ANYON NETWORKS FROM GEOMETRIC MODELS OF MATTER

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Abstract. This paper, completed in its present form by the second author after the first author passed away in 2019, describes an intended continuation of the previous joint work on anyons in geometric models of matter. This part outlines a construction of anyon tensor networks based on four-dimensional orbifold geometries and braid representations associated to surface-braids defined by multisections of the orbifold normal bundle of the surface of orbifold points.

1. Introduction: Geometric Models of Matter

This paper was planned as a continuation of our previous work [10]. The present version was completed by the second author after the first author passed away in January 2019. Because more than a year went by before the second author was able to put together an overview of that unfinished work, this resulting paper might not be faithful to how the first author would have originally envisioned it, but hopefully the main spirit of it and the general motivation is maintained. Since the work was unfinished, some of the discussions will appear here in the form of questions or sketches of ideas, rather than being fully elaborated. The current organization of the material presented in this paper follows closely the lecture given by the second author at the Newton Institute conference “Sir Michael Atiyah: Forays into Physics” in October 2019, organized by Maciej Dunajski and Nick Manton.

1.1. **Particle-like geometries.** Geometric models of matter were introduced by the first author and collaborators, [3], [5], [7], [8], [9], [18], [25], as a special class of 4-dimensional Riemannian manifolds that have “particle-like” properties. These manifolds include well known examples of gravitational instantons (self-dual Einstein metrics) such as the Taub-NUT and Atiyah–Hitchin manifolds, as well as gravitational instantons of type A_k and D_k. These manifolds can be seen as behaving like systems of charged particles, with quantum numbers specified by the geometry. These geometric models of matter can be thought of as a geometrization of the Skyrmions models of particles.

We will comment briefly on the main aspects of these geometric models. Skyrmions were originally developed to describe topologically stable field configurations in non-linear sigma models, in particular topological solitons of the pion field, [41]. The theory illustrates how quantum numbers from nuclear physics acquire geometric and topological meaning. For example, the baryon number B is an integer-valued topological charge given by the degree of a map $U : \mathbb{R}^3 \to SU(2)$. We refer the reader to [33] for a general discussion of the topology and geometry of skyrmion models.

The development of the skyrmion models also provided an initial source of evidence for the idea that certain geometries have "particle-like" properties. To give an example of this behavior, one can compare the fact that, in the case of magnetic skyrmions, the magnetic field lines can be topologically knotted in highly non-trivial ways with the fact that certain gravitational instantons with anti-self-dual Kerr-Schild metrics can have topologically knotted zero locus of their Weyl tensor, [40].

In [9], Atiyah, Manton and Schroers interpreted the signature $\tau(M)$ as playing the role of a baryon number, while an analog of the electric charge is provided by the self-intersection number of the surface at infinity, while in [7], for a different class of manifolds, Atiyah and Manton found an analog of baryon and lepton numbers in terms of both signature and Euler characteristic, with the signature measuring the difference between number of protons and number of neutrons. Among the striking phenomena observed by following this idea of particle-like geometries, is an analog of the "valley of stability" of atomic nuclei that manifests itself in the relation between the Chern class c_2 and the signature τ of algebraic surfaces (with related invariants such as c_1^2) subject to constraints such as the Bogomolov–Miyaoka–Yau inequality and the Noether inequality, see [7]. Similarly, in [3] the first author identified algebraic surfaces whose quantum numbers in the above sense match those of Helium.

These various geometries are referred to as "geometric models of matter" in the series of papers [3], [5], [7], [8], [9], [10], [18], [25]. A general principle in the description of the particle-like properties of these geometries is the idea that their "quantum numbers" should be topological quantities that are the result of the computation of an index theorem.

We discuss briefly some other aspects of these geometric models. The first involves passing from a static (4-dimensional) to a dynamic model, which is done in [5] by considering $(4+1)$-dimensional Ricci-flat spacetimes describing evolving Taub-NUT geometries. Indeed, the Taub-NUT geometry can be seen as a 4-dimensional section of the 5-dim Sorkin solution of the Kaluza–Klein monopole equations. More generally, the Campbell–Magaard embedding theorem shows that an arbitrary analytic Riemannian manifold M of dimension $\dim M = n$ can be locally embedded in a Ricci-flat Riemannian manifold of dimension $n+1$, although the embedding may only exist locally.

Another property that we need to discuss in view of the rest of this paper is forming composite systems. A way to consider composite systems is by allowing merging operations on the geometries that have an effect on their quantum numbers analogous to what one expects in composite systems of particles or quasi-particles.

Such an operation is provided by connected sums. While this composition of manifolds is a very natural and simple operation, it suggests that insisting that the particle-like geometries should be gravitational instantons (namely both Einstein and with (anti)self-dual Weyl tensor) is too restrictive. For example, there are well known obstructions to the existence of (anti)self-dual metrics on a connected sum of two (anti)self-dual 4-manifolds, which can be formulated in terms of a twistor space argument, [17].

More precisely, the twistor space $Z = Z(M)$ of a self-dual 4-manifold is a 3-dimensional complex manifold that fibers over M with $\mathbb{CP}^1$ fibers. A singular complex 3-manifold $\tilde{Z} = \tilde{Z}_1 \cup_{E_1 \simeq E_2} \tilde{Z}_2$ is obtained by blowing up the twistor spaces $Z_i = Z(M_i)$ along a $\mathbb{CP}^1$ fiber and gluing together the exceptional divisors. Donaldson and Friedman showed in [17] that if there is a smooth Kodaira–Spencer–Kuranishi deformation Z of $\tilde{Z}$, then Z is in fact the twistor space $Z(M)$ of a self-dual structure on the connected sum $M = M_1 \# M_2$. Obstructions to the deformation of $\tilde{Z}$ determine obstructions to the existence of a self-dual metric on the connected sum manifold. The construction of obstructions to the existence of self-dual metric on the connected sum was extended to the orbifold case or edge-cone metrics in [28], [31], [32].

Thus, while one can assume that the elementary building blocks of geometric models of matter that exhibit particle-like properties would be gravitational instantons, it is reasonable to relax these assumptions, regarding both the existence of an (anti)self-dual and an Einstein metric structure, on the composite systems obtained from such building blocks in general.

1.2. **Orbifold metrics and quasi-particle geometries.** In [10] we proposed an extension of the geometric models of matter approach, by showing that one can obtain geometries that behave like system of quasi-particles with fractional quantum numbers and with associated anyon representations arising from surface braids wrapped around 2-dimensional orbifold singularities in 4-dimensional geometries endowed with edge-cone metrics.

The main idea in this approach is that one wants to obtain two main properties of quasi-particles: fractional quantum numbers and anyon statistics. To this purpose, one considers geometries given by pairs (M, Σ) with M a smooth compact 4-dimensional manifold and Σ a smoothly embedded compact 2-dimensional surface. On these geometries one considers metrics that are orbifold edge-cone metrics on M with Σ as the set of orbifold points.

The use of orbifold geometries to obtain fractional quantum numbers was already used in the context of fractional quantum Hall effect models in [34], [35], [36], [37], and other related work by Mathai, the second author, and collaborators.

As in the case of this previous work, one obtains the fractional quantum numbers from Kawasaki index theorem for orbifolds, [27].

The braid representations that give rise to anyons, on the other hand, are obtained from surface braids determined by multisections of the orbifold normal bundle $\mathcal{N}(\Sigma)$ of Σ in M.

We review this construction more explicitly, since we will need it in the rest of the paper. We first review some results of Atiyah and LeBrun on edge-cone metrics from [6] and then we review the construction of anyon representations from our previous paper [10].

An edge-cone metric on (M, Σ) with cone angle $2\pi\beta$, $\beta \in \mathbb{R}^*_+$ has the form

$$g = d\rho^2 + \beta^2\rho^2(d\theta + u_j dx^j)^2 + w_{ij}dx^i dx^j + \rho^{1+\epsilon}h$$

where h is a symmetric tensor with continuous derivatives of all orders for vector fields with vanishing normal component along Σ. Thus, the geometry (M,Σ) is modelled on a 2-dimensional cone in directions transversal to Σ, and is smooth in the directions parallel to Σ.

For example, on the sphere (S^n, S^{n-2}), for $n \geq 3$, the standard round metric takes the form $dr^2 + \sin^2 r d\theta^2 + \cos^2 r\, g_{S^{n-2}}$, while a family of egde-cone metrics with angle $2\pi\beta$ can be obtained as $dr^2 + \beta^2 \sin^2 r d\theta^2 + \cos^2 r\, g_{S^{n-2}}$, see [1].

We are especially interested here in some simple cases of the Atiyah–Le Brun manifolds. The first case is (S^4, S^2) with edge-cone metric of angle $2\pi/\nu$. The complement $S^4 \smallsetminus S^2$ is conformally equivalent to $\mathbb{H}^3 \times S^1$ with the real hyperbolic space $\mathbb{H}^3$. Under this mapping the standard round metric on S^4 becomes $\mathrm{sech}^2\delta\ (h + d\theta^2)$ with h the hyperbolic metric and with $\delta : \mathbb{H}^3 \to \mathbb{R}$ the distance from a point. The tensor $ds^2 = \mathrm{sech}^2\delta\ (h + \beta^2 d\theta^2)$ then determines a family of edge-cone metrics with cone angle $2\pi\beta$.

The other kind of Atiyah–Le Brun manifolds we consider here is given by connected sums of projective planes $(\#^n\mathbb{CP}^2, \Sigma)$ with $\Sigma = \#^n\mathbb{CP}^1 \simeq S^2$. In this case (see [6]) for an open set $\mathcal{U} \subset \mathbb{H}^3$ and $V : \mathcal{U} \to \mathbb{R}^+$ a harmonic map for a metric h, consider the closed form $\star dV$. This has class $[\star dV/2\pi] = c_1(\mathcal{P}) \in H^2(\mathcal{U},\mathbb{R})$. Consider a line bundle $\mathcal{P}$ with curvature $d\theta = \star dV$, and a Riemannian metric on the total space of $\mathcal{P}$ of the form $g_0 = Vh + V^{-1}\theta^2$, with potential $V = \beta^{-1} + \sum_{i=1}^n G_{p_i}$ for points $p_i \in \mathbb{H}^3$ and Green functions G_{p_i}. This defines an edge-cone metrics $g = \beta(\mathrm{sech}^2\delta) g_0$. The metric completion of g on $\mathcal{P}$ gives $\#^n\mathbb{CP}^2$ with edge-cone angle $2\pi\beta$ along Σ. This construction of [6] gives a generalization of Abreu family of edge-cone metrics on $(\mathbb{CP}^2, \mathbb{CP}^1)$.

We are especially interested in the case of orbifolds, namely compact 4-dimensional manifolds M with an atlas of local uniformizing charts U_α homeomorphic to quotients $U_\alpha \simeq V_\alpha/G_\alpha$ of open sets $V_\alpha \subset \mathbb{R}^4$ by finite groups G_α. The manifold splits into two loci, $M = M_{sing} \cup M_{reg}$, of singular (orbifold) points and regular points, with $M_{sing} = \Sigma$ an embedded surface.

A good orbifold is a global quotient $M = X/G$ of a smooth 4-manifold X by a finite group G. The best case is when our geometries are good orbifolds and near Σ the local charts look like $\mathbb{C}^2/G_\nu$ with $G_\nu = \mathbb{Z}/\nu\mathbb{Z}$ and action $(w,\zeta) \mapsto (w, e^{2\pi i/\nu}\zeta)$. An edge-cone metrics with $\beta = 1/\nu$ is represented in local chart as a $\mathbb{Z}/\nu\mathbb{Z}$-invariant metric.

Atiyah and Le Brun showed in [6] that the orbifold Euler characteristics and an orbifold signature can be computed as an orbifold index theorem

$$\chi_{orb}(M) = \frac{1}{8\pi^2}\int_M \left(|W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dv(g)$$
$$= \chi(M) - (1 - \frac{1}{\nu})\chi(\Sigma)\,;$$
$$\tau_{orb}(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W_-|^2 \right) dv(g) = \tau(M) - \frac{1}{3}(1 - \frac{1}{\nu^2})[\Sigma]^2,$$

with $[\Sigma]^2$ the self-intersection number, that is, the Euler number of normal bundle of Σ in M, and with W the Weyl tensor, $W^\pm$ the self-dual and anti-self-dual parts, E the traceless part of Ricci tensor, and R the scalar curvature. These quantities $\chi_{orb}(M)$ and $\tau_{orb}(M)$ represent fractional quantum numbers for the orbifold (M, Σ), viewed as modeling a system of quasi-particles. The surface Σ of orbifold points plays the role of surface at infinity that contributes the electric charge to the matter content in the Atiyah–Manton–Schroers model of [9].

The normal bundle $\mathcal{N}(\Sigma)$ of the inclusion of Σ in M is an orbifold vector bundle. The fibers of $\mathcal{N}(\Sigma)$ are quotients $\mathbb{R}^2/G_\nu$ where $G_\nu = \mathbb{Z}/\nu\mathbb{Z}$ is the stabilizer of Σ. Thus, the role of self-intersection number in [9] is replaced here by the orbifold Euler number $\chi_{orb}(\mathcal{N}(\Sigma))$ of the normal bundle $\mathcal{N}(\Sigma)$. This assigns a fractional electric charge to the system of quasi-particles.

Again one sees in this case of edge-cone metrics that some of the properties of the basic building blocks are not always preserved under forming composite systems. For instance, there are possible obstructions to the Einstein condition, which are either topological obstructions or differentiable obstructions. Atiyah and LeBrun identified in [6] some topological obstructions through inequalities

$$2\chi(M) \pm 3\tau(M) \geq (1 - \frac{1}{\nu})(2\chi(\Sigma) \pm (1 + \frac{1}{\nu})[\Sigma]^2)$$

which need to hold for (M, Σ) to admit an Einstein edge-cone metric of cone angle $2\pi/\nu$. Differentiable obstructions also exist and have been identified by Le Brun using Seiberg–Witten theory, [30]. This gauge-theoretic approach shows, for instance, that if there is a symplectic form ω on M with Σ a symplectic submanifold with $(c_1(M) - (1 - 1/\nu)[\Sigma]) \cdot [\omega] < 0$ then for any $\ell \geq (c_1(M) - (1 - 1/\nu)[\Sigma])^2/3$ the pair (M', Σ) with $M' = M \#^\ell \overline{\mathbb{CP}^2}$ has no Einstein edge-cone metric. This shows that it is reasonable to expect that both the (anti)self-dual and the Einstein conditions may have to be relaxed for composite systems, even when they hold for all their elementary constituents.

1.3. **Anyons.** Bosons and fermions satisfy the statistics $|\psi_1\psi_2\rangle = \pm|\psi_2\psi_1\rangle$. Abelian anyons, in contrast, allow for a more general phase factor $|\psi_1\psi_2\rangle = e^{i\theta}|\psi_2\psi_1\rangle$. The more interesting case of nonabelian anyons involves braid representations, where the wave functions for a permuted order of particles are related by a linear map that correspond, through a representation of the braid group, to a braid effecting the reordering of the particles.

Anyons have found important applications in topological quantum computing, [20]. Unitary braid representations that span densely $SU(2^N)$ are universal for quantum computing, in the sense that they approximate arbitrary quantum gates for systems of N-qbits.

Anyons are systems of quasi-particles that are necessarily 2-dimensional. The reason for this constraint on dimensionality lies in the fact that unitary representations of braid groups $B_n(X)$ that are not reducible to representations of the symmetric group S_n (that is, to fermions and bosons) and are not reducible to wreath products of

$\pi_1(X)$ and S_n (generalized parastatistics) can happen only when X is a 2-dimensional surface.

Our previous work [10] considered the question of whether anyon systems of quasi-particles can be realized within the 4-dimensional geometric models of matter and whether the resulting anyon systems can be universal for quantum computing.

The construction of anyon systems and braid representations uses the 2-dimensional surface of orbifold points inside the 4-dimensional M. For a geometry (M, Σ) with $\dim M = 4$ and $\Sigma \subset M$ with $\dim \Sigma = 2$ the locus of orbifold points, one cannot use the braid groups $B_n(M \smallsetminus \Sigma)$ or $B_n(M)$ or $B_n^{orb}(M, \Sigma)$ (orbifold braid groups) because all of these would only give generalized parastatistics. Indeed, for any X with $\dim X \geq 3$, if X is simply connected, then the groups $B_n(X)$ are symmetric groups hence one only obtains fermions, bosons, and parastatistics, while if $\pi_1(X) \neq 1$, the braid groups are wreath products $B_n(X) = \pi_1(X)^n \rtimes S_n$, so that one obtains generalized parastatistics.

However, the braid groups $B_n(\Sigma)$ can give rise to anyon representations. In the case of a disc D^2 the Artin braid group has the explicit presentation

$$B_n(D^2) = B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{c} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle$$

In the case of a Riemann surface Σ there is also a similar explicit form of the presentation of the braid groups $B_n(\Sigma)$, which is due to Birman, [13]. In particular, these groups $B_n(\Sigma)$ are not wreath products, hence their representations do not fall back to the case of generalized parastatistics and one obtains genuine anyons.

To see how the anyons arise in terms of the geometry (M, Σ), one can consider a larger class of topological objects, the surface braids introduced by Viro and Kamada, [26]. A surface m-braid is obtained by considering a smooth 2-dimensional S, smoothly embedded in the product $D^2 \times D^2$ with the second projection $P_2 : D^2 \times D^2 \to D^2$ restricting to S as an m-fold branched cover $P : S \to D^2$. The preimage $\beta := P_2^{-1}(\partial D^2) \cap S \subset D^2 \times S^1$ is an ordinary closed m-braid. One can, for simplicitly, assume that β is the trivial braid. Let $b(S) \subset D^2$ be the set of branch points of the m-fold brached cover $P : S \to D^2$. The fundamental group $\pi_1(D^2 \smallsetminus b(S))$ maps to the ordinary braid group,

$$\rho_S : \pi_1(D^2 \smallsetminus b(S)) \to \pi_1(\mathrm{Conf}_m(D^2)) = B_m(D^2)\,.$$

This is obtained by taking paths $\gamma(t)$ in $D^2 \smallsetminus b(S)$ with

$$\rho_S(\gamma)(t) := P_1(S \cap P_2^{-1}(\gamma(t)))$$

with $P_i : D^2 \times D^2 \to D^2$ the projections, and view it as a path in $\mathrm{Conf}_m(D^2)$. The same procedure can be adapted to closed surface braids, with S smoothly embedded in $D^2 \times S^2$ with $P = P_2|_S : S \to S^2$ an m-fold branched covering, and to more general versions with a fixed surface Σ (possibly of genus $g(\Sigma) > 0$) and S smoothly embedded in $D^2 \times \Sigma$ with projection $P = P_2|_S : S \to \Sigma$ an m-fold branched covering.

The associated braid representation is obtained as $\rho_S(\gamma)(t) := P_1(S \cap P_2^{-1}(\gamma(t))$,

$$\rho_S : \pi_1(\Sigma \smallsetminus b(S)) \to \pi_1(\mathrm{Conf}_m(D^2)) = B_m(D^2)\,.$$

Similarly, we can consider the case where $\mathcal{F}$ is a disc-bundle over a closed surface Σ and S is smoothly embedded in $\mathcal{F}$ with $\pi : \mathcal{F} \to \Sigma$ restricting to m-fold branched cover $\pi|_S : S \to \Sigma$.

Consider the case of an orbifold geometry (M, Σ) with $\dim M = 4$ and $\dim \Sigma = 2$, where we assume M is a good orbifold $M = X/G$, for some finite group G. For simplicity we only look at the case where Σ is connected and $G = \mathbb{Z}/\nu\mathbb{Z}$. The normal bundle $\mathcal{N}(\Sigma)$ of $\Sigma \hookrightarrow M$ is an orbifold bundle, which is orbifold covered by the normal bundle $\mathcal{N}(\tilde{\Sigma})$ of the preimage $\tilde{\Sigma}$ in X. A lift to $\mathcal{N}(\tilde{\Sigma})$ of a generic section σ of $\mathcal{N}(\Sigma)$ gives a ν-fold branched covering S of Σ branched at finitely many points $b(S)$. Multisections of $\mathcal{N}(\tilde{\Sigma})$ are maps to a symmetric product $\mathrm{Sym}^\ell(F) = F^\ell/S_\ell$ of the fiber F. This gives an ℓ-fold branched covers of $\tilde{\Sigma}$ branched at the intersections with the diagonals, for the unit normal bundle with $F = D^2$. Thus, taking mutisections of the unit normal bundle $\mathcal{N}_1(\Sigma)$ gives $\nu\ell$-fold branched coverings S of Σ, hence $\ell\nu$-surface braids.

This construction is used in [10] to show that the geometry (M, Σ) supports braid representations that are universal for quantum computing. Consider the manifold (S^4, S^2) with the $2\pi/\nu$ edge-cone metric. Multisections of $\mathcal{N}_1(S^2)$ give $\nu\ell$-surface braids S, which are closed surface braids in $D^2 \times S^2$, and an associated braid representation $\rho_S : \pi_1(S^2 \smallsetminus b(S)) \to B_{\nu\ell}(D^2)$. Given a closed surface braid S obtained from a multisection of $\mathcal{N}_1(S^2)$ and a branch point $x_0 \in b(S)$, take a disc $D^2_b \subset S^2$ that is the complement of a small neighborhood of x_0. Restriction of the branched cover $P : S \to S^2$ to $D^2_b \subset S^2$ is also a $\nu\ell$-fold branched cover $P : \hat{S} \to D^2_b$ and $\hat{S} \subset D^2_f \times D^2_b$ is a surface braid, so $\hat{S} \cap (D^2_f \times \partial D^2_b)$ is an ordinary closed braid β, in general non-trivial in $B_{\nu\ell} = B_{\nu\ell}(D^2)$.

A braid system $(\beta_1, \dots, \beta_n)$ of $\hat{S}$ is the image under the braid representation $\rho_{\hat{S}} : \pi_1(D^2_b \smallsetminus b(\hat{S})) \to B_{\nu\ell}(D^2)$ of a set of generators $\gamma_1, \dots, \gamma_n$ of $\pi_1(D^2_b \smallsetminus b(\hat{S}))$. There is a characterization of braid systems such that $\hat{S} \cap (D^2_f \times \partial D^2_b)$ is a fixed closed braid $\beta \in B_{\nu\ell}(D^2)$: they are given by the n-tuples $(\beta_1, \dots, \beta_n) \in B_{\nu\ell}(D^2)^n$ such that each β_k is conjugate of a standard generator σ_i or σ_i^{-1} of the braid group, with $\beta_1 \cdots \beta_n = \beta$ in $B_{\nu\ell}(D^2)$. Equivalent braided surfaces (related by a fiber preserving diffeomorphism of $D^2_f \times D^2_b$ relative to $D^2_f \times \partial D^2_b$) correspond to braid systems $(\beta_1, \dots, \beta_n)$ related by Hurwitz action of B_n on $B^n_{\nu\ell}$

$$\sigma_i : (\beta_1, \dots, \beta_n) \mapsto (\beta_1, \dots, \beta_{i-1}, \beta_i \beta_{i+1} \beta_i^{-1}, \beta_i, \beta_{i+2}, \dots, \beta_n).$$

For $n = \nu\ell - 1$, consider surface braid with $n = \# b(\hat{S})$ such that braid system is standard set of generators $(\sigma_1, \dots, \sigma_n)$ of $B_{\nu\ell}(D^2)$. For such choice of $\hat{S}$ the braid representation maps to all of $B_{\nu\ell}(D^2)$. This last property can be used to show that the anyons obtained in this way can be universal for quantum computing.

The main examples of systems of anyons that are universal for quantum computing include Fibonacci anyons, the Jones representations of $B_n(D^2)$, and systems derived from TQFT, in particular from Chern–Simons theory at 5-th root of unity. A class of universal anyons for quantum computing is constructed in [21] using the topological modular functor of a TQFT in the sense of [4], which assigns to a 2-dimensional surface Σ with marked points a complex (hermitian) vector space $V(\Sigma)$ and to diffeomorphisms assigns (projective) unitary maps on $V(\Sigma)$. Consider the case where $\Sigma = (D^2, 3\ell)$ is a disc with 3ℓ marked points. Let $S_\ell = (\mathbb{C}^2)^{\otimes \ell}$ be the state space of ℓ-qbits. Construct a map $S_\ell \hookrightarrow V(D^2, 3\ell)$, so that the embedding intertwines the action of $B_{3\ell}$ on $V(D^2, 3\ell)$ by diffeomorphisms of D^2 preserving the set of marked points and the action of unitary operators on S_ℓ. In the case of a single qbit, the B_3 action on $V(D^2, 3) = \mathbb{C}^2$ gives the 1-qbit quantum gates. The 2-qbit CNOT gate is obtained via an approximation algorithm. One then uses the fact that arbitary gates in $SU(2^N)$ can be decomposed into tensor product of 1-qbit gates and CNOTs, to obtain that the system constructed in this way approximates aribitrarily well any gates in $SU(2^N)$.

Thus, we showed in [10] that the fact that the surface braids constructed using the orbifold geometry (M, Σ), in a simple case like (S^4, S^2) with the edge-cone metric of angle $2\pi/\nu$, give a braid representation that surjectively maps to all of $B_{\nu\ell}(D^2)$, hence they allow for the construction of a universal system of this general form. We will discuss this TQFT approach more in detail in the next section.

2. Networks of quasi-particle geometries

The purpose of the present paper is to show that the geometric models of quasi-particles constructed in our previous work [10] and recalled in the previous section can be combined to form anyon tensor networks, and to discuss the computational properties of such network configurations.

2.1. **Anyon tensor networks.**

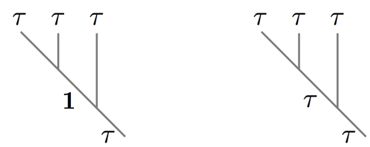
In recent years tensor networks were developed as an important computational tool in theoretical physics and quantum information theory. Combinatorially, they consist of a finite or countable collection of tensors connected by leg contractions. One denotes by $T_{i_1,\ldots,i_k}$ a tensor with k-legs, which visually one can represent with a node with k dangling half-edges, where index contraction corresponds to glueing legs together. This formalism allows for a graphical calculus and diagrammatic methods originally introduced by Roger Penrose for carrying out such tensor calculations. Quantum circuits can be described in terms of tensor networks and this provides a useful method for computing entangled quantum states. The tensor networks point of view has been especially useful recently in modelling a discretized version of the AdS/CFT holographic bulk/boundary correspondence, where the bulk space is discretized as a tensor network. The bulk tensor network computes a holographic quantum state at the boundary and this is used to address the Ryu–Takayanagi conjecture, which expresses the entanglement entropy of the boundary state in terms of the geometry (minimal curves and surfaces) in the bulk, see [38].

We are interested here in particular in the case of anyonic tensor networks. For a discussion of this general formalism we refer the reader to [11], [12], [14].

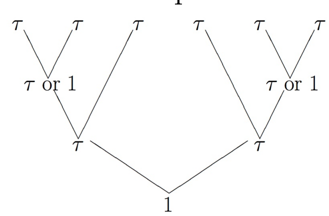
Before introducing the graphical description of anyon tensor networks, we recall the fusion and splitting trees of anyons. We do this using a simple example: the Fibonacci anyons, [42]. These are an anyon system with two types of particles, denoted by **1** and τ. They satisfy fusion rules

$$\mathbf{1} \otimes \tau = \tau, \quad \tau \otimes \mathbf{1} = \tau, \quad \tau \otimes \tau = \mathbf{1} \oplus \tau \,.$$

These are represented by fusion trees

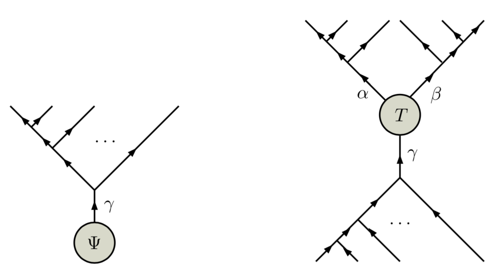


The ground state degeneracies for n τ-anyons can be seen by counting the number of fusion paths: one obtains the Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$, hence the name for this anyon model. The application of this model to quantum computing uses the fact that fusion trees with three τ-anyons simulate a single qbit, while for example one can use six τ-anyons to simulate two qbits as shown in the figure



Braiding these anyons simulates unitary transformation on this simulated space of qubits. The fusion and braiding of Fibonacci anyons is described by the F and R matrices. These operations satisfy an associativity condition expressed through pentagon and hexagon relations. In the case of Fibonacci anyons one can solve these explicitly for F and R. The resulting $R = \rho(\sigma_1)$ and $FRF^{-1} = \rho(\sigma_2)$ give the associated braid representation.

In anyon tensor networks, states $|\Psi\rangle$ are weighted superposition of anyon fusion/splitting trees with assigned anyon charge at the root, and operators are anyonic tensors T. Graphically these can be illustrated as



Anyonic matrix operators produce anyonic matrix product states via the anyon tensor network.

2.2. **TQFTs with corners.** Before continuing our discussion of anyon tensor networks, we need to recall briefly how one can extend the usual functorial setting of topological quantum field theory [4] to include the possibility of gluing 3-manifold along codimension zero parts of their boundary that in turn have boundaries. The formalism that takes care of this generalization was introduced in [23], [43]. We refer the reader to these references for more details and we only recall briefly the main idea and what we will need in our construction in §2.4 below.

In this setting, a 2-dimensional surface Σ is endowed with a DAP-decomposition (discs, annuli, and pairs of pants) that splits it into a finite collection of elementary surfaces of these three kinds, along with parameterization and labeling of the boundary curves. DAP-decompositions that are obtained from one another through a series of elementary moves (see Figure 2.2 of [23]) are equivalent. An extended surface is a triple $(\Sigma, \mathbb{D}, \ell)$ of a surface, a DAP-decomposition, and a labeling of the boundary components. An extended 3-manifold is a triple $(X, \mathbb{D}, n)$ of a 3-manifold, a DAP-decomposition $\mathbb{D}$ of its boundary ∂X and an integer n. We refer the reader to [23], [43] for the appropriate definition of extended morphisms and the geometric meaning of these extended data.

TQFTs on these data consist of a modular functor mapping the category of extended surfaces and morphisms to the category of finite dimensional complex vector spaces and an assignment to each extended 3-manifold $(X, \mathbb{D}, n)$ of a vector $Z(X, \mathbb{D}, n)$ (partition function) in the vector space $V(\partial X, \mathbb{D})$ of its boundary extended surface. The vector space assigned to an extended surface has the property that it splits as a tensor product over disjoint unions and that, if two extended surfaces are glued along a boundary component given by a curve C the vector spaces satisfy

$$V(\Sigma \cup_C \Sigma', \mathbb{D} \cup_C \mathbb{D}', (\ell, \ell')) = \bigoplus_a V(\Sigma, \mathbb{D}, (\ell, a)) \otimes V(\Sigma', \mathbb{D}', (\ell', a))$$

where ℓ, ℓ', a are the labels of the boundary components in $\Sigma \smallsetminus C$, $\Sigma' \smallsetminus C$ and C. If $(\Sigma, \mathbb{D}_1, \ell)$ and $(\Sigma', \mathbb{D}'_1, \ell')$ are two disjoint extended manifold in the boundary $\partial(M, \mathbb{D}, m)$ of an extended 3-manifold, one considers the vector space

$$\bigoplus_{\ell,\ell'} V(\Sigma, \mathbb{D}_1, \ell) \otimes V(\Sigma', \mathbb{D}'_1, \ell') \otimes V(\partial(M, \mathbb{D}, n) \smallsetminus ((\Sigma, \mathbb{D}_1, \ell) \cup (\Sigma', \mathbb{D}'_1, \ell'))$$

where the partition function can be written in the form (see Section 2 of [43] and Section 2 of [23]) $Z(M, \mathbb{D}, m) = \oplus_{\ell,\ell'} \sum_j \alpha_\ell^{(j)} \otimes \beta_{\ell'}^{(j)} \otimes \gamma_{\ell,\ell'}^{(j)}$, and one obtains for the glued manifold

$$Z((M, \mathbb{D}, m)_\varphi) = \oplus_\ell \sum_j \langle V(\varphi) \alpha_\ell^{(j)}, \beta_\ell^{(j)} \rangle \gamma_{\ell,\ell}^{(j)}. \tag{2.1}$$

We refer the reader to [23], [43] for a detailed discussion of the axioms of these TQFTs and their properties. In the following, to simplify notation, we will use a shorthand notation for the partition function $Z((M, \mathbb{D}, m)_\varphi)$ under gluing, by simply writing

$$Z((M, \mathbb{D}, m)_\varphi) = \langle V(\varphi) Z_{M,\Sigma}, Z_{M,\Sigma'} \rangle \, Z_{M,\partial M^0}$$

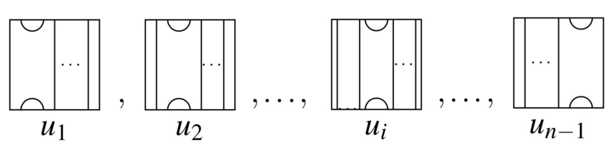
with $\partial M^0 = \partial M \smallsetminus (\Sigma \cup \Sigma')$ and we suppress the explicit notation for the DAP-decompositions and labels and the sums.

A similar construction of TQFTs with corners is possible using a category of extended surfaces with marked points, [24] (see also [2]). The case that is particularly interesting for us corresponds to the TQFTs based on Chern-Simons at roots of unity, used in [19], [21] to obtain a system universal for quantum computation. We have already recalled briefly the properties of those TQFTs in Section 1. While the quantum computing applications focus on the purely 2-dimensional part (the topological modular functor) based on surfaces with marked points, we will be interested also in the 3-manifold part of the TQFT. The relation of these TQFTs with corner and the Kaufmann bracket is analyzed in [23].

2.3. **Braid systems, Hurwitz action, and Temperley–Lieb algebra.** Before discussing our proposed construction of anyon tensor networks from geometric models of matter, we also need to recall some preliminary facts about the Jones representation that we will be using in the next subsection.

As we recalled above, given a surface braid S and the corresponding braid representation $\rho_S : \pi_1(\Sigma \smallsetminus b(S)) \to B_m(D^2)$ (with $m = \nu\ell$ depending on the orbifold normal bundle $\mathcal{N}(\Sigma)$ and on the multisection defining S, as discussed in the previous section), the associated braid system $(\beta_1, \dots, \beta_n)$ is the image in $B_m(D^2)^n$ of a set of generators $\{\gamma_1, \dots, \gamma_n\}$ of $\pi_1(\Sigma \smallsetminus b(S))$. As we discussed in [10] and recalled in the previous section, the condition that $m = n + 1$ and that ρ_S is such that $(\beta_1, \dots, \beta_n)$ is the standard set of generators $(\sigma_1, \dots, \sigma_n)$ of B_{n+1} can be used to ensure that the anyon systems obtained by this construction are universal for quantum computing. We assume then that $m = n+1$ and that the braid system $(\beta_1, \dots, \beta_n)$ is the standard set of generators σ_i of B_{n+1}. Thus, $\pi_1(\Sigma \smallsetminus b(S))$ maps surjectively to B_m.

We then consider the algebra homomorphism from the group algebra $\mathbb{F}[B_n]$ to the Temperley–Lieb algebra $TL_n(A)$, given by the Kauffman bracket. Here $\mathbb{F}$ is the field $\mathbb{C}(A)$ of rational functions in the variable A. The Temperley–Lieb algebra is generated by diagrams u_i



with the product operation given by vertical stacking of diagrams and with relations expressing far commutativity, braid relations and Hecke relations, and multiplication by $d = -A^2 - A^{-2}$ when a loop is removed. The Kauffman bracket is defined on braid diagram by undoing the braid crossings and replacing them with a sum of Temperley–Lieb diagrams. The image of the standard generators σ_i of the braid group B_n under the Kauffman brackets gives a set of invertible generators of the Temperley–Lieb algebra of the form $\langle \sigma_i \rangle = g_i = A\,\mathrm{id} + A^{-1}\, u_i$. We refer the reader to [15] for a more detailed survey.

Consider then the Hurwitz action of the braid group B_n on the set of braid systems $(\beta_1, \ldots, \beta_n)$, defined by

$$\sigma_i : (\beta_1, \ldots, \beta_{i-1}, \beta_i, \beta_{i+1}, \beta_{i+2}, \ldots, \beta_n) \mapsto (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \beta_{i+1}^{-1}\beta_i\beta_{i+1}, \beta_{i+1}, \ldots, \beta_n),$$

which preserves the Coxeter element given by the product $\beta_1\beta_2\cdots\beta_n$. If the initial braid systems $(\beta_1, \ldots, \beta_n)$ consists of the standard braid generators of B_{n+1}, then the Hurwitz action of B_n on B_{n+1}^n determines under the mapping defined by the Kauffman bracket an action of B_n by automorphisms of the Temperley–Lieb algebra, which maps the set $(g_1, \ldots, g_n)$ of invertible generators $g_i = \langle \sigma_i \rangle$ according to the Hurwitz action above. The case of the Hurwitz orbit of other braid systems $(\beta_1, \ldots, \beta_n)$ that are not the standard braid generators of B_m is more subtle. A linear basis of the Temperley–Lieb algebra can be constructed from these braid systems with the Hurwitz action as in [16], [45]. For our purposes, we will restrict our attention to the case where the braid system is standard.

The generic Temperley–Lieb algebra $TL_n(A)$ (where A is not specialized to a value in $\mathbb{C}$) is isomorphic to a sum of matrix algebras $\oplus_a M_{m_{n,a}}(\mathbb{F})$, where the ranks $m_{n,a}$ correspond to the decompositions of the Catalan numbers as a sums of squares $C_n = \sum_{a=0}^n m_{n,a}^2$. In the case of a specialization of A at a root of unity, as in the Jones representations associated to Chern–Simons topological field theories of order k, mentioned in the previous section, the Temperley–Lieb–Jones algebra $TLJ_n(A)$, which is a quotient of $TL_n(A)$ is isomorphic to a sum of matrix algebras $\oplus_a M_{m_{n,a}}(\mathbb{C})$ and one obtains in this way corresponding representations of the braid group,

$$\rho_{m,k,a} : B_m \to TLJ_n(A) \to M_{m_{n,a}}(\mathbb{C}).$$

Again, we refer the reader to [15] for a detailed exposition.

Through this identification with matrix algebras, the Hurwitz action on the standars braid system determines an induced action of B_n on the algebra $\oplus_a M_{m_{n,a}}(\mathbb{C})$. When restricting to a given block $M_{m_{n,a}}(\mathbb{C})$ this is given by inner automorphisms: for $\gamma \in B_n$, we denote by $T_{m_{n,a}}(\gamma)$ (or simply T_γ when the choice of the matrix block is understood) the resulting elements in $M_{m_{n,a}}(\mathbb{C})$ implementing then.

2.4. **Anyon networks from geometric models of matter.** The main question we are interested in here is whether one can realize anyon tensor networks in geometric models of matter, generalizing the construction of anyon systems described in [10] and recalled above. The idea here is to have building blocks given by orbifold geometries that realize good anyon systems, such as the Atiyah-LeBrun manifolds, and combine them geometrically in a way that can give rise to a realization of an anyon tensor network. There are several questions related to this problem. For example, whether the geometry that would support an anyon network constructed in this way can be in a “ground state”, satisfying the gravitational instanton condition, or whether the topology of the network of anyon tensors is reflected in the topology of the resulting geometric model. We illustrate some of these questions in simple models in §2.5.

Suppose given an orbifold (M, Σ) as described in the previous section, with orbifold normal bunlde $\mathcal{N}(\Sigma)$ and surface-braid multisections S defining associated braid representations $\rho_S : \pi_1(\Sigma \smallsetminus b(S)) \to B_m(D^2)$ (where as described above, m depends on the geometry of $\mathcal{N}(\Sigma)$ and S). Consider a choice of discs $D^2_i \subset \Sigma$, $i = 1, \dots, N$ and the corresponding surface braids S_i obtained from S by restriction. Let $\{\gamma_{i,1}, \dots, \gamma_{i,n_i}\}$ denote a set of generators of the group $\pi_1(D^2_i \smallsetminus b(S_i))$ and $\{\beta_{i,1}, \dots, \beta_{i,n_i}\}$ the corresponding braid systems.

Let (M_i, Σ_i), $i = 1, \dots, N$ be a collection of orbifold geometries, with the property that there are discs $\tilde{D}^2_i \subset \Sigma_i$ and multisections $\tilde{S}_i$ of the unit orbifold normal bundles $\mathcal{N}_1(\Sigma_i)$ with the following properties.

1. The restriction to $\tilde{D}^2_i$ of the orbifold normal bundle $\mathcal{N}(\Sigma_i)$ is isomorphic, as orbifold bundle, to the restriction to D^2_i of $\mathcal{N}(\Sigma)$.
2. The restrictions $\hat{S}_i \subset D^2_f \times \tilde{D}^2_i$ of $\tilde{S}_i$ to the base disc $\tilde{D}^2_i$ and the restrictions S_i of S to the base disc $D^2_i \subset \Sigma$ have the same number of branch points $\#b(\hat{S}_i) = \#b(S_i)$.

Then by the first hypothesis we can perform a gluing of (M, Σ) and the (M_i, Σ_i) with a fiber preserving diffeomorphisms φ_i of $D^2_f \times D^2_i$ and $D^2_f \times \tilde{D}^2_i$. The second hypothesis ensures the fundamental groups are abstractly isomorphic, $\pi_1(\tilde{D}^2_i \smallsetminus b(\hat{S}_i)) \simeq \pi_1(D^2_i \smallsetminus b(S_i))$. With the identification given by the map φ_i, the corresponding braid systems $\{\beta_{i,1}, \dots, \beta_{i,n_i}\}$ are related by the Hurwitz action. We denote the resulting geometry as $(M, \Sigma)\#^N_{\varphi_i, i=1}(M_i, \Sigma_i)$, or simply $(M, \Sigma)\#^N_{i=1}(M_i, \Sigma_i)$ leaving the transformations φ_i implicit. The gluing of the orbifold normal bundles and the multisections determines a new surface braid and an associated braid representation.

The braid representation $\rho_S : \pi_1(\Sigma \smallsetminus b(S)) \to B_m(D^2)$ of a multisection S of the unit orbifold normal bundle $\mathcal{N}_1(\Sigma)$ in an orbifold geometry (M, Σ) determines a representation by unitary operators on the vector space $\mathcal{V}$ spanned by the anyon fusion trees. This is obtained through the Jones representations $\rho_{k,m,a} : \mathcal{B}_m(D^2) \to TLJ_n(A) \to \mathrm{End}(\mathcal{V}_{m,k,a})$, where k is the level of the theory and a is the total charge. The intermediate representation is through the Temperley–Lieb–Jones algebra, as we recalled in the previous subsection, with the map given by the Kauffman bracket, and with the identification of the Temperley–Lieb–Jones algebra with a sum of matrix algebras.

We can then proceed to construct anyon tensor networks in the following way. First observe that, given a finite set of points B on a 2-dimensional surface Σ, we can find an open set $U \subset \Sigma$ that is topologically homeomorphic to a disc and that contains the set B. For example, we can take an embedded tree in Σ with B as set of vertices and thicken it with a sufficiently small thickness $\epsilon > 0$ so that the boundary has no self-intersection. Given an orbifold geometry (M, Σ) and a surface-braid S given by a multisections of the orbifold normal bundle $\mathcal{N}(\Sigma)$, let $B = b(S)$ be the finite set of branch points of the branched cover $S \to \Sigma$. Let 2^B be the set of all subsets of B. For all choices of a subset $A \in 2^B$, we have a representation $\rho_A : \pi_1(D^2 \smallsetminus A) \to \mathcal{B}_m(D^2)$, where m is fixed and depends only on $\mathcal{N}(\Sigma)$ and S, with compatibility given by the morphisms $\pi_1(D^2 \smallsetminus A) \to \pi_1(D^2 \smallsetminus A')$ induced by the inclusions $A \subset A'$. In particular the generators of all the $\pi_1(D^2 \smallsetminus A)$ can be viewed as a subset of the generators of $\pi_1(D^2 \smallsetminus b(S))$. Since we are considering a neighborhood of $b(S)$ that is homeomorphic to D^2 rather than the whole Σ, these groups are all free groups on a number of generators equal to the number of removed points.

Suppose given a finite graph G consisting of a set $V = V(G)$ of vertices, a set $E_{int} = E_{int}(G)$ of edges (or internal edges) and a set $E_{ext} = E_{ext}(G)$ of half-edges (or external edges). It will be convenient to think of an internal edge as a pair of half-edges that are matched under an involution and an external edge as a half-edge that is fixed by the involution, This equivalently describes G in terms of the set of vertices $V = V(G)$ and a set of half-edges or flags $F = F(G)$ with an involution. The valence $\mathrm{val}(v)$ of a vertex $v \in V$ is the number of both internal and external edges attached to v. We also consider a directed structure on G, so that edges and half edges carry an orientation. Each internal edge has a source and target vertex (possibly coincident) and each external edge has either a source or a target vertex. We correspondingly refer to edges and half-edges as incoming/outgoing at their target/source. We refer to such a graph G as the template of the network.

Assumptions: Consider then a collection (M_v, Σ_v) of orbifold geometries as above, parameterized by $v \in V$. These are chosen with the following properties:

(1) For all $v \in V$, the orbifold normal bundle $\mathcal{N}(\Sigma_v)$ comes with a choice of a multisection surface-braid S_v such that the set of branch points $b(S_v)$ of the branch cover $S_v \to \Sigma_v$ satisfies $\#b(S_v) = \ell \cdot \mathrm{val}(v)$, for some fixed $\ell \geq 2$, independent of the vertex.

(2) For $n_v = \mathrm{val}(v)$, let $\mathcal{P}_{n_v,\ell}(b(S_v))$ denote the set of partitions of $b(S_v)$ into a disjoint union

$$b(S_v) = B_{v,f_1} \sqcup \cdots \sqcup B_{v,f_{n_v}},$$

with $f_i \in F$ the flags attached to v, such that $\#B_{v,f_i} = \ell$ for all v, f_i. There is a partition $\mathbb{B} \in \mathcal{P}_{n_v,\ell}(b(S_v))$ with a corresponding choice of open neighborhoods homeomorphic to discs, $D^2_{v,f_i} \subset \Sigma_v$ with $B_{v,f_i} \subset D^2_{v,f_i}$, such that, if f_i attached to v and f'_i attached to v' are the two halves of an internal edge e_i, then the restrictions of the orbifold normal bundles to these discs are isomorphic

$$\varphi_{e_i} : \mathcal{N}(\Sigma_v)|_{D^2_{v,f_i}} \stackrel{\simeq}{\to} \mathcal{N}(\Sigma_{v'})|_{D^2_{v',f'_i}}. \tag{2.2}$$

(3) For all $v \in V$, the braid representations $\rho_{S_v} : \pi_1(\Sigma_v \smallsetminus b(S_v)) \to B_{m_v}(D^2)$ (with m_v determined by $\mathcal{N}(\Sigma_v)$ and S_v) take values in the same braid group $B_m(D^2)$ where $m = \ell + 1$, for ℓ as above.

(4) All the braid systems $(\beta_1, \ldots, \beta_\ell)$, given by the images in $B_m(D^2)$ of the generators of $\pi_1(D^2 \smallsetminus B_{v,f_i})$ are standard.

Construction: The procedure for constructing the anyon tensor network then goes as follows.

(1) Choose a partition of $b(S_v)$ into $n_v = \mathrm{val}(v)$ disjoint subsets as above, together with a choice of n_v open neighborhoods $D^2_{v,f_i} \subset \Sigma_v$ with $B_{v,f_i} \subset D^2_{v,f_i}$ with the properties specified above.

(2) For each pair of flags f_i, f_i' with $s(f_i) = v$ and $t(f_i') = v'$ that forms an edge e_i connecting v and v', we choose points $x_{v,f_i} \in D^2_{v,f_i} \smallsetminus b(S_v)$ and $x'_{v',f_i'} \in D^2_{v',f_i'} \smallsetminus b(S_{v'})$ where the connected sum is performed. As above let $(\tilde{M}_v, \tilde{\Sigma}_v)$ be a good covering of order ν of (M_v, Σ_v) with $\tilde{S}_v$ a multisection of $\mathcal{N}(\tilde{\Sigma}_v)$ corresponding to the ℓ-multisection S_v in $\mathcal{N}(\Sigma_v)$, so that $m = \nu\ell$ as before. The restriction of the unit normal bundle is $\mathcal{N}(\tilde{\Sigma}_v)|_{D^2_{v,f_i}} \simeq D^2_{v,f_i} \times D^2$. Above the point x_{v,f_i} we have m points in the fiber disc D^2 that correspond to $D^2 \cap \tilde{S}_v$, since x_{v,f_i} is not a branch point of the m-fold branched cover. The case of $x_{v',f_i'}$ is analogous. Thus, over these points we have a copy of the data (D^2, m) of a disc with m marked points. The topological quantum field theory of [21], given by the Chern-Simons modular functor at fifth root of unity, associates to the data (D^2, m) a vector space $V(D^2, m) =: \mathcal{V}_{v,f_i}$. Similarly, over $x_{v',f_i'}$ we have a similar datum, but with an orientation reversal due to the condition $t(f_i') = v'$, hence we obtain the dual vector space $\mathcal{V}^\vee_{v',f_i'} := V(D^2, m)^\vee$. The braid group B_m acts on $V(D^2, m)$ by diffeomorphisms of D^2 that preserve the m marked points. The gluing data φ_i in (2.2), in particular, that gives the gluing map, should be chosen so that x_{v,f_i} is mapped to $x_{v',f_i'}$ and the m marked points above x_{v,f_i} are matched with the m marked points above $x_{v',f_i'}$. We assign to the vertex v the vector space

$$\mathcal{V}_v := \bigotimes_{f_i\,:\,s(f_i)=v} \mathcal{V}_{v,f_i} \otimes \bigotimes_{f_i\,:\,t(f_i)=v} \mathcal{V}^\vee_{v,f_i} .$$

(3) The vector spaces $V(D^2, m)$ assigned to the disc with marked points by the TQFT of [21] can be expressed in terms of the Jones representation specialized at a root of unity $q = A^4 = e^{2\pi i/r}$. In particular, each component of the Jones representation that corresponds to the matrix algebra $M_{n_{m,a}}(\mathbb{C}) = \mathrm{End}(\mathcal{V}_{m,k,a})$, gives a corresponding summand $\mathcal{V}_{v,f_i,a} \simeq \mathcal{V}_{m,k,a}$ of the vector space $\mathcal{V}_{v,f_i}$, spanned by all the anyon fusion trees with total charge a (see §4.1.3 of [15]).

(4) The isomorphism φ_{e_i} of orbifold normal bundles in condition (2) above gives a transformation $T_{\varphi_{e_i}} \in \mathrm{End}(\mathcal{V}_{v,f_i})$, obtained as described in §2.3 above.

(5) Given any edge e_i that is a matching of two half-edges f_i, f_i', with vertices $v = s(e_i) = s(f_i)$ and $v' = t(e_i) = t(f_i')$, we obtain a pairing $\mathcal{V}^\vee_{v',f_i'} \times \mathcal{V}_{v,f_i} \to \mathbb{C}$ by $\alpha(T_{\varphi_{e_i}}(X))$ for $\alpha \in \mathcal{V}^\vee_{v',f_i'}$ and $X \in \mathcal{V}_{v,f_i}$. Thus, any pairing of flags f_i, f_i' connecting vertices v, v' by an edge e_i contracts the f_i index of the tensors in cV_v with the index f_i' of the tensors in $\mathcal{V}_{v'}$. Note that in the glued discs D^2_{v,f_i} and $D^2_{v',f_i'}$ the braid system is no longer standard, as it has been transformed by the Hurwitz action, but in all the discs corresponding to remaining external edges the braid system will still be standard.

(6) After performing all these index contractions with the matriecs $T_{\varphi_{e_i}}$ along the edges e_i, one obtains a resulting vector space

$$\mathcal{V}_G := \bigotimes_{f\in E_{ext}\,|\, s(f)\in V} \mathcal{V}_{s(f),f} \otimes \bigotimes_{f\in E_{ext}\,|\, t(f)\in V} \mathcal{V}^\vee_{t(f),f}$$

(7) Let (M_G, Σ_G) be the geometry resulting from the connected sums of the (M_v, Σ_v) at the D^2_{v,f_i} and D^2_{v',f'_i} with the identifications φ_{e_i} for $e_i = f_i \cup f'_i$, with a surface braid S_G that has $\#b(S_G) = \ell(\#E_{int} + \#E_{ext})$. The corresponding braid representation acts on the space $\mathcal{V}_G$.

(8) For all $v \in V$ and integers k_v, let $(X_v, \mathbb{D}_v, k_v)$ be a choice of an extended 3-manifold, with $X_v \subset \tilde{M}_v$ with $\partial X_v \supset \cup_i \{x_{v,f_i}\} \times D^2$ with m marked points, with the discs $\{x_{v,f_i}\} \times D^2 \subset \mathcal{N}(\tilde{\Sigma}_v)$ as above, and with the DAP-decomposition containing the curves $\{x_{v,f_i}\} \times \partial D^2$. Let $(Y_v, \mathbb{D}_{Y_v})$ denote the extended surface given by $\partial(X_v, \mathbb{D}_v, k_v) \smallsetminus \cup_i (\{x_{v,f_i}\} \times D^2, \{x_{v,f_i}\} \times \partial D^2)$. Then the TQFT above, extended to a TQFT with corners, assigns to any such choice of $(X_v, \mathbb{D}_v, n)$ a vector $Z(X_v, \mathbb{D}_v, k_v)$ in $\mathcal{V}_v \otimes V(Y_v, \mathbb{D}_{Y_v})$.

(9) Given an edge e_i matching two half-edges f_i, f'_i with $v = s(e_i) = s(f_i)$ and $v' = t(e_i) = t(f'_i)$, the gluing map (2.2) is used to glue the boundary components $(\{x_{v,f_i}\} \times D^2, \{x_{v,f_i}\} \times \partial D^2)$ of $(X_v, \mathbb{D}_v, k'_v)$ and $(\{x_{v',f'_i}\} \times D^2, \{x_{v',f'_i}\} \times \partial D^2)$ of $(X_{v'}, \mathbb{D}_{v'}, k_{v'})$. Written with our shorthand notation of Section 2.2, this gives a resulting vector

$$Z((X_{e_i}, \mathbb{D}_{e_i}, k_{e_i})_{\varphi_i}) = \langle V(\varphi_i) Z_{X_v, \mathbb{D}_v}, Z_{X_{v'}, \mathbb{D}_{v'}} \rangle Z_{X_{e_i}, \partial X^0_{e_i}}$$

in $\mathcal{V}_{e_i} \otimes V((Y_v, \mathbb{D}_{Y_v}) \cup_{\varphi_i} (Y_{v'}, \mathbb{D}_{Y_{v'}}))$, where $\mathcal{V}_{e_i} := \otimes_{f \neq f_i, f'_i} \mathcal{V}_{s(f),f} \otimes \mathcal{V}^\vee_{t(f),f}$ with $s(f), t(f)$ either v or v'. Performing all these identifications of pairs of flags to edges, one obtains a 3-dimensional geometry $(X_G, \mathbb{D}_G, n_G)$ with boundary $\partial(X_G, \mathbb{D}_G, n_G) = \cup_{f\in E_{ext}(G)} (\{x_{\partial(f),f}\} \times D^2, \{x_{\partial(f),f}\} \times \partial D^2) \cup (Y_G, \mathbb{D}_{Y_G})$ and a vector $Z(X_G, \mathbb{D}_G, n_G)$ in the vector space $\mathcal{V}_G \otimes V(Y_G, \mathbb{D}_{Y_G})$. One can then interpret this $Z(X_G, \mathbb{D}_G, n_G)$ as a tensor that takes as inputs vectors in $V(Y_G, \mathbb{D}_{Y_G})^\vee$ and computes as output a vector in $\mathcal{V}_G$, the vector space associated to the boundary E_{ext} of the network template G. By tracing out the "bulk indices" given by the $V(Y_G, \mathbb{D}_{Y_G})$ part of $Z(X_G, \mathbb{D}_G, n_G)$, one obtains a tensor in $\mathcal{V}_G$ which can be seen as taking inputs in $\otimes_{f\in E_{ext}\,|\, t(f)\in V} \mathcal{V}_{t(f),f}$ and computing outputs in $\otimes_{f\in E_{ext}\,|\, s(f)\in V} \mathcal{V}_{t(f),f}$ through the network.

2.5. **Geometric models supporting anyon tensor networks.** The simplest example of geometries supporting the construction of anyon tensor networks outlined in the previous subsection is obtained by considering as building blocks (M_v, Σ_v) the geometric models of matter given by the Atiyah–LeBrun manifolds $(\mathbb{CP}^2, \mathbb{CP}^1)$ with the Abreu edge-cone metrics, recalled in the first section of this paper, and a network template given by a finite tree T. The resulting anyon network is supported on a geometry $\#_T \mathbb{CP}^2$, which is a direct sum of $\#V(T)$ copies of $\mathbb{CP}^2$, with the self-dual edge-cone metrics presented in §5 of [6].

The simplest example that incorporates a nontrivial loop in the network template G is related to surfaces of class VII_0^+. Recall that a compact complex surface is of class VII if its first Betti number is one and its Kodaira dimension $-\infty$. It is of class VII_0 if it is also minimal, in the sense that it contains no nonsingular rational curve with self-intersection -1. The surfaces of class VII_0^+ are those that are of class VII_0 and have positive second Betti number, $m = b_2(X) > 0$. Such surfaces include, for example, the Inoué surfaces. Known class VII_0^+ surfaces are diffeomorphic to $(S^1 \times S^3)\#_m\overline{\mathbb{CP}^2}$. A variant of the Donaldson–Friedman twistor space argument [22] shows these surfaces admit hermitian anti-self-dual structures (self-dual on $(S^1 \times S^3)\#_m\mathbb{CP}^2$), and one can ask for edge-cone-metrics with this property. Topologically one can think of $(S^1 \times S^3)\#m\mathbb{CP}^2$ as a self-connected sum of $\#_m\mathbb{CP}^2$, which from the point of view of the underlying network templates corresponds to adding a loop to a tree.

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