KMS states of the BC system

1. For \( \beta \neq 1 \), KMS state, explicitly given by the formula
   \[
   \varphi_{\beta}(e^{(\theta)}) = \frac{e^{\beta H(\theta)}}{e^{\beta}}
   \]
   
   \[f_{\beta}(x) = \sum_{d|x} \frac{\mu(d)}{d^{\beta}}\]
   (Frobenius zeta function)
   \[\mu(d) = \begin{cases} 1 & \text{square-free positive integer} \\ 0 & \text{all other cases} \end{cases}\]

2. For \( 1 < \beta < \infty \), extremal \( \varphi_{\beta} \in \text{KMS}_{\beta} \) parameterized by \( \beta \in \mathbb{Z}^* \) (invertible \( \mathcal{Q} \)-lattice)
   \[\varphi_{\beta} = \mathbb{Z}^* \] (action free trans. of \( \mathbb{Z}^* \) symmetries)
   (positive energy rep.)

3. For \( \beta < 0 \),
   \[\varphi_{\beta}(e^{(\theta)}) = \frac{1}{\xi(\beta)} \zeta_{\beta}(\varphi(\theta))\]
   \[\zeta_{\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\beta} \] polylog function

4. At \( \beta = 0 \),
   \[\varphi_{0}(\mathcal{A}_g) = \mathcal{G}^\text{cycl} \] extension of \( \mathcal{Q} \)
   gen. by roots of 1

   \[\gamma \in \text{Gal}(\mathcal{Q}^{cycl}/\mathcal{Q})\]

   \[\gamma \varphi_{\beta}(\theta) = \varphi_{\beta}(\gamma \theta)\]

   \[\gamma \text{ realizing } \text{Gal}(\mathcal{Q}^{cycl}/\mathcal{Q}) \approx \mathbb{Z}^*\]
(Will give more general proof later classifying RNS
states for extensions of BC: except best point)

don't have
good analog of

cases

General problem in NT (Hilbert 12th problem)
given a number field \( K \) finite extension of \( \mathbb{Q} \)

\([K : \mathbb{Q}] = n\) (soln of polyn. eq. over \( \mathbb{Q} \))

Would like to give explicit \underline{families} sets of generators
for extensions \( \mathbb{K} \) of \( \mathbb{K} \) with abelian Galois group

(abelian extension) \( \) in partic. max abelian ext

\( \mathbb{K}^{ab} \) s.t. \( \text{Gal} (\mathbb{K}/\mathbb{Q}) = \text{Gal}(\mathbb{K}^{ab}/\mathbb{Q}) \)

abelianization of absolute Galois group

with explicit description
of Galois action on these generators

Kronecker-Weber theorem:

\( \mathbb{Q}^{ab} = \mathbb{Q}^{cycl} \)

\((\mathbb{Q}^{ab})^\text{tors} = \text{roots of unity} \)

\((\mathbb{Q}^{ab})^\text{tors} = \text{roots of multip. grp } a^\phi \)

as the generators

and Galois group action is action of

\( \mathbb{Q}^* \) = invertible homomorphism \( \mathbb{Q} \to \mathbb{Q} \)
We'll see another case: imaginary quadratic fields \( \mathcal{O}(\sqrt{d}) \) ded.

Idea: Would like to use NCG to construct

\[ \text{C^*alg. } (\mathcal{A}_K, \sigma_K) \quad \text{IK number field} \]

s.t. \( \mathcal{A}_K \text{ arithmetic} \) defined over IK

s.t. \( \mathcal{C}_0 \subset \text{KMS}_0 \) stable

\( \Phi(\mathcal{A}_K \text{ arithmetic}) \subset \mathcal{K}^{ab} \subset \mathcal{C} \)

some embedding

\[ \forall \Phi(a) = \Phi(\Theta_{\mathcal{K}^{ab}}(a)) \]

some group of symmetries

(isoms + endows) of \( (\mathcal{A}_K, \sigma_K) \)

If we could do this would solve Hilbert 12th problem (already for real quadratic fields \( \mathcal{O}(\sqrt{d}) \) not known)

* BC system for \( \mathcal{O} \)
* Imaginary quadratic fields
* General systems for fields but lack good \( \mathcal{A}_K \text{ arithmetic} \)

(Systems obtained from considering \( \mathcal{K} \)-lattices instead of \( \mathcal{O} \)-lattices)
Before generalizations of BC system

More arithmetical properties of BC algebra:

lifting from \( \mathcal{G} \) to \( \mathbb{Z} \):

\[
\frac{1}{n} \sum_{n} e(s) \quad \text{involves denominators}
\]

BC model over \( \mathbb{Z} \):

\[\tilde{\mu}, \tilde{\nu}, \tilde{\mu}^*, \tilde{\nu}^* \quad (\text{no longer adjoints of one another})\]

\[
\tilde{\mu}_n \tilde{\nu}_m = \tilde{\nu}_m \tilde{\mu}_n \\
\tilde{\mu}_n^* \tilde{\nu}_m^* = \tilde{\nu}_m^* \tilde{\mu}_n^* \\
\forall m, n \in \mathbb{N}
\]

\[
\tilde{\nu}_n \tilde{\mu}_n = \mathbb{A}_2 \\
\text{no longer a crossed product by a semigroup.}
\]

Still consider endomorphisms

\[
\sigma_n (e(r)) = e(nr) \quad \sigma_n : \mathbb{Z} \left[ \frac{\mathbb{Q}}{\mathbb{Z}} \right] \to \mathbb{Z} \left[ \frac{\mathbb{Q}}{\mathbb{Z}} \right]
\]

satisfies

\[
\tilde{\nu}_n^* x = \sigma_n(x) \tilde{\nu}_n^* \\
x \tilde{\nu}_n = \tilde{\nu}_n \sigma_n(x)
\]
but now if set
\[ \tilde{\rho}^*_n(x) = \tilde{\mu}_n \times \mu_n^* \]
no longer a ring endom of
\[ \mathbb{Z}[\mathbb{Q}^r] \]
bec because \( \mu_n^* \mu_n = n \neq \mathbb{1} \)
it replaces the relation \( \rho_n(x) = \mu_n \times \mu_n^* \) which involves denominators

\[ \hat{\rho}_n^* (e(y)) = \sum_{ns=r} e(s) \]

\[ \sigma_{nm} = \sigma_n \sigma_m \]
\[ \tilde{\rho}_m = \tilde{\rho}_n \tilde{\rho}_m \]
\[ \tilde{\rho}_m (\sigma_n^*(x)) = x^* \tilde{\rho}_m (y) \]
\[ \sigma_c (\tilde{\rho}_b^*(x)) = (b, c) \tilde{\rho}_b^* (\sigma_c^*(x)) \]
\[ b' = \frac{b}{(b, c)} \quad c' = \frac{c}{(b, c)} \]

To check these relations observe that
if \( E_n(x) = \{ y \in \mathbb{Q}^r : ny = x \} \)
\[ \text{then } E_{nm}(x) = \bigcup_{y_1 \in E_n(x)} E_m(y_1) \quad \text{and } E_{nm}(y_1) \cap E_{nm}(y_2) = \emptyset \text{ if } y_1 \neq y_2 \]
\[ = \tilde{\rho}_m (\tilde{\rho}_n (e(x))) = \tilde{\rho}_m (\bigoplus_{y \in E_n(x)} e(y)) = \sum_{y \in E_n(x)} \sum_{z \in E_m(y)} e(z) = \tilde{\rho}_m (e(x)) \]

for other rel, multiply by \( c = nc' \)
is \( n \)-to-\( 1 \) map \( E_b(\mathbb{Q}) \to E_b(c') \)
$A_2$: generated additively by

$$\tilde{\mu}_a \times \mu_b^* \quad (a,b) = 1 \quad x \in \mathbb{Z}[[\mathbb{Q}]]$$

(with $\tilde{\mu}_1 = \mu_1^* = 1$)

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The endomotive description of the BC algebra

Endo endomotive ring

$\mathbb{Q}[t,t^{-1}]$ ring

Spec is $\mathbb{G}_m \otimes \mathbb{Q}$

$S_k: \mathbb{P}(t,t^{-1}) \longrightarrow \mathbb{P}(k,k^*)$ ring homomorphisms

$1$ is fixed by all

inverse image under $t \mapsto k$ is:

$X_k = \text{Spec} \left( \mathbb{Q}[t,t^{-1}] / (t^{n-1}) \right)$ alg. variety (zero dim)

$X = \lim_{\rightarrow} X_k$ ordered by divisibility

maps of proj. system:

$u(n) = \text{class of } t \mod t^{n-1}$ (generator of $\mathbb{Q}[t,t^{-1}] / (t^{n-1})$)

$z_{m,n} (u(n)) = u(m)^a \quad a = \frac{m}{n}$ for $n \mid m$

Gives corresps direct limit of algebras $A = \lim_{\rightarrow} A_k$

$A_k = \mathbb{Q}[t,t^{-1}] / (t^{n-1}) \quad A_k = \mathbb{Q}[\mathbb{Z}/k\mathbb{Z}]$
So again \( A_k \setminus \{ \emptyset \} \) identified \( \mu(n) \) with \( e(\frac{1}{n}) \) in previous notation for generators

Also write \( \mu(x) \) for \( X \) and \( \mu(x) \) for \( X_n \) rep dim varieties pro-variety

"Field with one element"
(no such thing exists; if it has one element it ain't no field)

but can pretend such object exists and think?

of interpretation of counting of pts of varieties \( X \) over finite fields \( \mathbb{F}_q \) \( (q = p^n \) some \( p > 0 \) pos char. \)
when \( q \to 1 \)

Formal equality \( \mathbb{F}_1 \otimes_{\mathbb{F}_q} \mathbb{Z} = \mathbb{Z}[t, t^{-1}] / t^n - 1 \)

"extensions" \( \mathbb{F}_n \) "like" \( \mathbb{F}_q \) of \( \mathbb{F}_q \)

What this strange notation means is that "something" (variety, ...) defined over \( \mathbb{F}_q \) is a var. (--) def over \( \mathbb{Z} \)
with additional conditions that "descend" from \( \mathbb{Z} \) to \( \mathbb{F}_q \)

(philosophical concept, not mathematical definition)
A way to try to make this idea more rigorous

Soulé's approach to defining varieties over $\mathbb{F}_1$

Affine var. over $\mathbb{F}_1$,

$R = \text{cat. of rings (over } \mathbb{Z} \text{) gen. by the } A_n = \mathbb{Z}[t,t^{-1}]/t^n-1 \text{ and their tensor products (as } \mathbb{Z}\text{-modules)}$

$\mathcal{X} = \text{a "gadget" over } \mathbb{F}_1$

is a triple $(X, A_X, e_X)$

1. $X : R \to \text{Sets}$ a covariant functor
2. $A_X$ an algebra over $C$
3. A natural transformation $e_X$ from the functor $X$ to the functor

$R \mapsto \text{Hom}(A_X, R_C) \quad R_C = R \otimes \mathbb{Z}$

\[ \begin{array}{ccc}
R & \to & X(R) \\
\downarrow f & & \downarrow X(f) \\
R' & \to & X(R')
\end{array} \quad \quad \begin{array}{ccc}
\phi_R & : & \text{Hom}(R, R_C) \\
\downarrow & & \downarrow \phi_{X(R)} \\
\phi_{R'} & : & \text{Hom}(R', R_C)
\end{array} \]

\[ \text{Homorphism of gadgets} \]

Eq. $V = \text{Sp}(B)$ after $m, a \in \mathbb{Z}$

$X = X_V \text{ gadget } \quad X_V(R) = \text{Hom}_{\mathbb{Z}}(B, R)$

$A_{X_V} = B \otimes \mathbb{Z} C \quad e_{X_V}(f) = f \otimes 1 \quad \forall f \in X_V(R)$

Call $X_V = F_{(V)}$ functor from $\mathbb{Z}$-varieties to gadgets
Affine variety over $\mathbb{F}_p$ is a gadget $X$ st.

$\exists \ X_{Z}$ variety over $\mathbb{Z}$ and

immersion $i: X \rightarrow F(X_{Z})$ of gadget $st.$

$\forall \ V \text{ aff. over } \mathbb{Z} \quad \forall \ p: X \rightarrow F(V)$

$\exists \ \text{alg. morphism of varieties}$

$\varphi: X_{Z} \rightarrow V \quad s.t. \quad \varphi = f|_{X_{Z}} \circ i$

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Note: what is the gadget structure saying?

Think $R = \mathbb{Z}[\frac{1}{n}\mathbb{Z}]$ for some $n$.

Then $\text{Hom}(A_{\mathbb{C}}, R_{\mathbb{C}}) = X_{\mathbb{C}}(R_{\mathbb{C}})$

for $X_{\mathbb{C}} = \text{Sp}(A_{\mathbb{C}})$ $R_{\mathbb{C}}$-points of the affine scheme $X_{\mathbb{C}}$.

for $R_{\mathbb{C}} = \mathbb{C}[\frac{1}{n}\mathbb{Z}]$ the $R_{\mathbb{C}}$-points of $X_{\mathbb{C}}$ are the cyclotomic pts

( pts def. over a cycl. ext. if $\mathbb{Q}[\frac{1}{n}\mathbb{Z}]$ )

$$x(R) \rightarrow \text{Hom}(A_{\mathbb{C}}, R_{\mathbb{C}})$$

evaluation maps at cyclotomic points

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Idea: $X_{\mathbb{Z}}$ determined completely by its cyclotomic points.

(a finite "combinatorial" information)

$\Rightarrow X_{\mathbb{Z}}$ is coming from $\mathbb{F}_p$. 

that is some

idea of Soule's definition:
Example: the roots of unity

\( \mu^{(n)} \) as gadget

\( \mu^{(n)}(R) := \{ x \in R : x^n = 1 \} \)

\( \mu^{(n)}(R) = \text{Hom}_\mathbb{Z}(A_k, R) \)

aff. Var. over \( \mathbb{F}_i \)

\( \mu^{(\infty)}(R) := \lim_{\longrightarrow} \text{Hom}(A_k, R) = \text{Hom}(\mathbb{Z}[\frac{\mathbb{Q}}{\mathbb{Z}}], R) \)

pro-variety over \( \mathbb{F}_i \)

Then the homomorphisms of rings

\( \tau_{m,n} : A_m \rightarrow A_m \) when \( n \mid m \)

\( \tau_{m,n}(u/m) = u^{m/a} \)

\( a = \frac{m}{n} \)

determine the "extensions" \( F_n \subset F_m \)

\[ \Rightarrow \begin{cases} 
F_n \otimes F_i \mathbb{Z} = \mathbb{Z}[t, t^{-1}] / (t^n - 1) = A_n 
\end{cases} \]

So the abelian part \( \mathbb{Z}[\frac{\mathbb{Q}}{\mathbb{Z}}] \) of BC algebra
meaning of tower of extensions \( F_n \) of \( F_i \)

The maps of proj. system coming from \( \mathbb{Z} \) endomorphisms (Frobenius action)
A different approach to $\mathbb{F}_1$-geometry

(Borger) $\Lambda$-rings

$R$ ring (underlying abelian group torsion free)

+ action of $\mathbb{N}$ multiple semigroup by endomorphisms $s_p x^p$

so that $s_p(x) - x^p \in pR \quad \forall x \in R$

(that is : the action of $s_p$ is a lift of the Frobenius mod $p$)

a morphism of $\Lambda$-rings

$f : R \rightarrow R' \quad \text{st.} \quad f(s_p(x)) = s'_p(f(x)) \quad \forall x \in R$

i.e. $f(s_n(x)) = s'_n(f(x)) \quad \forall n \in \mathbb{N} \quad \forall x \in R$

Prototype case of $\Lambda$-ring

$R = \mathbb{Z}[t, t^{-1}]/t^{n+1} = A_n$

$s_k(P) (t, t^{-1}) = P(t^k, t^{-k})$

$\Rightarrow$ The BC algebra is direct limit of $\Lambda$-rings

meaning $\underset{k}{\lim} A_k \otimes \mathbb{Q} = \mathbb{Q}[s_2^-]$

with $s_k$'s giving $\mathcal{S}_k$-endom.

and compatible integral models $\rightarrow \mathbb{Z}$
Various generalizations of Bosi-Cames system:

1. $G_1 \approx G_2$ (two-dimensional $\mathbb{Q}$-lattices; elliptic curves, modular forms; Shimura variety of $G_2$)

2. Systems for general Shimura varieties + for number fields

3. How much $(G, \mathbb{Q})$ determines $K$? (anabelian geometry)

4. Function field case: $G_{SM}$ in positive characteristic

5. Drinfeld modules

6. "Multivariable" BC systems; $\Lambda$-rings; $F_1$-geometry

7. Discussion of real quadratic fields and noncommutative geometry

8. An analogy: 3-manifolds and surgery presentations
   - Loop quantum gravity: spin networks and foams

9. Endomorhies

   Bosi-Cames: "Hecke algebras, type III factors ..." Selecta 1995
   Cames-Marchali: "QSM of $\mathbb{Q}$-lattices" 2006
   Ha-Pigom: "Bosi-Cames, Marchali: Systems of Shimura varieties" 2005
   Cames-Marchali-Ramavashram: "RMS stiffo and complex multiplicity" 2005
   Cassemi-Marchali: "QSM over function fields" (ser. B Jwera's paper)
   Marchali: "NCG and Arithmatic" arXiv (ICM talk)
   Marchali: "Cyclotony & Endomorhies" Public number. Ulrum and analysis and
   etc.