathcal{C}$, $\chi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*)$

States from MZVs of cones

Algebra $\mathcal{A} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ and semigroup $S_{\mathcal{C}} \subset M_n(\mathbb{Z})^+$ preserving cone $\mathcal{C}$

$$\varrho\,\epsilon_v = (\ell_1(v)\cdots\ell_k(v))^{-1}\epsilon_v$$

on Hilbert space $\ell^2(\mathcal{C}^0\cap\mathbb{Z}^n)$ $\Rightarrow$ state

$$\mathrm{Tr}(a\varrho) = \sum_{v\in\mathcal{C}^0\cap\mathbb{Z}^n} \frac{\chi_a(v)}{\ell_1(v)\cdots\ell_k(v)}$$

character χ_a for a choice of $\alpha \in GL_n(\hat{\mathbb{Z}})$:

$$\chi_a(v) = \zeta_1^{k_1}\cdots\zeta_n^{k_n}$$

for $a = e(r_1)\otimes\cdots\otimes e(r_n)$ and $\zeta_i = \alpha(e(r_i)) \in \mathcal{Z}$.

Semigroup action and relations:

$$\mathrm{Tr}(a\varrho_s) = \sum_{v\in\mathcal{C}^0\cap\mathbb{Z}^n} \frac{\chi_a(v)}{\ell_1(s(v))\cdots\ell_k(s(v))}$$

$$\mathcal{R}(\zeta_{\mathcal{C}}(\ell_{1,i},\ldots,\ell_{k,i},\chi_{s(a_i)})) = 0 \Leftrightarrow \mathcal{R}(\zeta_{\mathcal{C}}(\ell_{1,i}\circ s,\ldots,\ell_{k,i}\circ s,\chi_{a_i})) = 0$$

The universal Witten–Reshetikhin–Turaev invariant (Habiro)

3-dimensional integral homology sphere M
surgery presentation $M = S^3_L$, algebraically split link $L = L_1 \cup \cdots \cup L_\ell$ in S^3 framing ± 1

$$S^3_L \cong S^3_{L'} \Leftrightarrow L \sim L' \text{ Fenn–Rourke moves}$$

Chern–Simons path integral (Witten) and quantum groups at roots of 1 (Reshetikhin–Turaev)

$$\tau(M) : \mathcal{Z} \to \mathbb{C}, \quad \tau_\zeta(M)$$

and Ohtsuki series

$$\tau^O(M) = 1 + \sum_{n=1}^{\infty} \lambda_n(M)(q-1)^n$$

Unified view (Habiro): $J_M(q) = J_L(q)$

$$J_M(q) \in \widehat{\mathbb{Z}[q]}$$

$$ev_\zeta(J_M(q)) = \tau_\zeta(M)$$

$$\tau_1(J_M(q)) = \tau^O(M)$$

Integral homology 3-spheres

$\mathbb{Z}hs =$ free ab group generated by orientation-preserving homeomorphism classes of integral homology 3-spheres
Ring with product $M_1 \# M_2$ connected sum

$$J_{M_1\#M_2}(q) = J_{M_1}(q) J_{M_2}(q), \quad J_{S^3}(q) = 1$$

$$J_{-M}(q) = J_M(q^{-1})$$

$\Rightarrow$ WRT ring homomorphism

$$J : \mathbb{Z}hs \to \widehat{\mathbb{Z}[q]}$$

Ohtsuki filtration

$$\mathbb{Z}hs = F_0 \supset F_1 \supset \cdots F_k \supset \cdots$$

F_k $\mathbb{Z}$-submodule spanned by

$$[M, L_1, \ldots, L_k] = \sum_{L' \subset \{L_1, \ldots, L_k\}} (-1)^{|L'|} M_{L'}$$

$L_i =$ alg split links ± 1-framed

Habiro conjecture

$$J : \widehat{\mathbb{Z}hs} \to \widehat{\mathbb{Z}[q]}$$

$\widehat{\mathbb{Z}hs} = \varprojlim_d \mathbb{Z}hs/F_d$ with $d : \mathbb{N} \to \mathbb{Z}_{\geq 0}$
$d(n)$-components of link have framing $\pm n$

Integral homology spheres and $\mathbb{F}_1$

$X_{\mathbb{Z}hs}(R) := \{\phi : \mathbb{Z}hs \to R \mid \exists \tilde{\phi} : \widehat{\mathbb{Z}[q]} \to R,\ \phi = \tilde{\phi} \circ J\}$
$J : \mathbb{Z}hs \to \widehat{\mathbb{Z}[q]}$ WRT invariant;
$X_{\mathbb{Z}hs}(R) =$ set of coarser R-valued invariants

$\mathcal{A}_X :=$ C^*-algebra completion of $\mathbb{Z}hs \otimes \mathbb{C}$ in norm induced by representations $E_{\zeta,f}$ of $\widehat{\mathbb{Z}[q]}$

$\Rightarrow X_{\mathbb{Z}hs}$ gadget over $\mathbb{F}_1$

Using that $\sigma \circ \phi : \mathbb{Z}hs \to \mathbb{C}$ factors through ev_ζ

Question: Using Habiro conjecture $X_{\widehat{\mathbb{Z}hs}}$ inductive limit of affine varieties over $\mathbb{F}_1$?

Question: Semigroup action on $\mathbb{Z}hs$?

$M = S^3_{L,m}$ with $L = L_1 \cup \cdots \cup L_k$ alg split w/ framing $\pm 1/m_i$ of L_i, $m = (m_1, \ldots, m_k)$

$$S^3_{L,m} \mapsto S^3_{L,mn}$$

is a semigroup action on $\mathbb{Z}hs$

$$S^3_{L,m} \# S^3_{L',m'} = S^3_{L \cup L',(m,m')}$$

But want $\sigma_n(M)$ homology sphere such that

$$\alpha_{\sigma_n(M)}(q) = \sigma_n(\alpha_M(q)) \in \widehat{\mathbb{Z}[q]}$$

for some invariant $\alpha : \mathbb{Z}hs \to \widehat{\mathbb{Z}[q]}$

Notice that for $M' = 1/m$-surgery on K in M

$$J_{M'}(q) = J_M(q) \mod (q^{2m} - 1)$$