Anyon Systems and Quantum Computation

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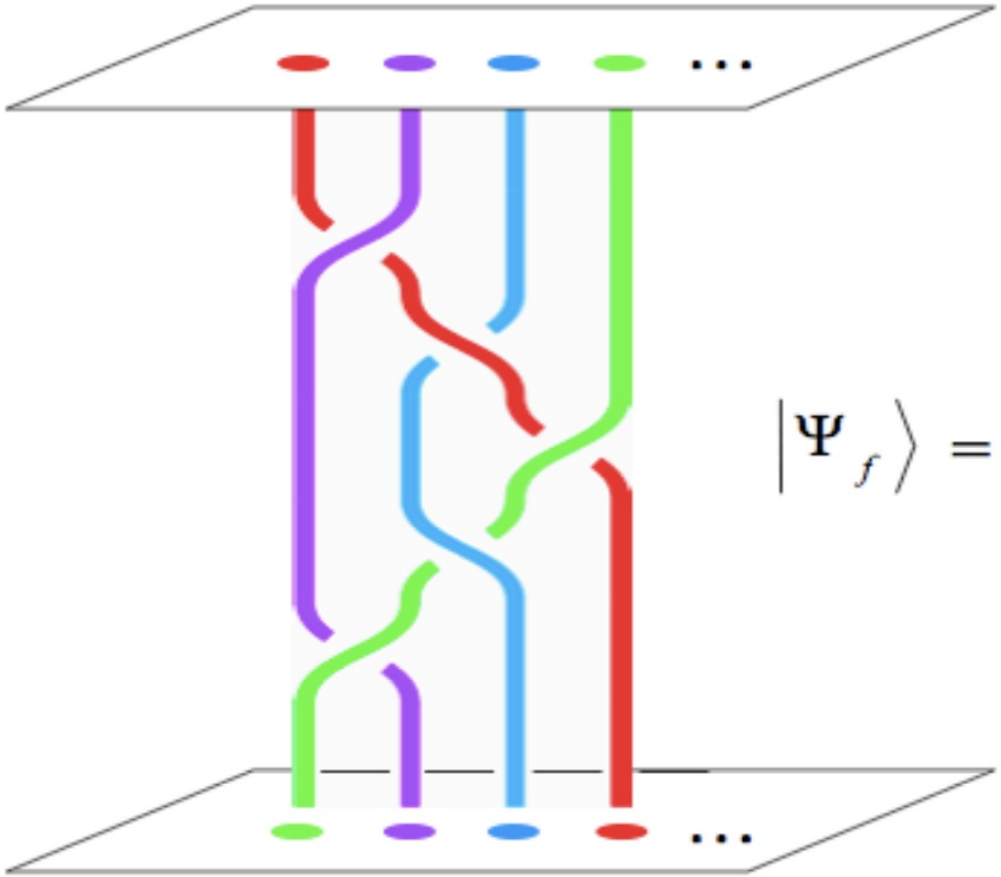
Based on:

- M. Marcolli, J. Napp, *Quantum computation and real multiplication*, Math. Comput. Sci. 9 (2015), no. 1, 63–84.
- M. Freedman, A. Kitaev, M. Larsen, Z. Wang, *Topological quantum computation*. Mathematical challenges of the 21st century (Los Angeles, CA, 2000). Bull. Amer. Math. Soc. (N.S.) 40 (2003), no. 1, 31–38.
- C. Nayak, S. Simon, A. Stern, M. Freedman, S. Das Sarma, *Non-abelian anyons and topological quantum computation*, Rev. Modern Phys., Vol. 80 (2008), no. 3, 1083–1159.

Anyons



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Kareljan Schoutens and Nick Bonesteel, illustration of non-abelian anyons

Qbits and quantum gates

- qbits $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- entangled states (two qbits)

$$\beta_{00} \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- quantum gates: analog of classical logical gates implemented by unitary operators on qubits
- single-qubit X and Z gates

$$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

bit flip $X|0\rangle = |1\rangle$ and phase flip $Z|+\rangle = |-\rangle$ for $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

- set of quantum gates is universal if any unitary operator can be approximated to arbitrary accuracy by a finite sequence of gates in the set

A useful property

- a universal set of quantum gates:
  1. the 1-qbit gates given by elements of $SU(2)$
  2. the CNOT gate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- arbitrary gates in $SU(2^N)$ can always be decomposed as tensor products of CNOTs and 1-qbit gates
- to show some set of uintaries (for example unitary representations of certain braid groups) are universal for quantum computing, it suffices to show that they approximate with arbitrary precision all the 1-qbit gates and CNOT

Braids and statistics

- system of n identical particles on a manifold X are described by configuration space

$$\mathrm{Conf}_n(X) := F_n(X)/S_n$$

with complements of the diagonals $F_n(X) = X^n \setminus \Delta$

$$F_n(X) = \{(x_1, \ldots, x_n) \in X^n \,|\, x_i \neq x_j,\ \forall i \neq j,\ i,j = 1, \ldots, n\}$$

and symmetric group S_n acting freely on $F_n(X)$

- braid groups of X are the fundamental groups

$$B_n(X) := \pi_1(\mathrm{Conf}_n(X))$$

- higher homotopy groups $\pi_i(F_n(X)) = \pi_i(\mathrm{Conf}_n(X))$ for $i \geq 2$ and fundamental groups form exact sequence

$$1 \to \pi_1(F_n(X)) \to \pi_1(\mathrm{Conf}_n(X)) \to S_n \to 1$$

higher dimensional case

- if X is a manifold of dimension $m = \dim X > 2$ then
  1. $\pi_1(F_n(X)) = \pi_1(X)^n$
  2. braid group is a wreath product

$$B_n(X) = \pi_1(X) \wr S_n = \pi_1(X)^n \rtimes S_n$$

  3. higher homotopy groups

$$\pi_i(\mathrm{Conf}_n(X)) = \pi_1(X)^n, \qquad \text{for } 2 \leq i \leq m-2$$

- set of irreducible unitary representations of the braid group $B_n(X) = \pi_1(\mathrm{Conf}_n(X))$ labels inequivalent quantizations of the classical system of n particles

Fermions, Bosons, and Parastatistics

- fermions and bosons correspond to 1-dimensional representations of S_n (sign nontrivial or trivial)
- parastatistics are higher dimensional representations of the symmetric groups S_n
- if X with $\dim X \geq 3$ and simply connected $\pi_1(X) = 1$ then $B_n(X) = S_n$ and quantization gives only fermions, bosons, and parastatistics
- if $\dim X \geq 3$ and $\pi_1(X) \neq 1$ also generalized parastatistics coming from higher dim representations of wreath products $\pi_1(X) \wr S_n$
- in dimension $\dim X \geq 3$ there are no other possible quantizations

Braid groups in dimension 2

- the braid group $B_n(X)$ for $\dim X = 2$ is *not* simply a wreath product: explicit presentation (Birman)
- Σ closed surface of genus g presentation of $B_n(\Sigma)$
  - generators σ_i, $i = 1, \dots n-1$, a_ℓ, b_ℓ, $\ell = 1, \dots, g$
  - Artin braid relations

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i, && \text{when } |i-j| \geq 2 \\ &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, && i = 1, \dots, n-2, \end{aligned}$$

  - commutator relation

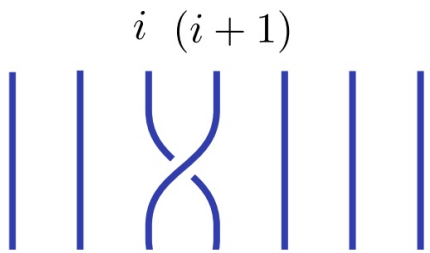
$$\prod_{\ell=1}^{g} [a_\ell, b_\ell^{-1}] = \sigma_1\sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1$$

  - intertwining relations

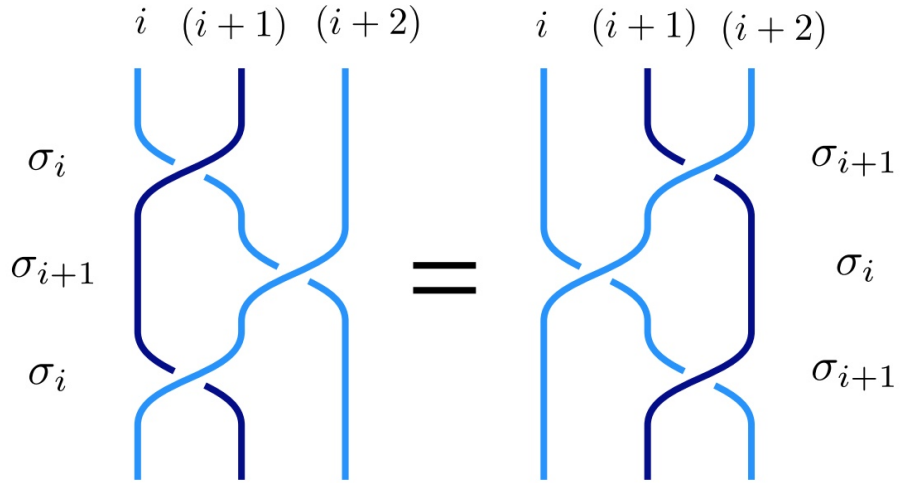
$$a_\ell\sigma_i = \sigma_i a_\ell, \qquad b_\ell\sigma_i = \sigma_i b_\ell, \quad i \neq 1,$$

$$\begin{aligned} &\sigma_1^{-1}a_\ell\sigma_1^{-1}a_\ell = a_\ell\sigma_1^{-1}a_\ell\sigma_1^{-1} && && \sigma_1^{-1}b_\ell\sigma_1^{-1}b_\ell = b_\ell\sigma_1^{-1}b_\ell\sigma_1^{-1} && \\ &\sigma_1^{-1}a_\ell\sigma_1^{-1}b_\ell = b_\ell\sigma_1^{-1}a_\ell\sigma_1 && && \sigma_1^{-1}a_\ell\sigma_1 a_r = a_r\sigma_1^{-1}a_\ell\sigma_1 && \ell < r, \\ &\sigma_1^{-1}b_\ell\sigma_1 b_r = b_r\sigma_1^{-1}b_\ell\sigma_1 && \ell < r, && \sigma_1^{-1}a_\ell\sigma_1 b_r = b_r\sigma_1^{-1}a_\ell\sigma_1 && \ell < r, \\ &\sigma_1^{-1}b_\ell\sigma_1 a_r = a_r\sigma_1^{-1}b_\ell\sigma_1 && \ell < r, && \end{aligned}$$

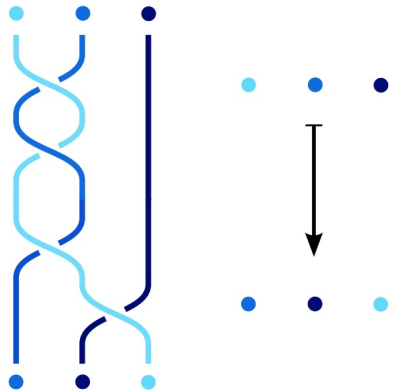
Standard generator σ_i



Braid relations



Surjection $\mathbf{B}_n \longrightarrow S_n$



Anyons

- so in the case of $\dim X = 2$ different class of statistics
- 2-dimensional quasiparticles with nontrivial braiding statistics
- two different cases Σ of genus $g > 0$ and $\Sigma = S^2$ of genus zero
- abelianizations

$$\pi_1(\mathrm{Sym}^n(\Sigma))^{ab} = \pi_1(\Sigma)^{ab} = H^1(\Sigma, \mathbb{Z})$$

$$B_n(\Sigma)^{ab} = H^1(\Sigma, \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \quad \text{for } g > 0$$

with $S_n^{ab} = \mathbb{Z}/2\mathbb{Z}$

- so 1-dimensional representations of $B_n(\Sigma)$ for genus $g > 0$ are characters of $H^1(\Sigma, \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$ ($\sigma_i = \pm 1$: only bosons and fermions, and generators a_i, b_i act like phases)
- but in genus zero get different representations (anyons)

- braid groups $B_n(S^2)$: generators σ_i with braid relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i, && \text{when } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, && i = 1, \dots, n-2, \end{aligned}$$

  and additional relation

$$\sigma_1\sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2\sigma_1 = 1$$

- element $\sigma_1\sigma_2 \cdots \sigma_{n-1}$ has finite order $2n$ (while for higher genus no elements of finite order)
- also $\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2$ has finite order $2(n-1)$ and $\sigma_1\sigma_2 \cdots \sigma_{n-3}\sigma_{n-2}^2$ has order $2(n-2)$ and all finite order elements are conjugates of powers of these three
- abelianization

$$B_n(S^2)^{ab} = \mathbb{Z}/2(n-1)\mathbb{Z}$$

- now depends on n and $B_n(S^2)^{ab}$ has 1-dimensional representations that don't factor through $S_n^{ab} = \{\text{sign} = \pm 1\}$
- these are fractional statistics: (abelian) anyons
- restricted spectrum depending on n (because reps of $\mathbb{Z}/2(n-1)\mathbb{Z}$), but also interesting higher dimensional representations

- T. Einarsson, *Fractional statistics on compact surfaces*, Modern Physics Letters B, Vol.5 (1991) N.10, 675–686.
- T.D. Imbo, J. March-Russell, *Exotic statistics on surfaces*, Physics Letters B, Vol.252 (1990) N.1, 84–90.
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- J. Birman, *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. 22 (1969) 213–238.
- P. Bellingeri, *On presentations of surface braid groups*, J. Algebra 274 (2004), no. 2, 543–563.

General anyon systems

- in these 1-dimensional anyon representations swapping two identical anyons induces a nontrivial phase shift of the wavefunction

$$\psi \mapsto e^{\pi i \theta} \psi \,, \qquad \text{with} \qquad \theta \in \mathbb{R} \setminus \mathbb{Z}$$

- anyons with non-abelian braiding statistics (higher-dimensional representations of the braid group):

$$\psi_\alpha \mapsto \sum_\beta \rho_{\alpha\beta} \psi_\beta$$

with ρ a unitary square matrix with dimension corresponding to the degeneracy of the system

Fusion rules

- anyon system: a list of the different particle types and fusion rules that assign to a pair of anyons a combination of resulting anyons

$$x_i \otimes x_j = \oplus_k N_{ij}^k \; x_k$$

  non-negative integers N_{ij}^k specify the admissible fusion channels

- one of the anyon types corresponds to the vacuum $\mathbf{1}$, (ground state of the system) and satisfies

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1} \quad \text{and} \quad \mathbf{1} \otimes x_i = x_i \otimes \mathbf{1} = x_i$$

  for all anyon types in the system

- label the anyon types $\{x_0, \ldots, x_N\}$ so that $x_0 = \mathbf{1}$
- each anyon x_i has a dual x_i^*, also one of the anyons in the list (vacuum $\mathbf{1}$ is self-dual)

- fusion coefficients satisfy

$$N_{0j}^{k} = \delta_{jk}, \quad N_{ij}^{0} = \delta_{ij'}$$

$$N_{ij}^{k} = N_{ji}^{k} = N_{ik'}^{j'} = N_{i'j'}^{k'},$$

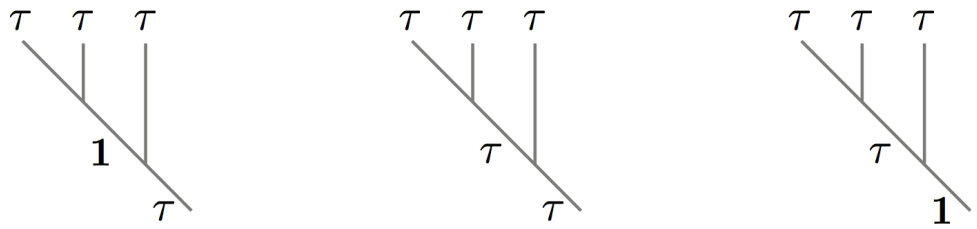
  with $x_{i'} = x_i^*$ dual anyon
- if N_i matrix with $(N_i)_{jk} = N_{ij}^{k}$, then $N_{i'} = N_i^t$ transpose matrix
- Notation: $\Lambda_i = \mathrm{diag}(\lambda_{ij})$ diagonal matrix with entries eigenvalues λ_{ij} of N_i, and S_i matrix with columns corresponding eigenvectors

Fibonacci anyons

- two possible particles, denoted by $x_0 = \mathbf{1}$ and $x_1 = \tau$
- fusion rules

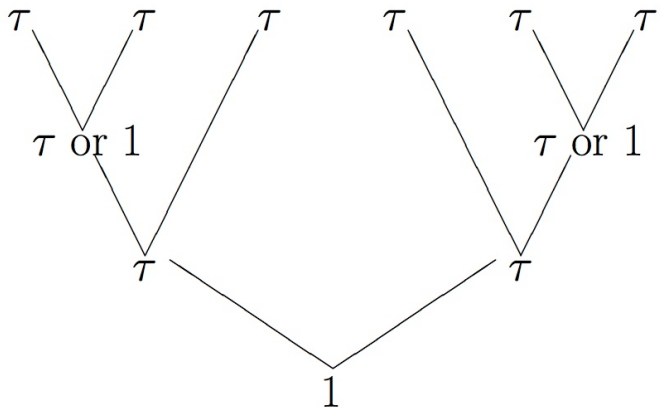
$$x_0 \otimes x_0 = x_0, \quad x_0 \otimes x_1 = x_1 \otimes x_0 = x_1, \quad \text{and} \quad x_1 \otimes x_1 = x_0 \oplus x_1$$

Example of fusion trees:

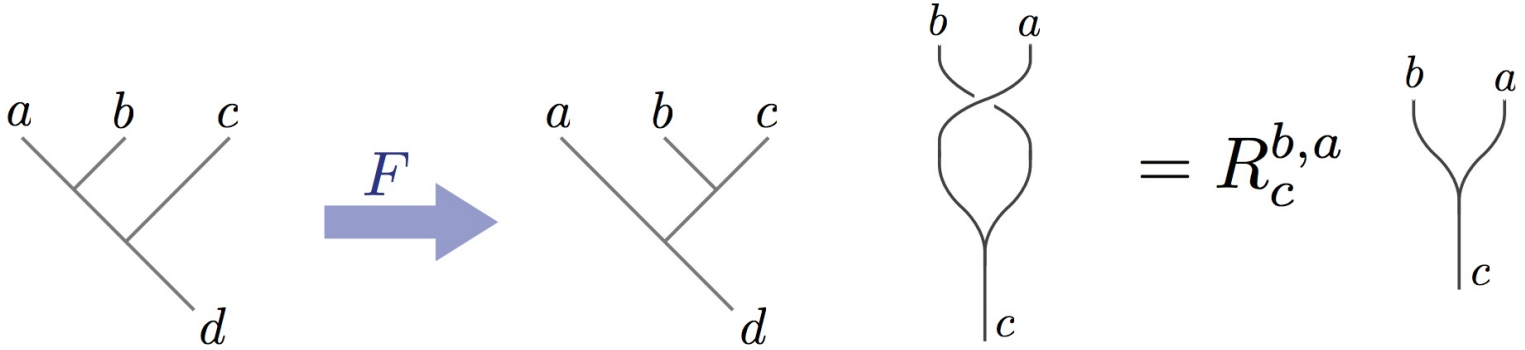


- L. Hormozi, G. Zikos, N. Bonesteel, S. Simon, *Topological quantum compiling*, Phys. Rev. B 75 (2007) 165310.
- A.Y. Kitaev, *Anyons in an exactly solved model and beyond*, Annals of Physics, Vol. 321 (2006) 2–111.

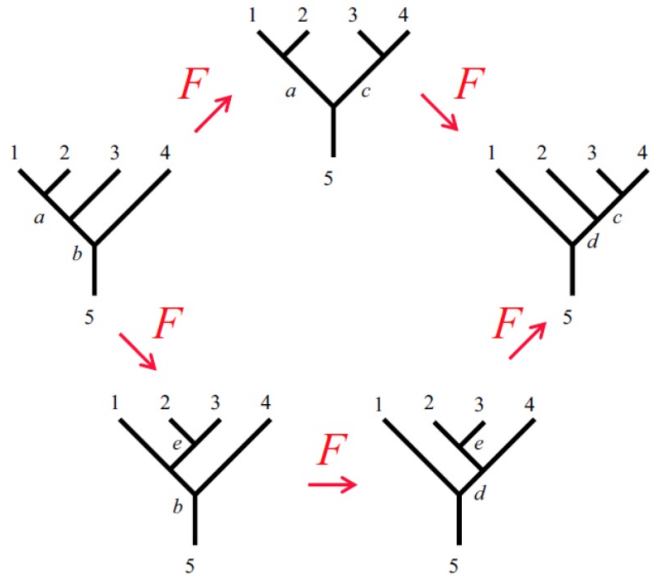
- three τ-anyons fusion trees simulate a single qbit; six τ-anyons simulate two qbits

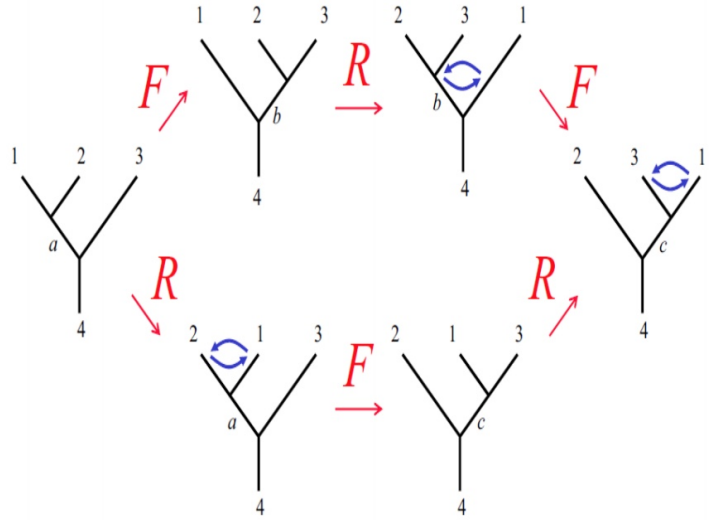


- braiding these anyons simulates unitary transformation on this simulated qubit
- F and R matrices



- associativity: pentagon and hexagon relations, solve for F, R for Fibonacci





$$(F^5_{12c})^d_a (F^5_{a34})^c_b = \sum_e (F^d_{234})^c_e (F^5_{1e4})^d_b (F^b_{123})^e_a$$

$$\sum_b (F^4_{231})^c_b R^4_{1b} (F^4_{123})^b_a = R^c_{13} (F^4_{213})^c_a R^a_{12}$$

- $R = \rho(\sigma_1)$ and $FRF^{-1} = \rho(\sigma_2)$ give the braid representation

Categorical formulation of anyon systems

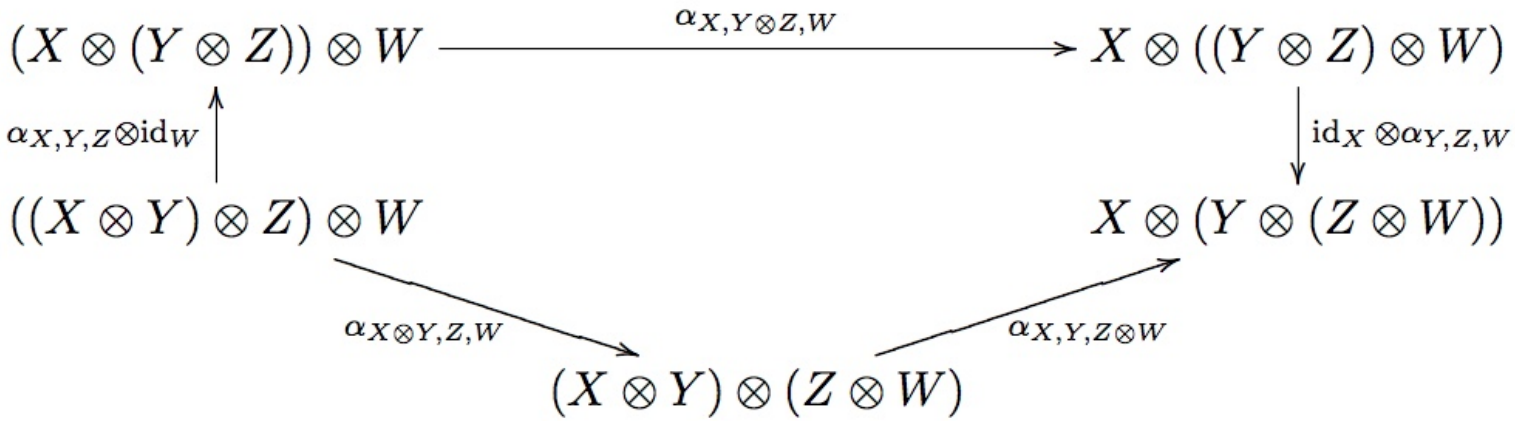
- main categorical setting for topological quantum computing modular tensor categories; also categorical structures: braided monoidal category and fusion categories (categorification of fusion rules and fusion rings)

- category $\mathcal{C}$ is semi-monoidal if it is endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and with natural associativity isomorphisms

$$\tau_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ satisfying the pentagon relation

$$(\tau_{X,Y,Z} \otimes 1_W) \circ \tau_{X,Y \otimes Z,W} \circ (1_X \otimes \tau_{Y,Z,W}) = \tau_{X \otimes Y,Z,W} \circ \tau_{X,Y,Z \otimes W}$$


$(X \otimes (Y \otimes Z)) \otimes W$
$\alpha_{X,Y\otimes Z,W}$
$X \otimes ((Y \otimes Z) \otimes W)$
$\alpha_{X,Y,Z}\otimes \mathrm{id}_W$
$\mathrm{id}_X \otimes \alpha_{Y,Z,W}$
$((X \otimes Y) \otimes Z) \otimes W$
$X \otimes (Y \otimes (Z \otimes W))$
$\alpha_{X\otimes Y,Z,W}$
$\alpha_{X,Y,Z\otimes W}$
$(X \otimes Y) \otimes (Z \otimes W)$

- category $\mathcal{C}$ is braided semi-monoidal if it also has natural symmetry isomorphisms

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$$

for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ satisfying the hexagon relation

$$\tau_{X,Y,Z} \circ \sigma_{X \otimes Y,Z} \circ \tau_{X,Y,Z} = (\sigma_{X,Z} \otimes 1_Y) \circ \tau_{X,Z,Y} \circ (1_X \otimes \sigma_{Y,Z})$$

- the braiding is symmetric if $\sigma_{X,Y} \circ \sigma_{Y,X} = 1_{X \otimes Y}$ for all $X, Y \in \mathrm{Obj}(\mathcal{C})$
- in non-symmetric cases: transparent object X if $\sigma_{X,Y} \circ \sigma_{Y,X} = 1_{X \otimes Y}$ for all objects Y

- category $\mathcal{C}$ is monoidal (or tensor) if semi-monoidal and has *unit object* $\mathbf{1} \in \mathrm{Obj}(\mathcal{C})$ with natural isomorphisms

$$\lambda_X : \mathbf{1} \otimes X \to X \quad \text{and} \quad \rho_X : X \otimes \mathbf{1} \to X$$

satisfying triangle relation

$$1_X \otimes \lambda_Y = (\rho_X \otimes 1_Y) \circ \tau_{X,\mathbf{1},Y}$$

- given semi-monoidal category $\mathcal{C}$, unit augmentation $\mathcal{C}^+$ a monoidal category given bycoproduct $\mathcal{C} \amalg I$, where I is the trivial group seen as a category with a single object $\mathbf{1}$ and morphism $1_{\mathbf{1}}$, and with the tensor product $\otimes$ on $\mathcal{C}$ extended by $- \otimes \mathbf{1} = \mathbf{1} \otimes - = Id_{\mathcal{C} \amalg I}$
- category $\mathcal{C}$ is braided monoidal if it is monoidal and braided semi-monoidal, with the compatibility between the braiding and the unit given by

$$\lambda_X \circ \sigma_{X,\mathbf{1}} = \rho_X.$$

- $\mathcal{C}$ is abelian if
  1. preadditive: $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are abelian groups
  2. with zero object (both initial and terminal): 0 for every other object C unique morphism $0 \to C$ (initial) and unique morphism $C \to 0$ (terminal)
  3. has finite direct sums and direct products
  4. every morphism has a kernel and a cokernel
  5. epimorphisms and monomorphisms are *normal*:
     - epimorphism $f : X \to Y$: if $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2 : Y \not\varnothing Z$
     - monomorphism = epimorphism in opp category
     - normal: every epimorphism is a cokernel of some morphism and every monomorphism is a kernel of some morphism
     - example: in Grps monomorphisms normal only if image is normal subgroup
- category $\mathcal{C}$ is $\mathbb{C}$-*linear* if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are $\mathbb{C}$-vector spaces, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$

- simple object in an abelian category: object C not isomorphic to zero object, whose every subobject is isomorphic to zero or to C
- category $\mathcal{C}$ is semisimple if abelian category where every object can be written as a direct sum of simple objects
- $\mathcal{C}$ is finite if there are only finitely many simple objects up to isomorphism
- if $\mathcal{C}$ is a monoidal category and $\mathbb{C}$-linear finite semisimple, one assumes also that the unit object $\mathbf{1}$ is a simple object
- fusion category: a $\mathbb{C}$-linear finite semisimple rigid tensor category where all the Hom spaces are finite dimensional
- modular tensor categories: a special class of fusion categories where the only *transparent objects* are multiples of the unit $\mathbf{1}$ (opposite end with respect to symmetric case)
- modular tensor categories encode topological structure of rational 2D conformal field theory

Fusion rules and categorification

- anyon systems described as particle types and fusion rules: discrete data $\Rightarrow$ algebraic structure fusion ring
- underlying categorical setting: categorification of fusion rules and fusion ring
- Modular Data (not always from a modular tensor category)
  - modular fusion rule
    1. a set $\{x_i\}_{i=1,\dots,N}$ of anyon types
    2. charge conjugation matrix $C_{ij} = \delta_{ii'}$ where $x_{i'} = x_i^*$ is the dual
    3. set of $N \times N$-matrices $N_i = (N_{ij}^k)$, for $i = 1, \dots, N$ with entries in $\mathbb{Z}_{\geq 0}$ and with a common basis of eigenvectors

$$N_i = \tilde{S}\, \Lambda_i\, \tilde{S}^\dagger$$

       with $\Lambda_i = \mathrm{diag}(\lambda_{ij})$ diagonal matrix of eigenvalues of N_i and columns of $\tilde{S}$ corresponding eigenvectors
  - diagonal matrix T with $T_{ij} = \delta_{ij}\theta_j$ and $\theta_i \in U(1)$, satisfying $(ST)^3 = D_+ S_{00}\, S^2$, where $S = D^{-1}\tilde{S}$ with $D = (\sum_i d_i^2)^{1/2}$ with d_i the Perron–Frobenius eigenvalue of N_i, and $D_+ = \sum_i \theta_i d_i^2$

- fusion ring $\mathcal{R}$ associated to a modular fusion rule: polynomial ring $\mathbb{Z}[x_1, \ldots, x_N]$ modulo ideal generated by relations

$$x_i x_j = \sum_k N_{ij}^k x_k$$

- categorification problem: constructing a braided monoidal category (or a fusion category) $\mathcal{C}$ with $K_0(\mathcal{C}) = \mathcal{R}$
- Note: an abelian monoidal category $\mathcal{C}$ has Grothendieck group $K_0(\mathcal{C})$ that also has ring structure
- only few cases come from modular tensor categories
- E. Rowell, R. Stong, Z. Wang, *On classification of modular tensor categories*, Comm. Math. Phys. Vol. 292 (2009), no. 2, 343–389.

How good are anyon systems?

- What makes an anyon system good?

Properties for quantum computation

- quantum computation from anyon systems:
  - anyon system $\Longrightarrow$ an associated braid group
  - a braid group $\Longrightarrow$ unitary representations
  - does unitary representation span *densely* the group $SU(2^N)$?
  - $SU(2^N) \Longrightarrow$ quantum gates for a system of N qbits
- if density in $SU(2^N)$ holds: anyon system is *universal for quantum computing*

Examples of anyon systems universal for quantum computing

- Fibonacci anyons
- Jones unitary representations of $B_n = B_n(D^2)$
- TQFT: Chern–Simons theory at 5-th root of unity

(Jones representation at $q = e^{\pm 2\pi i/5}$)

- general fact about quantum gates: arbitary gates in $SU(2^N)$ can be decomposed into tensor product of 1-qbit gates and CNOTs

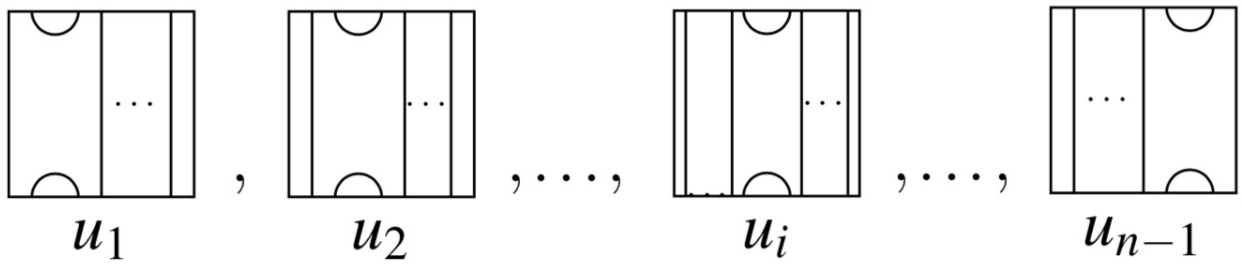
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Chern-Simons (Freedman–Larsen–Wang)

- topological modular functor of a TQFT:
  - 2D surface $\Sigma \Rightarrow$ complex (hermitian) vector space $V(\Sigma)$
  - diffeomorphisms $\Rightarrow$ (projective) unitary maps on $V(\Sigma)$
- consider $\Sigma = (D^2, 3\ell)$ disc with 3ℓ marked points
  - $S_\ell = (\mathbb{C}^2)^{\otimes \ell}$ state space of ℓ qbits
  - construct a map $S_\ell \hookrightarrow V(D^2, 3\ell)$
  - embedding intertwines action of $B_{3\ell}$ on $V(D^2, 3\ell)$ by diffeos of D^2 preserving set of marked points and action of unitary operators on S_ℓ
  - B_3 action on $V(D^2, 3) = \mathbb{C}^2$ (single qbit) gives the 1-qbit quantum gates
  - the 2-qbit CNOT gate is obtained via an approximation algorithm

Jones representation

- Temperley–Lieb algebra $TL_n(A)$



relations: far commutativity, braid relations, Hecke relations
multiplication by $d = -A^2 - A^{-2}$ when a loop removed in composition (vertical stacking of diagrams)

- Construction of the Jones unitary representation:
  - braid group B_n, group algebra $\mathbb{F}[B_n]$
  - mapping $\mathbb{F}[B_n]$ to Temperley-Lieb algebra $TL_n(A)$ (using Kauffman bracket)
  - inner product on $TL_n(A)$ given by Markov trace (closing diagram and counting loops)
  - identify $TL_n(A)$ with a sum of matrix algebras

- generic Temperley–Lieb algebra $TL_n(A)$ (where A is not specialized to a value in $\mathbb{C}$) is isomorphic to a sum of matrix algebras $\oplus_a M_{m_{n,a}}(\mathbb{F})$, where the ranks $m_{n,a}$ correspond to the decompositions of the Catalan numbers as a sums of squares $C_n = \sum_{a=0}^{n} m_{n,a}^2$
- for a specialization of A at a root of unity (as in the Jones representations associated to Chern–Simons topological field theories of order k) Temperley–Lieb–Jones algebra $TLJ_n(A)$ is a quotient of $TL_n(A)$
- at a root of unity Temperley–Lieb–Jones algebra $TLJ_n(A)$ is isomorphic to a sum of matrix algebras $\oplus_a M_{m_{n,a}}(\mathbb{C})$
- representations of the braid group

$$\rho_{m,k,a} : B_m \to TLJ_n(A) \to M_{m_{n,a}}(\mathbb{C}).$$

Chern-Simons and Jones representation
(Freedman–Larsen–Wang)

- Chern-Simons modular functor (at level r) constructed using irreducible representations of quantum groups (Reshetikhin–Turaev)
- construction of hermitian inner products that make CS modular functor unitary
- expression in terms of Jones representation $q = A^4 = e^{2\pi i/r}$ at fifth root of unity $r = 5$

Anyon systems and AF algebras

- fusion rules for anyon systems determine the data of a stationary Bratteli diagram
- Bratteli diagrams determine AF (approximately finite dimensional) C^*-algebras
- algebra comes with a representation on a Hilbert space that is isomorphic to the state space of the anyon system described in terms of fusion paths
- the K_0 group of the AF algebra explicit expression in terms of the quantum dimensions of the anyon system
- also endowed with a product structure induced by the fusion ring of the anyon system

Bratteli diagrams

- Bratteli diagram is an infinite directed graph $\Gamma = (V, E)$
  - vertex set has a partitioning $V = \cup_{n \geq 0} V_n$
  - edge set has a partitioning $E = \cup_{n \geq 0} E_n$, where E_n is the set of oriented edges with source in V_n and target in V_{n+1}
- V_0 consists of a single vertex
- incidence matrices φ_n of the Bratteli diagrams are given by

$$(\varphi_n)_{ij} = \#\{e \in E_n \,|\, s(e) = v_i \in V_n,\ t(e) = v_j \in V_{n+1}\}$$

- telescoping: Bratteli diagram Γ form a new Γ': given sequence $0 < m_1 < \cdots < m_k < \cdots$, new diagram Γ' has $V'_n = V_{m_n}$ and edge set E'_n given by all the possible directed paths in the original diagram Γ with source in V_{m_n} and target in $V_{m_{n+1}}$
- incidence matrices of telescoping diagrams given by product

$$\varphi'_n = \prod_{k=m_n}^{m_{n+1}-1} \varphi_k$$

Bratteli diagrams and AF algebras

- AF algebra $\mathbb{A}$ is a C^*-algebra given by a direct limit

$$\mathbb{A} = \varinjlim_n \mathbb{A}_n$$

  of finite dimensional C^*-algebras $\mathbb{A}_n$
- Wedderburn's theorem: direct sums of matrix algebras over $\mathbb{C}$

$$\mathbb{A}_n = \bigoplus_{k=1}^{N_n} \mathcal{M}_{r_k}(\mathbb{C})$$

- then AF algebra can be described by Bratteli diagram with $\#V_n = N_n$ and vertices in V_n decorated by the direct summands $\mathcal{M}_{r_k}(\mathbb{C})$
- embedding $\mathbb{A}_n \hookrightarrow \mathbb{A}_{n+1}$ determined by incidence matrix φ_n: matrix algebra $\mathcal{M}_{r_i}(\mathbb{C})$ of vertex $v_i \in V_n$ embedded in matrix algebra $\mathcal{M}_{r_j}(\mathbb{C})$ of vertex $v_j \in V_{n+1}$ with multiplicity $(\varphi_n)_{ij}$

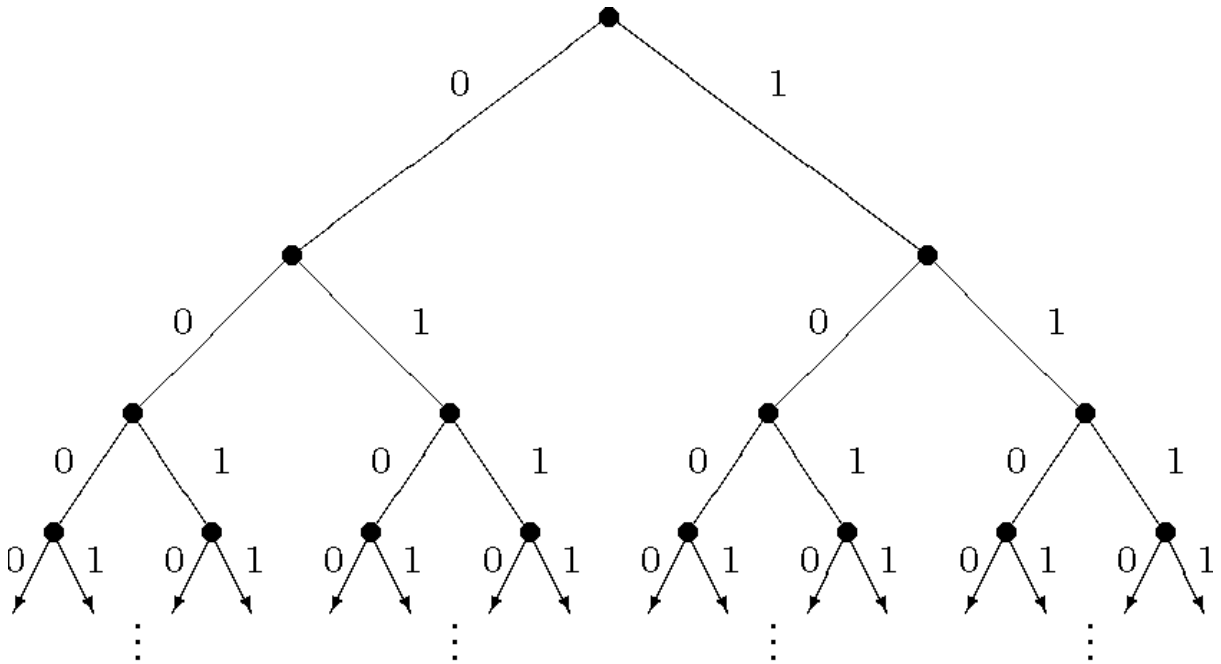
$$\mathcal{M}_{r_i}(\mathbb{C})^{\oplus(\varphi_n)_{ij}} \hookrightarrow \mathcal{M}_{r_j}(\mathbb{C})$$

AF algebras (noncommutative Cantor sets)

C^*-algebras approximated by finite dimensional algebras (direct limits of a direct system of finite dimensional algebras)

determined by Bratteli diagram: $\mathcal{F}_k$ = fin dim algebras $\phi_{k,k+1}$ embeddings with specified *multiplicities*

$C(\Lambda)$ = commutative AF algebra corresponding to the diagram



Example: Fibonacci algebra

Fibonacci AF algebra $\mathcal{F}_n = M_{F_{n+1}}(\mathbb{C}) \oplus M_{F_n}(\mathbb{C})$, embeddings from partial embedding

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Fibonacci Cantor set from the interval $I = [0, 4]$ remove F_{n+1} open intervals $J_{n,j}$ of lengths $\ell_n = 1/2^n$, according to the rule:

- isomorphism of Bratteli diagrams $\Gamma = (V, E)$ and $\Gamma' = (V', E')$: bijection of V and V' and E and E' preserving grading and intertwining source and target maps
- Bratteli diagrams up to the equivalence relation generated by isomorphisms and telescoping
- equivalence class of the Bratteli diagram under this equivalence relation is a complete isomorphism invariant of AF algebra
- Bratteli diagram Γ is stationary if $\#V_n = N$ and $\varphi_n = \varphi$ for all $n \geq 1$
- Bratteli diagram is simple if there is a telescoping Γ' such that all entries of the incidence matrices φ'_n are positive

- Bratteli $\Gamma \Rightarrow$ Hilbert space $\mathcal{H}$ direct limit of the system of finite dimensional Hilbert spaces

$$\mathcal{H}_n = \oplus_{k=1}^{N_n} \mathbb{C}^{r_k}$$

- with embeddings $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$ determined by the incidence matrices φ_n of Γ (need suitable normalization factors so embeddings are isometries)

Fibonacci anyons and AF algebra

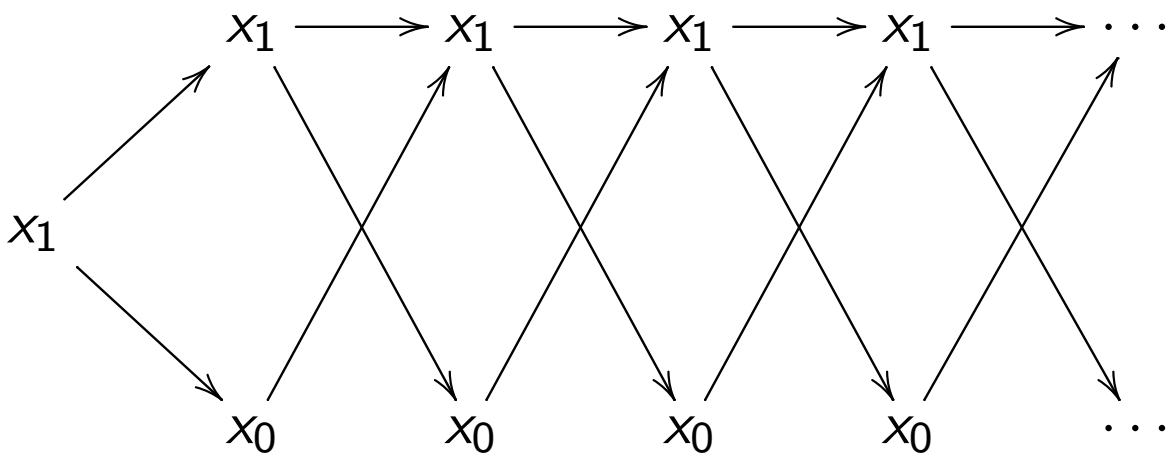
- fusion properties of a system of Fibonacci anyons with an arbitrarily large number of particles
- encoded in the Fibonacci AF C^* algebra given by the direct limit $\mathbb{A} = \varinjlim_n \mathbb{A}_n$ of matrix algebras

$$\mathbb{A}_n = M_{\mathrm{Fib}(n)}(\mathbb{C}) \oplus M_{\mathrm{Fib}(n-1)}(\mathbb{C})$$

  with $\mathrm{Fib}(n)$ the n-th Fibonacci number
- embeddings $\phi_n : \mathbb{A}_n \hookrightarrow \mathbb{A}_{n+1}$ implemented by matrix describing basic fusion rules

$$\varphi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- possible fusion paths in a system of an arbitrary number of particles can be seen as a diagram



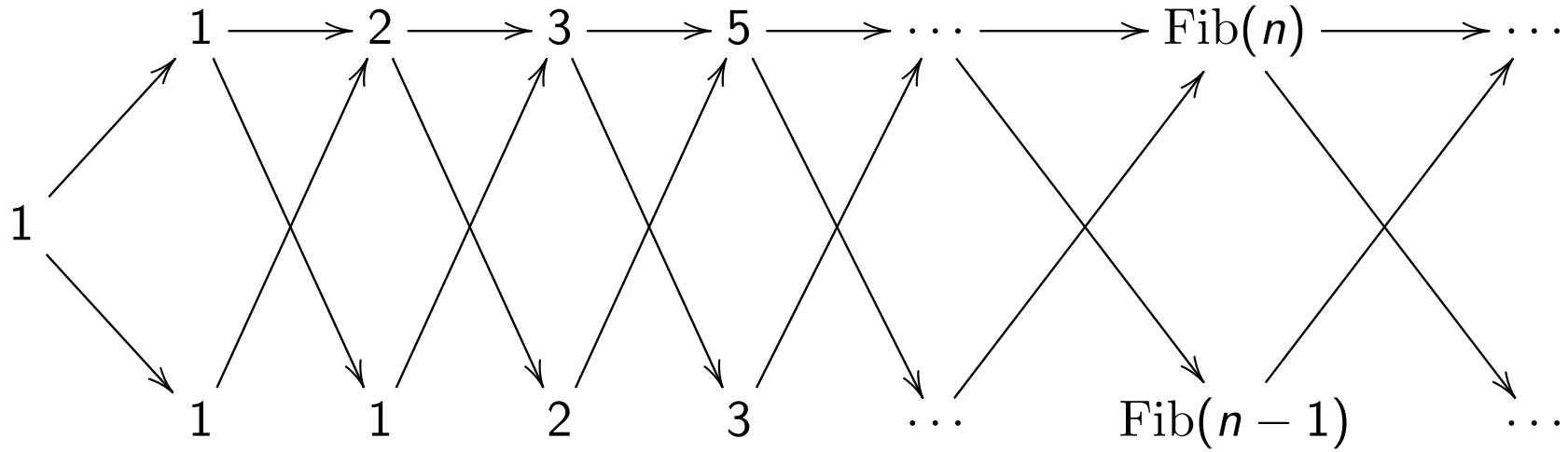
  where $x_1 = \tau$ Fibonacci anyon and $x_0 = \mathbf{1}$
- this diagram depicts how an x_1 particle fused with an x_1 particle can result in an x_0 or x_1 particle, respectively, with probabilities $p(x_0) = \tau^{-2}$ and $p(x_1) = \tau^{-1}$, with $\tau = (1 + \sqrt{5})/2$ the golden ratio
- but an x_0 fused with an x_1 can result only in an x_1
- system with N anyons: N columns in corresponding Bratteli diagram

- Hilbert space defined by Bratteli diagram agrees with the Hilbert space of the anyon system
- basis of the Hilbert space of the system is given by the set of fusion paths
- fusion tree basis for fusions of N Fibonacci anyons (that is, counting of all possible labelings of the fusion paths) has cardinality $\mathrm{Fib}(N)$
- can identify the Hilbert space $\mathcal{H}_N = \mathbb{C}^{\mathrm{Fib}(N)}$ of a system of N Fibonacci anyons system with the Hilbert space

$$\mathbb{C}^{\mathrm{Fib}(N)} \simeq \mathbb{C}^{\mathrm{Fib}(N-1)} \bigoplus \mathbb{C}^{\mathrm{Fib}(N-2)}$$

  constructed after N steps of the Bratteli diagram
- here replace the labeling x_0 and x_1 on the nodes with the dimensions of the corresponding Hilbert space

- diagram is given by



- this diagram as describes direct system of Hilbert spaces $\mathcal{H}_N$ that give a representation of the Fibonacci AF algebra
- Fibonacci AF algebra $\mathbb{A} = \overline{\cup_{n \geq 1} \mathbb{A}_n}$

$$\mathbb{A}_n = M_{\mathrm{Fib}(n)}(\mathbb{C}) \oplus M_{\mathrm{Fib}(n-1)}(\mathbb{C})$$

- direct limit over the stationary Bratteli diagram with

$$\varphi_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- Fibonacci AF algebra $\mathbb{A} = \mathbb{A}_\tau$ is algebra of operators on Hilbert space of anyon system $\mathcal{H} = \varinjlim_n \mathcal{H}_n$

AF algebras of anyon systems

- data $\{x_i\}_{i=0,\dots,N-1}$ and N_{ij}^k of an anyon system
- for each i form a stationary Bratteli diagram Γ_i:
  - vertices $\#V_n = N$ for all $n \geq 1$
  - single vertex of V_0 labelled by x_0
  - N vertices of V_n labelled by $\{x_i\}_{i=0,\dots,N-1}$
  - incidence matrices $\varphi_{i,n} = \varphi_i = N_i$
- AF algebras $\mathbb{A}_i = \varinjlim_n \mathbb{A}_{i,n}$

$$\mathbb{A}_{i,n} = \mathcal{M}_{d_{i,n}}(\mathbb{C})^{\oplus N}$$

acting on Hilbert spaces $\mathcal{H}_i = \varinjlim_n \mathcal{H}_{i,n}$

- Hilbert space $\mathcal{H} = \oplus_i \mathcal{H}_i$ with $\mathcal{H}_i$ determined by the Bratteli diagram Γ_i is isomorphic to the Hilbert space of the anyon system
- embeddings $\mathcal{H}_{i,n} \hookrightarrow \mathcal{H}_{i,n+1}$ are determined by (normalized) incidence matrices $\varphi_{i,n} = \varphi_i$ of Γ_i
- Hilbert space $\mathcal{H}_{i,n} = \oplus_{j=0}^{N-1} \mathcal{H}_{i,n,j}$ has dimension

$$d_{i,n} = \dim \mathcal{H}_{i,n} = \sum_{j_1,\dots,j_n} N_{ij_1}^{j_2} \cdots N_{ij_n}^{j_{n-1}}$$

- individual component $\mathcal{H}_{i,n,j}$ of dimension

$$d_{i,n,j} = \dim \mathcal{H}_{i,n,j} = \sum_{j_1,\dots,j_{n-1}} N_{ij_1}^{j_2} \cdots N_{ij}^{j_{n-1}}$$

- total Hilbert space $\mathcal{H}$ direct limit of the $\oplus_i \mathcal{H}_{i,n}$ of dim $d_n = \sum_i d_{i,n}$
- this counting of dimensions agrees counting of anyon fusion paths

K-theory of C^*-algebras

- $K_0(\mathcal{A})$: idempotents (for C^*-algebras $\Leftrightarrow$ projections: $p \mapsto P = pp^*(1-(p-p^*)^2)^{-1})$, $P \sim Q$ iff $P = X^*X$ $Q = XX^*$, $X =$ partial isometry $(X = XX^*X)$; stable equivalence: $P \sim Q$, $P \in \mathcal{M}_n(\mathcal{A})$ $Q \in \mathcal{M}_m(\mathcal{A})$, $\exists R$ proj $P \oplus R \sim Q \oplus R$ $\Rightarrow K_0(\mathcal{A})^+$, Grothendieck group $= K_0(\mathcal{A})$
- $K_1(\mathcal{A})$:

$$\mathrm{GL}_n(\mathcal{A}) = \text{ invertible elements in } \mathcal{M}_n(\mathcal{A})$$

$$\mathrm{GL}_n^0(\mathcal{A}) = \text{ identity component}$$

$$\mathrm{GL}_n(\mathcal{A}) \to \mathrm{GL}_{n+1}(\mathcal{A}) \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathrm{GL}_n(\mathcal{A})/\mathrm{GL}_n^0(\mathcal{A}) \to \mathrm{GL}_{n+1}(\mathcal{A})/\mathrm{GL}_{n+1}^0(\mathcal{A})$$

direct limit $= K_1(\mathcal{A})$

- AF algebras: $K_1(\mathcal{A}) = 0$; K_0-group scaled ordered group

$$(K_0(\mathbb{A}), K_0(\mathbb{A})^+, [1_{\mathbb{A}}]) = \varinjlim_{\varphi_n} (K_0(\mathbb{A}_n), K_0(\mathbb{A}_n)^+, [1_{\mathbb{A}_n}])$$

Number field

- relation of AF algebra to number field generated by the eigenvalues of fusion matrices
- algebraic integer λ let $\mathbb{K}_\lambda = \mathbb{Q}[\lambda]$ with $\mathcal{O}_\lambda$ the ring of integers

$$L_\lambda := \mathcal{O}_\lambda[\lambda^{-1}]$$

- torsion-free abelian groups of finite rank G_1 and G_2 are quasi-isomorphic if there exist morphisms $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_1$ with $f_1 \circ f_2 = n\, id_{G_2}$ and $f_2 \circ f_1 = n\, id_{G_1}$ for some $n \in \mathbb{N}$
- K_0-group $K_0(\mathbb{A}_i)$ of the AF algebra $\mathbb{A}_i$ of the anyon system is quasi-isomorphic to

$$L_{\mathbb{A}_i} := \bigoplus_j L_{\lambda_{ij}}$$

λ_{ij} eigenvalues of fusion matrix N_i

- for stationary Bratteli diagram Γ_i with $\varphi_{i,n} = \varphi_i$, and $\mathbb{A}_{i,n} = \oplus_{j=0}^{N-1} M_{d_{i,n,j}}(\mathbb{C})$ have $K_0(\mathbb{A}_{i,n}) = \mathbb{Z}^N$

$$(K_0(\mathbb{A}_i), K_0(\mathbb{A}_i)^+, [1_{\mathbb{A}_i}]) = \varinjlim_{\varphi_i} \mathbb{Z}^N$$

- $\varphi : \mathbb{Z}^N \to \mathbb{Z}^N$ injective homomorphism
- $\mathrm{Spec}(\varphi)$ the set of eigenvalues of φ, algebraic integers
- direct limit is quasi-isomorphic to

$$G = \varinjlim_{\varphi} \mathbb{Z}^N \simeq \bigoplus_{\lambda \in \mathrm{Spec}(\varphi)} L_\lambda^{\oplus n_\lambda}$$

  $n_\lambda =$ dim of Jordan block $J_{n_\lambda}(\lambda)$ in Jordan normal form of φ
- use matrix $\varphi_i = N_i$ equal to

$$N_i = \tilde{S}\, \Lambda_i\, \tilde{S}^\dagger \quad \text{with } \Lambda_i = \mathrm{diag}(\lambda_{ij}) \text{ eigenvalues}$$

- M. Dugas, *Torsion-free abelian groups defined by an integral matrix*, International Journal of Algebra, Vol.6 (2012) N.2, 85–99

- abelian group $L_{\mathbb{A}_i} = \bigoplus_j L_{\lambda_{ij}}$ also has a ring structure
- coming eigenvalues λ_{ij} of the fusion matrices satisfying a Verlinde formula
- Verlinde formula

$$\lambda_{aj}\lambda_{bj} = \sum_c N_{ab}^c \lambda_{c,j}$$

- for fixed j same multiplicative structure as in the fusion ring of the anyon system

$$\mathbb{Z}[x_0, \ldots, x_{N-1}]/(x_i x_j - \sum_k N_{ij}^k x_k)$$

- Converse? No, not every data of an AF algebra and its Bratteli diagrams can determine an anyon system
- especially interesting class of algebras: noncommutative tori with real multiplication and AF algebra embeddings

Noncommutative tori

- quantum torus of modulus $\theta \in \mathbb{R}$ is the universal C^*-algebra $\mathcal{A}_\theta$ generated by two unitaries U, V satisfying the commutation relation

$$UV = e^{2\pi i\theta} VU$$

- NC torus is *not* an AF algebra (it has nontrivial $K_1(\mathcal{A}_\theta) \simeq \mathbb{Z}^2$)
- but C^*-algebra $\mathcal{A}_\theta$ of quantum torus can always be embedded into an AF algebra $\mathbb{A}_\theta$ so that the embedding induces isomorphism on K_0 preserving positive cone
- AF algebra $\mathbb{A}_\theta$ determined by continued fraction expansion of $\theta = [c_0, c_1, c_2, \ldots]$

Continued Fractions

- $k_1, \ldots, k_n \in \mathbb{N}$ and $n \geq 1$

$$[k_1, \ldots, k_n] := \frac{1}{k_1 + \frac{1}{k_2 + \ldots \frac{1}{k_n}}} = \frac{P_n(k_1, \ldots, k_n)}{Q_n(k_1, \ldots, k_n)}$$

P_n, Q_n polynomials with integer coefficients calculated inductively from relations

$$Q_{n+1}(k_1, \ldots, k_n, k_{n+1}) = k_{n+1} Q_n(k_1, \ldots, k_n) + Q_{n-1}(k_1, \ldots, k_{n-1}),$$

$$P_n(k_1, \ldots, k_n) = Q_{n-1}(k_2, \ldots, k_n)$$

put formally $Q_{-1} = 0, Q_0 = 1$

- also have the relation

$$[k_1, \ldots, k_{n-1}, k_n + x_n] =$$

$$\frac{P_{n-1}(k_1, \ldots, k_{n-1})\, x_n + P_n(k_1, \ldots, k_n)}{Q_{n-1}(k_1, \ldots, k_{n-1})\, x_n + Q_n(k_1, \ldots, k_n)} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} (x_n)$$

where action of GL_2 by fractional linear transformations

$$z \mapsto \frac{az + b}{cz + d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z)$$

- for $\alpha \in (0,1)$ irrational, unique sequence of integers $k_n(\alpha) \geq 1$ such that α is the limit of $[\, k_1(\alpha), \ldots, k_n(\alpha)\,]$ as $n \to \infty$
- unique sequence $x_n(\alpha) \in (0,1)$ such that

$$\alpha = [\, k_1(\alpha), \ldots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)\,]$$

  for each $n \geq 1$
- rational numbers $\alpha \in \mathbb{Q}$ have a finite continued fraction expansion $\alpha = [\, k_1(\alpha), \ldots, k_n(\alpha)\,]$ for some $n \in \mathbb{N}$ and an ambiguity in the continued fraction representation $[k_1, \ldots, k_n] = [k_1, \ldots, k_n - 1, 1]$ (resolve ambiguity by taking shortest)

- convergents of the continued fractions expansion
  - irrational α

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & k_1(\alpha) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n(\alpha) \end{pmatrix} (x_n(\alpha)).$$

  - successive numerators and denominators

$$p_n(\alpha) := P_n(k_1(\alpha), \ldots, k_n(\alpha)),\ q_n(\alpha) := Q_n(k_1(\alpha), \ldots, k_n(\alpha))$$

  - so that $p_n(\alpha)/q_n(\alpha)$ is the sequence of convergents of α.
  - associated matrix

$$g_n(\alpha) := \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix}$$

Continued fraction AF algebra

- $q_n = q_n(\theta)$ the denominators of the successive convergents of the continued fraction expansion of θ
- finite dim algebras

$$\mathbb{A}_{n,\theta} = M_{q_n}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C})$$

- embeddings $\mathbb{A}_{n-1,\theta} \hookrightarrow \mathbb{A}_{n,\theta}$ are given by

$$\varphi_n = \begin{pmatrix} 0 & 1 \\ 1 & c_n \end{pmatrix}$$

$$M_{q_{n-1}}(\mathbb{C}) \oplus M_{q_{n-2}}(\mathbb{C}) \to M_{q_{n-1}}(\mathbb{C}) \oplus \left(M_{q_{n-2}}(\mathbb{C}) \oplus M_{q_{n-1}}(\mathbb{C})^{\oplus c_n}\right) \subset M_{q_n}(\mathbb{C})$$

with c_n the n-th digit of the continued fraction expansion, satisfying

$$q_n = c_n q_{n-1} + q_{n-2}$$

noncommutative tori with real multiplication

- quantum tori $\mathcal{A}_\theta$ with irrational modulus $\theta \in \mathbb{R} \setminus \mathbb{Q}$ that is a quadratic irrationality
- $\theta \in \mathbb{K} \setminus \mathbb{Q}$, with $\mathbb{K}$ a real quadratic field
- fixed point of some hyperbolic element $\varphi \in \mathrm{SL}_2(\mathbb{Z})$

$$\varphi(\theta) = \frac{a\theta + b}{c\theta + d} = \theta, \qquad \text{where} \quad \varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- continued fraction expansion of θ is eventually periodic
- noncommutative tori with real multiplication have many similarities with theory of elliptic curves with complex multiplication (Manin)

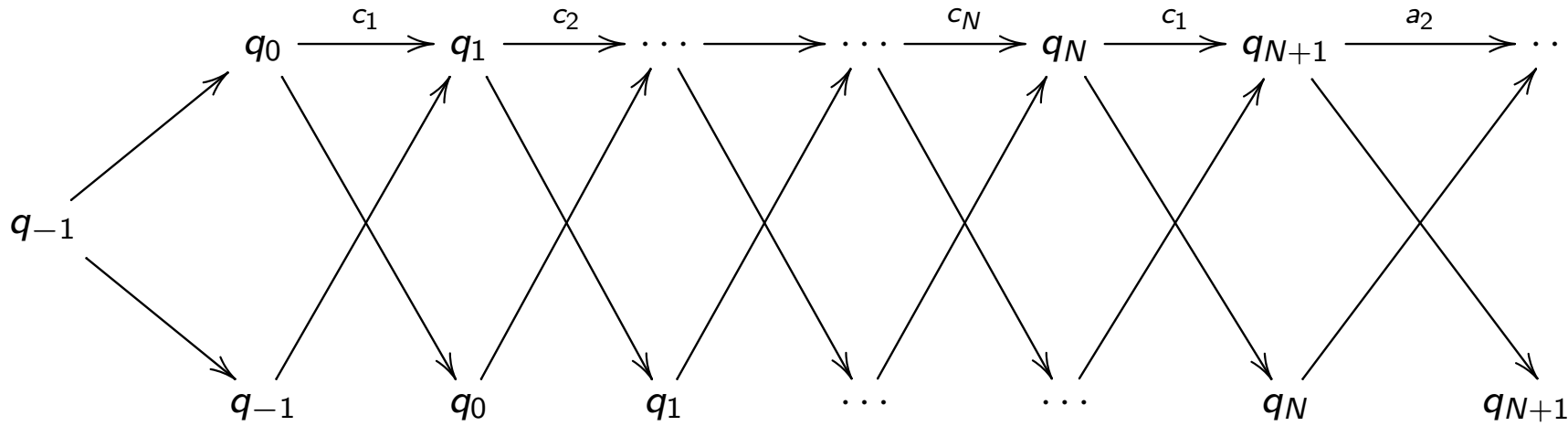
- $\mathrm{Red} = \cup_{n\geq 1}\mathrm{Red}_n$ semigroup of reduced matrices in $\mathrm{GL}_2(\mathbb{Z})$

$$\mathrm{Red}_n = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n \end{pmatrix} \mid k_1, \ldots, k_n \in \mathbb{N} \right\}$$

- reduced matrices are hyperbolic and every conjugacy class g of hyperbolic matrices in $\mathrm{GL}_2(\mathbb{Z})$ contains reduced representatives, all of which have the same length $\ell(g)$
- number of such representatives is $\ell(g)/k(g)$ with $k(g)$ largest integer such that $g = h^{k(g)}$ for some h
- θ is a quadratic irrationality: AF algebra $\mathbb{A}_\theta$ has stationary Bratteli diagram with $\#V_n = 2$ for all $n \geq 1$ and incidence matrix $\varphi_n = \varphi \in \mathrm{Red} \subset \mathrm{GL}_2(\mathbb{Z})$ that fixes the modulus,

$$\varphi(\theta) = \frac{a\theta + b}{c\theta + d} = \theta, \quad \text{where} \quad \varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

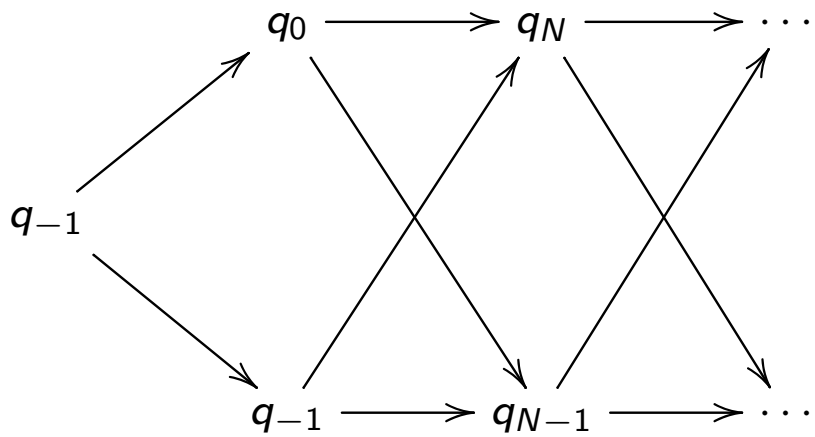
- for any irrational θ have $\mathbb{A}_\theta$ AF algebra with $\mathbb{A}_n = \mathcal{M}_{q_n}(\mathbb{C}) \oplus \mathcal{M}_{q_{n-1}}(\mathbb{C})$ with Bratteli diagram Γ_θ



with $q_{-1} = q_0 = 1$ and $q_n = c_n q_{n-1} + q_{n-2}$

- if $\theta \in \mathbb{R}$ quadratic irrationality $\exists$ hyperbolic matrix $g \in \mathrm{SL}_2(\mathbb{Z})$ with $g \cdot \theta = \theta$ and eventually periodic cont fraction period $[c_1, \ldots, c_N]$
- telescope the Bratteli diagram Γ_θ to a new Γ'_θ that collapses together N successive steps from V_1 to V_N, and then the next N steps and so on, so that in Γ'_θ we have $\varphi'_{\ell N+n} = \prod_{k=1}^{N} \varphi_k$, for all $\ell \geq 0$

- resulting diagram isomorphic to stationary diagram



with incidence matrix $g \in \mathrm{Red} \subset \mathrm{GL}_2(\mathbb{Z})$

$$g = \begin{pmatrix} 0 & 1 \\ 1 & c_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & c_N \end{pmatrix}$$

- in case of $\theta = \tau = (1+\sqrt{5})/2$ golden ratio, continued fraction expansion $\tau = [1,1,1,1,1,\ldots]$, get Fibonacci AF C^*-algebra $\mathbb{A}$

Embedding of the quantum torus in the AF algebra

- generators U and V of the quantum torus $\mathcal{A}_\theta$ are approximated by elements U_n and V_n in $\mathcal{M}_{q_n}(\mathbb{C})$ satisfying the relation

$$U_n\, V_n = \exp(2\pi i \frac{p_n}{q_n})\, V_n\, U_n,$$

  with p_n/q_n the successive quotients of the continued fraction expansion approximation of the modulus θ of the quantum torus.

- the elements U_n and V_n are explicitly given by $q_n \times q_n$-matrices: clock and shift matrices
- approximate generators U_n and V_n act on the Hilbert space $\mathbb{C}^{q_n}$ of the system as (compositions of) phase shifter gates with phase ξ_n and downshift permutation gates on q_n elements

- clock and shift matrices

$$U_n = \begin{pmatrix} 1 & & & & \\ & \xi_n & & & \\ & & \xi_n^2 & & \\ & & & \ddots & \\ & & & & \xi_n^{q_n-1} \end{pmatrix}, \quad \text{with} \quad \xi_n = \exp(2\pi i \frac{p_n}{q_n})$$

$$V_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Bimodules and quantum tori with real multiplication

- action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$ by fractional linear transformations

$$g(\theta) = \frac{a\theta + b}{c\theta + d}, \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

- for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, given $g \in \mathrm{GL}_2(\mathbb{Z})$ considers Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}) = \mathcal{S}(\mathbb{R})^c$ with an action of generators U and V of $\mathcal{A}_\theta$ given by

$$(fU)(t,s) = f(t - \frac{c\theta + d}{c}, s - 1)$$

$$(fV)(t,s) = \exp(2\pi i(t - \frac{ad}{c}))f(t,s)$$

- bimodule structure: also (left) action of $\mathcal{A}_{g(\theta)}$ with generators U' and V'

$$(U'f)(t,s) = f(t - \frac{1}{c}, s - a)$$

$$(V'f)(t,s) = \exp(2\pi i(\frac{t}{c\theta + d} - \frac{s}{c}))f(t,s)$$

- completion in norm $\|f\| = \|_{\mathcal{A}_{g(\theta)}}\langle f, f\rangle\|^{1/2}$
- these bimodule denoted by $E_g(\theta)$ called basic bimodules
- basic bimodules satisfy

$$\mathrm{Hom}_{\mathcal{A}_\theta}(E_g(\theta), E_h(\theta)) \simeq E_{hg^{-1}}(g(\theta))$$

- for θ quadratic irrationality, take a $g \in \mathrm{GL}_2(\mathbb{Z})$ with $g(\theta) = \theta$
- can assume g has non-negative entries and $\det(g) = -1$
- for θ quadratic irrationality and $g(\theta) = \theta$ can also assume $c\theta + d \geq 0$

Real multiplication

- for θ quadratic irrationality and $g(\theta) = \theta$ basic module $E_g(\theta)$ is an $\mathcal{A}_\theta$–$\mathcal{A}_\theta$ bimodule and one can form tensor products over $\mathcal{A}_\theta$

$$E_g(\theta) \otimes_{\mathcal{A}_\theta} E_g(\theta) \otimes_{\mathcal{A}_\theta} \cdots \otimes_{\mathcal{A}_\theta} E_g(\theta) \simeq E_{g^n}(\theta)$$

- bimodule $E_g(\theta)$ generates the nontrivial self Morita equivalences of the torus $\mathcal{A}_\theta$
- the bimodule $E_g(\theta)$ gives real multiplication structure to $\mathcal{A}_\theta$
- self-Morita equivalence: think of the bimodule $E_g(\theta)$ as an "endomorphism" of the quantum torus $\mathcal{A}_\theta = \mathcal{A}_{g(\theta)}$
- like elliptic curves with complex multiplication that have additional homomorphisms coming from imaginary quadratic field, here extra endomorphisms (in the form of bimodules/correspondences) coming from real quadratic field

Category of modules and anyon system

- subcategory $\mathcal{M}_\theta$ of the category of finite projective (right) modules over $\mathcal{A}_\theta$ generated by the basic module $E_g(\theta)$ and by $\mathcal{A}_\theta$, seen as a module over itself
- category $\mathcal{M}_\theta$ is monoidal, with tensor product $\otimes_{\mathcal{A}_\theta}$, with the object $\mathcal{A}_\theta = \mathbf{1}$ as the unit
- $K_0(\mathcal{A}_\theta) = K_0(\mathbb{A}_\theta) = \mathbb{Z} + \mathbb{Z}\theta$
- basic module $E_g(\theta)$ has dimension (given by the range of the trace on K_0) $c\theta + d$
- under the identification $K_0(\mathcal{A}_\theta) = \mathbb{Z} + \mathbb{Z}\theta$, given by the von Neumann trace of the quantum torus, we have $[E_g(\theta)] = c\theta + d$
- unit element satisfies $[\mathcal{A}_\theta] = [E_1(\theta)] = 1$

- also have $[E_{g^2}(\theta)] = c'\theta + d'$, where

$$g^2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- so get

$$[E_{g^2}(\theta)] = c(a+d)\theta + d^2 + bc = \mathrm{Tr}(g)\,(c\theta + d) - \det(g)$$

$$= \mathrm{Tr}(g)\,[E_g(\theta)] - \det(g)[\mathcal{A}_\theta]$$

- if all entries of g are non-negative and $\det(g) = -1$, get

$$[E_{g^2}(\theta)] = \mathrm{Tr}(g)\,[E_g(\theta)] + [\mathbf{1}]$$

- form an anyon system with two anyon types $x_0$and x_1, where $X_0 = \mathbf{1}$ and $X_1 = E_g(\theta)$
- with the tensor product of copies of $E_g(\theta)$ get that corresponding classes $x_0 = [X_0]$ and $x_1 = [X_1]$ in $K_0(\mathcal{A}_\theta)$ generate a fusion ring with

$$x_0 x_i = x_i x_0 = x_i \quad \text{for } i = 0, 1 \text{ and } \quad x_1 x_1 = \mathrm{Tr}(g)\, x_1 + x_0$$

- $\mathcal{M}_\theta$ is a categorification of this fusion ring
- fusion matrix

$$N_1 = \begin{pmatrix} \mathrm{Tr}(g) & 1 \\ 1 & 0 \end{pmatrix}$$

- eigenvalues of N_1 same as eigenvalues of g, solutions of the characteristic polynomial

$$\det(g - \lambda I) = \lambda^2 - \mathrm{Tr}(g)\lambda + \det(g) = \det(N_1 - \lambda I) = 0$$

- eigenvalues satisfy the relation $\lambda_1\lambda_2 = \det(g) = -1$, and the eigenvectors of N_1 satisfy the conditions $y = \lambda x$ and $x(1 + \mathrm{Tr}(g)\lambda - \lambda^2) = 0$
- unitary matrix of eigenvectors

$$\tilde{S} = \frac{1}{(1+\lambda^2)^{1/2}} \begin{pmatrix} 1 & \lambda \\ \lambda & -1 \end{pmatrix}$$

with λ an eigenvalue of the matrix g

- so have

$$g \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} a\theta + b \\ c\theta + d \end{pmatrix} = (c\theta + d) \begin{pmatrix} \theta \\ 1 \end{pmatrix}$$

since $g(\theta) = \theta$, hence $\lambda = c\theta + d$ is an eigenvalue of g

- matrix $\tilde{S}$ of the modular fusion rules

$$\tilde{S} = \frac{1}{(1+(c\theta+d)^2)^{1/2}} \begin{pmatrix} 1 & c\theta + d \\ c\theta + d & -1 \end{pmatrix}$$

Fibonacci anyons in terms of bimodules

- basic bimodule $E_g(\tau)$ satisfies

$$[E_{g^2}(\tau)] = [E_g(\tau) \otimes_{\mathcal{A}_\theta} E_g(\tau)] = [E_g(\tau)] + [\mathbf{1}]$$

- fusion ring generated by $x_0 = [\mathcal{A}_\tau] = [\mathbf{1}]$ and $x_1 = [E_g(\tau)]$ satisfies fusion rules of the Fibonacci anyons

$$x_0 x_i = x_i x_0 = x_i \quad \text{and} \quad x_1 x_1 = x_1 + x_0$$

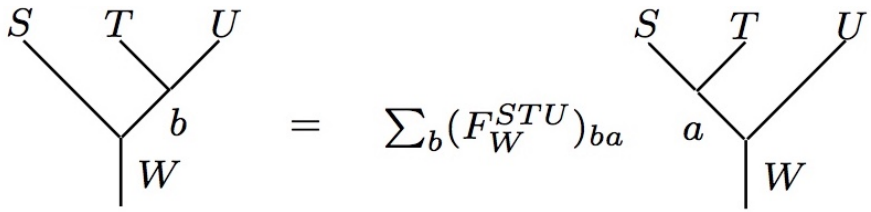
- ring structure agrees with product on $K_0(\mathcal{A}_\tau) = \mathbb{Z} + \mathbb{Z}\tau$ seen as algebraic integers in $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, with $[E_g(\tau)] = \tau$ and $[\mathbf{1}] = 1$ and $[E_{g^2}(\tau)] = \tau + 1 = \tau^2$
- modular fusion rules of the real multiplication anyons arise from a modular tensor category only in the case of the Fibonacci anyons
- from classification of modular tensor categories
  E. Rowell, R. Stong, Z. Wang, *On classification of modular tensor categories*, Comm. Math. Phys. Vol. 292 (2009), no. 2, 343–389
- all other RM quantum tori *weaker* forms of anyon systems

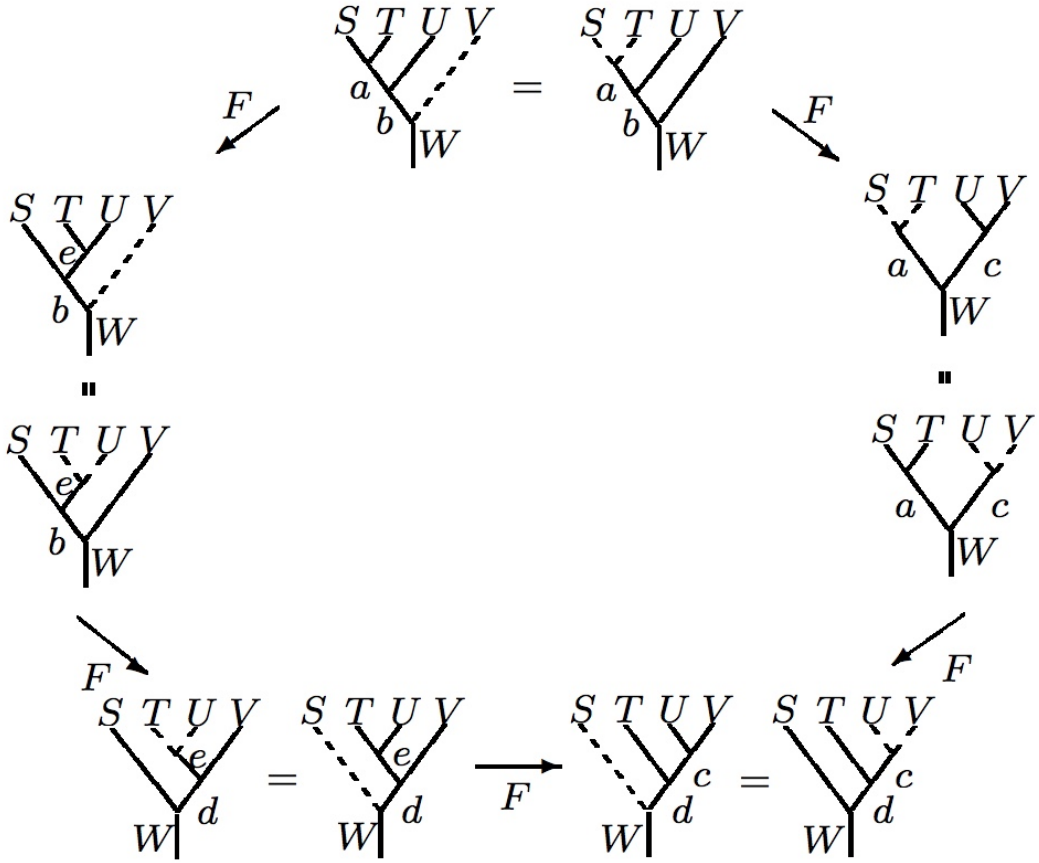
F-matrices and basic modules

- F-matrices of anyon systems determined by natural transformations

$$\bigoplus_c \mathrm{Hom}(x_u, x_c \otimes x_k) \otimes \mathrm{Hom}(x_c, x_i \otimes x_j) \xrightarrow{F_u^{ijk}} \bigoplus_d \mathrm{Hom}(x_u, x_i \otimes x_d) \otimes \mathrm{Hom}(x_d, x_j \otimes x_k)$$

- left-hand-side: $\mathrm{Hom}(x_u, (x_i \otimes x_j) \otimes x_k)$
- right-hand-side: $\mathrm{Hom}(x_u, x_i \otimes (x_j \otimes x_k))$
- pentagon relation: solve with given fusion rules to find entries of F





$$(F_W^{STc})_{da}(F_W^{aUV})_{cb} = \sum_e (F_d^{TUV})_{ce}(F_W^{SeV})_{db}(F_b^{STU})_{ea}$$

case of Fibonacci anyons

$$(F_1^{\tau\tau 1})_{\tau 1}(F_1^{1\tau\tau})_{1\tau} = \sum_{e\in\{1,\tau\}} (F_\tau^{\tau\tau\tau})_{1e}(F_1^{\tau e\tau})_{\tau\tau}(F_\tau^{\tau\tau\tau})_{e1}$$

$$F = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \phi^{-1} & \sqrt{\phi^{-1}} \\ 0 & \sqrt{\phi^{-1}} & -\phi^{-1} \end{array}\right]$$

(note here use notation ϕ for golden ratio $(1+\sqrt{5})/2$ instead of τ)

case of quantum tori with real multiplication

- θ quadratic irrationality and $g \in \mathrm{GL}_2(\mathbb{Z})$ with $g(\theta) = \theta$
- $X_0 = \mathcal{A}_\theta$ and $X_1 = E_g(\theta)$
- with $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{A}_\theta}$ and $\otimes = \otimes_{\mathcal{A}_\theta}$

$$\bigoplus_c \mathrm{Hom}(X_u, X_c \otimes X_k) \otimes \mathrm{Hom}(X_c, X_i \otimes X_j) \xrightarrow{F_u^{ijk}} \bigoplus_d \mathrm{Hom}(X_u, X_i \otimes X_d) \otimes \mathrm{Hom}(X_d, X_j \otimes X_k)$$

- then have

$$F_u^{ijk} : E_h(\theta) \oplus E_h(\theta) \to E_h(\theta) \oplus E_h(\theta)$$

- here $h = g^k$, where k is the difference between the number of upper indices equal to 1 and the number of lower indices equal to 1 in F_u^{ijk}

- to check this: all indices i, j, k, u take value either zero or one
- sixteen resulting cases all checked similarly using

$$\mathrm{Hom}_{\mathcal{A}_\theta}(E_g(\theta), E_h(\theta)) \simeq E_{hg^{-1}}(g(\theta))$$

- Example: for F_0^{101} we have as source

$$\mathsf{Hom}(X_0, X_0 \otimes X_1) \otimes \mathsf{Hom}(X_0, X_1 \otimes X_0) \oplus \mathsf{Hom}(X_0, X_1 \otimes X_1) \otimes \mathsf{Hom}(X_1, X_1 \otimes X_0)$$

$$= E_g(\theta) \otimes_{\mathcal{A}_\theta} E_g(\theta) \oplus E_{g^2}(\theta) \otimes_{\mathcal{A}_\theta} \mathcal{A}_\theta = E_{g^2}(\theta) \oplus E_{g^2}(\theta)$$

and as target

$$\mathsf{Hom}(X_0, X_1 \otimes X_0) \otimes \mathsf{Hom}(X_0, X_0 \otimes X_1) \oplus \mathsf{Hom}(X_0, X_1 \otimes X_1) \otimes \mathsf{Hom}(X_1, X_1 \otimes X_0)$$

$$= E_g(\theta) \otimes_{\mathcal{A}_\theta} E_g(\theta) \oplus E_{g^2}(\theta) \otimes_{\mathcal{A}_\theta} \mathcal{A}_\theta = E_{g^2}(\theta) \oplus E_{g^2}(\theta)$$

- Note: if $g \in \mathrm{GL}_2(\mathbb{Z})$ nonnegative entries and $\det(g) = -1$ with $g(\theta) = \theta$, then modules $E_{g^2}(\theta)$ and $E_g(\theta)^{\oplus \mathrm{Tr}(g)} \oplus \mathcal{A}_\theta$ have same class $[E_{g^2}(\theta)] = \mathrm{Tr}(g)\,[E_g(\theta)] + 1$ in $K_0(\mathcal{A}_\theta)$
- then get more familiar-looking form with F_u^{ijk} an $m \times m$-matrix for

$$m = N_{0k}^u N_{ij}^0 + N_{1k}^u N_{ij}^1 = N_{i0}^u N_{jk}^0 + N_{i1}^u N_{jk}^1$$

with $N_{0j}^k = N_{j0}^k = \delta_{jk}$ and

$$N_{1j}^k = \begin{pmatrix} \mathrm{Tr}(g) & 1 \\ 1 & 0 \end{pmatrix}$$

- in particular in this way recover Fibonacci anyon solution $F_0^{111} = 1$ and with $\tau = (1 + \sqrt{5})/2$

$$F_1^{111} = \begin{pmatrix} \tau^{-1} & \tau^{-1/2} \\ \tau^{-1/2} & -\tau^{-1} \end{pmatrix}$$

Pentagon relations in quantum tori and quantum dilogarithm

- interpret the F-matrices as homomorphisms between sums of basic modules
- matrix elements $(F_u^{ijk})_a^b$ are homomorphims

$$(F_u^{ijk})_a^b \in \mathrm{Hom}_{\mathcal{A}_\theta}(E_h(\theta), E_h(\theta)) \simeq \mathcal{A}_\theta$$

- so can interpret pentagon equation

$$(F_{u_5}^{i_a j_3 k_4})_b^c \, (F_{u_5}^{i_1 j_2 k_c})_a^d = \sum_e (F_{u_b}^{i_1 j_2 k_3})_a^e \, (F_{u_5}^{i_1 j_e k_4})_b^d \, (F_{u_d}^{i_2 j_3 k_4})_e^c$$

  as an equation in $\mathcal{A}_\theta$
- seek solutions from elements of the algebra that are known to satisfy other types of pentagon relations
- construct a family of approximate solutions, related to the Faddeev–Kashaev quantum dilogarithm, through the approximation of the generators of the quantum torus $\mathcal{A}_\theta$ by elements of the AF algebra $\mathbb{A}_\theta$

quantum dilogarithm

- quantized version of Rogers pentagon identity for dilogarithm function, to which it reduces in limit of quantization parameter $q \to 1$
- dilogarithm (or Euler dilogarithm)

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} du = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \qquad |z| < 1$$

- Rogers dilogarithm

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(1-z)\log(z)$$

- pentagon identity

$$L(x) + L(y) - L(xy) = L(\frac{x-xy}{1-xy}) + L(\frac{y-xy}{1-xy})$$

- Weyl algebra generated by invertibles U and V with the relation $UV = qVU$, for a fixed $q \in \mathbb{C}^*$ with $|q| < 1$
- Faddeev–Kashaev quantum dilogarithm as an element in a completion of the Weyl algebra

$$\Psi_q(x) = (x; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k x)$$

- satisfies the pentagon identity

$$\Psi_q(V)\ \Psi_q(U) = \Psi_q(U)\ \Psi_q(-VU)\ \Psi_q(V)$$

- infinite product no longer convergent when the parameter q is on the unit circle $|q| = 1$
- so not directly defined as an element in the C^*-algebra of the quantum torus $\mathcal{A}_\theta$

- there is a way to extend the quantum dilogarithm to $q =$ a root of unity, so that it still satisfies pentagon relation
- for ζ a root of unity of order N, the quantum dilogarithm is defined using the function $\Phi_\zeta(x)$ of the form

$$\Phi_\zeta(x) = (1 - x^N)^{(N-1)/2N} \prod_{k=1}^{N-1} (1 - \zeta^k x)^{-k/N}$$

- for $q = \exp(2\pi i p_n/q_n) = \xi_n$, consider the completion of the Weyl algebra to be the rational quantum torus, with generators $U = uU_n$ and $V = vV_n$, with $u, v \in S^1$ and with U_n and V_n the $q_n \times q_n$ clock and shift matrices
- the rational noncommutative torus is a bundle of matrix algebras $M_{q_n}(\mathbb{C})$ over the commutative torus T^2 of coordinates (u, v): almost-commutative geometry

- for $\zeta = \xi_n = \exp(2\pi i p_n/q_n)$ one obtains the pentagon relation

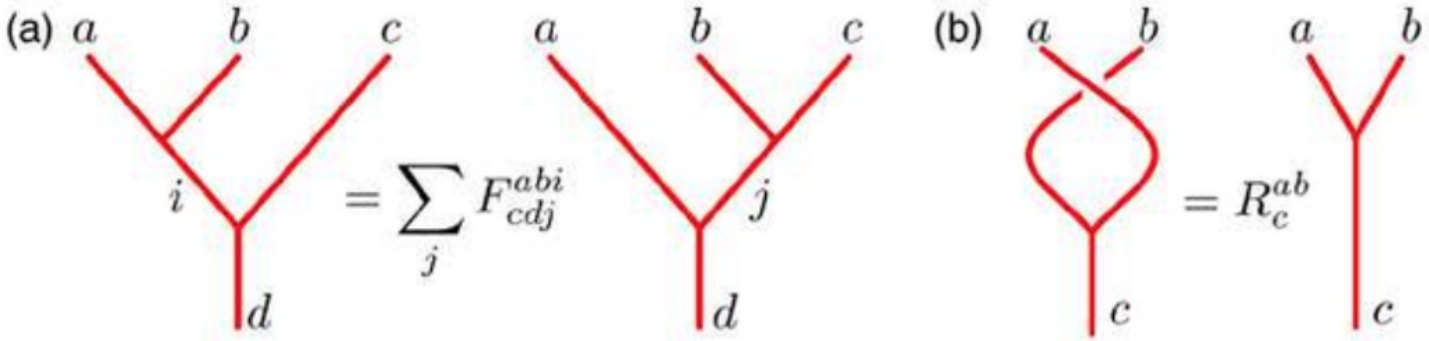
$$\Phi_{\xi_n}(vV_n) \cdot \Phi_{\xi_n}(uU_n) =$$
$$\Phi_{\xi_n}\left(\frac{u}{(1-v^{q_n})^{1/q_n}}U_n\right) \cdot$$
$$\Phi_{\xi_n}\left(\frac{-uv}{(1-u^{q_n}-v^{q_n})^{1/q_n}}U_nV_n\right) \cdot$$
$$\Phi_{\xi_n}\left(\frac{v}{(1-u^{q_n})^{1/q_n}}V_n\right)$$

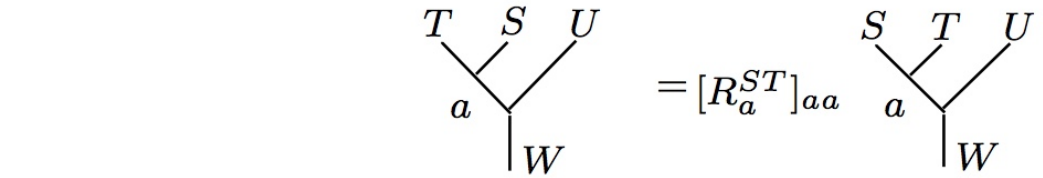
- for fixed choice of u, v, with $u^{q_n} \neq 1$ and $v^{q_n} \neq 1$ regard these $\Phi_{\xi_n}(vV_n)$, $\Phi_{\xi_n}(uU_n)$ etc as elements of the matrix algebra $\mathcal{M}_{q_n}(\mathbb{C})$ in terms of the approximate generators of the quantum torus $\mathcal{A}_\theta$ embedded in the AF algebra $\mathbb{A}_\theta$
- as elements of $\mathbb{A}_\theta$ they are operators acting on the Hilbert space of anyon fusion paths
- general strategy: look for solution of pentagon for F with entries either zero or functions of $\Phi_{\xi_n}(vV_n)$, $\Phi_{\xi_n}(uU_n)$

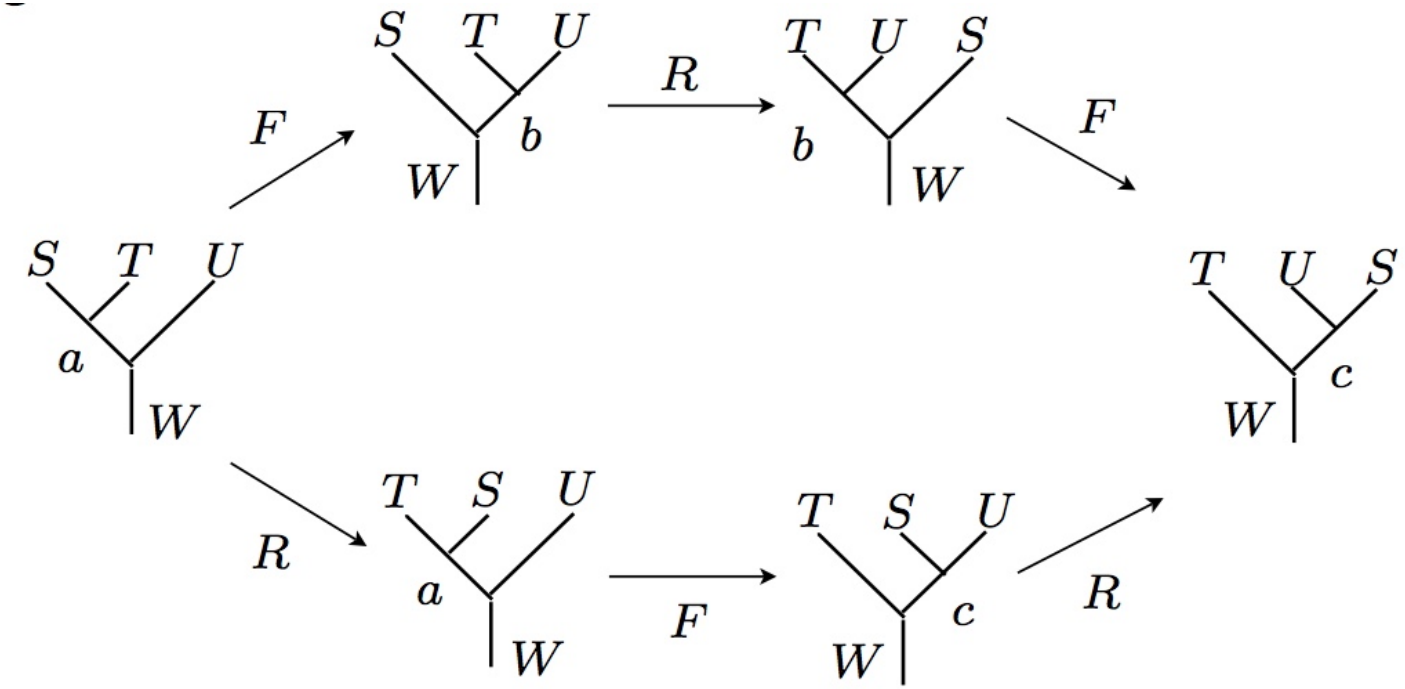
R-matrices and hexagon relation

- hexagon relation involving the F-matrix and the braiding R-matrix

$$\sum_b (F_{u_4}^{i_2 j_3 k_1})_a^d \, R_{u_4}^{i_1 j_b} \, (F_{u_4}^{i_1 j_2 k_3})_a^b = R_{u_c}^{i_1 j_3} \, (F_{u_4}^{i_2 j_1 k_3})_a^c \, R_{u_a}^{i_1 j_2}$$

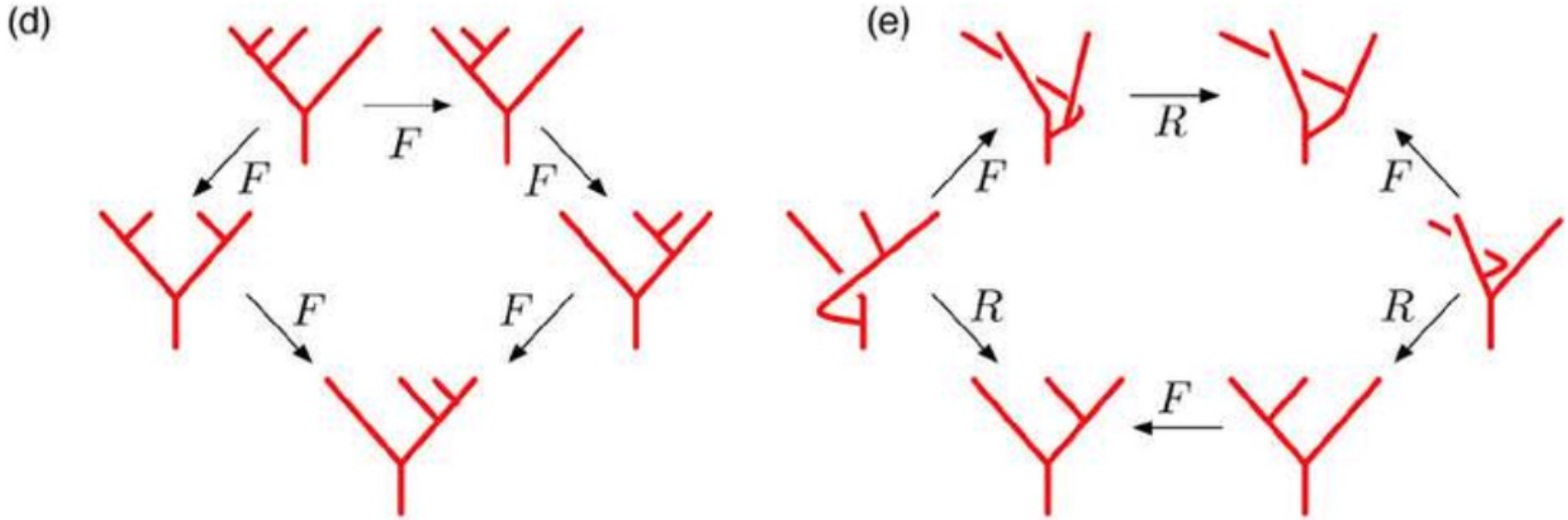






$$R_c^{SU}(F_W^{TSU})_{ca}R_a^{ST} = \sum_b (F_W^{TUS})_{bc}R_W^{Sb}(F_W^{STU})_{ba}.$$

(d): F-matrix hexagon; (e): reinterpretation of the R-matrix hexagon



for Fibonacci anyons R-matrix determined by the F-matrix and hexagon equation

$$R = \begin{pmatrix} \exp(\frac{4\pi i}{5}) & 0 \\ 0 & -\exp(\frac{2\pi i}{5}) \end{pmatrix}$$

Dimension functions and braiding

- case of the Fibonacci anyons: unitary operators $B_{i,n}$ describing the transformation the braid group element σ_i affects on the physical Hilbert space for an n anyon system
- for $i > n-1$, let $B_{i,n} = 1$
- B_∞ infinite braid group; the $\{\sigma_i, \mid i \in \mathbb{Z}_+\}$ generate B_∞
- can always determine $B_{i,n}$ for any i, n through the matrices R and $F^{-1}RF$
- Fibonacci case: for $\mathcal{A}_\tau$ the K_0 group, with its positive cone, is given by

$$(K_0(\mathcal{A}_\tau), K_0^+(\mathcal{A}_\tau)) = (\mathbb{Z}^2, \{(n,m) \in \mathbb{Z}^2 \mid n + \tau m \geq 0\})$$

- range of the trace on $K_0(\mathcal{A}_\tau)$ is given by the subgroup (pseudolattice) $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{R}$

description of the ordered K_0-group in terms of dimension functions

- dimension function on a Bratteli diagram Γ is any $\mathbb{Z}$-valued function defined for almost all vertices satisfying the equation

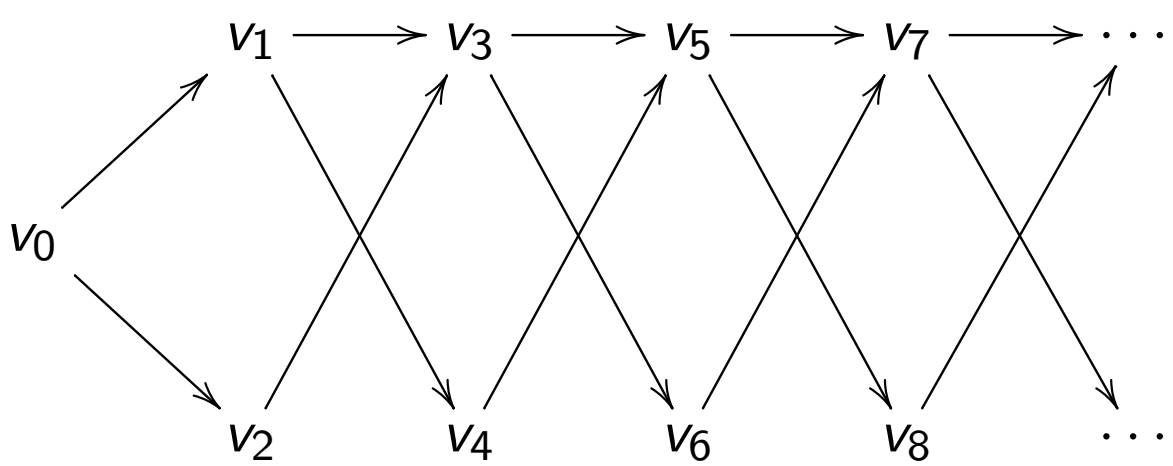
$$f(\Lambda) = \sum_{\lambda:\, \lambda \nearrow \Lambda} f(\lambda)\varkappa(\lambda, \Lambda), \quad \Lambda \in \Gamma$$

  with $\varkappa$ the multiplicity of the edge from λ to Λ
- two functions f_1, f_2 identified if they differ only on a finite set of vertices
- function is virtually positive if nonnegative on almost all vertices
- for an AF algebra $\mathbb{A}$ determined by a Bratteli diagram Γ, the group $K_0(\mathbb{A})$ is isomorphic to the group of dimension functions on Γ, and the cone $K_0^+(\mathbb{A})$ is isomorphic to the group of virtually positive elements
- S. Kerov, *Asymptotic representation theory of the symmetric group and its applications in analysis*, Translations of Mathematical Monographs, Vol. 219. American Mathematical Society, 2003.

Example of Fibonacci anyons

- Γ Bratteli diagram of the Fibonacci anyons system
- dimension functions on Γ are of the form

$f =$



where the $v_i \in \mathbb{Z}$ satisfy $v_{2k+1} = v_{2k} + v_{2k-1}$ and $v_{2k+2} = v_{2k-1}$ for $k > 0$ and $v_2 = v_0$

- since for Fibonacci AF algebra edges in Γ all have multiplicity one, so dimension functions satisfy

$$f(\Lambda) = \sum_{\lambda:\, \lambda \nearrow \Lambda} f(\lambda), \quad \Lambda \in \Gamma$$

dimension functions and infinite braid group

- embedding of the group of dimension functions on Γ as a subgroup of the infinite braid group B_∞

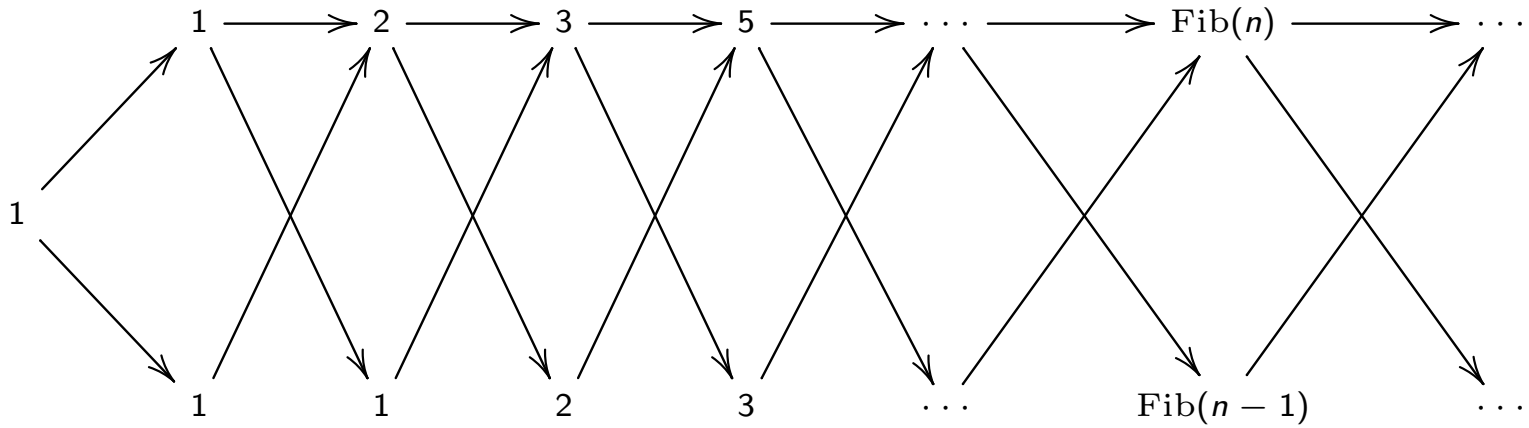
$$\varphi : \text{dimension functions on } \Gamma \to B_\infty$$

$$\varphi(f) = \sigma_1^{v_0}\sigma_3^{v_1}\sigma_5^{v_2}\cdots\sigma_{2n+1}^{v_n}\cdots$$

- this determines a representation of $K_0(\mathcal{A}_\tau)$ in B_∞

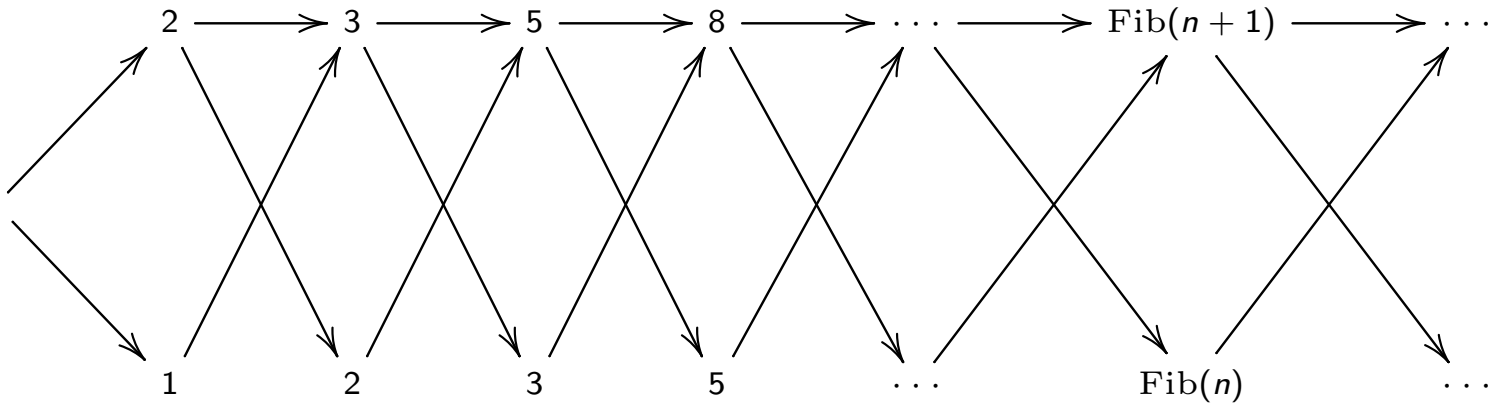
- on generators: dimension function corresponding to $1 \in K_0(A)$ is

$f_1(\Gamma) =$



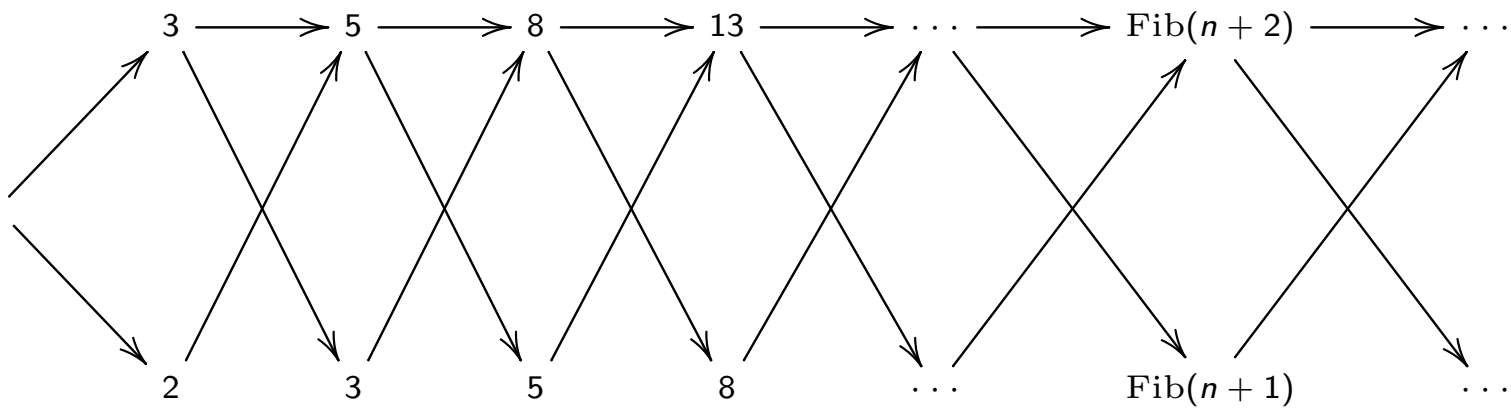
- corresponding element in B_∞ is the product $\varphi(f_1) = \sigma = \prod_n \sigma_{2n+1}^{\mathrm{Fib}(n)}$
- from generator f_1 and action of the shift of the continued fraction expansion get

$f_{1/\tau}(\Gamma) =$



- so dimension function corresponding to τ is

$f_\tau(\Gamma) =$



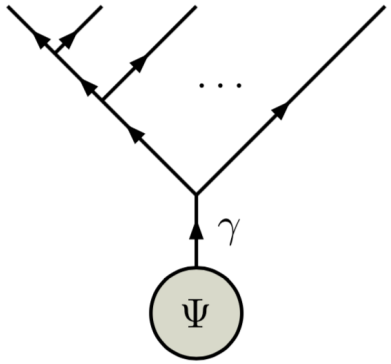
- subgroup of B_∞ is then abelian subgroup generated by $\varphi(f_1)$ and $\varphi(f_\tau)$

Tensor networks

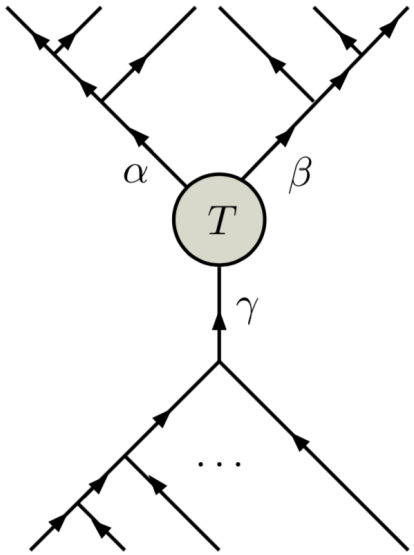
- finite or countable collection of tensors connected by leg contractions (tensor $T_{i_1,\ldots,i_k}$ with k-legs; index contraction corresponds to glueing legs)
- graphical calculus diagrammatic methods (Roger Penrose)
- quantum circuits as tensor networks
- useful method for computing entangled quantum states
- holographic bulk/boundary correspondence with bulk space discretized by tensor networks: entanglement entropy of boundary state and geometry (minimal curves/surfaces) in bulk space, Ryu–Takayanagi conjectures

Anyon Tensor Networks

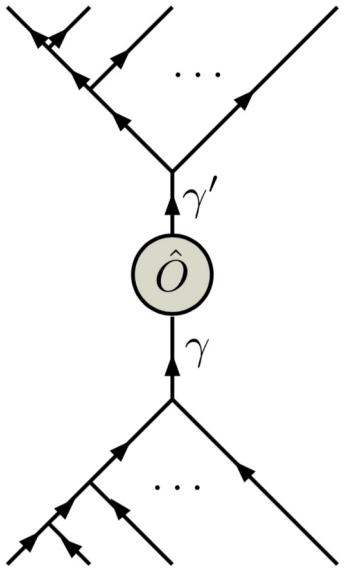
- states $|\Psi\rangle$ weighted superposition of anyon fusion/splitting trees with assigned anyon charge at the root



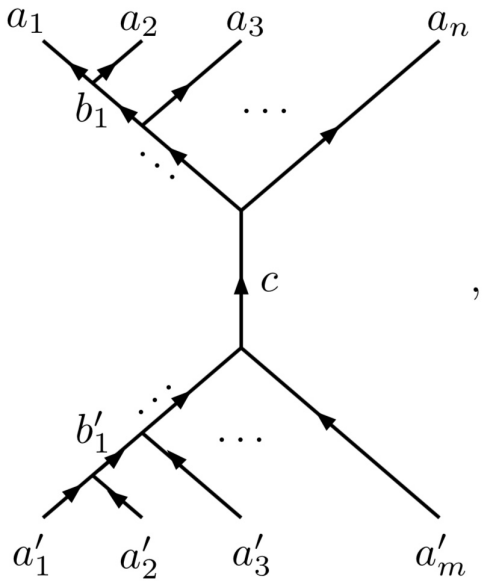
- operators: anyonic tensors



- anyonic matrix operators



$$\hat{O} = \sum_{\substack{b_1,\dots,b_{n-2} \\ b'_1,\dots,b'_{m-2} \\ c}} O^{b_1 \dots b_{n-2}\ c}_{b'_1 \dots b'_{m-2}}$$



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Recent work on anyon tensor networks

- J. Berger, T.J. Osborne, *Perfect tangles*, arXiv:1804.03199
- J.C. Bridgeman, S.D. Bartlett, A.C. Doherty, *Tensor Networks with a Twist: Anyon-permuting domain walls and defects in PEPS*, arXiv:1708.08930
- B.M. Ayeni, *Studies of braided non-Abelian anyons using anyonic tensor networks*, arXiv:1708.06476

Geometric construction of anyon systems and anyon tensor networks in 4-manifolds with surfaces of orbifold points

- M. Atiyah, M. Marcolli, *Anyons in geometric models of matter*. J. High Energy Phys. 2017, no. 7, 076, front matter+23 pp.
- M. Atiyah, M. Marcolli, *Anyon networks from geometric models of matter*, Q. J. Math. 72 (2021), no. 1-2, 717–733.

Geometric Models of Matter (Atiyah, Manton, Schroers, *et al.*)

- certain 4-dim Riemannian manifolds with self-dual Weyl tensor behave "like" elementary particles
- gravitational instantons: Taub-NUT, Atiyah–Hitchin, gravitational instantons of types A_k and D_k
- dynamical models: $(4+1)$-dimensional Ricci-flat spacetimes describing evolving Taub-NUT geometries (Atiyah–Franchetti–Manton)
- other more general classes of 4-manifold with "particle properties": algebraic surfaces with c_2 and c_1^2 as "lepton/baryon numbers", Enriques-Kodaira classification as "valley of stability" (Atiyah–Manton)
- geometrization of the skyrmion model of particles (topological solitons in non-linear sigma models with pion fields π combined with a field σ to $SU(2)$-valued scalar field, proposed as models of nucleons)

Quantum numbers of geometric models of matter

- Atiyah–Manton–Schroers: the signature $\tau(M)$ is interpreted as a baryon number; the electric charge is determined by the self-intersection number of the surface at infinity
- Atiyah–Manton: baryon and lepton numbers are expressed in terms of both signature and Euler characteristic (signature measuring difference between number of protons and number of neutrons)

General principle: quantum numbers of "particle-like" manifolds should be topological quantities computed by an index theorem

Composite systems

• merging operations on geometries that can be seen as composite systems of particles or quasi-particles

• connected sum operation: existence of self-dual metric on a connected sum of two self-dual 4-manifolds depends on a twistor space argument (Donaldson–Friedman)

• twistor space $Z = Z(M)$ of a self-dual 4-manifold is a 3-dimensional complex manifold that fibers over M with $\mathbb{CP}^1$ fibers

• singular complex 3-manifold $\tilde{Z} = \tilde{Z}_1 \cup_{E_1 \simeq E_2} \tilde{Z}_2$ by blowup of twistor spaces $Z_i = Z(M_i)$ along a $\mathbb{CP}^1$ fiber and gluing exceptional divisors

• Donaldson–Friedman: if $\exists$ smooth Kodaira–Spencer–Kuranishi deformation Z of $\tilde{Z}$ then Z is twistor space $Z(M)$ of a self-dual structure on the connected sum $M = M_1 \# M_2$

• Constraints on the formation of composite systems of geometric models of matter (relaxing the self-duality hypothesis? suggested in more recent Atiyah–Manton)

Dynamical models

• 4-dim gravitational instanton (self-dual Riemannian manifolds with an Einstein metric) is seen in these geometric models of matter as a static "particle-like" object

• made dynamical by embedding in a $(4+1)$-dimensional Ricci-flat geometry (Atiyah–Franchetti–Schroers)

• example: Taub-NUT geometry as 4-dim section of the 5-dim Sorkin solution of the Kaluza–Klein monopole equations

• Campbell–Magaard embedding: an arbitrary analytic Riemannian manifold M of dimension $\dim M = n$ can be locally embedded in a Ricci-flat Riemannian manifold of dimension $n+1$, but embedding may only exist locally

Building geometric models of systems of quasi-particles

Want fractional quantum numbers and anyon statistics

Main ideas:

- consider pairs (M, Σ) with M a smooth compact 4-dimensional manifold and Σ a smoothly embedded compact 2-dimensional surface
- consider metrics (edge-cone metrics) on M with Σ as set of orbifold points
- obtain fractional quantum numbers from Kawasaki index theorem for orbifolds
- get anyons and braid representations from surface braids determined by multisections of the orbifold normal bundle $\mathcal{N}(\Sigma)$ of Σ in M

Edge-cone metrics (Atiyah–Le Brun)

- edge-cone metric on (M, Σ) cone angle $2\pi\beta$, $\beta \in \mathbb{R}_+^*$

$$g = d\rho^2 + \beta^2\rho^2(d\theta + u_j dx^j)^2 + w_{ij} dx^i dx^j + \rho^{1+\epsilon} h$$

h symmetric tensor with continuous derivatives all orders for vector fields with vanishing normal component along Σ

- modelled on a 2-dimensional cone in directions transversal to Σ, smooth in the directions parallel to Σ

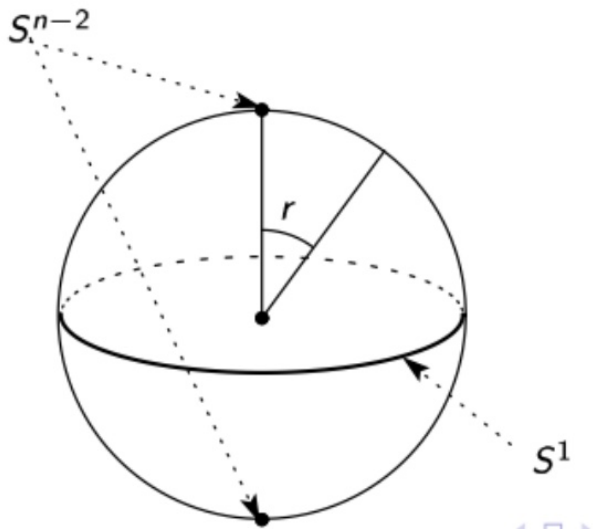
Example: edge cone metric on the sphere (S^n, S^{n-2}) $n \geq 3$

$$S^n - (S^{n-2} \sqcup S^1) = (0, \tfrac{\pi}{2}) \times S^1 \times S^{n-2} \ni (r, \theta, x)$$

$$g_S = dr^2 + \sin^2 r \, d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} =: h_1 : \textbf{standard round metric}$$

$$h_\beta := dr^2 + \beta^2 \sin^2 r \, d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} \quad \cong \quad h_1 : \text{loc. isom. } (\beta > 0)$$

edge-cone Einstein metric of cone angle $2\pi\beta$ on (S^n, S^{n-2})



Kazuo Akutagawa, *Computations of the orbifold Yamabe invariant*, Math. Z. 271 (2012), 611–625

Atiyah–Le Brun manifold

- (S^4, S^2) with edge-cone metric of angle $2\pi/\nu$
  - complement $S^4 \setminus S^2$ conformally equivalent to $\mathbb{H}^3 \times S^1$ (hyperbolic $\mathbb{H}^3$)
  - standard round metric on S^4 becomes $\operatorname{sech}^2\delta\ (h + d\theta^2)$ with h hyperbolic metric and $\delta : \mathbb{H}^3 \to \mathbb{R}$ distance from a point
  - $ds^2 = \operatorname{sech}^2\delta\ (h + \beta^2 d\theta^2)$ family of edge-cone metrics with cone angle $2\pi\beta$

Atiyah–Le Brun manifold: *connected sums of projective planes* $(\#^n\mathbb{CP}^2, \Sigma)$ with $\Sigma = \#^n\mathbb{CP}^1 \simeq S^2$

- $\mathcal{U} \subset \mathbb{H}^3$ open, $V : \mathcal{U} \to \mathbb{R}^+$ harmonic for metric h
- closed form $\star dV$; class $[\star dV/2\pi] = c_1(\mathcal{P}) \in H^2(\mathcal{U}, \mathbb{R})$; line bundle $\mathcal{P}$
- θ connection on $\mathcal{P}$ with curvature $d\theta = \star dV$
- Riemannian metric on total space of $\mathcal{P}$

$$g_0 = Vh + V^{-1}\theta^2$$

- potential $V = \beta^{-1} + \sum_{i=1}^n G_{p_i}$ points $p_i \in \mathbb{H}^3$ and Green functions G_{p_i}
- edge-cone metrics $g = \beta(\operatorname{sech}^2\delta)g_0$
- metric completion of g on $\mathcal{P}$ gives $\#^n\mathbb{CP}^2$ with edge-cone angle $2\pi\beta$ along Σ
- generalization of Abreu family of edge-cone metrics on $(\mathbb{CP}^2, \mathbb{CP}^1)$

Orbifolds and edge-cone metrics (Atiyah–Le Brun)

- compact 4-dimensional M with atlas of local uniformizing charts U_α homeomorphic $U_\alpha \simeq V_\alpha/G_\alpha$ to quotients of open sets $V_\alpha \subset \mathbb{R}^4$ by finite groups G_α
- $M = M_{sing} \cup M_{reg}$ singular (orbifold) points and regular points with $M_{sing} = \Sigma$ embedded surface
- good orbifold: global quotient $M = X/G$ smooth 4-manifold X and finite group G
- near Σ local chart $\mathbb{C}^2/G_\nu$ with $G_\nu = \mathbb{Z}/\nu\mathbb{Z}$ and $(w, \zeta) \mapsto (w, e^{2\pi i/\nu}\zeta)$
- edge-cone metrics with $\beta = 1/\nu$, represented in local chart as a $\mathbb{Z}/\nu\mathbb{Z}$-invariant metric

Orbifolds as geometric models of systems of quasi-particles

- quantum numbers from Kawasaki index theorem for orbifolds
- Atiyah–Le Brun orbifold Euler characteristic and an orbifold signature

$$\chi_{orb}(M) = \frac{1}{8\pi^2}\int_M \left(|W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2\right)\, dv(g)$$

$$= \chi(M) - (1-\frac{1}{\nu})\chi(\Sigma)$$

$$\tau_{orb}(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W_-|^2\right)\, dv(g) = \tau(M) - \frac{1}{3}(1-\frac{1}{\nu^2})[\Sigma]^2$$

with $[\Sigma]^2$ self-intersection number (Euler number of normal bundle of Σ in M), W Weyl tensor, $W^{\pm}$ self-dual and anti-self-dual part, E traceless part of Ricci tensor, R scalar curvature

- $\chi_{orb}(M)$ and $\tau_{orb}(M)$ fractional quantum numbers for the orbifold (M, Σ), viewed as modeling a system of quasi-particles
- surface Σ of orbifold points plays role of surface at infinity that contributes the electric charge to the matter content in the Atiyah–Manton–Schroers model
- normal bundle $\mathcal{N}(\Sigma)$ of the inclusion of Σ in M is an *orbifold vector bundle*
- fibers of $\mathcal{N}(\Sigma)$ are quotients $\mathbb{R}^2/G_\nu$ where $G_\nu = \mathbb{Z}/\nu\mathbb{Z}$ is the stabilizer of Σ
- role of self-intersection number becomes orbifold Euler number $\chi_{orb}(\mathcal{N}(\Sigma))$ of the normal bundle $\mathcal{N}(\Sigma)$
- fractional electric charge of the system of quasi-particles

Constraints on composite systems

• possible obstructions existence of self-dual structures on connected sums (obstructions to smooth deformation giving twistor space as above)

• possible obstructions to Einstein condition (topological obstructions or differentiable obstructions)

- *topological obstructions* (Atiyah-Le Brun): inequalities

$$2\chi(M) \pm 3\tau(M) \geq (1 - \frac{1}{\nu})(2\chi(\Sigma) \pm (1 + \frac{1}{\nu})[\Sigma]^2)$$

  need to hold for (M, Σ) to admit an Einstein edge-cone metric of cone angle $2\pi/\nu$

- *differentiable obstructions* (Le Brun): from Seiberg–Witten
  if symplectic form ω on M with Σ symplectic submanifold with $(c_1(M) - (1 - 1/\nu)[\Sigma]) \cdot [\omega] < 0$ then for any $\ell \geq (c_1(M) - (1 - 1/\nu)[\Sigma])^2/3$ the pair (M', Σ) with $M' = M\#^{\ell}\overline{\mathbb{CP}^2}$ has no Einstein edge-cone metric

• relax self-dual Einstein conditions for composite systems: require only for "elementary constituents"

Anyons when $\dim(X) = 2$

- for $X = D^2$ Artin braid group

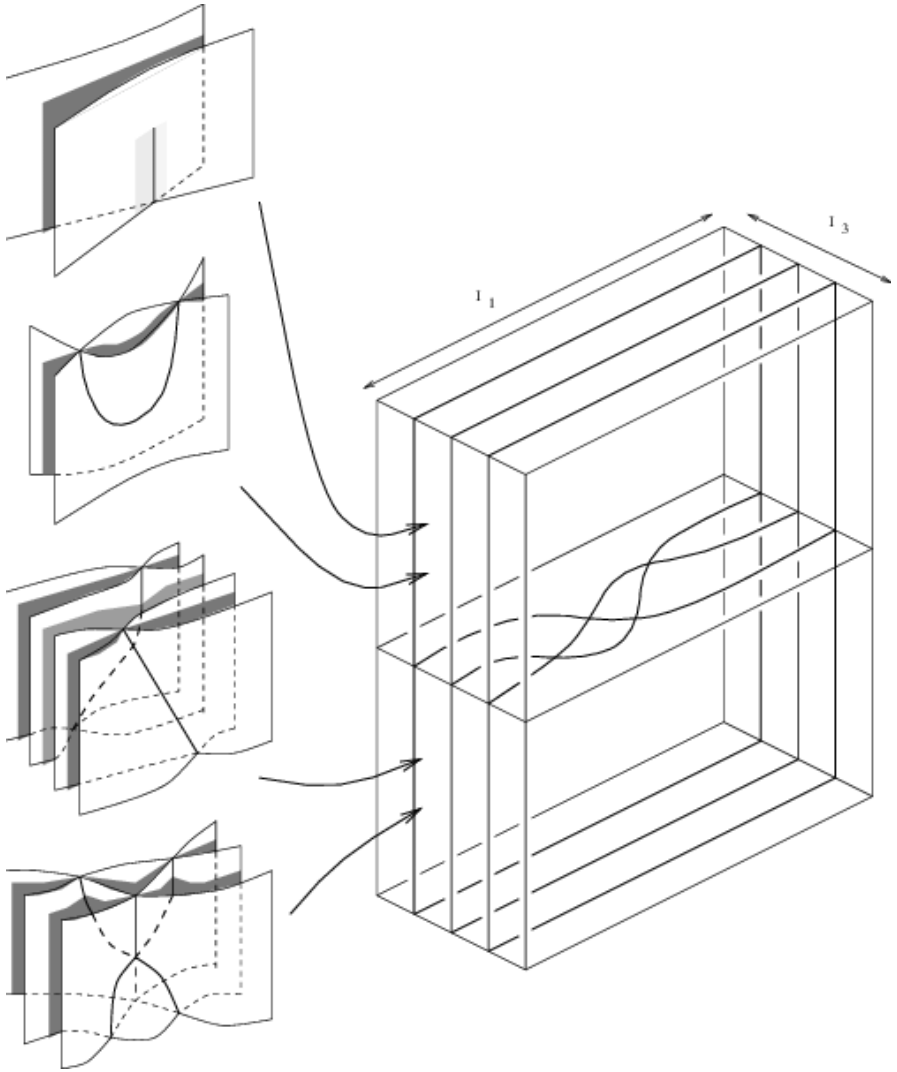
$$B_n(D^2) = B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle$$

- for $X = \Sigma$ a Riemann surface Birman presentations of $B_n(\Sigma)$ as discussed before

Case of geometric models: (M, Σ) with $\dim M = 4$ and $\Sigma \subset M$ with $\dim \Sigma = 2$ locus of orbifold points

- Cannot use $B_n(M \setminus \Sigma)$ or $B_n(M)$ or $B_n^{orb}(M, \Sigma)$ (orbifold braid groups) because those only give generalized parastatistics
- still possible to obtain anyons

Surface Braids (Viro, Kamada)



• J. Scott Carter, Seiichi Kamada, Masahico Saito, *Alexander Numbering of Knotted Surface Diagrams*, Proc. Amer. Math. Soc. 128 (2000), no. 12, 3761–3771

Surface Braids (Viro, Kamada)

- surface m-braid: smooth 2-dimensional S, smoothly embedded in $D^2 \times D^2$ with second projection $P_2 : D^2 \times D^2 \to D^2$ restricting to S as an m-fold branched cover $P : S \to D^2$
- preimage $\beta := P_2^{-1}(\partial D^2) \cap S \subset D^2 \times S^1$ ordinary closed m-braid
- Note: sometimes assumed that β trivial braid
- $b(S) \subset D^2$ set of branch points of the m-fold brached cover $P : S \to D^2$
- Braid representation: fundamental group $\pi_1(D^2 \setminus b(S))$

$$\rho_S : \pi_1(D^2 \setminus b(S)) \to \pi_1(\mathrm{Conf}_m(D^2)) = B_m(D^2)$$

- Construction: paths $\gamma(t)$ in $D^2 \setminus b(S)$

$$\rho_S(\gamma)(t) := P_1(S \cap P_2^{-1}(\gamma(t)))$$

seen as a path in $\mathrm{Conf}_m(D^2)$
(with $P_i : D^2 \times D^2 \to D^2$ projections)

Variants of Surface Braids

- closed surface braids: S smoothly embedded in $D^2 \times S^2$ with $P = P_2|_S : S \to S^2$ an m-fold branched covering
- more general version: fixed surface Σ (possibly of genus $g(\Sigma) > 0$) and S smoothly embedded in $D^2 \times \Sigma$ with projection $P = P_2|_S : S \to \Sigma$ an m-fold branched covering
- braid representation: $\rho_S(\gamma)(t) := P_1(S \cap P_2^{-1}(\gamma(t))$

$$\rho_S : \pi_1(\Sigma \setminus b(S)) \to \pi_1(\mathrm{Conf}_m(D^2)) = B_m(D^2)$$

- further case: $\mathcal{F}$ a disc-bundle over closed surface Σ and S smoothly embedded in $\mathcal{F}$ with $\pi : \mathcal{F} \to \Sigma$ restricting to m-fold branched cover $\pi|_S : S \to \Sigma$

Sections and multisections of the orbifold unit normal bundle

• (M, Σ) orbifold geometry, $\dim M = 4$ and $\dim \Sigma = 2$, good orbifold $M = X/G$ some finite group G (assume Σ connected and $G = \mathbb{Z}/\nu\mathbb{Z}$)

• normal bundle $\mathcal{N}(\Sigma)$ of $\Sigma \hookrightarrow M$ is an orbifold bundle, orbifold covered by normal bundle $\mathcal{N}(\tilde{\Sigma})$ of preimage $\tilde{\Sigma}$ in X

• lift to $\mathcal{N}(\tilde{\Sigma})$ of a generic section σ of $\mathcal{N}(\Sigma)$ gives ν-fold branched covering S of Σ branched at finitely many points $b(S)$

• multisections of $\mathcal{N}(\tilde{\Sigma})$ are maps to $\mathrm{Sym}^{\ell}(F) = F^{\ell}/S_{\ell}$ of fiber F; ℓ-fold branched covers of $\tilde{\Sigma}$ branched at intersections with diagonals (for unit normal bundle $F = D^2$)

• combining these: taking mutisection of unit normal $\mathcal{N}_1(\Sigma)$ get $\nu\ell$-fold branched coverings S of Σ, hence $\ell\nu$-surface braids

Focus on the Atiyah-LeBrun manifold (S^4, S^2) with $2\pi/\nu$ edge cone metric

- multisections of $\mathcal{N}(S^2)$ give $\nu\ell$-surface braids S (closed surface braids in $D^2 \times S^2$)
- associated braid representation $\rho_S : \pi_1(S^2 \setminus b(S)) \to B_n(D^2)$
- given a closed surface braid S obtained from a multisection of $\mathcal{N}(S^2)$ and a branch point $x_0 \in b(S)$, take a disc $D_b^2 \subset S^2$ that is the complement of a small neighborhood of x_0
- restriction of branched cover $P : S \to S^2$ to $D_b^2 \subset S^2$ is also a $\nu\ell$-fold branched cover $P : \hat{S} \to D_b^2$ and $\hat{S} \subset D_f^2 \times D_b^2$ is a surface braid, so $\hat{S} \cap (D_f^2 \times \partial D_b^2)$ is an ordinary closed braid β, in general non-trivial in $B_{\nu\ell} = B_{\nu\ell}(D^2)$
- the braid system $(\beta_1, \ldots, \beta_n)$ of $\hat{S}$ is the image under the braid representation $\rho_{\hat{S}} : \pi_1(D_b^2 \setminus b(\hat{S})) \to B_{\nu\ell}(D^2)$ of set of generators $\gamma_1, \ldots, \gamma_n$ of $\pi_1(D_b^2 \setminus b(\hat{S})$ (basepoint on ∂D_b^2)

Standard braid system

• characterization of braid systems such that $\hat{S} \cap (D^2_f \times \partial D^2_b)$ is a fixed closed braid $\beta \in B_{\nu\ell}(D^2)$: n-tuples $(\beta_1, \dots, \beta_n) \in B_{\nu\ell}(D^2)^n$ such that each β_k is conjugate of a standard generator σ_i or σ_i^{-1} of the braid group, with $\beta_1 \cdots \beta_n = \beta$ in $B_{\nu\ell}(D^2)$

• equivalent braided surfaces (related by a fiber preserving diffeomorphism of $D^2_f \times D^2_b$ relative to $D^2_f \times \partial D^2_b$) correspond to braid systems $(\beta_1, \dots, \beta_n)$ related by Hurwitz action of B_n on $B^n_{\nu\ell}$

$$\sigma_i : (\beta_1, \dots, \beta_n) \mapsto (\beta_1, \dots, \beta_{i-1}, \beta_i \beta_{i+1} \beta_i^{-1}, \beta_i, \beta_{i+2}, \dots, \beta_n)$$

• for $n = \nu\ell - 1$ consider surface braid with $n = \#b(\hat{S})$ such that braid system is standard set of generators $(\sigma_1, \dots, \sigma_n)$ of $B_{\nu\ell}(D^2)$

• for such choice of $\hat{S}$ braid representation obtains all $B_{\nu\ell}(D^2)$

• this last property can be used to show that the anyons obtained in this way can be universal for quantum computing (by composition with the Freedman–Larsen–Wang TQFT construction)

- given surface braid S and corresponding braid representation

$$\rho_S : \pi_1(\Sigma \setminus b(S)) \to B_m(D^2)$$

- $m = \nu\ell$ depending on the orbifold normal bundle $\mathcal{N}(\Sigma)$ and on the multisection defining S
- for $m = n + 1$ and $(\beta_1, \ldots, \beta_n)$ standard set of generators $(\sigma_1, \ldots, \sigma_n)$ have $\pi_1(\Sigma \setminus b(S))$ mapping surjectively to B_m
- then use algebra homomorphism from $\mathbb{F}[B_m]$ to the Temperley–Lieb algebra $TL_m(A)$, given by the Kauffman bracket
- $\mathbb{F}$ field $\mathbb{C}(A)$ of rational functions in the variable A
- image of the standard generators σ_i under Kauffman bracket gives set of invertible generators of Temperley–Lieb algebra

$$\langle \sigma_i \rangle = g_i = A\,\mathrm{id} + A^{-1}\,u_i$$

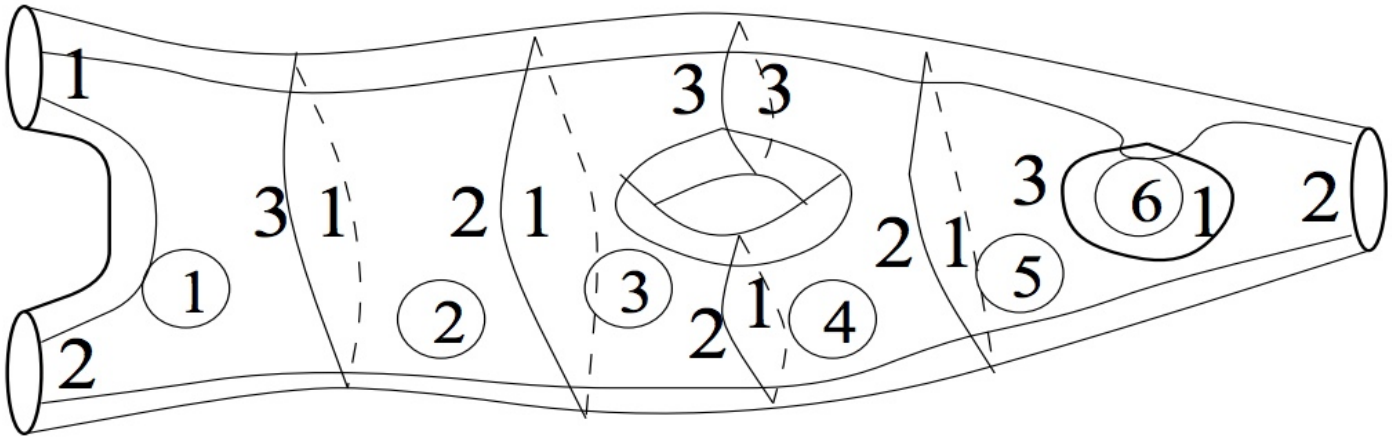
- Hurwitz action of B_n on B^n_{n+1} determines under Kauffman action of B_n by automorphisms of the Temperley–Lieb algebra
- at roots of unity identify Temperley–Lieb–Jones quotient algebra $TLJ_n(A)$ with sum of matrix algebras $\oplus_a M_{m_{n,a}}(\mathbb{C})$
- representations of the braid group

$$\rho_{m,k,a} : B_m \to TLJ_n(A) \to M_{m_{n,a}}(\mathbb{C}).$$

- Hurwitz action on the standars braid system determines an induced action of B_n on the algebra $\oplus_a M_{m_{n,a}}(\mathbb{C})$
- restricting to blocks $M_{m_{n,a}}(\mathbb{C})$ inner automorphisms $T_{m_{n,a}}(\gamma)$ for $\gamma \in B_n$
- obtain in this way quantum gates: universality from universality of Jones representation

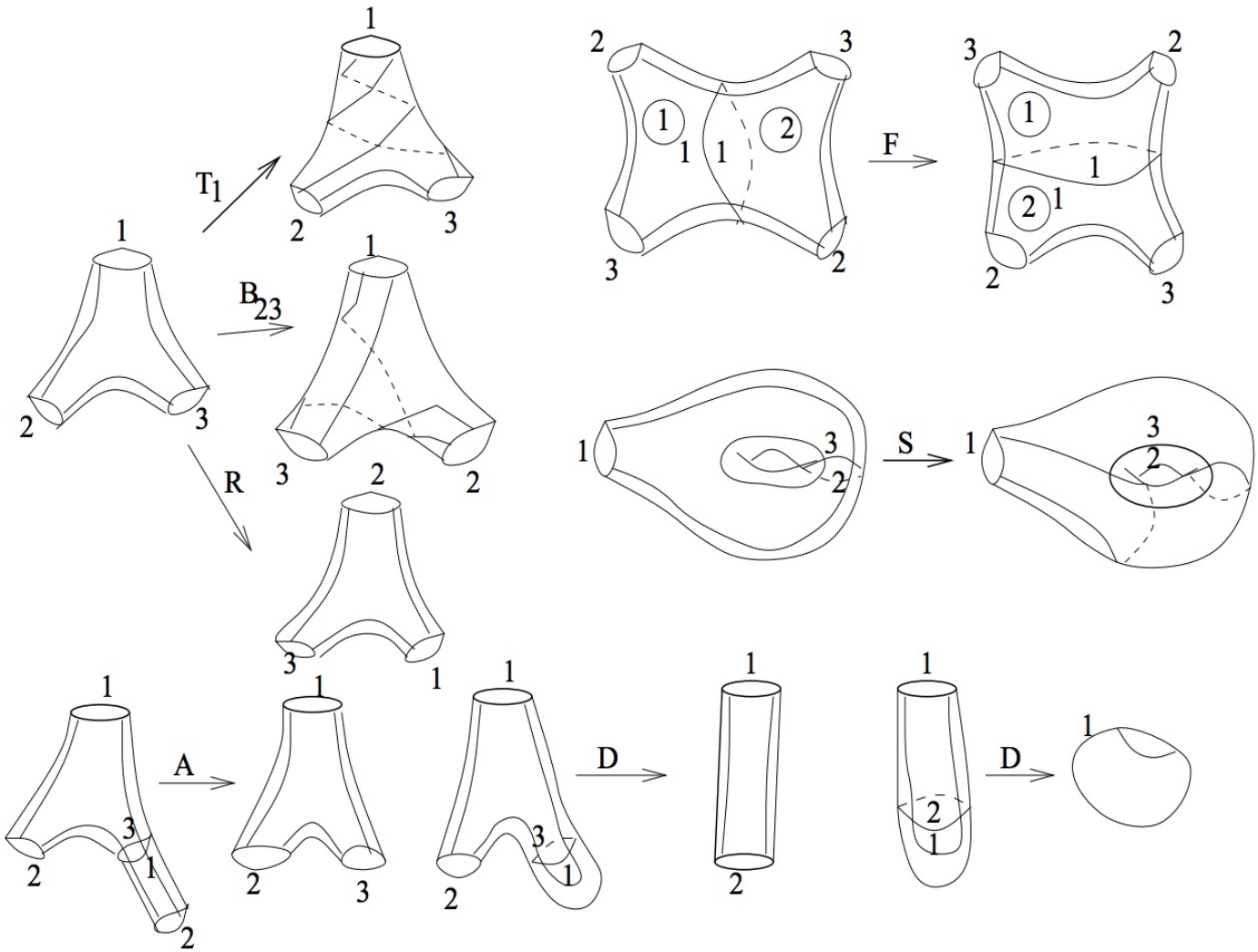
TQFT with corners

- usual functorial setting of topological quantum field theory (Atiyah)
- include the possibility of gluing 3-manifold along codimension zero parts of their boundary
- 2-dimensional surface Σ is endowed with a DAP-decomposition (discs, annuli, and pairs of pants)
- up to equivalence by a set of elementary moves
- extended surface is a triple $(\Sigma, \mathbb{D}, \ell)$ of a surface, a DAP-decomposition, and a labeling of the boundary components
- extended 3-manifold is a triple $(X, \mathbb{D}, n)$ of a 3-manifold, a DAP-decomposition $\mathbb{D}$ of its boundary ∂X and an integer n
- an associated notion of "extended morphisms" of these data



DAP decomposition of a surface

R. Gelca, *Topological quantum field theory with corners based on the Kauffman bracket*, Comment. Math. Helv. 72 (1997), no. 2, 216–243



elementary moves of DAP decompositions

TQFT on extended surfaces and 3-manifolds

- functor mapping the category of extended surfaces and morphisms to the category of finite dimensional complex vector spaces (called modular functor)
- an assignment to each extended 3-manifold $(X, \mathbb{D}, n)$ of a vector $Z(X, \mathbb{D}, n)$ (partition function) in the vector space $V(\partial X, \mathbb{D})$ of its boundary extended surface
- vector space assigned to an extended surface has the property that it splits as a tensor product over disjoint unions and that, if two extended surfaces are glued along a boundary component given by a curve C the vector spaces satisfy

$$V(\Sigma \cup_C \Sigma', \mathbb{D} \cup_C \mathbb{D}', (\ell, \ell')) = \bigoplus_a V(\Sigma, \mathbb{D}, (\ell, a)) \otimes V(\Sigma', \mathbb{D}', (\ell', a))$$

where ℓ, ℓ', a are the labels of the boundary components in $\Sigma \setminus C$, $\Sigma' \setminus C$ and C.

- if $(\Sigma, \mathbb{D}_1, \ell)$ and $(\Sigma', \mathbb{D}'_1, \ell')$ are two disjoint extended manifold in the boundary $\partial(M, \mathbb{D}, m)$ of an extended 3-manifold, one considers the vector space

$$\bigoplus_{\ell,\ell'} V(\Sigma, \mathbb{D}_1, \ell) \otimes V(\Sigma', \mathbb{D}'_1, \ell') \otimes V(\partial(M, \mathbb{D}, n) \smallsetminus ((\Sigma, \mathbb{D}_1, \ell) \cup (\Sigma', \mathbb{D}'_1, \ell'))$$

- with partition function

$$Z(M, \mathbb{D}, m) = \oplus_{\ell,\ell'} \sum_j \alpha_\ell^{(j)} \otimes \beta_{\ell'}^{(j)} \otimes \gamma_{\ell,\ell'}^{(j)}$$

- for the glued manifold:

$$Z((M, \mathbb{D}, m)_\varphi) = \oplus_\ell \sum_j \langle V(\varphi)\alpha_\ell^{(j)}, \beta_\ell^{(j)} \rangle$$

- shorthand notation for partition function $Z((M, \mathbb{D}, m)_\varphi)$ under gluing:

$$Z((M, \mathbb{D}, m)_\varphi) = \langle V(\varphi) Z_{M,\Sigma}, Z_{M,\Sigma'} \rangle \, Z_{M,\partial M^0}$$

with $\partial M^0 = \partial M \smallsetminus (\Sigma \cup \Sigma')$ suppressing explicit DAP-decompositions, labels, and sums

- axioms of these TQFTs and their properties:
  - R. Gelca, *Topological quantum field theory with corners based on the Kauffman bracket*, Comment. Math. Helv. 72 (1997), no. 2, 216–243
  - K. Walker, *On Witten's 3-manifolds invariants*, unpublished notes, 1991 (revised version 2003) 133 pages, available at `http://citeseerx.ist.psu.edu/`
- other constructions:
  - J.E. Andersen, K. Ueno, *Geometric construction of modular functors from conformal field theory*, J. Knot Theory Ramifications 16 (2007), no. 2, 127–202.
  - J. Grove, *Constructing TQFTs from modular functors*, J. Knot Theory Ramifications 10 (2001), no. 8, 1085–1131.

Gluing of orbifold geometries

- (M, Σ) with orbifold normal bundle $\mathcal{N}(\Sigma)$ and surface-braid multisections S defining associated braid representations
- choice of discs $D_i^2 \subset \Sigma$, $i = 1, \ldots, N$ and the corresponding surface braids S_i obtained from S by restriction
- (M_i, Σ_i), $i = 1, \ldots, N$ be a collection of orbifold geometries, with discs $\tilde{D}_i^2 \subset \Sigma_i$ and multisections $\tilde{S}_i$ of unit orbifold normal bundles $\mathcal{N}_1(\Sigma_i)$ with
  1. orbifold bundle isomorphism of restrictions
  $$\mathcal{N}(\Sigma_i)|_{\tilde{D}_i^2} \simeq \mathcal{N}(\Sigma)|_{D_i^2}$$
  2. restrictions $\hat{S}_i = \tilde{S}_i|_{\tilde{D}_i^2} \subset D_f^2 \times \tilde{D}_i^2$ and $S_i = S|_{D_i^2 \subset \Sigma}$ have the same number of branch points $\#b(\hat{S}_i) = \#b(S_i)$
- hypotheses ensures fundamental groups are abstractly isomorphic and braid systems related by Hurwitz action
- then can perform a gluing $(M, \Sigma)\#^N_{\varphi_i, i=1}(M_i, \Sigma_i)$
- resulting gluing of the orbifold normal bundles and the multisections determines a new surface braid and an associated braid representation

Building anyon tensor networks

- template graph G for the network
- collection $\{(M_v, \Sigma_v)\}_{v \in V(G)}$ of orbifold geometries
  1. normal bundle $\mathcal{N}(\Sigma_v)$ has multisection surface-braid S_v with branch points $b(S_v)$ of branch cover $S_v \to \Sigma_v$ with
  $$\#b(S_v) = \ell \cdot \mathrm{val}(v), \quad \text{some fixed } \ell \geq 2 \text{ indep of } v$$
  2. for $n_v = \mathrm{val}(v)$, $\exists$ partition $b(S_v) = B_{v,f_1} \sqcup \cdots \sqcup B_{v,f_{n_v}}$ with isomorphic orbifold normal bundles
  $$\varphi_{e_i} : \mathcal{N}(\Sigma_v)|_{D^2_{v,f_i}} \xrightarrow{\simeq} \mathcal{N}(\Sigma_{v'})|_{D^2_{v',f'_i}}$$
  on disk neighborhoods $D^2_{v,f_i} \subset \Sigma_v$ with $B_{v,f_i} \subset D^2_{v,f_i}$
  3. braid representations ρ_{S_v} take values in same braid group $B_{\ell+1}(D^2)$
  4. all braid systems $(\beta_1, \ldots, \beta_\ell)$ of $\pi_1(D^2 \smallsetminus B_{v,f_i})$ are standard
- these assumptions allow gluing of geometries (M_v, Σ_v) along edges of template graph

- assign to the vertex v the vector space

$$\mathcal{V}_v := \bigotimes_{f_i\,:\,s(f_i)=v} \mathcal{V}_{v,f_i} \otimes \bigotimes_{f_i\,:\,t(f_i)=v} \mathcal{V}^{\vee}_{v,f_i}$$

  where $\mathcal{V}^{\vee}_{v',f_i'} := V(D^2, m)^{\vee}$ (from the TQFT of the Chern-Simons modular functor at fifth root of unity)
- Jones representation specialized at a root of unity $q = A^4 = e^{2\pi i/r}$: each component matrix algebra $M_{n_{m,a}}(\mathbb{C}) = \mathrm{End}(\mathcal{V}_{m,k,a})$ corresponds to summand $\mathcal{V}_{v,f_i,a} \simeq \mathcal{V}_{m,k,a}$ of $\mathcal{V}_{v,f_i}$
- isomorphism φ_{e_i} of orbifold normal bundles gives $T_{\varphi_{e_i}} \in \mathrm{End}(\mathcal{V}_{v,f_i})$: index contractions with matrices $T_{\varphi_{e_i}}$ along edges e_i

- resulting vector space

$$\mathcal{V}_G := \bigotimes_{f \in E_{ext} \mid s(f) \in V} \mathcal{V}_{s(f),f} \otimes \bigotimes_{f \in E_{ext} \mid t(f) \in V} \mathcal{V}^{\vee}_{t(f),f}$$

- glued geometry (M_G, Σ_G) has a surface braid S_G with $\#b(S_G) = \ell(\#E_{int} + \#E_{ext})$ and braid representation acting on $\mathcal{V}_G$
- tensors of the tensor network: extended 3-manifold $(X_v, \mathbb{D}_v, k_v)$ with $X_v \subset \tilde{M}_v$ with $\partial X_v \supset \cup_i \{x_{v,f_i}\} \times D^2$ with m marked points, DAP-decomposition containing the curves $\{x_{v,f_i}\} \times \partial D^2$ (points x_{v,f_i} where connected sums performed)
- TQFT with corners assigns to $(X_v, \mathbb{D}_v, n)$ vector $Z(X_v, \mathbb{D}_v, k_v)$ in $\mathcal{V}_v \otimes V(Y_v, \mathbb{D}_{Y_v})$ with $(Y_v, \mathbb{D}_{Y_v})$ extended surface

$$\partial(X_v, \mathbb{D}_v, k_v) \setminus \cup_i (\{x_{v,f_i}\} \times D^2, \{x_{v,f_i}\} \times \partial D^2)$$

- after performing all gluings according to edges of G obtain 3-dimensional geometry $(X_G, \mathbb{D}_G, n_G)$ with boundary

$$\partial(X_G, \mathbb{D}_G, n_G) = \cup_{f \in E_{ext}(G)}(\{x_{\partial(f),f}\} \times D^2, \{x_{\partial(f),f}\} \times \partial D^2) \cup (Y_G, \mathbb{D}_{Y_G})$$

  and vector $Z(X_G, \mathbb{D}_G, n_G)$ in vector space $\mathcal{V}_G \otimes V(Y_G, \mathbb{D}_{Y_G})$
- interpret $Z(X_G, \mathbb{D}_G, n_G)$ as a tensor taking input vectors in $V(Y_G, \mathbb{D}_{Y_G})^\vee$ computing outputs in $\mathcal{V}_G$
- tracing out the "bulk indices" in $V(Y_G, \mathbb{D}_{Y_G})$ one obtains a tensor in $\mathcal{V}_G$
- this can be seen as a code with inputs in $\otimes_{f \in E_{ext} \,|\, t(f) \in V} \mathcal{V}_{t(f),f}$ and outputs in $\otimes_{f \in E_{ext} \,|\, s(f) \in V} \mathcal{V}_{t(f),f}$ or as a state in $\mathcal{V}_G$
- Questions: conditions ensuring perfect tensor (maximal entanglement) conditions? good holographic properties of the networks?