2-dimensional \( \mathbb{Q} \)-lattices

\[ \Lambda, \phi \]  
\( \Lambda \subset \mathbb{C} \cong \mathbb{R}^2 \)

\( \{ e_1, e_2 \} \) basis of \( \mathbb{C} \) as \( \mathbb{R} \)-vector space

\( \text{GL}_2^+(\mathbb{R}) \) acts on \( \mathbb{C} \) by \( \mathbb{R} \)-linear transformations

for \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \)

and \( \Lambda_0 = \mathbb{Z} e_1 + \mathbb{Z} e_2 = \mathbb{Z} + i\mathbb{Z} \) standard lattice

\( M_2(\mathbb{Z}) = \text{Hom} (\mathbb{Q}^2, \mathbb{Q}^2) \)

or equivalently \( \rho \in M_2(\mathbb{Z}) \)

\[ \rho : \mathbb{Q}^2 \rightarrow \mathbb{Q}^2 / \Lambda_0 \]

\[ \rho (a) = \rho (a)e_1 + \rho (a)e_2 \]

Then describe \((\Lambda, \phi)\) 2-dim \( \mathbb{Q} \)-lattices as

\[ (\Lambda, \phi) = (\Lambda^{-1} \Lambda_0, \phi') \quad \text{for some} \quad \alpha \in \text{GL}_2^+(\mathbb{R}) \quad \rho \in M_2(\mathbb{Z}) \]

\( (\alpha^{-1} \Lambda_0, \alpha^{-1} \phi') = (\beta^{-1} \Lambda_0, \beta^{-1} \phi') \) \( \text{iff} \)

\[ \beta \alpha^{-1} \Lambda_0 = \Lambda_0 \quad \text{and} \quad \rho \alpha^{-1} \phi' = \phi' \]

\[ \Rightarrow \quad \beta \alpha^{-1} = \gamma \in \text{SL}_2(\mathbb{Z}) \quad \text{and} \quad \phi' = \gamma \phi \quad \gamma \in \text{SL}_2(\mathbb{Z}) \]

\[ (\beta, \phi') = (\gamma \alpha, \phi') \]

So 2-dim \( \mathbb{Q} \)-lattices

\[ \Lambda \setminus (M_2(\mathbb{Z}) \times \text{GL}_2^+(\mathbb{R})) \]

\( \text{diagonal action of} \ \text{SL}_2(\mathbb{Z}) \)
\[ a = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R}) \]

\[ a(x e_1 + y e_2) = (ax + by)e_1 + (cx + dy)e_2 \]

\( \mathbb{C}^* \equiv \lambda = a + ib = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \)

acts as above

\[ \lambda(\lambda, \phi) = (\lambda \lambda, \lambda \phi) \text{ scaling action of } \mathbb{C}^* \times \text{GL}_2^+(\mathbb{R}) \]

i.e.

\[ (\alpha, \rho) \rightarrow (\lambda^{-1} \alpha, \rho) \]

Then 2-dim \( \mathbb{C} \)-lattices up to scaling

\[ \pi \left( \begin{pmatrix} M_2(\mathbb{Z}) \times \text{GL}_2^+(\mathbb{R}) \end{pmatrix}_{\mathbb{C}^*} \right) = \pi \left( \begin{pmatrix} M_2(\mathbb{Z}) \times H^1 \end{pmatrix}_{\text{diag. action of } \pi} \right) \]

in fact

\[ \text{GL}_2^+(\mathbb{R}) \text{ acts on upper half plane } \mathbb{H} \]

by fractional linear transformations

\[ z \mapsto \frac{az + b}{cz + d} \]

stabilizer of point \( z = i \) is \( \mathbb{C}^* \)

so map \( \alpha \rightarrow \alpha(i) \)

\[ \text{GL}_2^+(\mathbb{C}) \rightarrow \mathbb{H} \] identifies \( \mathbb{H} = \text{GL}_2^+(\mathbb{C})_{\mathbb{C}^*} \)

\( \mathbb{C} \)-lattices up to scale \( \sim \) pairs \((\rho, \phi)\)

\[ (\lambda, \phi) = (\lambda (\mathbb{Z} + \mathbb{Z} i), \lambda (\rho, \gamma \rho^*) ) \]
Commensurability relation

Partially defined action of $\text{GL}_2^+(\mathbb{Q})$

if $(x, \phi) \longleftrightarrow (p, \alpha) \iff p \in M_2(\mathbb{Z})$ we $\text{GL}_2^+(\mathbb{Q})$

then \( g \in \text{GL}_2^+(\mathbb{Q}) \) s.t. \( gf \in M_2(\mathbb{Z}) \)

gives commensurable lattice \((gp, gw)\)

but notice: orbits of \( \Gamma = \text{SL}_2(\mathbb{Z}) \)

so more precisely start with groupoid

\[ \tilde{G}_2 = \left\{ (g, p, \alpha) \in \text{GL}_2^+(\mathbb{Q}) \times M_2(\mathbb{Z}) \times \text{GL}_2^+(\mathbb{R}) : gp \in M_2(\mathbb{Z}) \right\} \]

units of groupoid \( \tilde{G}_2 = M_2(\mathbb{Z}) \times \text{GL}_2^+(\mathbb{R}) \)

then take a quotient by an action of \( \Gamma \times \Gamma \)

\( \Gamma = \text{SL}_2(\mathbb{Z}) \)

\[ \tilde{G}_2 \simeq \Gamma \setminus \tilde{G}_2 \]

\[ (g, p, \alpha) \mapsto (g, p, \alpha) \mapsto (g_1g_2^{-1}, p_2, \alpha_2 \alpha_1) \]

it is the groupoid of the commensurability relation:

\[ (g, p, \alpha) \mapsto \left( (\alpha \gamma \gamma^{-1}, \alpha \gamma p), (\alpha \gamma \gamma^{-1}, \alpha \gamma p) \right) \]

\( \Gamma \) class mod \( \Gamma \times \Gamma \)

\[ f \cdot f_2 (g, p, \alpha) = \sum_{h \in \Gamma \setminus \tilde{G}_2} f_1 (gh^{-1}, hp, h \alpha) f_2 (h, p, \alpha) \]

\( \text{Groupoid algebra} \)
When considering 2-dim \( \mathbb{Q} \)-lattices mod \( C^* \) scaling would like to take \( \mathcal{G}_2/C^* \)

but composition law of groupoid not well defined in the quotient

example: take point \( z = i \in \mathbb{H} \) and \( \mathbb{Q} \)-lattices

\[
\begin{align*}
(\Lambda, \phi) &= (\mathbb{Z} + 2\mathbb{Z}, 0) \\
(\Lambda', \phi') &= (\mathbb{Z} + i\mathbb{Z}, 0)
\end{align*}
\]

they are commensurable

composition: \( (\Lambda, \phi), (\Lambda', \phi') \circ (\Lambda', \phi') \)

in \( \mathcal{G}_2 \) is \( (\Lambda', \phi') \) ok

in \( \mathcal{G}_2 \) also consider pair

\[
( i(\Lambda, \phi), i(\Lambda', \phi') )
\]

can still compose with \( (\Lambda', \phi'), (\Lambda, \phi) \)

because \( i \) is symm. of \( \phi' \) (autom.)

so \( i(\Lambda', \phi') = (\Lambda', \phi') \)

composition in \( \mathcal{G}_2 \) is \( i(\Lambda, \phi), (\Lambda, \phi) ) \neq (\Lambda, \phi), (\Lambda, \phi) \)

not a unit of \( \mathcal{G}_2 \) unit of \( \mathcal{G}_2 \)

but in \( \mathcal{G}_2/C^* \) \( ( i(\Lambda, \phi), i(\Lambda', \phi') ) \) becomes equal to \( (\Lambda, \phi), (\Lambda, \phi) ) \)

so composition not well defined

Can get around this problem and still obtain convolution algebra for \( \mathbb{Q} \)-latt mod scaling, just not groupoid algebra.
the same convention product

\[(f_1 \ast f_2)(g,\rho,\alpha) = \sum_{h \in \text{GL}_2^+(\mathbb{Q}) \setminus \text{GL}_2^+(\mathbb{Q})} f_1(gh^{-1}, \rho, h(z)) f_2(h, \rho, \alpha)\]

but applied to functions \(p \in \text{GL}_2^+(\mathbb{Q}) \times (\mathbb{M}_2(\mathbb{Z}) \times \text{GL}_2^+(\mathbb{R}))\)

that are not of compact support but are invariant under \(x\) and compact support under \(G_x\) \(x \in \mathbb{A}\)

\(C^*\)-actions on \(\text{GL}_2^+(\mathbb{R})\)

i.e. \(f(g, \rho, x \lambda) = x^k f(g, \rho, \alpha)\)

homogeneous of weight \(k\)

choose those that homogeneous of weight 0 (invariant)

\[(f_1 \ast f_2)(g,\rho,\alpha) = \sum_{h \in \text{GL}_2^+(\mathbb{Q}) \setminus \text{GL}_2^+(\mathbb{Q})} f_1(gh^{-1}, \rho, h(z)) f_2(h, \rho, \alpha)\]

\[
f^x(g,\rho,\alpha) = \frac{f(g^{-1}, gp, g(z))}{f(g, \rho, \alpha)}
\]

\(\mathbb{A}\) resulting algebra

Representations: \(y \in \mathbb{M}_2(\mathbb{Z}) \times \mathbb{H}\)

\(y = (\rho, \tau)\)

\(G_y = \{ g \in \text{GL}_2^+(\mathbb{Q}) : gp \in \mathbb{M}_2(\mathbb{Z}) \}\)

\((\pi_y(f) \pi_x)(g) = \sum_{h \in \text{GL}_2^+(\mathbb{Q}) \setminus \text{GL}_2^+(\mathbb{Q})} f(gh^{-1}, \rho, h(z)) \pi_x(h)\)

\(\pi_y(g)\) \(\pi_x\) Reps of \(\mathbb{A}\)
If \( f = \sup \|x_y(f)\| \)
\[ f \text{ compact support} \implies \sup < \infty \]

Note that \( \mathcal{M}_0(\mathbb{R}) \times \mathbb{R} \) is not compact
So \( f(g, p, z) = 1_{\mathbb{R}} \) for comp. support
So \( A_2 \) is not a unital algebra

**Time evolution:**
\[ \sigma_t(f)(g, p, z) = \det(g) f(g, p, z) \]
again as in 1-dim case
\[ \det(g) = \frac{\text{covol}(\Lambda')}{\text{covol}(\Lambda)} \]

**Symmetries:**
\[ \text{Gl}^+(\mathbb{Q}) \cdot \text{Gl}_2(\mathbb{Z}) = \text{Gl}_2(A_0, f) \]

Note
\[ \text{Gl}^+(\mathbb{Q}) \cap \text{Gl}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z}) \]

Action of \( \text{Gl}_2(\mathbb{Z}) \):
\[ \gamma: (\Lambda, \phi) \mapsto (\Lambda, \phi \circ \gamma) \]
\[ \gamma: \mathbb{Q}_+^2 \to \mathbb{Q}_+^2 \]
preserves commensurability: \((\Lambda, \phi) \sim (\Lambda', \phi') \implies (\Lambda, \phi \circ \gamma) \sim (\Lambda', \phi' \circ \gamma)\)
on algebra automorphisms
\[ \gamma(f)(g, p, z) = f(g, p, z) \gamma \text{ compatible with time evolution} \]
\[ \Theta_n(f)(g, p, z) = \begin{cases} \int_{(g, p, 1) \in \text{det}(m)^{-1}, z} f(g, p, 1) \text{det}(m)^{-1} \quad &\text{if } g, p \text{ in } M_n(\mathbb{Z}) \text{ and } \text{det}(m)^{-1} \in M_n(\mathbb{Z}) \\ 0 &\text{otherwise} \end{cases} \]

acting by endomorphisms

Of these certain endom are inner

\[ \forall n \in \mathbb{N} \text{ c } M_n^+(\mathbb{Z}) \quad \text{inner endom.} \]

\[ \Theta_n(f) = \mu_n f \mu_n^* \quad \text{matrix} \]

\[ \mu_n(g, f, z) = \begin{cases} 1 &g \in \text{P}(n) \\ 0 &\text{otherwise} \end{cases} \]

\[ \text{partially def acti of } \mathbb{Q}^+ \subset \text{GL}_2^+(\mathbb{Q}) \]

\[ \text{} = \text{symmetries mod inner } \text{GL}_2^+(\mathbb{Q}) : \text{GL}_2(\mathbb{Q}) \]

\[ \text{Autmorphisms of the modular field} \]

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Modular field (field of modular functions)

Converse groups level N:

\[ \Gamma(N) = \left\{ \gamma \in \text{P} : \gamma \equiv 1 \text{ mod } N \right\} \]

Notation:

\[ \alpha \in \text{GL}_2^+(\mathbb{R}) \]

\[ (f \mid_\alpha) (z) = \text{det}(\alpha)^{-1} f\left( \frac{aq + b}{cz + d} \right) (c z + d)^{-k} \]

for \( \alpha = (a \ b) \]

\[ (c \ d) \]
Modular functions of level $N$:

$f$ holomorphic function on $\mathbb{H}$ s.t.

$$f|_\gamma = f \quad \forall \gamma \in \Gamma(N)$$

$F_N$ field of most functions of level $N$

$F = \bigcup F_N$ field of mod functions

- There are explicit generators for $\mathfrak{o} F_N$

- $F$ can be embedded (in many ways) as a subfield of $\mathbb{C}$

- Embeddings $F \subseteq \mathbb{C}$ parameterized by invertible 2-dim $\mathbb{Q}$-lattices $(\Lambda, \psi)$

- $\text{Aut}(F) = \mathfrak{q}_+^* \backslash \text{Gle}(\mathfrak{a}_q, f)$

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Analysis of arithmetic subalgebras of BC system will involve modular functions here