Anyons, networks, and codes in geometric models of matter

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"Sir Michael Atiyah: Forays into Physics"
Newton Institute, Cambridge, 2019

This talk is based on:

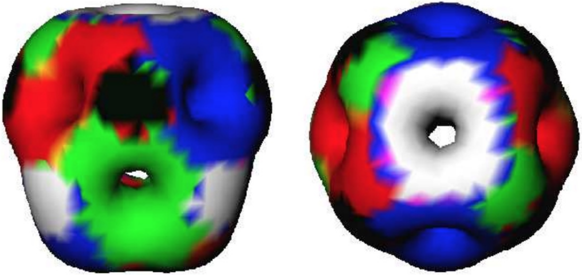
- Michael F. Atiyah, Matilde Marcolli *Anyons in geometric models of matter*, J. High Energy Physics, 07 (2017) 076
- Michael F. Atiyah, Matilde Marcolli *Anyon networks from geometric models of matter*, unfinished preprint

Geometric Models of Matter (Atiyah, Manton, Schroers, *et al.*)

- certain 4-dim Riemannian manifolds with self-dual Weyl tensor behave "like" elementary particles
- gravitational instantons: Taub-NUT, Atiyah–Hitchin, gravitational instantons of types A_k and D_k
- dynamical models: $(4+1)$-dimensional Ricci-flat spacetimes describing evolving Taub-NUT geometries (Atiyah–Franchetti–Manton)
- other more general classes of 4-manifold with "particle properties": algebraic surfaces with c_2 and c_1^2 as "lepton/baryon numbers", Enriques-Kodaira classification as "valley of stability" (Atiyah–Manton)
- geometrization of the skyrmion model of particles (topological solitons in non-linear sigma models with pion fields π combined with a field σ to $SU(2)$-valued scalar field, proposed as models of nucleons)

Skyrmions

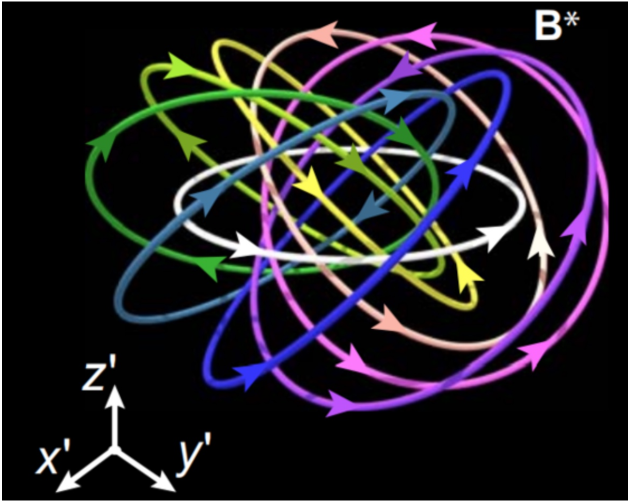
• N.S. Manton, *Classical Skyrmions – Static Solutions and Dynamics*, Mathematical Methods in the Applied Sciences, Vol.35 (2012) N.10, 1188–1204



baryon number B integer-valued topological charge: degree of a map $U : \mathbb{R}^3 \to SU(2)$; skyrmions with $B = 6$ in figure

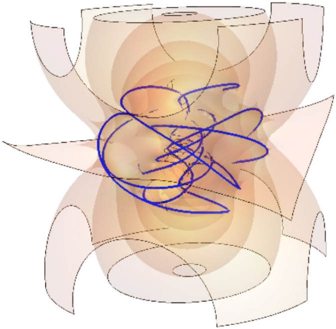
Skyrmions

- in magnetic skyrmions knotted magnetic field lines



Gravitational Instantons (recent examples)

• Snigdh Sabharwal and Jan Willem Dalhuisen, *Anti-self-dual spacetimes, gravitational instantons and knotted zeros of the Weyl tensor*, Journal of High Energy Physics, 07 (2019) 004



Gravitational instantons with anti-self-dual Kerr-Schild metrics with knotted zeros of the Weyl tensor

Quantum numbers of geometric models of matter

- Atiyah–Manton–Schroers: the signature $\tau(M)$ is interpreted as a baryon number; the electric charge is determined by the self-intersection number of the surface at infinity
- Atiyah–Manton: baryon and lepton numbers are expressed in terms of both signature and Euler characteristic (signature measuring difference between number of protons and number of neutrons)

General principle: quantum numbers of "particle-like" manifolds should be topological quantities computed by an index theorem

Composite systems

• merging operations on geometries that can be seen as composite systems of particles or quasi-particles

• connected sum operation: existence of self-dual metric on a connected sum of two self-dual 4-manifolds depends on a twistor space argument (Donaldson–Friedman)

• twistor space $Z = Z(M)$ of a self-dual 4-manifold is a 3-dimensional complex manifold that fibers over M with $\mathbb{CP}^1$ fibers

• singular complex 3-manifold $\tilde{Z} = \tilde{Z}_1 \cup_{E_1 \simeq E_2} \tilde{Z}_2$ by blowup of twistor spaces $Z_i = Z(M_i)$ along a $\mathbb{CP}^1$ fiber and gluing exceptional divisors

• Donaldson–Friedman: if $\exists$ smooth Kodaira–Spencer–Kuranishi deformation Z of $\tilde{Z}$ then Z is twistor space $Z(M)$ of a self-dual structure on the connected sum $M = M_1 \# M_2$

• Constraints on the formation of composite systems of geometric models of matter (relaxing the self-duality hypothesis? suggested in more recent Atiyah–Manton)

Dynamical models

• 4-dim gravitational instanton (self-dual Riemannian manifolds with an Einstein metric) is seen in these geometric models of matter as a static "particle-like" object

• made dynamical by embedding in a $(4+1)$-dimensional Ricci-flat geometry (Atiyah–Franchetti–Schroers)

• example: Taub-NUT geometry as 4-dim section of the 5-dim Sorkin solution of the Kaluza–Klein monopole equations

• Campbell–Magaard embedding: an arbitrary analytic Riemannian manifold M of dimension $\dim M = n$ can be locally embedded in a Ricci-flat Riemannian manifold of dimension $n+1$, but embedding may only exist locally

Building geometric models of systems of quasi-particles

Want fractional quantum numbers and anyon statistics

Main ideas:

- consider pairs (M, Σ) with M a smooth compact 4-dimensional manifold and Σ a smoothly embedded compact 2-dimensional surface
- consider metrics (edge-cone metrics) on M with Σ as set of orbifold points
- obtain fractional quantum numbers from Kawasaki index theorem for orbifolds
- get anyons and braid representations from surface braids determined by multisections of the orbifold normal bundle $\mathcal{N}(\Sigma)$ of Σ in M

Edge-cone metrics (Atiyah–Le Brun)

• edge-cone metric on (M, Σ) cone angle $2\pi\beta$, $\beta \in \mathbb{R}_+^*$

$$g = d\rho^2 + \beta^2\rho^2(d\theta + u_j dx^j)^2 + w_{ij}dx^i dx^j + \rho^{1+\epsilon}h$$

h symmetric tensor with continuous derivatives all orders for vector fields with vanishing normal component along Σ

• modelled on a 2-dimensional cone in directions transversal to Σ, smooth in the directions parallel to Σ

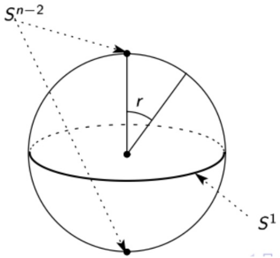
Example: edge cone metric on the sphere (S^n, S^{n-2}) $n \geq 3$

$$S^n - (S^{n-2} \sqcup S^1) = (0, \tfrac{\pi}{2}) \times S^1 \times S^{n-2} \ni (r, \theta, x)$$

$$g_S = dr^2 + \sin^2 r\, d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} =: h_1 : \textbf{standard round metric}$$

$$h_\beta := dr^2 + \beta^2 \sin^2 r\, d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} \quad \cong \quad h_1 : \text{loc. isom. } (\beta > 0)$$

edge-cone Einstein metric of cone angle $2\pi\beta$ on (S^n, S^{n-2})



Kazuo Akutagawa, *Computations of the orbifold Yamabe invariant*, Math. Z. 271 (2012), 611–625

Atiyah–Le Brun manifold

- (S^4, S^2) with edge-cone metric of angle $2\pi/\nu$
  - complement $S^4 \setminus S^2$ conformally equivalent to $\mathbb{H}^3 \times S^1$ (hyperbolic $\mathbb{H}^3$)
  - standard round metric on S^4 becomes $\mathrm{sech}^2\delta\ (h + d\theta^2)$ with h hyperbolic metric and $\delta : \mathbb{H}^3 \to \mathbb{R}$ distance from a point
  - $ds^2 = \mathrm{sech}^2\delta\ (h + \beta^2 d\theta^2)$ family of edge-cone metrics with cone angle $2\pi\beta$

Atiyah–Le Brun manifold: *connected sums of projective planes* $(\#^n \mathbb{CP}^2, \Sigma)$ with $\Sigma = \#^n \mathbb{CP}^1 \simeq S^2$

- $\mathcal{U} \subset \mathbb{H}^3$ open, $V : \mathcal{U} \to \mathbb{R}^+$ harmonic for metric h
- closed form $\star dV$; class $[\star dV/2\pi] = c_1(\mathcal{P}) \in H^2(\mathcal{U}, \mathbb{R})$; line bundle $\mathcal{P}$
- θ connection on $\mathcal{P}$ with curvature $d\theta = \star dV$
- Riemannian metric on total space of $\mathcal{P}$

$$g_0 = Vh + V^{-1}\theta^2$$

- potential $V = \beta^{-1} + \sum_{i=1}^n G_{p_i}$ points $p_i \in \mathbb{H}^3$ and Green functions G_{p_i}
- edge-cone metrics $g = \beta(\mathrm{sech}^2 \delta) g_0$
- metric completion of g on $\mathcal{P}$ gives $\#^n \mathbb{CP}^2$ with edge-cone angle $2\pi\beta$ along Σ
- generalization of Abreu family of edge-cone metrics on $(\mathbb{CP}^2, \mathbb{CP}^1)$

Orbifolds and edge-cone metrics (Atiyah–Le Brun)

• compact 4-dimensional M with atlas of local uniformizing charts U_α homeomorphic $U_\alpha \simeq V_\alpha / G_\alpha$ to quotients of open sets $V_\alpha \subset \mathbb{R}^4$ by finite groups G_α

• $M = M_{sing} \cup M_{reg}$ singular (orbifold) points and regular points with $M_{sing} = \Sigma$ embedded surface

• good orbifold: global quotient $M = X/G$ smooth 4-manifold X and finite group G

• near Σ local chart $\mathbb{C}^2/G_\nu$ with $G_\nu = \mathbb{Z}/\nu\mathbb{Z}$ and $(w, \zeta) \mapsto (w, e^{2\pi i/\nu}\zeta)$

• edge-cone metrics with $\beta = 1/\nu$, represented in local chart as a $\mathbb{Z}/\nu\mathbb{Z}$-invariant metric

Orbifolds as geometric models of systems of quasi-particles

- quantum numbers from Kawasaki index theorem for orbifolds
- Atiyah–Le Brun orbifold Euler characteristic and an orbifold signature

$$\chi_{orb}(M) = \frac{1}{8\pi^2}\int_M \left(|W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) \, dv(g)$$

$$= \chi(M) - (1-\frac{1}{\nu})\chi(\Sigma)$$

$$\tau_{orb}(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W_-|^2 \right) \, dv(g) = \tau(M) - \frac{1}{3}(1-\frac{1}{\nu^2})[\Sigma]^2$$

with $[\Sigma]^2$ self-intersection number (Euler number of normal bundle of Σ in M), W Weyl tensor, $W^{\pm}$ self-dual and anti-self-dual part, E traceless part of Ricci tensor, R scalar curvature

• $\chi_{orb}(M)$ and $\tau_{orb}(M)$ fractional quantum numbers for the orbifold (M, Σ), viewed as modeling a system of quasi-particles

• surface Σ of orbifold points plays role of surface at infinity that contributes the electric charge to the matter content in the Atiyah–Manton–Schroers model

• normal bundle $\mathcal{N}(\Sigma)$ of the inclusion of Σ in M is an *orbifold vector bundle*

• fibers of $\mathcal{N}(\Sigma)$ are quotients $\mathbb{R}^2/G_\nu$ where $G_\nu = \mathbb{Z}/\nu\mathbb{Z}$ is the stabilizer of Σ

• role of self-intersection number becomes orbifold Euler number $\chi_{orb}(\mathcal{N}(\Sigma))$ of the normal bundle $\mathcal{N}(\Sigma)$

• fractional electric charge of the system of quasi-particles

Constraints on composite systems

• possible obstructions existence of self-dual structures on connected sums (obstructions to smooth deformation giving twistor space as above)

• possible obstructions to Einstein condition (topological obstructions or differentiable obstructions)

- *topological obstructions* (Atiyah-Le Brun): inequalities

$$2\chi(M) \pm 3\tau(M) \geq (1 - \frac{1}{\nu})(2\chi(\Sigma) \pm (1 + \frac{1}{\nu})[\Sigma]^2)$$

  need to hold for (M, Σ) to admit an Einstein edge-cone metric of cone angle $2\pi/\nu$

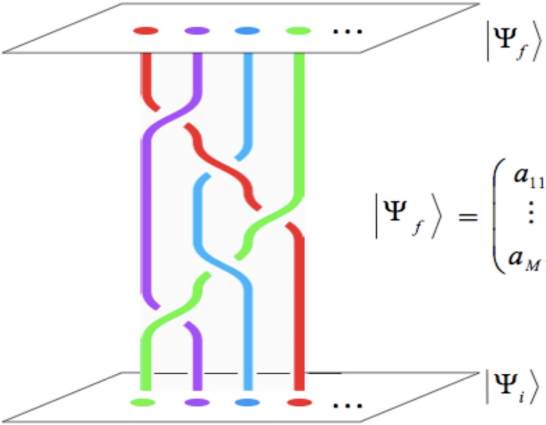
- *differentiable obstructions* (Le Brun): from Seiberg–Witten if symplectic form ω on M with Σ symplectic submanifold with $(c_1(M) - (1 - 1/\nu)[\Sigma]) \cdot [\omega] < 0$ then for any $\ell \geq (c_1(M) - (1 - 1/\nu)[\Sigma])^2/3$ the pair (M', Σ) with $M' = M \#^\ell \overline{\mathbb{CP}^2}$ has no Einstein edge-cone metric

• relax self-dual Einstein conditions for composite systems? require only for "elementary constituents"?

Anyons

- bosons/fermions statistics $|\psi_1\psi_2\rangle = \pm|\psi_2\psi_1\rangle$
- abelian anyons $|\psi_1\psi_2\rangle = e^{i\theta}|\psi_2\psi_1\rangle$
- nonabelian anyons: braid representations
- anyons and quantum computing: unitary braid representations that span densely $SU(2^N)$ are universal for quantum computing, approximate arbitrary quantum gates for system of N-qbits

Anyons



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Kareljan Schoutens and Nick Bonesteel, illustration of non-abelian anyons

Anyons are 2-dimensional systems of quasi-particles

- unitary representations of braid groups $B_n(X)$
- not reducible to representations of symmetric S_n (fermions/bosons)
- not reducible to wreath products of $\pi_1(X)$ and S_n (generalized parastatistics)
- can happen only for X 2D surface

Question: can anyon systems of quasi-particles be realized within the 4-dimensional geometric models of matter? can they be universal for quantum computing?

Braid groups and Anyons

Relation between braid groups, fractional statistics, and anyons

Configuration spaces: X smooth manifold, $F_n(X) = X^n \setminus \Delta$ complement of diagonals

$$F_n(X) = \{(x_1, \dots, x_n) \in X^n \,|\, x_i \neq x_j, \forall i \neq j, \; i,j = 1, \dots, n\}$$

free action of symmetric group S_n on $F_n(X)$

$$\mathrm{Conf}_n(X) := F_n(X)/S_n$$

Braid groups: $B_n(X) := \pi_1(\mathrm{Conf}_n(X))$

$$1 \to \pi_1(F_n(X)) \to B_n(X) \to S_n \to 1$$

for $\dim X > 2$ one has $\pi_1(F_n(X)) = \pi_1(X)^n$ so wreath product

$$B_n(X) = \pi_1(X)^n \rtimes S_n$$

• for a system of n identical particles on a smooth manifold X, with configuration space $\mathrm{Conf}_n(X)$ the set of irreducible unitary representations of the braid group $B_n(X)$ labels inequivalent quantizations of the classical system

• these can have different possible statistics including bosons and fermions, parastatistics, generalized parastatistics, and anyons

• parastatistics are higher dimensional representations of symmetric groups (fermions and bosons are one-dimensional representations)

• Example: X simply connected with $\dim X \geq 3$, then just fermions, bosons, and parastatistics as $B_n(X)$ are symmetric groups

• *generalized parastatistics*: case of $\dim X \geq 3$ but $\pi_1(X) \neq 1$, so representations of wreath product $B_n(X) = \pi_1(X)^n \rtimes S_n$

• anyons: only in the case where $B_n(X)$ is not a wreath product, so for $\dim X = 2$

Anyons when $\dim(X) = 2$

• for $X = D^2$ Artin braid group

$$B_n(D^2) = B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle$$

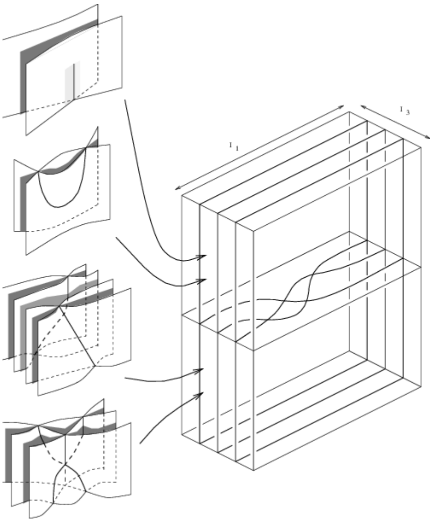
• for $X = \Sigma$ a Riemann surface also have explicit presentations of $B_n(\Sigma)$ (Birman) Note: these groups $B_n(\Sigma)$ *not* wreath products

Case of geometric models: (M, Σ) with $\dim M = 4$ and $\Sigma \subset M$ with $\dim \Sigma = 2$ locus of orbifold points

• Cannot use $B_n(M \smallsetminus \Sigma)$ or $B_n(M)$ or $B_n^{orb}(M, \Sigma)$ (orbifold braid groups) because those only give generalized parastatistics

• still possible to obtain anyons

Surface Braids (Viro, Kamada)



• J. Scott Carter, Seiichi Kamada, Masahico Saito, *Alexander Numbering of Knotted Surface Diagrams*, Proc. Amer. Math. Soc. 128 (2000), no. 12, 3761–3771

Surface Braids (Viro, Kamada)

- surface m-braid: smooth 2-dimensional S, smoothly embedded in $D^2 \times D^2$ with second projection $P_2 : D^2 \times D^2 \to D^2$ restricting to S as an m-fold branched cover $P : S \to D^2$
- preimage $\beta := P_2^{-1}(\partial D^2) \cap S \subset D^2 \times S^1$ ordinary closed m-braid
- Note: sometimes assumed that β trivial braid
- $b(S) \subset D^2$ set of branch points of the m-fold brached cover $P : S \to D^2$
- Braid representation: fundamental group $\pi_1(D^2 \smallsetminus b(S))$

$$\rho_S : \pi_1(D^2 \smallsetminus b(S)) \to \pi_1(\mathrm{Conf}_m(D^2)) = B_m(D^2)$$

- Construction: paths $\gamma(t)$ in $D^2 \smallsetminus b(S)$

$$\rho_S(\gamma)(t) := P_1(S \cap P_2^{-1}(\gamma(t)))$$

seen as a path in $\mathrm{Conf}_m(D^2)$
(with $P_i : D^2 \times D^2 \to D^2$ projections)

Variants of Surface Braids

• closed surface braids: S smoothly embedded in $D^2 \times S^2$ with $P = P_2|_S : S \to S^2$ an m-fold branched covering

• more general version: fixed surface Σ (possibly of genus $g(\Sigma) > 0$) and S smoothly embedded in $D^2 \times \Sigma$ with projection $P = P_2|_S : S \to \Sigma$ an m-fold branched covering

• braid representation: $\rho_S(\gamma)(t) := P_1(S \cap P_2^{-1}(\gamma(t))$

$$\rho_S : \pi_1(\Sigma \smallsetminus b(S)) \to \pi_1(\mathrm{Conf}_m(D^2)) = B_m(D^2)$$

• further case: $\mathcal{F}$ a disc-bundle over closed surface Σ and S smoothly embedded in $\mathcal{F}$ with $\pi : \mathcal{F} \to \Sigma$ restricting to m-fold branched cover $\pi|_S : S \to \Sigma$

Sections and multisections of the orbifold unit normal bundle

- (M, Σ) orbifold geometry, $\dim M = 4$ and $\dim \Sigma = 2$, good orbifold $M = X/G$ some finite group G (assume Σ connected and $G = \mathbb{Z}/\nu\mathbb{Z}$)
- normal bundle $\mathcal{N}(\Sigma)$ of $\Sigma \hookrightarrow M$ is an orbifold bundle, orbifold covered by normal bundle $\mathcal{N}(\tilde{\Sigma})$ of preimage $\tilde{\Sigma}$ in X
- lift to $\mathcal{N}(\tilde{\Sigma})$ of a generic section σ of $\mathcal{N}(\Sigma)$ gives ν-fold branched covering S of Σ branched at finitely many points $b(S)$
- multisections of $\mathcal{N}(\tilde{\Sigma})$ are maps to $\mathrm{Sym}^\ell(F) = F^\ell/S_\ell$ of fiber F; ℓ-fold branched covers of $\tilde{\Sigma}$ branched at intersections with diagonals (for unit normal bundle $F = D^2$)
- combining these: taking mutisection of unit normal $\mathcal{N}_1(\Sigma)$ get $\nu\ell$-fold branched coverings S of Σ, hence $\ell\nu$-surface braids

Focus on the Atiyah-LeBrun manifold (S^4, S^2) with $2\pi/\nu$ edge cone metric

- multisections of $\mathcal{N}(S^2)$ give $\nu\ell$-surface braids S (closed surface braids in $D^2 \times S^2$)
- associated braid representation $\rho_S : \pi_1(S^2 \smallsetminus b(S)) \to B_n(D^2)$
- given a closed surface braid S obtained from a multisection of $\mathcal{N}(S^2)$ and a branch point $x_0 \in b(S)$, take a disc $D^2_b \subset S^2$ that is the complement of a small neighborhood of x_0
- restriction of branched cover $P : S \to S^2$ to $D^2_b \subset S^2$ is also a $\nu\ell$-fold branched cover $P : \hat{S} \to D^2_b$ and $\hat{S} \subset D^2_f \times D^2_b$ is a surface braid, so $\hat{S} \cap (D^2_f \times \partial D^2_b)$ is an ordinary closed braid β, in general non-trivial in $B_{\nu\ell} = B_{\nu\ell}(D^2)$
- the braid system $(\beta_1, \dots, \beta_n)$ of $\hat{S}$ is the image under the braid representation $\rho_{\hat{S}} : \pi_1(D^2_b \smallsetminus b(\hat{S})) \to B_{\nu\ell}(D^2)$ of set of generators $\gamma_1, \dots, \gamma_n$ of $\pi_1(D^2_b \smallsetminus b(\hat{S})$ (basepoint on ∂D^2_b)

Standard braid system

• characterization of braid systems such that $\hat{S} \cap (D^2_f \times \partial D^2_b)$ is a fixed closed braid $\beta \in B_{\nu\ell}(D^2)$: n-tuples $(\beta_1, \ldots, \beta_n) \in B_{\nu\ell}(D^2)^n$ such that each β_k is conjugate of a standard generator σ_i or σ_i^{-1} of the braid group, with $\beta_1 \cdots \beta_n = \beta$ in $B_{\nu\ell}(D^2)$

• equivalent braided surfaces (related by a fiber preserving diffeomorphism of $D^2_f \times D^2_b$ relative to $D^2_f \times \partial D^2_b$) correspond to braid systems $(\beta_1, \ldots, \beta_n)$ related by Hurwitz action of B_n on $B^n_{\nu\ell}$

$$\sigma_i : (\beta_1, \ldots, \beta_n) \mapsto (\beta_1, \ldots, \beta_{i-1}, \beta_i \beta_{i+1} \beta_i^{-1}, \beta_i, \beta_{i+2}, \ldots, \beta_n)$$

• for $n = \nu\ell - 1$ consider surface braid with $n = \#b(\hat{S})$ such that braid system is standard set of generators $(\sigma_1, \ldots, \sigma_n)$ of $B_{\nu\ell}(D^2)$

• for such choice of $\hat{S}$ braid representation obtains all $B_{\nu\ell}(D^2)$

How good are anyon systems constructed through geometric models (M, Σ)?

• What makes an anyon system good?
Properties for quantum computation

• quantum computation from anyon systems:

- anyon system $\Longrightarrow$ an associated braid group
- a braid group $\Longrightarrow$ unitary representations
- does unitary representation span *densely* the group $SU(2^N)$?
- $SU(2^N) \Longrightarrow$ quantum gates for a system of N qbits

• if density in $SU(2^N)$ holds: anyon system is *universal for quantum computing*

Examples of anyon systems universal for quantum computing

• Fibonacci anyons

• Jones unitary representations of $B_n = B_n(D^2)$

• TQFT: Chern–Simons theory at 5-th root of unity
(Jones representation at $q = e^{\pm 2\pi i/5}$)

• general fact about quantum gates: arbitary gates in $SU(2^N)$ can be decomposed into tensor product of 1-qbit gates and CNOTs

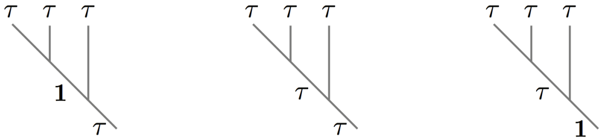
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Fibonacci anyons (simpler case)

- two types of particles, denoted $\mathbf{1}$ and τ
- fusion rules

$$\mathbf{1} \otimes \tau = \tau, \quad \tau \otimes \mathbf{1} = \tau, \quad \tau \otimes \tau = \mathbf{1} \oplus \tau$$

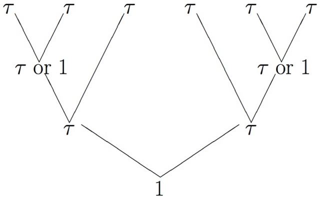
- fusion trees



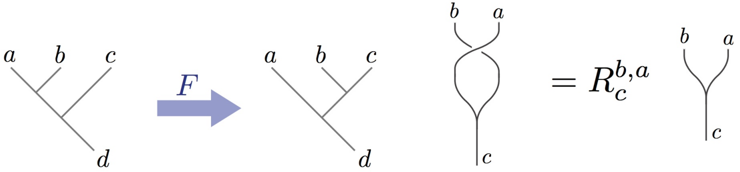
- ground state degeneracies for n τ-anyons from counting number of fusion paths: Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$

• Simon Trebst, Matthias Troyer, Zhenghan Wang, Andreas W.W. Ludwig, *A short introduction to Fibonacci anyon models*, Prog. Theor. Phys. Supp. 176, 384 (2008)

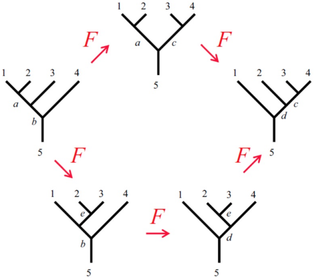
- three τ-anyons fusion trees simulate a single qbit; six τ-anyons simulate two qbits



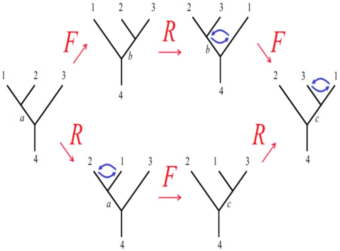
- braiding these anyons simulates unitary transformation on this simulated qubit
- F and R matrices



- associativity: pentagon and hexagon relations, solve for F, R for Fibonacci



$$(F^5_{12c})^d_a (F^5_{a34})^c_b = \sum_e (F^d_{234})^c_e (F^5_{1e4})^d_b (F^b_{123})^e_a$$



$$\sum_b (F^4_{231})^c_b R^4_{1b} (F^4_{123})^b_a = R^c_{13} (F^4_{213})^c_a R^a_{12}$$

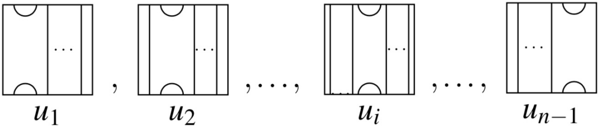
- $R = \rho(\sigma_1)$ and $FRF^{-1} = \rho(\sigma_2)$ give the braid representation

Chern-Simons (Freedman–Larsen–Wang)

- topological modular functor of a TQFT:
  - 2D surface $\Sigma \Rightarrow$ complex (hermitian) vector space $V(\Sigma)$
  - diffeomorphisms $\Rightarrow$ (projective) unitary maps on $V(\Sigma)$
- consider $\Sigma = (D^2, 3\ell)$ disc with 3ℓ marked points
  - $S_\ell = (\mathbb{C}^2)^{\otimes \ell}$ state space of ℓ qbits
  - construct a map $S_\ell \hookrightarrow V(D^2, 3\ell)$
  - embedding intertwines action of $B_{3\ell}$ on $V(D^2, 3\ell)$ by diffeos of D^2 preserving set of marked points and action of unitary operators on S_ℓ
  - B_3 action on $V(D^2, 3) = \mathbb{C}^2$ (single qbit) gives the 1-qbit quantum gates
  - the 2-qbit CNOT gate is obtained via an approximation algorithm

Jones representation

- Temperley–Lieb algebra $TL_n(A)$



relations: far commutativity, braid relations, Hecke relations
multiplication by $d = -A^2 - A^{-2}$ when a loop removed in composition (vertical stacking of diagrams)

- Construction of the Jones unitary representation:
  - braid group B_n, group algebra $\mathbb{F}[B_n]$
  - mapping $\mathbb{F}[B_n]$ to Temperley-Lieb algebra $TL_n(A)$ (using Kauffman bracket)
  - inner product on $TL_n(A)$ given by Markov trace (closing diagram and counting loops)
  - identify $TL_n(A)$ with a sum of matrix algebras

Chern-Simons and Jones representation
(Freedman–Larsen–Wang)

- Chern-Simons modular functor (at level r) constructed using irreducible representations of quantum groups (Reshetikhin–Turaev)
- construction of hermitian inner products that make CS modular functor unitary
- expression in terms of Jones representation $q = A^4 = e^{2\pi i/r}$ at fifth root of unity $r = 5$

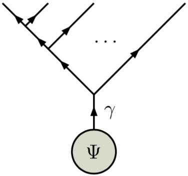
Case of Geometric Models of Matter (Atiyah–LeBrun manifold)

Tensor networks

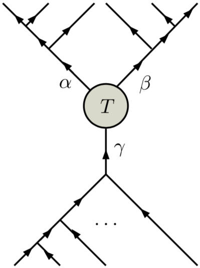
- finite or countable collection of tensors connected by leg contractions (tensor $T_{i_1,\ldots,i_k}$ with k-legs; index contraction corresponds to glueing legs)
- graphical calculus diagrammatic methods (Roger Penrose)
- quantum circuits as tensor networks
- useful method for computing entangled quantum states
- holographic bulk/boundary correspondence with bulk space discretized by tensor networks: entanglement entropy of boundary state and geometry (minimal curves/surfaces) in bulk space, Ryu–Takayanagi conjectures

Anyon Tensor Networks

- states $|\Psi\rangle$ weighted superposition of anyon fusion/splitting trees with assigned anyon charge at the root



- operators: anyonic tensors



Recent work on anyon tensor networks

- J. Berger, T.J. Osborne, *Perfect tangles*, arXiv:1804.03199
- J.C. Bridgeman, S.D. Bartlett, A.C. Doherty, *Tensor Networks with a Twist: Anyon-permuting domain walls and defects in PEPS*, arXiv:1708.08930
- B.M. Ayeni, *Studies of braided non-Abelian anyons using anyonic tensor networks*, arXiv:1708.06476

• Question: can realize anyon tensor networks in geometric models of matter? Main idea/strategy:

- use building blocks given by geometric models of matter that realize good anyon systems (Atiyah-LeBrun manifolds)
- identify network models that can be realized via non-obstructed gluing
- effect on the anyon systems of gluing via anyonic tensors
- geometric construction of anyonic tensors

Network structure in geometric models of matter

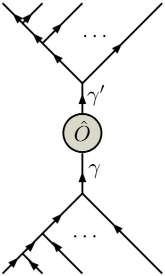
- tree-shaped networks: use as geometric model the connected sums of projective planes $\#_T \mathbb{CP}^2$ along a tree T with Atiyah–LeBrun edge-cone metrics
- non-tree networks: more subtle case: *self-connected sum*
- self-connected sum: delete two points on a 4-manifold M, with small disjoint 4-balls B_1 and B_2 around them, identify S^3 boundaries ∂B_1 and ∂B_2 in $M \setminus (B_1 \cup B_2)$
- (M, g) self-dual with Z twistor space, variant of the Donaldson-Friedman construction gives condition for existence of smooth twistor space $\tilde{Z}$ of a self-connected sum of M
- example: Inoué surfaces
- general problem: existence of unobstructed structures on (self-)connected sums along a specified non-tree network (e.g. of $\mathbb{CP}^2$'s or S^4's)

• twistor space construction for self-connected sums

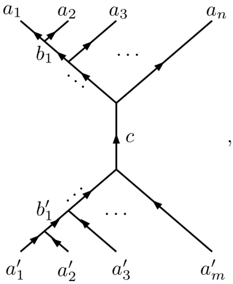
- A. Fujiki, *Anti-self-dual Hermitian structures on Inoue surfaces via twistor method*, preprint

Anyonic tensors in geometric models of matter (Sketch)

- anyonic matrix operators



$$\hat{O} = \sum_{\substack{b_1,\dots,b_{n-2}\\ b'_1,\dots,b'_{m-2}\ c}} O^{b_1\dots b_{n-2}\ c}_{b'_1\dots b'_{m-2}}$$



,

- produce via anyon networks anyonic matrix product states
- gluing along connected sums of multisections of orbifold normal bundles: braid representations

$$\pi_1(D^2 \smallsetminus b(S)) \star_{\mathbb{Z}} \pi_1(D^2 \smallsetminus b(S')) \to B_m(D^2)$$

- effect of inserting anyonic matrix operators achieved geometrically by gluing multisections of orbifold normal bundle via a non-trivial transformation

Further questions

- what kind of anyon tensor networks are realizable in geometric models of matter
- what kind of matrix product states or other quantum states do they compute?
- are there universal examples? (can simulate tensor networks)

Conclusion: geometric models of matter are 4-dimensional geometries that "behave like" particle systems; they also can be adapted to behave like systems of quasi-particles like anyons, including capturing some quantum computational properties