Algebraic Geometry of Segmentation and Tracking

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Geometry of Neuroscience

Geometry of lines in 3-space and Segmentation and Tracking

This lecture is based on the papers:

Reference:

- Marco Pellegrini, *Ray shooting and lines in space. Handbook of discrete and computational geometry*, pp. 599–614, CRC Press Ser. Discrete Math. Appl., CRC, 1997
- Thorsten Theobald, *An enumerative geometry framework for algorithmic line problems in* $\mathbb{R}^3$, SIAM J. Comput. Vol.31 (2002) N.4, 1212–1228
- Frank Sottile and Thorsten Theobald, *Line problems in nonlinear computational geometry*, Contemp. Math. Vol.453 (2008) 411–432.

General Question: *Computational Geometry Problem* efficiently find intersections of a large number of rays (flow of light) and the objects of a scene

Aspects of the problem

- *Vision*: Segmentation and Tracking
- *Robotics*: moving objects in 3-space without collisions
- *Computer Graphics*: rendering realistic images simulating the flow of light

Coordinates on lines in 3-space

1. By pairs of planes: four parameters (a, b, c, d)

$$\ell = \begin{cases} y = az + b \\ x = cz + d \end{cases}$$

2. By pairs of points: two reference planes $z = 1$ and $z = 0$, intersection of a non-horizontal line ℓ $(x_0, y_0, 0)$ and $(x_1, y_1, 1)$ determine ℓ: four parameters (x_0, y_0, x_1, y_1)

Plücker coordinates of lines in 3-space

homogeneous coordinates: coordinates (x, y, z) with $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$

$$(x_0 : x_1 : x_2 : x_3) \quad \text{with} \quad (x_0, x_1, x_2, x_3) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3)$$

for $\lambda \neq 0$ scalar (*projective coordinates* on $\mathbb{P}^3$)

3. Plücker coordinates: $a = (a_0, a_1, a_2, a_3)$ and $b = (b_0, b_1, b_2, b_3)$

$$\ell = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

with $a_0, b_0 > 0$

homogeneous Plücker coordinates

Take all 2×2 minors of the 2×4 matrix above and compute determinants

$$\xi_{ij} = \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$$

Plücker point

$$P(\ell) = (\xi_{01}, \xi_{02}, \xi_{03}, \xi_{12}, \xi_{31}, \xi_{23})$$

homogeneous coordinates of a point in $\mathbb{P}^5$

Plücker relations: coordinates ξ_{ij} satisfy relation

$$\xi_{01}\xi_{23} + \xi_{02}\xi_{31} + \xi_{03}\xi_{12} = 0$$

only homogeneous coordinates in $\mathbb{P}^5$ that satisfy this relation come from lines ℓ in 3-space

The Klein Quadric

$$\mathcal{K} = \{P = (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^5 \,|\, x_0 x_5 + x_1 x_4 + x_2 x_3 = 0\}$$

Plücker hyperplanes: vector $v(\ell) = (\xi_{01}, \xi_{02}, \xi_{03}, \xi_{12}, \xi_{31}, \xi_{23})$

$$h(\ell) = \{P \in \mathbb{P}^5 \,|\, v(\ell) \cdot P = 0\}$$

Twistor Theory

• The idea of transforming problems about lines in 3-spaces into points and hyperplanes in $\mathbb{P}^5$ via the Klein quadric $\mathcal{K}$ goes back to physics: Penrose's twistor approach to general relativity

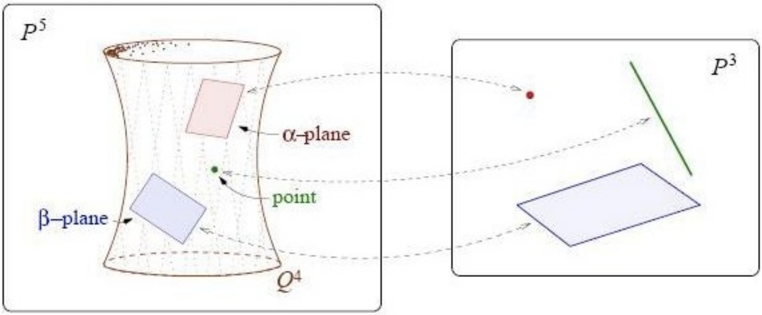
• 4-dimensional vector space T (twistor space); projectivized $\mathbb{P}(T) = G(1, T) \sim \mathbb{P}^3$; Klein quadric $\mathcal{K}$ is embedding $G(2, T) \hookrightarrow \mathbb{P}^5$ Grassmannian of 2-planes in 4-space T

• Penrose Twistor Transform:

$$\mathbb{P}^3 = G(1, T) \longleftarrow F(1, 2, T) \longrightarrow G(2, T) = \mathcal{K}$$

$F(1, 2, T)$ flag varieties with projection maps

• the Klein quadric has rulings by two families of planes (α-planes and β-planes)



• the α-planes are the images under the second projection of the fibers of the first projection in the Penrose transform diagram

$$\mathbb{P}^3 = G(1;T) \longleftarrow F(1,2;T) \longrightarrow G(2;T)$$

• the β-planes similarly from dual Penrose diagram

$$G(3;T^*) \longleftarrow F(2,3;T^*) \longrightarrow G(2;T)$$

• In these planes every line is a light ray: two $\mathbb{P}^1$'s in the base of a light cone $C(\infty)$, same as $\mathbb{P}(\mathcal{S}_\infty) \times \mathbb{P}(\tilde{\mathcal{S}}_\infty)$, where $\mathcal{S}_\infty$ 2-dim spinor space over vertex ∞ of the light cone $C(\infty)$

• Grassmannian $G(2,T)$ is *compactified and complexified* Minkowski space with big cell as complexified spacetime

Only reviewed as motivation: we will not be using Twistor Theory

Projectivized and complex

• Why use projective spaces $\mathbb{P}^3$ and $\mathbb{P}^5$ instead of affine spaces $\mathbb{A}^4$ and $\mathbb{A}^6$?

Projective algebraic geometry works better than affine (because compactness)

$$\mathbb{P}^N = \mathbb{A}^N \cup \mathbb{P}^{N-1}$$

big cell $\mathbb{A}^N$ and (projective) hyperplane $\mathbb{P}^{N-1}$ at infinity

• Why use complex geometry $\mathbb{P}^N(\mathbb{C})$ for a real geometry problem?

Complex algebraic geometry works better than real (polynomials always have the correct number of solutions: intersections, etc.)

Set of real points $\mathbb{P}^N(\mathbb{R})$ of complex projective spaces $\mathbb{P}^N(\mathbb{C})$; similarly $\mathcal{K}(\mathbb{C})$ complex projective algebraic variety and $\mathcal{K}(\mathbb{R})$ its real points

General idea: formulate problems in projective algebraic geometry; solve for complex algebraic varieties; restrict to real points

Use of Plücker coordinates

- problems about lines in 3-space transformed into problems about hyperplanes and points in $\mathbb{P}^5$
- why better? a lot of tools about algebraic geometry of hyperplane arrangements
- disadvantages? five parameters instead of four (can increase running time of algorithms)

However: known that even if computational complexity of a hyperplane arrangement of n hyperplanes in $\mathbb{P}^5$ is $\mathcal{O}(n^5)$, its intersection of $\mathcal{K}$ has computational complexity only $\mathcal{O}(n^4 \log n)$

The geometric setup: a scene in 3-space becomes a hyperplane arrangement in $\mathbb{P}^5$

• suppose given a configuration of objects in 3-space: assume *polyhedra* (can always approximate smooth objects by polyhedra, through a mesh)

• triangulate polyhedra and extends edges of the triangulation to infinite lines ℓ

• each such line ℓ determines a Plücker hyperplane $h(\ell)$ in $\mathbb{P}^5$

• get a hyperplane arrangement in $\mathbb{P}^5$

Ray Shooting Problem

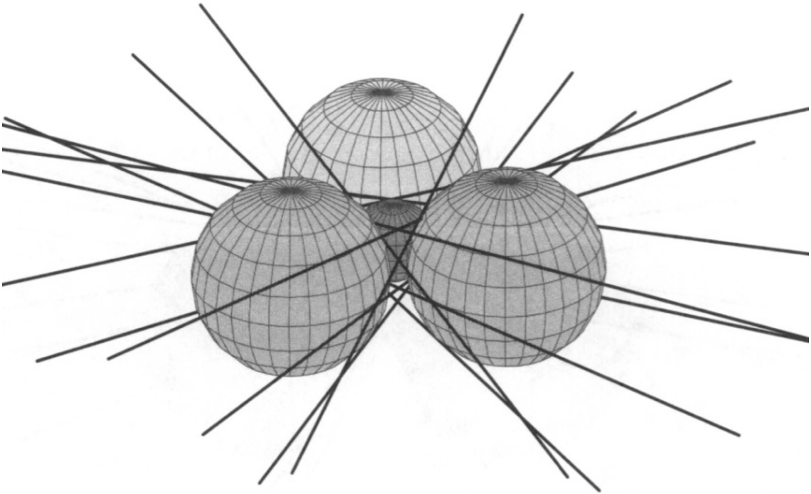
- used for hidden surface removal, detecting and computing intersections of polyhedra
- Given a collection of polyhedra $\mathcal{P}$ in 3-space
- given a point P and a direction $\vec{V}$
- want to identify the *first object* $\mathcal{P}$ *intersected* by a ray originating at P pointing in the direction $\vec{V}$
- consider a triangle τ of the triangulation of $\mathcal{P}$: a line ℓ passes through τ iff the point $P(\ell)$ in $\mathbb{P}^5$ is in the intersection of the (real) half-spaces $h^+(\ell_1) \cap h^+(\ell_2) \cap h^+(\ell_3)$ or $h^-(\ell_1) \cap h^-(\ell_2) \cap h^-(\ell_3)$ determines by the hyperplanes $h(\ell_1)$, $h(\ell_2)$, $h(\ell_3)$ of the three boundary lines ℓ_i of τ
- to find solution to ray-shooting problem: locate $P(\ell)$ check list of triangles τ associated to the corresponding cell of the hyperplane arrangement

Another Problem: which bodies from a given scene cannot be seen from *any* location outside the scene

• Geometric formulation: *determining common tangents to four given bodies in* $\mathbb{R}^3$

• for polyhedra: common tangents means common transversals to edge lines; for smooth objects tangents

Model Result: Four spheres in $\mathbb{R}^3$ (centers not all aligned) have at most 12 common tangent lines; there are configurations that realize 12



Four spheres with coplanar centers and 12 common tangent lines
(figure from Sottile–Theobald)

How to get the geometric formulation

- Partial Visibility:

- $C \subset \mathbb{R}^3$ *convex body* (bounded, closed, convex, with inner points)

- set $\mathcal{C}$ of convex bodies C in $\mathbb{R}^3$ (a scene)

- a body C is *partially visible* if $\exists P \in C$ and $\vec{V}$ such that ray from P in direction $\vec{V}$ does not intersect any other C' in $\mathcal{C}$

unobstructed view of at least some points of C from some viewpoint outside the scene: *visibility ray*

- reduce collection $\mathcal{C}$ be removing all C not partially visible

- if there is a visibility ray for C can continuously move it (translate, rotate) until line is tangent to at least two bodies in $\mathcal{C}$ (one of which can be C): reached boundary of the visibility region for C

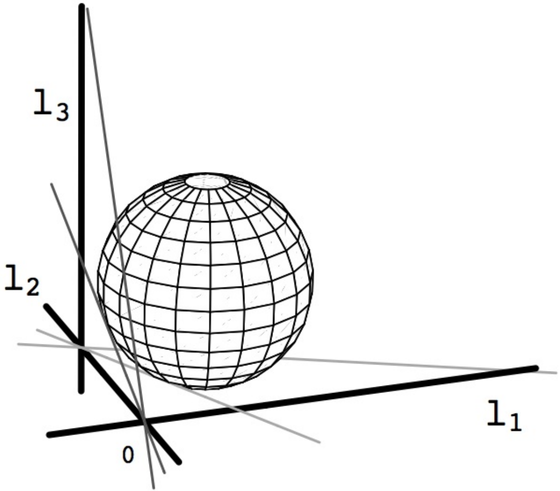
- Consider set $\mathcal{T}(\mathcal{C})$ of lines in $\mathbb{R}^3$ that intersect all bodies $C \in \mathcal{C}$
- Lines in $\mathbb{R}^3$ have four parameters; open condition so $\mathcal{T}(\mathcal{C})$ is a 4-dimensional set
- $\mathcal{T}(\mathcal{C})$ *semialgebraic set* (defined by algebraic equalities and inequalities) with boundary $\partial\mathcal{T}(\mathcal{C})$ containing lines that are tangent to at least one $C \in \mathcal{C}$
- *combinatorial structure* of the set $\mathcal{T}(\mathcal{C})$: faces determined by sets of lines tangent to a fixed subset of bodies in $\mathcal{C}$
- because $\mathcal{T}(\mathcal{C})$ is 4-dimensional, have faces of dimensions $j \in \{0, 1, 2, 3\}$ given by lines tangent to $4 - j$ bodies

Lines and Spheres

- Case C polyhedra: *linear computational geometry*
- Case C smooth (e.g. spheres): *nonlinear computational geometry*

Core problem: find configurations of lines tangent to k spheres and transversal to $4 - k$ lines in $\mathbb{R}^3$

Refer to all cases as "common tangents"



One sphere and three lines with with four common tangents (figure from Theobald)

Bounds on number of common tangents

Fact: (Theobald)
Given k spheres and transversal to $4-k$ lines in $\mathbb{R}^3$; if only finitely many common tangents, then maximal number is

$$N_k = \begin{cases} 2 & k=0 \\ 4 & k=1 \\ 8 & k=2 \\ 12 & k=3,\ k=4 \end{cases}$$

In each case there are configuration realizing the bound

• need Bézout: $\{f_i(x_0,\dots,x_n)\}_{i=1}^n$ homogeneous polynomials degrees d_i with finite number of common zeros in $\mathbb{P}^N$ then number of zeros (with multiplicity) at most $d_1 d_2 \cdots d_n$

• Bézout gives $N_0 \le 2$, $N_1 \le 4$, $N_2 \le 8$

• For instance, to get $N_1 \leq 4$: common tangents to three lines and one sphere means

1. three linear equations

$$\xi_{01}\xi'_{23} - \xi_{02}\xi'_{13} + \xi_{03}\xi'_{12} + \xi_{12}\xi'_{03} - \xi_{13}\xi'_{02} + \xi_{23}\xi'_{01} = 0$$

   which express the fact that the line ℓ with $P(\ell) = (\xi_{01}, \xi_{02}, \xi_{03}, \xi_{12}, \xi_{31}, \xi_{23})$ and the fixed line ℓ' (one of the three given lines) with $P(\ell') = (\xi'_{01}, \xi'_{02}, \xi'_{03}, \xi'_{12}, \xi'_{31}, \xi'_{23})$ intersect in $\mathbb{P}^3$

2. one equation

$$P(\ell)^t \, (\wedge^2 Q) \, P(\ell) = 0$$

   expressing the fact that the line ℓ with $P(\ell) = (\xi_{01}, \xi_{02}, \xi_{03}, \xi_{12}, \xi_{31}, \xi_{23})$ is tangent to the quadric Q

3. and the Plücker relations that restrict to the Klein quadric $\mathcal{K}$

$$\xi_{01}\xi_{23} + \xi_{02}\xi_{31} + \xi_{03}\xi_{12} = 0$$

hence in total using Bézout get $N_1 \leq 4$.

- linear operator defined as

$$\wedge^2 : M_{m\times n}(\mathbb{R}) \to M_{\binom{m}{2}\times\binom{n}{2}}(\mathbb{R})$$

$$(\wedge^2 A)_{I,J} = \det(A_{[I,J]})$$

where $I \subset \{1,\dots,m\}$ with $\#I = 2$ and $J \subset \{1,\dots n\}$ with $\#J = 2$ and $A_{[I,J]}$ the 2×2 minor of A with rows and columns I and J

Note: the equation $P^t\,(\wedge^2 Q)\,P = 0$ for tangency of line and quadric comes from the fact that ℓ tangent to quadric Q iff 2×2 matrix $L^t QL$ is singular, where L is the 4×2 matrix representing ℓ in Plücker form

$$L(\ell) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

condition that $L^T QL$ is singular:

$$\det(L^t QL) = (\wedge^2 L^t)(\wedge^2 Q)(\wedge^2 L) = (\wedge^2 L)^t(\wedge^2 Q)(\wedge^2 L)$$

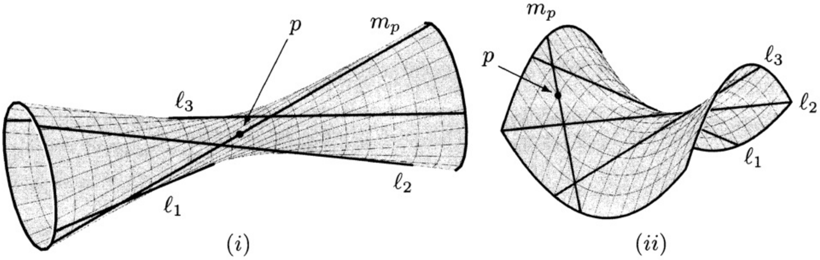
where $\wedge^2 L = P(\ell)$ identifying 6×1 matrix as vector in $\mathbb{P}^5$

• in the case of a sphere in $\mathbb{R}^3$ of radius r and center (c_1, c_2, c_3) explicit equation

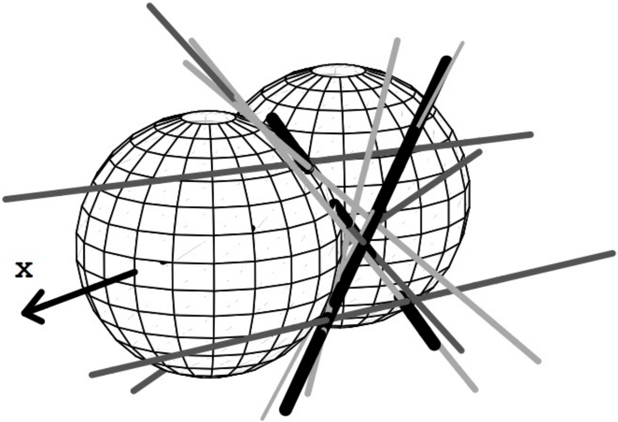
$$\begin{pmatrix} p_{01} \\ p_{02} \\ p_{03} \\ p_{12} \\ p_{13} \\ p_{23} \end{pmatrix}^T \begin{pmatrix} c_2^2 + c_3^2 - r^2 & -c_1 c_2 & -c_1 c_3 & c_2 & c_3 & 0 \\ -c_1 c_2 & c_1^2 + c_3^2 - r^2 & -c_2 c_3 & -c_1 & 0 & c_3 \\ -c_1 c_3 & -c_2 c_3 & c_1^2 + c_2^2 - r^2 & 0 & -c_1 & -c_2 \\ c_2 & -c_1 & 0 & 1 & 0 & 0 \\ c_3 & 0 & -c_1 & 0 & 1 & 0 \\ 0 & c_3 & -c_2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{01} \\ p_{02} \\ p_{03} \\ p_{12} \\ p_{13} \\ p_{23} \end{pmatrix}$$

because quadric is $(x_1 - c_1 x_0)^2 + (x_2 - c_2 x_0)^2 + (x_3 - c_3 x_0)^2 = r^2 x_0^2$ in homogeneous coordinates

- Case of $N_1 \leq 4$ as above
- Case $N_2 \leq 8$ very similar argument
- Case $N_0 \leq 2$ also similar: common transversals to four lines in 3-space classical enumerative geometry problem, known 2 can be achieved (with $\mathbb{R}$-solutions) so $N_0 = 2$
- $N_1 = 4$ a construction achieving the maximum shown in previous figure
- $N_2 = 8$: explicit construction of a configuration of two lines and two balls with eight common tangent lines



configurations of four lines: ℓ_1, ℓ_2, ℓ_3 on a ruling of a hyperboloid; lines transversal to them second ruling; a fourth line either two or zero (real) intersections with hyperboloid (from Sottile–Theobald)



configuration of two lines and two balls with eight common tangent lines (figure from Theobald)

bound $N_3 \leq 12$

• H hyperplane in $\mathbb{P}^5$ characterizing lines transversal to given $\ell = \ell_1$

$$H = \{P' \,|\, \xi_{01}\xi'_{23} - \xi_{02}\xi'_{13} + \xi_{03}\xi'_{12} + \xi_{12}\xi'_{03} - \xi_{13}\xi'_{02} + \xi_{23}\xi'_{01} = 0\}$$

points of $\mathcal{K} \cap H$ correspond to such lines

• quadrics Q_2, Q_3, Q_4 and corresponding $\wedge^2 Q_i$:

$$P(\ell')^t \, (\wedge^2 Q_i) \, P(\ell') = 0$$

equations characterizing tangency to Q_i for $P(\ell') \in H$

• so far would get a total degree 6 (three quadratic equations and one linear) but these are not independent and there are intersections of multiplicity two which leads to a counting of 12 to get the multiplicity counting

• quadric Q of a sphere $(x_1 - c_1 x_0)^2 + (x_2 - c_2 x_0)^2 + (x_3 - c_3 x_0)^2 = r^2 x_0^2$ in homogeneous coordinates in $\mathbb{P}^3$

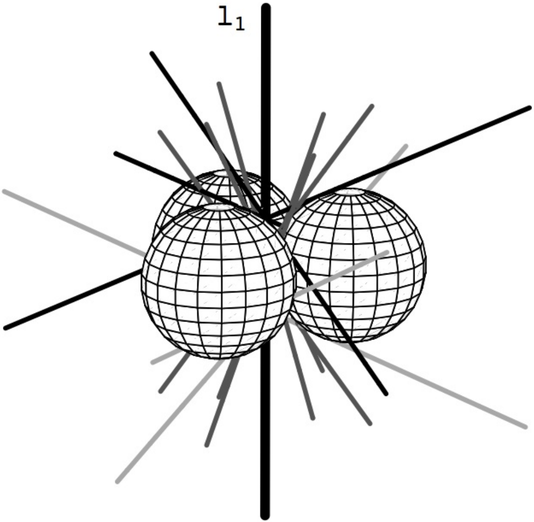
• intersects the plane $\mathbb{P}^2$ at infinity in the conic $x_1^2 + x_2^2 + x_3^2 = 0$

• point $p = (0, \zeta_1, \zeta_2, \zeta_3)$ on the conic: tangent to conic at p in plane $\mathbb{P}^2$ has Plücker coordinates $(0, 0, 0, \zeta_3, -\zeta_2, \zeta_1)$

• Plücker vector of a tangent to conic in the $\mathbb{P}^2$ at infinity is also contained in $\wedge^2 Q_2$, $\wedge^2 Q_3$, $\wedge^2 Q_4$

• use this to conclude that the tangent hyperplanes to the quadrics $\wedge^2 Q_2$, $\wedge^2 Q_3$, $\wedge^2 Q_4$ and $\mathcal{K}$ all contain a common 2-dimensional subspace

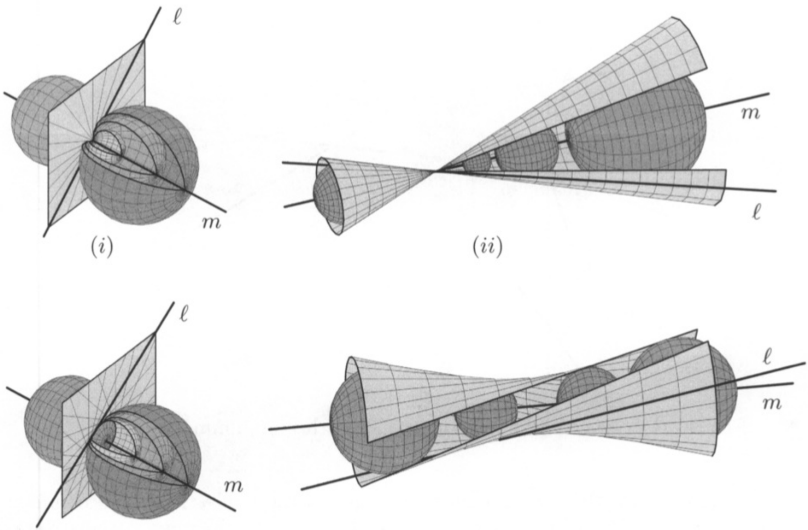
• conclude that multiplicity of intersection at least 2, which gives the correct counting of max number of tangents



configuration of one line and three balls with twelve common tangent lines (figure from Theobald)

• a similar argument for the bound $N_4 \leq 12$ (see Theobald and Sottile-Theobald)

• further step: give a characterization of the configuration with infinitely many common tangents like four collinear centers and equal radii (or collinear centers and inscribed in same hyperboloid), see Theobald and Sottile-Theobald



configurations of four spheres with infinitely many common tangents (Sottile–Theobald)

Conclusions

the geometric argument on maximal number of common tangents allows for an estimate of the algorithmic complexity of the visibility problem: solving polynomial equations of the given degree (eg a degree 12 equation for the case of 4-spheres)