Feynman integrals and motives

Matilde Marcolli

Plenary talk: AMS Meeting, UC Riverside, November 2013 based on:

- O.Ceyhan, M.Marcolli, *Feynman integrals and motives of configuration spaces*, Communications in Mathematical Physics: Vol.313, N.1 (2012), 35–70, arXiv:1012.5485
- O.Ceyhan, M.Marcolli, *Feynman integrals and periods in configuration spaces*, arXiv:1207.3544
- O.Ceyhan, M.Marcolli, *Algebraic renormalization and Feynman integrals in configuration spaces*, arXiv:1308.5687

General question: express Feynman integral computations (and some direct mathematical generalizations) as computations of periods of algebraic varieties

Period: integral of an algebraic differential form on an algebraic variety over a chain defined by algebraic equations

$$\int_\sigma \omega$$

(in general transcendental number but “obtained from algebraic data")

- What kind of periods?
- What kind of *motives*?

Quantum Field Theory: perturbative (massless) scalar field theory

$$S(\phi) = \int \mathscr{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

in D dimensions, with Lagrangian density

$$\mathscr{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathscr{L}_{int}(\phi)$$

Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_\Gamma \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

Two different settings for Feynman integral computations:
Momentum space: momentum variables k_e with $e \in E_\Gamma$
$\Gamma(\phi)$ built from edge-propagators

$$\frac{1}{(m^2 + \|k_e\|^2)}$$

Configuration space: position variables x_v with $v \in V_\Gamma$
$\Gamma(\phi)$ built from propagators:

$$G_{0,\mathbb{R}}(x_{s(e)} - x_{t(e)}) = \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}}, \quad \text{where} \quad D = 2\lambda + 2$$

or massive

$$G_{m,\mathbb{R}}(x_{s(e)} - x_{t(e)}) = \frac{m^\lambda}{(2\pi)^{(\lambda+1)}} \|x_{s(e)} - x_{t(e)}\|^{-\lambda} \mathscr{K}_\lambda(m\|x_{s(e)} - x_{t(e)}\|)$$

with $\mathscr{K}_\nu(z)$ modified Bessel function

Dual pictures:

• $G_{0,\mathbb{R}}(x_{s(e)} - x_{t(e)})$ Green function of Laplacian; $G_{m,\mathbb{R}}(x_{s(e)} - x_{t(e)})$ fundamental solution of Helmholtz equation $(\Delta + m^2)G = \delta$

• Fourier transform: (test functions $\varphi \in \mathscr{S}(\mathbb{R}^D)$)

$$(\widehat{G_{0,\mathbb{R}} \star \varphi})(k) = \frac{4\pi^{D/2}}{\Gamma(\lambda)} \frac{1}{\|k\|^2} \widehat{\varphi}(k)$$

$$(\widehat{G_{m,\mathbb{R},} \star \varphi})(k) = \frac{1}{(m^2 + \|k\|^2)} \widehat{\varphi}(k)$$

Algebraic formulation: extend from real to complex variables using a quadratic form instead of the Euclidean norm

Momentum space Feynman amplitude ($n = \# E_\Gamma$)

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

quadratic form

$$q_e(k_e) = \sum_{j=1}^{D} k_{e,j}^2 + m^2$$

Configuration space Feynman amplitude (massless; $m = \# V_\Gamma$)

$$U(\Gamma) = \int \frac{1}{(Q_1 \cdots Q_n)^\lambda} d^D x_{v_1} \cdots d^D x_{v_m}$$

$$Q_e(x_{s(e)}, x_{t(e)}) = \sum_{j=1}^{D} (x_{s(e),j} - x_{t(e),j})^2$$

- Advantages: *get an algebraic differential form*
- Disadvantages: *singular on a hypersurface* (whose motive is difficult to control)

Analytic formulation: extend from real to complex variables using the Euclidean norm

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\| x_{s(e)} - x_{t(e)} \|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

over chain of integration $\sigma_\Gamma = \mathbb{R}^{\# V_\Gamma}$

- Advantages: *Singular on diagonals* (motive will be easy to control)
- Disadvantages: *not an algebraic differential form* (only smooth)

Recent results:

• Using *algebraic formulation in momentum space*

• Earlier conjectures: periods would be $\mathbb{Q}(2\pi i)$-combinations of multiple zeta values (periods of mixed Tate motives)

• New results show *explicit* non-mixed-Tate periods:

- Dzmitry Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, arXiv:1006.3533
- Francis Brown, Oliver Schnetz, *A K3 in phi4*, arXiv:1006.4064.
- Francis Brown, Dzmitry Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056

Configuration space picture

X smooth projective variety with a dense $\mathbb{A}^D$ (e.g. $X = \mathbb{P}^D$)

We look at ***two different problems***

• Real case: the *analytic formulation* of the Feynman amplitude (physically motivated case)

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

with $\sigma_\Gamma = X(\mathbb{R})^{\# V_\Gamma}$

• $\mathscr{C}^\infty$-differential form on X^{V_Γ} with singularities along diagonals

• not a closed form

• Complex case: a *complexification* of the previous problem (mathematical generalization)

$Z = X \times X$ with projection $p : Z \to X$, $p : z = (x, y) \mapsto x$

$$\omega_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$

• closed form

• chain of integration:

$$\sigma^{(Z,y)} = X^{V_\Gamma} \times \{y = (y_v)\} \subset Z^{V_\Gamma} = X^{V_\Gamma} \times X^{V_\Gamma}$$

for a fixed $y = (y_v \,|\, v \in V_\Gamma)$

Relation to Green functions:

• Green function of real Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda + 2$:

$$G_{\mathbb{R}}(x, y) = \frac{1}{\|x - y\|^{2\lambda}}$$

• On $\mathbb{A}^D(\mathbb{C})$ complex Laplacian

$$\Delta = \sum_{k=1}^{D} \frac{\partial^2}{\partial x_k \partial \bar{x}_k}$$

has Green form

$$G_{\mathbb{C}}(x, y) = \frac{-(D-2)!}{(2\pi i)^D \|x - y\|^{2D-2}}$$

real and complex amplitudes modeled on these two cases

Different methods:

• real case: explicit computation of (regularized) integral

$$\int_{\sigma_\Gamma} \omega_\Gamma$$

using expansion of Green function in Gegenbauer polynomials

• complex case: cohomological method, pullback $\omega_\Gamma^{(Z)}$ to a compactification of configuration space where cohomologous to algebraic form with log poles; regularize to separate poles from chain of integration; explicitly compute the motive

Explicit computations of Feynman amplitudes (real case):

Step 1: explicit chains in X^{V_Γ}

- Acyclic orientations: Γ no looping edges, $\Omega(\Gamma)$ set of acyclic orientations; Stanley: $(-1)^{V_\Gamma} P_\Gamma(-1)$ acyclic orientations where $P_\Gamma(t)$ chromatic polynomial
- orientation $\mathbf{o} \in \Omega(\Gamma) \Rightarrow$ partial ordering of vertices $w \geq_{\mathbf{o}} v$
- chain with boundary $\partial\Sigma_{\mathbf{o}} \subset \cup_{e \in E_\Gamma} \Delta_e$

$$\Sigma_{\mathbf{o}} := \{(x_v) \in X^{V_\Gamma}(\mathbb{R}) \ : \ r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}$$

middle dimensional relative homology class

$$[\Sigma_{\mathbf{o}}] \in H_{|V_\Gamma|}(X^{V_\Gamma}, \cup_{e \in E_\Gamma} \Delta_e)$$

- $\Sigma_{\mathbf{o}} \smallsetminus \cup_v \{r_v = 0\}$ bundle fiber $(S^{D-1})^{V_\Gamma}$ base

$$\overline{\Sigma}_{\mathbf{o}} = \{(r_v) \in (\mathbb{R}_+^*)^{V_\Gamma} \ : \ r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}$$

Step 2: Gegenbauer polynomials

- Generating function and orthogonality ($|t| < 1$ and $\lambda > -1/2$)

$$\frac{1}{(1-2tx+t^2)^\lambda} = \sum_{n=0}^\infty C_n^{(\lambda)}(x) t^n$$

$$\int_{-1}^1 C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)\,(1-x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}$$

- $D = 2\lambda + 2$ Newton potential expansion in Gegenbauer polynomials:

$$\frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} = \frac{1}{\rho_e^{2\lambda}(1 + (\frac{r_e}{\rho_e})^2 - 2\frac{r_e}{\rho_e}\omega_{s(e)} \cdot \omega_{t(e)})^\lambda}$$

$$= \rho_e^{-2\lambda} \sum_{n=0}^\infty (\frac{r_e}{\rho_e})^n C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}),$$

with $\rho_e = \max\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and $r_e = \min\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and with $\omega_v \in S^{D-1}$

Step 3: angular and radial integrals

- on chain of integration $\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$ Feynman integral becomes (Version N.1)

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\Sigma_{\mathbf{o}}} \prod_{e\in E_\Gamma} r_{t_{\mathbf{o}}(e)}^{-2\lambda} \left(\sum_n \left(\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}}\right)^n C_n^{(\lambda)}(\omega_{s_{\mathbf{o}}(e)} \cdot \omega_{t_{\mathbf{o}}(e)}) \right) dV$$

with positive integers $m_{\mathbf{o}}$ (multiplicities) and volume form $dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v \, d\omega_v$

- angular integrals:

$$\mathscr{A}_{(n_e)_{e\in E_\Gamma}} = \int_{(S^{D-1})^{V_\Gamma}} \prod_e C_{n_e}^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v$$

- radial integrals:

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \prod_{e\in E_\Gamma} \mathscr{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) \prod_v r_v^{D-1} dr_v$$

$$\mathscr{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) = r_{t_{\mathbf{o}}(e)}^{-2\lambda} \sum_{n_e} \mathscr{A}_{n_e} \left(\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}}\right)^{n_e}$$

Example: polygons and polylogarithms

- Γ polygon with k edges, $D = 2\lambda + 2$:

$$\mathscr{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)} \right)^k \cdot \dim \mathscr{H}_n(S^{2\lambda+1})$$

$\mathscr{H}_n(S^{2\lambda+1})$ space of harmonic functions deg n on $S^{2\lambda+1}$ (Gegenbauer polynomial and zonal spherical harmonics)

- when $D = 4$, Feynman amplitude:

$$(2\pi^2)^k \sum_{\mathbf{o}} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \mathrm{Li}_{k-2}\Big(\prod_i \frac{r_{w_i}^2}{r_{v_i}^2}\Big) \prod_v r_v \, dr_v$$

polylogarithm functions

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

vertices v_i, w_i sources and tails of oriented paths of $\mathbf{o}$

Problem: computations intractable very quickly for larger graphs!

• Can reduce to trivalent vertices and use triple integrals of harmonic functions: Gaunt coefficients $\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} Y^{(n_3)}_{\ell_3} \rangle_D$ Racah's factorization in terms of *isoscalar factors*

$$\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}, Y^{(n_3)}_{\ell_3} \rangle_D = \begin{pmatrix} n_1 & n_2 & n_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix}_{D:D-1} \langle Y^{(n'_1)}_{\ell'_1}, Y^{(n'_2)}_{\ell'_2}, Y^{(n'_3)}_{\ell'_3} \rangle_{D-1}$$

$\ell_i = (n'_i, \ell'_i)$ with $n'_i = m_{D-2,i}$ and $\ell'_i = (m_{D-3,i}, \ldots, m_{1,i})$

• There are general explicit (but complicated) expressions for the isoscalar factors

• For $D = 4$ “leading term" involving multiple series related to MVZs:

Mordell–Tornheim multiple series:

$$\zeta_{MT,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R}_P^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}$$

$$\mathscr{R} = \mathscr{R}_P^{(k)} := \{(n_1,\ldots,n_k)\,|\,n_i > 0,\ \ i = 1,\ldots,k\}$$

Apostol–Vu multiple series:

$$\zeta_{AV,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R}_{MP}^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}$$

$$\mathscr{R} = \mathscr{R}_{MP}^{(k)} := \{(n_1,\ldots,n_k)\,|\,n_k > \cdots > n_2 > n_1 > 0\}$$

BUT: possible occurrences of non-mixed-Tate terms in larger graphs!

Different behavior of the complex case

Step 1: Graph configuration spaces

$$Conf_\Gamma(X) = X^{V_\Gamma} \smallsetminus \bigcup_{e \in E_\Gamma} \Delta_e$$

• Wonderful compactifications: compactify $Conf_\Gamma(X)$ to a smooth projective algebraic variety $\overline{Conf}_\Gamma(X)$ so that

$$\mathscr{D}_\Gamma = \overline{Conf}_\Gamma(X) \smallsetminus Conf_\Gamma(X)$$

is a normal crossings divisor

• For $Z = X \times X$ take $F(X,\Gamma) \simeq \mathrm{Conf}_\Gamma(X) \times X^{V_\Gamma}$ with $\Delta_e^{(Z)} \cong \Delta_e \times X^{V_\Gamma}$ and compactify to $\overline{F(X,\Gamma)}$ in the same way

• General method: realize $\overline{Conf_\Gamma}(X)$ or $\overline{F(X,\Gamma)}$ as a sequence of blowups of X^{V_Γ} (or Z^{V_Γ}) along a collection of dominant transforms of diagonals

• Equivalent description: closure in

$$Conf_\Gamma(X) \hookrightarrow \prod_{\gamma \in \mathscr{G}_\Gamma} \mathrm{Bl}_{\Delta_\gamma} X^{V_\Gamma}$$

with $\mathscr{G}_\Gamma$ subgraphs induced (all edges of Γ between subset of vertices) and 2-vertex-connected

• Fulton–MacPherson configuration spaces (= complete graph case of $Conf_\Gamma(X)$); more general arrangements of subvarieties: DeConcini–Procesi, Li Li

• strata of $\mathscr{D}_\Gamma$ parameterized by forests of nested subgraphs (as in Fulton–MacPherson case)

Motives of algebraic varieties (Grothendieck) Universal cohomology theory for algebraic varieties (with realizations)

- *Pure motives*: smooth projective varieties with correspondences

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{/\sim,\mathbb{Q}}(X,Y)\,p$$

Algebraic cycles mod equivalence (rational, homological, numerical), composition

$$\mathrm{Corr}(X,Y)\times\mathrm{Corr}(Y,Z)\to\mathrm{Corr}(X,Z)$$

$$(\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha)\bullet\pi^*_{Y,Z}(\beta))$$

intersection product in $X\times Y\times Z$; with projectors $p^2=p$ and $q^2=q$ and Tate twists $\mathbb{Q}(m)$ with $\mathbb{Q}(1)=\mathbb{L}^{-1}$

Numerical pure motives: $\mathscr{M}_{num,\mathbb{Q}}(k)$ semi-simple abelian category (Jannsen)

• *Mixed motives*: varieties that are possibly singular or not projective (much more complicated theory!) Triangulated category $\mathscr{DM}$ (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

• *Mixed Tate motives* $\mathscr{DMT} \subset \mathscr{DM}$ generated by the $\mathbb{Q}(m)$

Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Motives and the Grothendieck ring of varieties

• Difficult to determine explicitly in the triangulated category of *mixed motives*

• Simpler invariant (universal Euler characteristic for motives): class $[X_\Gamma]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \setminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$
(K_0 group of category of pure motives: virtual motives)

Universal Euler characteristics:
Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \setminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

values in a commutative ring $\mathscr{R}$ is same thing as a ring homomorphism

$$\chi : K_0(\mathscr{V}) \to \mathscr{R}$$

Examples:

- Topological Euler characteristic
- Couting points over finite fields

Motives of configuration spaces – Key ingredient: Blowup formulae

- For mixed motives (Voevodsky category):

$$\mathfrak{m}(\mathrm{Bl}_V(Y)) \cong \mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} \mathfrak{m}(V)(k)[2k]$$

- For Grothendieck classes Bittner relation

$$[\mathrm{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V]([\mathbb{P}^{\mathrm{codim}_Y(V)-1}] - 1)$$

exceptional divisor E

- Conclusion: the motive of $\overline{Conf_\Gamma}(X)$ and of $\overline{F(X,\Gamma)}$ is mixed Tate if X is mixed Tate.

Smooth and algebraic forms

• de Rham cohomology of a smooth quasi-projective varieties computed using algebraic differential forms (Grothendieck)

• if complement of normal crossings divisor can use forms with log poles (Deligne)

$$H^*(\mathscr{U}) \simeq \mathbb{H}^*(\mathscr{X}, \Omega^*_{\mathscr{X}}(\log(\mathscr{D})))$$

• $\mathscr{X}$ smooth projective variety $\dim_{\mathbb{C}} m$; $\mathscr{D}$ normal crossings divisor; $\mathscr{U} = \mathscr{X} \smallsetminus \mathscr{D}$; ω smooth closed differential form deg$m$ on $\mathscr{U}$;
$\Rightarrow \exists$ algebraic differential form η log poles along $\mathscr{D}$, with $[\eta] = [\omega] \in H^m_{dR}(\mathscr{U})$

• Conclusion: $\exists$ algebraic form $\eta_\Gamma^{(Z)}$ with log poles along union of D_γ, cohomologous to $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ on $\tilde{\sigma}_\Gamma^{(Z,y)}$

• Warning: motive over $\mathbb{Q}$, but algebraic form may be over larger field! (work in progress: show form defined over $\mathbb{Q}(2\pi i)$ using Bochner–Martinelli kernel and Green forms)

Regularization problem

• $\eta_\Gamma^{(Z)}$ algebraic differential form; $\tilde{\sigma}_\Gamma^{(Z,y)}$ algebraic cycle: Feynman integral becomes

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)} \smallsetminus (\mathscr{D}_\Gamma \cap \tilde{\sigma}_\Gamma^{(Z,y)})} \eta_\Gamma^{(Z)}$$

would be a period... but divergent!!
(because of intersection $\mathscr{D}_\Gamma \cap \tilde{\sigma}_\Gamma^{(Z,y)}$ of chain with divisor)

• need a regularization and renormalization procedure to eliminate divergences

• Different methods: (1) principal value and residue currents; (2) deformation to the normal cone; (3) algebraic renormalization via Hopf algebras and Rota–Baxter algebras

• Focus on (3)

Algebraic renormalization

Two step procedure:

• Regularization: replace divergent integral $U(\Gamma)$ by function with poles

• Renormalization: pole subtraction with consistency over subgraphs (Hopf algebra structure)

• Kreimer, Connes–Kreimer, Connes–M.: Hopf algebra of Feynman graphs and BPHZ renormalization method in terms of Birkhoff factorization and differential Galois theory

• Ebrahimi-Fard, Guo, Kreimer: algebraic renormalization in terms of Rota–Baxter algebras

Connes–Kreimer Hopf algebra $\mathscr{H} = \mathscr{H}(\mathscr{T})$ (depends on theory)

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathscr{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Algebraic renormalization (Connes-Kreimer; Ebrahimi-Fard, Guo, Kreimer)

• Rota–Baxter algebra of weight $\lambda = -1$: $\mathcal{R}$ commutative unital algebra; $T : \mathcal{R} \to \mathcal{R}$ linear operator with

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy)$$

• Example: $T =$ projection onto polar part of Laurent series

• T determines splitting $\mathcal{R}_+ = (1-T)\mathcal{R}$, $\mathcal{R}_- =$ unitization of $T\mathcal{R}$; both $\mathcal{R}_\pm$ are algebras

• Feynman rule $\phi : \mathcal{H} \to \mathcal{R}$ commutative algebra homomorphism from CK Hopf algebra $\mathcal{H}$ to Rota–Baxter algebra $\mathcal{R}$ weight -1

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$$

• Note: ϕ does *not know* that $\mathcal{H}$ Hopf and $\mathcal{R}$ Rota-Baxter, only commutative algebras

• Birkhoff factorization $\exists \phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathscr{H}, \mathscr{R}_\pm)$

$$\phi = (\phi_- \circ S) \star \phi_+$$

where $\phi_1 \star \phi_2(X) = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$

• Connes-Kreimer inductive formula for Birkhoff factorization:

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X''))$$

$$\phi_+(X) = (1 - T)(\phi(X) + \sum \phi_-(X')\phi(X''))$$

where $\Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''$

• Recovers what known in physics as BPHZ renormalization procedure

Back to configuration spaces:

• Can use *same* configuration space and compactification for all graphs with fixed number of vertices (cost: more blowups)

• Smooth variety $\mathscr{Y}$ with normal crossings divisor $\mathscr{D}$; form with log poles $\eta \in \Omega^k_{\mathscr{Y}}(\log \mathscr{D})$ and intersection $\mathscr{D}_J = D_{j_1} \cap \cdots D_{j_r}$ of components of $\mathscr{D}$

$$\int_\Sigma \mathrm{Res}_{\mathscr{D}_J}(\eta) = \frac{1}{(2\pi i)^r} \int_{\mathscr{L}_{\mathscr{D}_J}(\Sigma)} \eta$$

$\mathrm{Res}_{\mathscr{D}_J}(\eta) =$ iterated Poincaré residue
$\mathscr{L}_{\mathscr{D}_J}(\Sigma) =$ Leray coboundary

Rota-Baxter algebra for $(\mathscr{Y}, \mathscr{D})$

- even forms with log poles $\Omega^{\rm even}_{\mathscr{Y}}(\log \mathscr{D})$: commutative algebra
- polar part operator

$$T(\eta) = \sum_{j=1}^{n} \frac{df_j}{f_j} \wedge {\rm Res}_{D_j}(\eta)$$

f_j = local equation for D_j

- $(\Omega^{\rm even}_{\mathscr{Y}}(\log \mathscr{D}), T)$ = Rota–Baxter algebra of weight -1

$$T(\eta \wedge T(\xi)) + T(T(\eta) \wedge \xi) - T(\eta) \wedge T(\xi) = T(\eta \wedge \xi)$$

• obtain Rota-Baxter algebra of configuration spaces

• Regularization: given a Feynman graph Γ and the (non-holomorphic closed) form $\omega_\Gamma^{(Z)}$: pull back to wornderful compactification and replace by cohomologous algebraic form η_Γ with log poles

• algebraic Feynman rules: the assignment

$$\phi : \Gamma \mapsto \omega_\Gamma^{(Z)} \mapsto \eta_\Gamma$$

defines a morphism of commutative algebras from the Hopf algebra of Feynman graphs to the Rota–Baxter algebra of configuration spaces

• Renormalization: apply BPHZ to this algebraic Feynman rule

Birkhoff factorization

$$\phi_-(\Gamma) = -T(\eta_\Gamma + \sum_{\gamma\subset\Gamma} \phi_-(\gamma) \wedge \eta_{\Gamma/\gamma})$$

$$\phi_+(\Gamma) = (1-T)(\eta_\Gamma + \sum_{\gamma\subset\Gamma} \phi_-(\gamma)\wedge\eta_{\Gamma/\gamma}) = \eta_{\Gamma,\mathscr{D}} + \sum_{\gamma\subset\Gamma} (\phi_-(\gamma)\wedge\eta_{\Gamma/\gamma})_{\mathscr{D}},$$

with $\eta_{\mathscr{D}} = \eta - T(\eta)$

Renormalized integral

$$\int_{\tilde{\sigma}_{\Gamma,\mathbb{C}}} \eta_{\Gamma,\mathscr{D}} + \sum_{\gamma\subset\Gamma} (\phi_-(\gamma) \wedge \eta_{\Gamma/\gamma})_{\mathscr{D}}$$

free of divergences and integral of algebraic differential form: genuine period