A COMPACTIFICATION OF IGUSA VARIETIES

ELENA MANTOVAN

Abstract. We investigate the notion of Igusa level structure for a one-dimensional Barsotti-Tate group over a scheme X of positive characteristic and compare it to Drinfeld's notion of level structure. In particular, we show how the geometry of the Igusa covers of X is useful for studying the geometry of its Drinfeld covers (e.g. connected and smooth components, singularities).

Our results apply in particular to the study of the Shimura varieties considered in [3]. In this context, they are higher dimensional analogues of the classical work of Igusa for modular curves and of the work of Carayol for Shimura curves. In the case when the Barsotti-Tate group has constant p-rank, this approach was carried-out by Harris and Taylor in [3].

Contents

1. Introduction

In the theory of modular curves, an important role is played by Igusa curves ([4]). These are moduli spaces of elliptic curves in positive characteristic p which can be identified with the smooth components of the reduction modulo p of modular curves. As schemes, they naturally arise as smooth compactifications of finite étale Galois covers of the ordinary loci of the reduction of modular curves of level prime to p.

[0]Department of Mathematics, CalTech, Pasadena, CA 91125 - mantovan@caltech.edu

This classical approach was extended by Carayol ([1]) to study the bad reduction of Shimura curves, and later by Harris and Taylor ([3]) to the context of some simple Shimura varieties of (PEL)-type. (PEL)-type Shimura varieties arise as moduli spaces of polarized abelian varieties endowed with additional structures. In [3], Harris and Taylor isolate some simple Shimura varieties whose reduction modulo a chosen prime w of positive characteristic could successfully be studied via higher-dimensional analogues of Igusa curves. Their key assumption is that the deformation theory of the abelian varieties classified by the Shimura varieties is controlled by one-dimensional Barsotti-Tate groups. As the additional structures on the abelian varieties induce additional structures on the associated Barsotti-Tate groups, these are endowed with a compatible action of the ring of integers $\mathcal{O}_K$ of a local field $K/\mathbb{Q}_p$ (and are accordingly called compatible Barsotti-Tate $\mathcal{O}_K$-modules).

In this higher-dimensional context, the classical stratification of the reduction of modular curves into ordinary locus and supersingular points is replaced by the p-rank stratification of the reduction of Shimura varieties, whose strata are defined as the locally closed reduced subschemes where the p-rank of the p-divisible part of the abelian varieties is constant. Following classical Igusa theory, Harris and Taylor formulate some new Igusa moduli problems in terms of level structure on the étale part of the pertinent Barsotti-Tate $\mathcal{O}_K$-modules . These are representable when restricted to the p-rank strata of the reduction of a Shimura variety of level prime to w , and moreover finite étale and Galois (thus, in particular, also smooth). Further more, each p-rank stratum of a Shimura variety with bad reduction at w is the disjoint union of some distinguished smooth subvarieties (not necessarily connected), each isomorphic (up to a inseparable map) to the Igusa variety of the same level over the corresponding p-rank stratum of a certain Shimura variety with good reduction. The explanation of how these smooth subvarieties piece together inside the bad reduction of the Shimura variety remains unaddressed.

In this paper we provide an answer to this question, by introducing a notion of Igusa level structure which does not require constant p-rank. This allows us to extend the Igusa covers over the ordinary stratum (i.e. the maximal p-rank stratum) to the whole Shimura variety. We prove that these compactified Igusa covers are finite flat smooth covers over the underlying Shimura variety with good reduction. Further more, in the cases of bad reduction, we show that each smooth component of the Shimura varieties is isomorphic (up to an inseparable map) to the corresponding Igusa variety. An explicit description of the intersections in terms of the local coordinates at each point shows that these intersections are in general not transversal, not even reduced. These results give an immediate understanding of the number of geometrically connected components of the Shimura varieties in positive characteristic. In the last section, we extend our analysis to the geometrically connected components of the Shimura varieties in characteristic zero. In particular, we show that the number of geometrically connected components of (the generic fibers of) the Shimura varieties is independent of the level at w.

In an attempt to maintain notations and exposition as simple and self-contained as possible, we take an "axiomatic approach" to the above questions and reformulate them in a general context, for a scheme X of characteristic p and a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{G}$ over X. In this context, we introduce the classical notion of Drinfeld covers and a generalization of Harris' and Taylor's notion of Igusa covers, and explain how the Igusa covers can be used to describe the geometry of the Drinfeld covers. The connection with the Shimura varieties studied in [3] is made explicit in the last section.

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2. Preliminaries

We provide a short overview of the theory of one-dimensional Barsotti-Tate $\mathcal{O}_K$-modules , for $\mathcal{O}_K$ the ring of integers of a p-adic local field K, following [7],[2]. We exclusively focus on those aspects of the theory which are relevant for this paper.

2.1. **Barsotti-Tate $\mathcal{O}_K$-modules.** Let K be a finite extension of $\mathbb{Q}_p$. We denote by $\mathcal{O}_K$ the ring of integers of K, by $\mathcal{P}_K$ its prime ideal and by $\pi \in \mathcal{P}_K$ a uniformizer of $\mathcal{O}_K$. We write $\mathbb{F} = \mathcal{O}_K/\mathcal{P}_K$ for the residue field of $\mathcal{O}_K$, $q = p^f$ for the cardinality of $\mathbb{F}$. Finally, let $\hat{K}^{nr}$ be the maximal unramified extension of K, $\mathcal{O}_{\hat{K}^{nr}}$ its ring of integers and k its residue field. In the following, for any $\mathbb{F}$-scheme Y, we write $F = F_Y : Y \to Y$ for its q-Frobenius.

Given an integer $m \geq 1$, a finite flat $\mathcal{O}_K/\mathcal{P}_K^m$-module over a scheme S is a finite flat group scheme G/S, endowed with a faithful action of $\mathcal{O}_K/\mathcal{P}_K^m$. A Barsotti-Tate $\mathcal{O}_K$-module over a scheme S is a Barsotti-Tate group H/S, together with an embedding $\mathcal{O}_K \hookrightarrow \mathrm{End}\,(H)$. In the following, all the Barsotti-Tate $\mathcal{O}_K$-modules are assumed compatible and one-dimensional (see [3], p. 59 for a definition).

2.1.1. By combining the classical theory of Barsotti-Tate groups ([7]) with Drinfeld's theory of elliptic modules ([2], by an elliptic modules we mean a formal one-dimensional Barsotti-Tate $\mathcal{O}_K$-module), one obtains the following classification.

Proposition 1. (1) *For any $g \geq 1$ there is a unique one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_K$-module $\Sigma_{K,g}$ over k of height g.*
(2) *Every one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over k is of the form $\Sigma_{K,g} \times (K/\mathcal{O}_K)^h$, for some g and h (the integer h is called the p-rank of the Barsotti-Tate $\mathcal{O}_K$-module).*

2.1.2. Let H/S be a Barsotti-Tate $\mathcal{O}_K$-module over a locally noetherian $\mathcal{O}_K$-scheme S where p is locally nilpotent. We regard the p-rank of H as a function on the closed points of S. It is a result of Messing ([7]) that the p-rank is a lower semicontinuous function. In particular, there exists a stratification of the reduced fiber of S, by closed reduced subschemes,

$$0 \subset S^{[0]} \subset \cdots \subset S^{[h]} \subset \cdots \subset S^{\mathrm{red}}$$

(which is called the *p-rank stratification* of S, associated with H) such that over each open stratum $S^{(h)} = S^{[h]} - S^{[h-1]}$ the Barsotti-Tate $\mathcal{O}_K$-module H has constant p-rank equal to h. Further more, over each $S^{(h)}$ there exists an exact sequence of Barsotti-Tate $\mathcal{O}_K$-modules

$$0 \to H^0 \to H \to H^{et} \to 0$$

where H^0 is formal and H^{et} is ind-étale of height h (see [3], Lemma II.1.1, pp. 60-62, and Corollary II.1.2, p. 62).

2.1.3. Let H_0/k be a Barsotti-Tate $\mathcal{O}_K$-module . By a deformation of H_0 to a local ring A, with residue field k, we mean a pair (H, j) where $H/\mathrm{Spec}\,A$ is a compatible Barsotti-Tate $\mathcal{O}_K$-module and $j : H_0 \to H \times_A k$ an isomorphism of Barsotti-Tate $\mathcal{O}_K$-modules. By the deformation functor of H_0 we mean the set-valued functor from artinian local $\mathcal{O}_K$-algebras with residue field k, which sends an algebra A to the set of isomorphism classes of deformations of H_0 over A,

It is a result of Drinfeld that the deformation functor of $\Sigma_{K,g}$ is pro-represented by a complete noetherian local ring $\mathcal{R}_g = \mathcal{R}_{K,g}$ with residue field k,

$$\mathcal{R}_{K,g} \simeq \mathcal{O}_{\hat{K}^{nr}}[[T_1, \dots, T_{g-1}]]$$

([2], Proposition 4.2, pp. 570-572). We write $(\tilde{\Sigma}_{K,g}, \tilde{j}))$ for the universal deformation of $\Sigma_{K,g})$ over $\mathcal{R}_{K,g}$. Then, the deformation functor of $H_{g,h} = \Sigma_{K,g} \times (K/\mathcal{O}_K)^h$ is represented by the formal $\mathcal{R}_{K,g}$-scheme $\mathrm{Hom}\,(TH_{g,h}, \tilde{\Sigma}_{K,g})$, which is formally smooth of dimension h over $\mathcal{R}_{K,g}$ (by $TH_{g,h}$ we denote the Tate module of $H_{g,h}$) (see [2], Proposition 4.5, p. 547, and [3], p. 64). We write $\mathcal{R}_{g,h}$ for the $\mathcal{R}_{K,g}$-algebra pro-representing the above functor and $(\mathcal{H}_{g,h}, \tilde{j})$ for the universal deformation of $H_{g,h}$ over $\mathcal{R}_{g,h}$.

2.2. **Drinfeld level structure.** In [2] Drinfeld introduces a notion of level structure for elliptic modules; in [3], Harris and Taylor extend this notion to the context of Barsotti-Tate $\mathcal{O}_K$-modules . Their definition is based on Katz's and Mazur's notion of a full set of sections for a finite flat commutative group-scheme. A proof of the equivalence between Drinfeld's original definition and Katz's and Mazur's one can be found in [3], Corollary II.2.3, pp. 79-80.

2.2.1. Let S be a scheme and Z a finite flat S-scheme of finite presentation and rank $N \geq 1$. By definition, a full set of sections of Z/S is a set of N points (not necessarily distinct) $P_1, \dots P_N \in Z(S)$ such that for every affine S-scheme $\mathrm{Spec}\,R$ and for every $f \in B = H^0(Z_R, \mathcal{O})$, $\det(T - f) = \prod_{i=1}^{N}(T - f(P_i)) \in R[T]$ ([9], section 1.8.2, p.33). In the case when Z is a S-group scheme, for A a finite abelian abstract group of order equal to the rank of Z, Katz and Mazur also define the notion of an A-generator of Z/S, as the datum of a group morphism $\phi : A \to Z(S)$ such that the set of points $\{\phi(a) \mid a \in A\}$ is a full set of sections of Z/S ([9], section 1.10.5, p. 44).

Definition 2. *Let G be a finite flat $\mathcal{O}_K/\mathcal{P}_K^m$-module, of rank q^{mn}, over a scheme S. A Drinfeld structure of G/S is a morphism of $\mathcal{O}_K/\mathcal{P}_K^m$-modules*

$$\alpha : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to G(S)$$

such that α is a $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$-generator of G/S.

Let H/S be a compatible Barsotti-Tate $\mathcal{O}_K$-module of constant height n over a scheme S. A Drinfeld structure of level m on H/S is a Drinfeld structure on $H[\mathcal{P}_K^m]/S$ (see [3]*, Section II.2, p. 73.)*

In the following, we also regard a Drinfeld structure on G/S as a morphism of S-groups schemes $\alpha : \underline{(\mathcal{P}_K^{-m}/\mathcal{O}_K)}_S^n \to G$.

It follows from Proposition 1.9.1 in [9] (p. 38) that the set-valued functor on S-schemes, which maps a scheme T/S to the sets of Drinfeld structures of G/S is represented by a finite S-scheme. We call the representing S-scheme the *Drinfeld cover* of S associated with G. For $G = H[\mathcal{P}_K^m]$ and H/S a Barsotti-Tate $\mathcal{O}_K$-module , we call it the *Drinfeld cover of level m* of S associated with H/S.

Proposition 3. *Let G/S be a finite flat $\mathcal{O}_K/\mathcal{P}_K^m$-module, of rank q^{mn}, over a connected scheme S, and suppose given a short exact sequence*

$$0 \to G_1 \to G \to G_2 \to 0$$

of finite flat $\mathcal{O}_K/\mathcal{P}_K^m$-modules over S, with G_2 finite étale.

Then, a morphism of $\mathcal{O}_K/\mathcal{P}_K^m$-modules

$$\alpha : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to G(S)$$

is a Drinfeld structure on G if and only if there exists a direct $\mathcal{O}_K/\mathcal{P}_K^m$-summand $M \subset (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$ such that:

1. *$\alpha_{|M} : M \to G_1(S)$ is a Drinfeld structure on G_1;*
2. *α induces an isomorphism of S-group schemes*

$$\alpha_2 : \underline{(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n/M}_S \to G_2.$$

Further more, if we fix a choice of a complement of M in $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, then when G/S admits a Drinfeld structure there exists a canonical splitting over S

$$G \simeq G_1 \times G_2.$$

On the other hand, given such a splitting over S, the data of Drinfeld structures on G_1 and G_2, as in conditions (1) and (2) above, give rise to a unique Drinfeld structure on G/S which restricts to the given ones on G_1 and G_2.

Proof. The first part of the statement follows from Proposition 1.11.2 and Lemma 1.8.3 in [9]. We consider the second part of the statement. A choice of a complement of M is equivalently to a choice of a section of the natural projection

$$s : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n/M \to (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n.$$

With abuse of notations, we denote the corresponding morphism of étale S-group scheme also by s. To s we associate the group-theoretic section $\sigma = \alpha \circ s \circ \alpha_2^{-1} : G_2 \to G$ which gives rise to the splitting in the statement.

On the other hand, given s and σ as above, we define

$$\alpha = (\iota \times \sigma) \circ (\alpha_1 \times \alpha_2) \circ (i \times s)^{-1} : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to G(S),$$

for ι (resp. i) the natural inclusions of G_1 (resp. M) in G (resp. $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$). It follows from Proposition 1.11.3 in [9] that, when α_1, α_2 are Drinfeld structures on G_1, G_2 respectively, α is a Drinfeld structure on G. □

The above result applies in particular to Drinfeld structures on a Barsotti-Tate $\mathcal{O}_K$-module H/S together with a short-exact sequence of Barsotti-Tate $\mathcal{O}_K$-modules $0 \to H_1 \to H \to H_2 \to 0$ with H_2 ind-étale (e.g. on a Barsotti-Tate $\mathcal{O}_K$-module of constant p-rank and its formal-étale short exact sequence, as in Parts (4) and (5) of Lemma II.2.1 in [3], pp. 73–75).

2.2.2. It follows from the proposition 3, that in the case when S is not connected, the datum of a Drinfeld structure of level m of a Barsotti-Tate $\mathcal{O}_K$-module H/S, fitting in an exact sequence $0 \to H_1 \to H \to H_2 \to 0$ with H_2 ind-étale, defines a canonical decomposition of S as disjoint union of closed subschemes (not all necessarily non-empty), indexed by the direct summands of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, which are free of rank equal to the height of H_1 over $\mathcal{O}_K/\mathcal{P}_K^m$. In the following, we write S_M for the closed (and open) subscheme of S associated with a direct summand M.

2.2.3. In [2] Drinfeld studies the notion of level structure on deformations of the Barsotti-Tate $\mathcal{O}_K$-module $H_{g,h}$. For any integer $m \geq 1$, he considers the set-valued functor from artinian local $\mathcal{O}_K$-algebras with residue field k, which sends an algebra A to the set of isomorphism classes of deformations of $H_{g,h}$ over A together with a structure of level m, and proves that it is pro-represented by a regular complete noetherian local ring $\mathcal{R}_{g,h,m}$ with residue field k, which is finite flat over $\mathcal{R}_{g,h}$. Moreover, $\mathcal{R}_{g,h,m}$ is formally smooth of dimension h over $\mathcal{R}_{K,g,m} = \mathcal{R}_{g,0,m}$ (see [2], Proposition 4.3, p. 572, and Proposition 4.5, p. 574).

In the case of the formal Barsotti-Tate $\mathcal{O}_K$-module $\Sigma_{K,g}$, let

$$\tilde{\alpha} : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^g \to \tilde{\Sigma}_{K,g}(\mathcal{R}_{K,g,m})$$

denote the universal structure of level m on $\tilde{\Sigma}_{K,g}/\mathcal{R}_{K,g,m}$, and $\{e_0^{(m)}, \dots, e_{g-1}^{(m)}\}$ the canonical basis of the $\mathcal{O}_K/\mathcal{P}_K^m$-module $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^g$. We fix a parameter X of $\tilde{\Sigma}_{K,g}/\mathcal{R}_{K,g}$, and denote by f_π the power series corresponding to multiplication by the uniformizer $\pi \in \mathcal{O}_K$ on $\tilde{\Sigma}_{K,g}$. Then the elements $\theta_i^{(m)} = X(\tilde{\alpha}(e_i^{(m)})) \in \mathcal{R}_{K,g,m}$, $0 \leq i \leq g-1$, form a system of local parameters of $\mathcal{R}_{K,g,m}$.

2.2.4. As an application of Drinfeld's result in deformation theory one can deduce the following two results regarding the geometry of Drinfeld covers.

Proposition 4. *Let X be a noetherian $\mathcal{O}_K$-scheme and $\mathcal{G}$ a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over X, of constant height n.*

(1) *Suppose X satisfies "the versality condition": at each closed point $x \in X$ the natural map from the formal completion of X at x to the formal space of deformations of $\mathcal{G}(x)$ is an isomorphism. Then the Drinfeld covers X_m of X are regular and flat, for all $m \geq 1$.*
(2) *Suppose X is a reduced k-scheme and $\mathcal{G}/X$ has constant p-rank equal to h. For any direct summand M of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$, let $X_{m,M}$ be the associated closed (and open) subscheme of X_m defined in section 2.2.2, with respect to the formal-étale short exact sequence of $\mathcal{G}/X$. Then, the scheme $X_{m,M}$ is finite flat over X, of degree $\#P_M(\mathcal{O}_K/\mathcal{P}_K^m)/\#GL_{n-h}(\mathcal{O}_K/\mathcal{P}_K^m)$ (where $P_M(\mathcal{O}_K/\mathcal{P}_K^m)$ denotes the parabolic subgroup of $GL_n(\mathcal{O}_K/\mathcal{P}_K^m)$ attached to M).*

Proof. The statement is a reformulation of some results in [3] (Lemma III.4.1, part (4), p.112; Lemma II.2.4, p.82; Lemma III.4.6, p.116). We sketch a proof.

For the first part, both regularity and flatness can be detected on the formal completion at closed points. Moreover, after replacing X by a faithfully flat extension, we may assume it is the spectrum of a complete local ring with algebraically closed residue field. Then the statement is an immediate consequence of Drinfeld's result in deformation theory.

For the second part, again it suffices to check the statement for the formal completions at closed points, and moreover, without loss of generality, we may assume the versality condition of part (1), since the general case would then follow from this special case. Thus, finiteness and flatness follow respectively from Proposition 1.9.1 in [9] (p. 38) and part (1), and it is enough to verify that the degree is indeed the one in the statement (which is done in [3] Lemma III.4.6, p.116). □

2.3. **Igusa level structure.** In [3] (Section IV.1, p. 121), Harris and Taylor introduce the notion of Igusa varieties in the context of certain simple Shimura varieties. In this section, we recall their definition and results but we reformulate them away from the context in which they arise. We remark that their work apply exclusively to Barsotti-Tate $\mathcal{O}_K$-modules of constant p-rank. Also, in [3] there are two kinds of Igusa varieties; in this paper we refer only to the first kind.

2.3.1. Let S be a k-scheme and H a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over S, of constant height n and constant p-rank h. Then, H fits in a short exact sequence of Barsotti-Tate $\mathcal{O}_K$-modules

$$0 \to H^0 \to H \to H^{et} \to 0$$

where H^0 is formal of height $n-h$ and H^{et} is ind-étale of height h.

Definition 5. *For any positive integer m, we call an isomorphism of $\mathcal{O}_K$-modules*

$$j : \underline{(\mathcal{P}_K^{-m}/\mathcal{O}_K)}_S^h = \underline{(K/\mathcal{O}_K)}_S^h[\mathcal{P}_K^m] \to H^{et}[\mathcal{P}_K^m]$$

an Igusa structure of level m on H/S.

For any integer $m \geq 1$, we call the *Igusa cover* of level m of S the S-scheme I_m which represents the set-valued functor on S-schemes mapping a scheme T/S to the sets of Igusa structures of level m on H/S (and write j_m for the universal isomorphism over I_m). The S-scheme I_m is a finite étale $GL_h(\mathcal{O}_K/\mathcal{P}_K^m)$-torsor (see [9], Proposition 1.10.13, Part (2), p. 47).

2.3.2. The notion of Igusa covers is useful to study the geometry of Drinfeld covers.

We maintain the above notations and assumptions. Let S_m be the Drinfeld cover of level m of S. Then, S_m decomposes as $S_m = \coprod_M S_{m,M}$ where M ranges among the direct $\mathcal{O}_K$-summands of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$ (see section 2.2.2).

For any such M, write $P_M(\mathcal{O}_K/\mathcal{P}_K^m)$ for the parabolic subgroup of $GL_n(\mathcal{O}_K/\mathcal{P}_K^m)$ corresponding to M, and we choose an epimorphism of kernel M

$$p_M : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \twoheadrightarrow (\mathcal{P}_K^{-m}/\mathcal{O}_K)^h.$$

Proposition 6. *Assume S is reduced. Then the map $j_M = F^{(n-h)m} \circ j_m \circ p_M$*

$$j_M : \underline{(\mathcal{P}_K^{-m}/\mathcal{O}_K)}_S^n \twoheadrightarrow \underline{(\mathcal{P}_K^{-m}/\mathcal{O}_K)}_S^h = \underline{(K/\mathcal{O}_K)}_S^h[\mathcal{P}^m] \to H^{et}[\mathcal{P}^m] \to H^{et\,(q^{(n-h)m})}[\mathcal{P}^m]$$

defines a morphism $j_M^* : I_m \to S_{m,M}$, which is an isomorphism and fits in the following commutative diagram.

$$\begin{array}{ccc} I^{(h)} & \xrightarrow{j_M^*} & S_{m,M} \\ \downarrow & & \downarrow \\ S & \xrightarrow{F^{(n-h)m}} & S \end{array}$$

Proof. This statement is a reformulation of a result in [3] (Section IV.1, p. 124). We sketch a proof. The assumption that S is reduced implies that I_m is reduced, thus the natural projection $H \to H^{et}$ gives rise to an isomorphism

$$H^{(q^{(n-h)m})}[\mathcal{P}^m](I_m) \simeq H^{et\,(q^{(n-h)m})}[\mathcal{P}^m](I_m).$$

Composing its inverse with the map j_M in the statement, we obtain a morphism

$$\alpha : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to H^{et\,(q^{(n-h)m})}[\mathcal{P}^m](I_m) \simeq H^{(q^{(n-h)m})}[\mathcal{P}^m](I_m).$$

Proposition 3 implies that α is a Drinfeld structure of level m on $H^{(q^{(n-h)m})}$, which moreover vanishes on M by construction. It follows that it defines a morphism $j_M^* : I_m \to S_{m,M}$ which makes the above diagram commute. It is an easy observation that j_M^* is a monomorphism. Finally, both schemes are finite flat over the scheme S in the bottom right corner of the diagram, thus to conclude that j_M^* is an isomorphism it suffices to check that they have the same degree. □

Corollary 7. *Maintaining the above notations. Assume S is smooth.*
Then, both the Igusa and Drinfeld covers of S are smooth of the same dimension.

3. Compactified Igusa covers

In [3] Harris and Taylor define the Igusa covers in the case when the p-rank of the Barsotti-Tate $\mathcal{O}_K$-module is constant over the base. In this section, we investigate the possibility of defining Igusa covers over a general base in positive characteristic, i.e. when the p-rank is not necessarily constant. Since for any connected k-scheme X and any Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{G}$ over X, the locus where the p-rank of $\mathcal{G}$ is maximal is a dense open, we regard this question as the problem of extending the Igusa covers defined over such open to the whole X, which is why we refer to the generalized Igusa covers as compactified (or proper) Igusa covers. We also investigate how the Igusa covers can be used to describe the geometry of the corresponding Drinfeld covers in general, extending the results of proposition 6.

3.1. Deformations with bounded p-rank.

Before considering the question of defining Igusa covers in general, we address a question in deformation theory of Barsotti-Tate $\mathcal{O}_K$-modules in positive characteristic.

It follows from the definition that the p-rank h of a Barsotti-Tate $\mathcal{O}_K$-module is always bounded by its height, more precisely $h \leq n-1$ if n is the height. On the other hand, we also know that it increases under deformation (see section 2.1.2). In the following, we study the deformation theory of Barsotti-Tate $\mathcal{O}_K$-modules in equal characteristic, when a (strict) bound on the p-rank is imposed.

For any g, h, we write $R_{g,h}$ (resp. $R_{K,g}$) for the rings $\mathcal{R}_{g,h} \otimes_{\mathcal{O}_{\hat{K}^{nr}}} k$ (resp. $\mathcal{R}_{K,g} \otimes_{\mathcal{O}_{\hat{K}^{nr}}} k$).

Proposition 8. *Let n, h, h' be positive integers, $0 \leq h \leq h' \leq n-1$, and write $g = n - h \geq g' = n - h'$.*

1. *$n = g$: the set-valued functor from reduced complete noetherian local k-algebras, with residue field k, which sends an algebra A to the set of deformations of $\Sigma_{K,g}$ over A of p-rank less than or equal to h', is represented by a reduced complete noetherian local k-algebra $R_{K,g}^{g'}$, formally smooth of dimension $h' = g - g'$. $R_{K,g}^{g'}$ naturally arises as a quotient of $R_{K,g}$ and the isomorphism $R_{K,g} \simeq k[[T_1, \dots, T_{g-1}]]$ induces an isomorphism $R_{K,g}^{g'} \simeq k[[T_{g'}, \dots, T_{g-1}]]$.*

   We write $\tilde{\Sigma}_{K,g}^{g'}$ for the universal deformation of $\Sigma_{K,g}$ over $R_{K,g}^{g'}$.
2. *$n \neq g$: the set-valued functor from reduced complete noetherian local k-algebras, with residue field k, which sends an algebra A to the set of deformations of $H_{g,h} = \Sigma_{K,g} \times (K/\mathcal{O}_K)^h$ over A of p-rank less than or equal to*

h', is represented by the formal $R^{g'}_{K,g}$-scheme $Hom(TH_{g,h}, \tilde{\Sigma}^{g'}_{K,g})$, which is formally smooth of dimension h over $R^{g'}_{K,g}$.

We write $R^{g'}_{g,h}$ for the $R^{g'}_{K,g}$-algebra representing the above functor and $\mathcal{H}^{g'}_{g,h}$ for the universal deformation of $H_{g,h}$ over it.

Proof. We start by considering the case of a formal Barsotti-Tate $\mathcal{O}_K$-module (i.e. $h = 0$, $n = g$). It follows from section 2.1 that for any h' the above functor is represented by a reduced quotient of $R_{K,g}$. Moreover, for $h' = n-1$ ($g' = 1$) we have $R^1_{K,g} = R_{K,g}$, and also for $h' = h$ ($g' = g$) $R^g_{K,g} = k$.

Let us consider the case of a general h'. Bounding the p-rank of a Barsotti-Tate $\mathcal{O}_K$-module from above is equivalent to bounding the height of its formal part from below. Then, $R^{g'}_{K,g}$ is the unique reduced complete local quotient of $R_{K,g} \otimes_{\mathcal{O}_{\hat{K}^n r}} k$ with the following universal property:

- for any (reduced) complete noetherian local ring R and k-morphism $\phi : R_{K,g} \otimes_{\mathcal{O}_{\hat{K}^n r}} k \to R$ (not necessarily local), the Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{G}^0_R = \phi^*(\tilde{\Sigma}_{K,g})^0$ has height greater than or equal to $g' = n - h'$ if and only if the morphism ϕ factors through the quotient $R_{K,g} \otimes_{\mathcal{O}_{\hat{K}^n r}} k \to R^{g'}_{K,g}$.

We recall that the height of the formal part of a Barsotti-Tate $\mathcal{O}_K$-module can be read off the power series representing the multiplication by a uniformizer of $\mathcal{O}_K$, namely as the q-logarithm of the exponent of the lowest non-vanishing monomial of the power series.

We fix a parameter X of $\tilde{\Sigma}_{K,g}$ and an isomorphism $R_{K,g'} \simeq k[[T_1, \ldots, T_{g-1}]]$. Then, over $R_{K,g'}$, the power series f_π representing the multiplication by the uniformizer $\pi \in \mathcal{O}_K$ on $\tilde{\Sigma}_{K,g}$ satisfies the congruence relation

$$f_\pi \equiv T_1 X^q + T_2 X^{q^2} + \cdots + T_{g-1} X^{q^{g-1}} \bmod X^{q^g} (T_1 \ldots, T_{g-1})^2$$

(see [2], Section 4, Part (A), p. 570–572). Therefore, for any k-morphism $\phi : k[[T_1, \ldots, T_{g-1}]] \to R$, the multiplication by π on $\phi^*(\tilde{\Sigma}_{K,g})$ is defined by the power series $\phi^*(f_\pi)$ satisfying the congruence relation

$$\phi^*(f_\pi) \equiv \phi(T_1) X^q + \phi(T_2) X^{q^2} + \cdots + \phi(T_{g-1}) X^{q^{g-1}}$$

modulo the ideal $X^{q^g}(\phi(T_1) \ldots, \phi(T_{g-1}))^2$.

We deduce that in the reduced quotient $R^{g'}_{K,g}$ the parameters $T_1, \ldots, T_{g'-1}$ necessarily all vanish. On the other hand, the quotient $k[[T_1, \ldots, T_{g-1}]]/(T_1, \ldots T_{g'-1})$ is a reduced complete local ring and moreover the restriction of $\tilde{\Sigma}^0_{K,g}$ to such quotient has height equal to g at the generic point. Thus, the isomorphism $R_{K,g'} \simeq k[[T_1, \ldots, T_{g-1}]]$ induces an isomorphism $R^{g'}_{K,g} \simeq k[[T_{g'}, \ldots, T_{g-1}]]$.

Finally, it is an easy consequence of the result of section 2.1.3 that the case of a general Barsotti-Tate $\mathcal{O}_K$-module follows directly from the case of a formal Barsotti-Tate $\mathcal{O}_K$-module .

□

3.2. **Drinfeld covers.** We establish some notations and results for Drinfeld covers over a general base. Let X be a reduced k-scheme and $\mathcal{G}$ a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over X, of constant height n and maximal p-rank h. (We remark that an upper bound on the p-rank always exists).

For any $m \geq 0$, let X_m be the Drinfeld cover of X of level m, and write

$$\alpha : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to \mathcal{G}[\mathcal{P}_K^m](X_m).$$

for the universal Drinfeld structure of level m on $\mathcal{G}/X_m$.

Let M be a direct summand of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$ over $\mathcal{O}_K/\mathcal{P}^m$. We consider the set-valued functor on X-scheme which sends a X-scheme T to the set of Drinfeld structures

$$\alpha_T : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to \mathcal{G}[\mathcal{P}_K^m](T)$$

which vanish on M. The lemma below proves that this functor is represented by a closed subscheme $X_{m,M}$ of X_m.

Lemma 9. *Let S be a scheme, G and H two finite, flat, commutative group schemes over S, and $f : G \to H$ a morphism of S-group schemes. Then, there exists a closed, finitely presented subscheme W in S such that, for any S-scheme T, the base change morphism $f_T : G_T \to H_T$ vanishes if and only if the morphism $T \to S$ factors as $T \to W \hookrightarrow S$.*

Proof. We prove the lemma by arguments as in Chapter 1 of [9]. Since the question is Zariski local on S, we may assume $S = \mathrm{Spec}\,(R)$, $G = \mathrm{Spec}\,(A)$ and $H = \mathrm{Spec}\,(B)$, where A and B are two R-algebras which as R-modules are free, of finite rank N and N' respectively. We choose $\{b_1, \ldots b_{N'}\}$ (resp. $\{a_1, \ldots, a_N\}$) an R-basis on B (resp. A), such that the group identity on H/S corresponds to the morphism $B \to R$ which sends b_1 to 1 and b_i to 0 for all $i > 1$.

Let $f^* : B \to A$ be the morphism between the affine algebras induced by f. Then, f^* corresponds to a matrix $C = (c_{ij}) \in M_{N' \times N}(R)$, and the subscheme W is the closed subscheme of S cut by the ideal $(c_{ij} \mid j > 1)$. □

Let $X^{(h)}$ denote the open subscheme of X where the p-rank of $\mathcal{G}$ is maximal, and write $X_m^{(h)}$ for its Drinfeld cover of level m. Since the p-rank of $\mathcal{G}$ on $X^{(h)}$ is constant, $X_m^{(h)}$ decomposes as the disjoint union of closed subschemes $X_{m,M}^{(h)}$ indexed by the direct summands of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$ over $\mathcal{O}_K/\mathcal{P}_K^m$.

Remark 10. *Maintaining the above notations.*

(1) $X_{m,M} \cap X_m^{(h)} = X_{m,M}^{(h)}$.
(2) $(X_m)^{red} = \bigcup_M X_{m,M}$.

The first remark is a simple consequence of the definition. As for the second, for any closed point x of X_m, let us consider the map

$$\alpha(x) : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to \mathcal{G}_x[\mathcal{P}_K^m](k(x)) \simeq \mathcal{G}_x^{et}[\mathcal{P}_K^m](k(x)).$$

Since the height of $\mathcal{G}_x^{et}$ is less than or equal to h, it follows that $\ker \alpha(x)$ is a direct summand of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$ of rank at least $n-h$. Therefore, for any x there exists at least one (and exactly one if $x \in X_m^{(h)}$) direct summand M of rank $n-h$ contained in $\ker \alpha(x)$.

Proposition 11. *For all m and M as above. The morphism $X_{m,M} \to X$ is finite flat of degree $\#P_M(\mathcal{O}_K/\mathcal{P}_K^m)/\#GL_{n-h}(\mathcal{O}_K/\mathcal{P}_K^m)$.*

Proof. The above properties can be detected on the formal completions at closed points. Moreover, we already know that the result holds over the open $X^{(h)}$ of X

(see proposition 4). Thus, it suffices to study the formal completion of $X_{m,M}$ at all geometric closed points x of X_m such that $h(x) < h$.

Let x be a geometric closed point of $X_{m,M}$, such that $h(x) = h' < h$, and denote by $\bar{x}$ its image in X. We fix an isomorphism $\mathcal{G}^0(x) = \mathcal{G}^0(\bar{x}) \simeq \Sigma_{K,g'}$ ($g' = n - h'$). Then, the natural map from the formal completion of X at $\bar{x}$ to the formal deformation space of $\mathcal{G}(x)$ gives rise to a morphism of k-algebras

$$\phi(x) : R^g_{g',h'} = k[[X_1, \dots, X_{h'}; T_g, \dots, T_{g'-1}]] \to \mathcal{O}^\wedge_{X,\bar{x}}$$

which allows us to identify the Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{G}_x = \mathcal{G}/\mathcal{O}^\wedge_{X,\bar{x}}$ with the pushforward under $\phi(x)$ of the universal Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{H} = \mathcal{H}^g_{g',h'}$,

$$0 \to \tilde{\Sigma}^g_{K,g'} \to \mathcal{H} \to \mathcal{G}^{et}_x \to 0.$$

Without loss of generality we may assume that $\phi(x)$ is an isomorphism (the general case would then follow from this special case).

By definition, the complete local ring of $X_{m,M}$ at x is the unique finite local ring over $\mathcal{O}^\wedge_{X,\bar{x}}$ where $\mathcal{G}_x$ is endowed with a universal Drinfeld level structure α which lifts $\alpha(x)$ and vanishes on M. We choose an epimorphism

$$\mathrm{p} : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \twoheadrightarrow (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{h'}$$

which vanishes on the submodule M, together with a splitting s of p. Then, it follows from proposition 3 that the datum of a Drinfeld structure α of level m on $\mathcal{G}$ over $\mathcal{O}^\wedge_{X_{m,M},\bar{x}}$ is equivalent to the following data:

(1) a structure of level m on $\mathcal{G}^{et}_x$

$$\alpha^{et} : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{h'} \to \mathcal{G}^{et}_x[\mathcal{P}^m](\mathcal{O}^\wedge_{X_{m,M},x});$$

(2) a structure of level m on $\tilde{\Sigma}^g_{K,g'}$ of the form

$$\alpha^0 : \ker \mathrm{p} \to \tilde{\Sigma}^g_{K,g'}[\mathcal{P}^m](\mathcal{O}^\wedge_{X_{m,M},x}),$$

which vanishes on M;

(3) a splitting of the short exact sequence

$$0 \to \tilde{\Sigma}^g_{K,g'}[\mathcal{P}^m] \to \mathcal{G}_x[\mathcal{P}^m] \to \mathcal{G}^{et}_x[\mathcal{P}^m] \to 0.$$

Moreover, the condition that the level structure α on $\mathcal{G}_x$ reduces, modulo the maximal ideal of $\mathcal{O}^\wedge_{X_{m,M},x}$, to $\alpha(x)$ uniquely determines the datum (1). With this in mind, we proceed to explicitly compute $\mathcal{O}^\wedge_{X_{m,M},x}$ as a finite extension of $\mathcal{O}^\wedge_{X,\bar{x}}$.

First we construct the finite extension B of $\mathcal{O}^\wedge_{X,\bar{x}}$ over which the Barsotti-Tate $\mathcal{O}_K$-module $\tilde{\Sigma}^g_{K,g'}$ is endowed with the universal level structure of the kind as in datum (2). Then, we define the extension B' of B over which the group scheme $\mathcal{G}_x[\mathcal{P}^m]$ splits as $\mathcal{G}^{et}_x[\mathcal{P}^m] \times \tilde{\Sigma}^g_{K,g'}[\mathcal{P}^m]$. The above considerations imply $B' = \mathcal{O}^\wedge_{X_{m,M},x}$.

We choose an isomorphism $\ker \mathrm{p} \simeq (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'}$ such that the epimorphism $\ker \mathrm{p} \twoheadrightarrow (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'-g}$ which maps $e_i \mapsto 0$ for $0 \le i \le g-1$ and $e_i \mapsto e'_{i-g}$ $g \le i \le g'-1$ (for $e_0, \dots, e_{g-1}$ and $e'_0, \dots, e'_{g'-g}$ the canonical basis of the modules $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'}$ and $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'-g}$ respectively) vanishes on M.

Then Drinfeld's result reviewed in section 2.2.3 implies that

$$B = k[[X_1, \dots X_{h'}; \theta^{(m)}_g, \dots \theta^{(m)}_{g'-1}]]$$

and is finite and flat over $\mathcal{O}^\wedge_{X,\bar{x}} \simeq k[[X_1, \dots X'_h; T_g, \dots T_{g'-1}]]$. We claim

$$B' = k[[Y_1^{(m)}, \dots Y_{h'}^{(m)}]][[\theta_g^{(m)}, \dots \theta_{g'-1}^{(m)}]] \supset B = k[[X_1, \dots X_{h'}; \theta_g^{(m)}, \dots \theta_{g'-1}^{(m)}]],$$

where the parameters $Y_i^{(m)}$ $(0 \leq i \leq h')$ satisfy the recursive equations:

$$f_\pi(Y_i^{(m)}) = Y_i^{(m-1)},\ Y_i^{(0)} = X_i \qquad \forall i.$$

In fact, let us consider the following diagram

$$\begin{array}{ccccc} \mathrm{Ext}^1(\mathcal{G}_x^{et}, \tilde{\Sigma}) & \xrightarrow{\cdot\pi^m} & \mathrm{Ext}^1(\mathcal{G}_x^{et}, \tilde{\Sigma}) & \xrightarrow{res} & \mathrm{Ext}^1(\mathcal{G}_x^{et}[\mathcal{P}_K^m], \tilde{\Sigma}[\mathcal{P}_K^m]) \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}(\pi^{-m}T, \tilde{\Sigma}) & \xrightarrow{res} & \mathrm{Hom}(T, \tilde{\Sigma}) & & \end{array}$$

where the first row is exact and the two vertical maps are isomorphisms (we write $T = T\mathcal{G}_x^{et}$ and $\tilde{\Sigma} = \tilde{\Sigma}^g_{K,g'}$).

We deduce that the existence of a splitting as in condition (4) is equivalent to the existence of a morphism

$$\beta' : \pi^{-m}T \to \tilde{\Sigma}$$

whose restriction $\beta = \beta'_{|T} : T \to \tilde{\Sigma}$ corresponds, under the above identifications, to $\mathcal{G}_x$ regarded as an extension of $\mathcal{G}_x^{et}$ by $\tilde{\Sigma}$.

Let $v_1, \dots v_{h'}$ be the basis of the $\mathcal{O}_K$-module T, corresponding to the parameters X_i of $\mathcal{O}^\wedge_{X,\bar{x}}$ (i.e. $X_i = X(\beta(v_i))$, for X the chosen parameter on $\tilde{\Sigma}$). The existence of a morphism β' which restricts to β (and thus, of a splitting isomorphism as in condition (4)) is equivalent to the existence of solutions $Y_i^{(m)}$ to the equations

$$f_{\pi^m}(Y) = X_i.$$

In fact, for a set of solutions $\{Y_1^{(m)}, \dots, Y_{h'}^{(m)}\}$, the morphism β' defined by the conditions $X(\beta'(\pi^{-m}v_i)) = Y_i^{(m)}$, for all i, obviously satisfies the condition $\beta'_{|T} = \beta$.

We remark that the ring $B' = \mathcal{O}^\wedge_{\tilde{X}^{[h]}_{m,M},x}$ is indeed regular, finite and flat over B, and thus over $\mathcal{O}^\wedge_{X,x}$. □

The above proof also implies the following proposition.

Proposition 12. *Maintaining the above notations. Suppose that $\mathcal{G}/X$ satisfies "the versality condition":*

- *at each point $x \in X$ the natural map from the formal completion of X at x to the formal space of deformations in equal characteristic of $\mathcal{G}(x)$ with p-rank bounded by h is an isomorphism (thus, in particular X is smooth).*

Then, the scheme $X_{m,M}$ is a smooth k-scheme.

3.3. **Igusa covers.** We extend Harris' and Taylor's notion of Igusa structure to the case when the p-rank of the Barsotti-Tate $\mathcal{O}_K$-module is not constant.

Let X be a reduced k-scheme and $\mathcal{G}$ a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over X, of height n and maximal p-rank h.

We denote by $F : \mathcal{G} \to \mathcal{G}^{(q)}$ the q-th power of Frobenius morphism of $\mathcal{G}$ $(q = \#\mathbb{F})$. Then F is an endomorphism of Barsotti-Tate $\mathcal{O}_K$-modules and not just of the underlying Barsotti-Tate groups. For any integer $r \geq 0$, we write $\mathcal{G}[F^r]$ for the kernel of the isogeny F^r. For $r = (n-h)m$, $\mathcal{G}[F^{(n-h)m}] \subset \mathcal{G}[\mathcal{P}_K^m]$ and the quotient

$\mathcal{G}[\mathcal{P}_K^m]/\mathcal{G}[F^{(n-h)m}]$ is a finite flat group scheme over X, of order q^{hm}, with inherits a structure of $\mathcal{O}_K$-module.

Definition 13. *For any integer $m \geq 1$, we call an Igusa structure of level m on $\mathcal{G}/X$ the datum of a Drinfeld structure on $\mathcal{G}[\mathcal{P}_K^m]/\mathcal{G}[F^{(n-h)m}]$.*

We define the Igusa cover of level m of X, I_m/X, to be the reduced subscheme underlying the Drinfeld cover of X associated with $\mathcal{G}[\mathcal{P}_K^m]/\mathcal{G}[F^{(n-h)m}]$.

3.3.1. It is a simple remark that in the case of constant p-rank equal to h the newly defined Igusa covers of X agree with the previous ones. Indeed, in this case, the Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{G}$ fits in a formal-étale exact sequence

$$0 \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{G}^{et} \to 0.$$

By restricting the above sequence to the corresponding $\mathcal{P}_K^m$-torsion subgroups, we obtain a short exact sequence of finite flat $\mathcal{O}_K/\mathcal{P}_K^m$-modules

$$0 \to \mathcal{G}^0[\mathcal{P}_K^m] \to \mathcal{G}[\mathcal{P}_K^m] \to \mathcal{G}^{et}[\mathcal{P}_K^m] \to 0$$

where $\mathcal{G}^0[\mathcal{P}_K^m] = \mathcal{G}[F^{(n-h)m}]$, since $\mathcal{G}^0$ is one-dimensional formal of constant height $n-h$ (see [3], Lemma II.2.1, Part (2), p. 74). In particular, the above sequence gives rise to a canonical isomorphism of finite étale $\mathcal{O}_K/\mathcal{P}_K^m$-modules

$$\mathcal{G}[\mathcal{P}_K^m]/\mathcal{G}[F^{(n-h)m}] \simeq \mathcal{G}^{et}[\mathcal{P}_K^m].$$

The existence of this canonical isomorphism implies that the two definitions agree.

It follows, in particular, that the pullback of the (compactified) Igusa cover I_m/X over the open $X^{(h)}$ (where the p-rank of $\mathcal{G}$ is maximal) can be identified with the (open) Igusa cover of $I_m/X^{(h)}$ defined by Harris and Taylor.

Proposition 14. *Maintaining the notations as above. Let m be a positive integer. The Igusa cover of level m is finite flat over X, of degree $\#GL_h(\mathcal{O}_K/\mathcal{P}_K^m)$.*

Proof. Since I_m restricted to $X^{(h)}$ can be identified with the "old" Igusa cover, it is, in particular, finite étale of the correct degree over $X^{(h)}$ (see section 2.3.1). Therefore, it suffices to study the restriction of I_m over $X - X^{(h)}$.

Let x be a geometric closed point in I_m, such that the height of $\mathcal{G}(x)$ is $h' = h(x) < h$. We denote by $\bar{x}$ its image in X, and by $\alpha(x)$ the corresponding level structure on $\mathcal{G}(x)[\mathcal{P}_K^m]/\mathcal{G}(x)[F^{mg}]$. We also write $(\mathcal{G}_x, \alpha_x)$ for the restriction of $\mathcal{G}$ over $\mathcal{O}^\wedge_{I_m,x}$ together with its Igusa structure. We fix an isomorphism $\mathcal{G}(x) \simeq H_{g',h'}$ ($g' = n - h'$). The natural map from the formal completion of X at $\bar{x}$ to the formal deformation space of $H_{g',h'}$ gives rise to a morphism of k-algebras

$$\phi(x) : R^g_{g',h'} \to \mathcal{O}^\wedge_{X,\bar{x}}.$$

Without loss of generality we may assume $\phi(x)$ is an isomorphism, and identify $\mathcal{O}^\wedge_{I_m,x}$ with the finite $R^g_{g',h'}$-algebra B_m representing the set-valued functor from reduced complete noetherian local k-algebras, with residue field k, which sends an algebra A to the set of deformations over A of $H_{g',h'}$, of p-rank less than or equal to h, together with an Igusa structure of level m.

We claim that the B_m is finite and flat over $R^g_{g',h'}$, of degree $\#GL_h(\mathcal{O}_K/\mathcal{P}_K^m)$.

Let $\mathcal{H} = \mathcal{H}^g_{g'h'}$ be the universal Barsotti-Tate $\mathcal{O}_K$-module defined over $R^g_{g',h'}$. For simplicity, we also write $R = R^g_{g',h'}$ and $\tilde{\Sigma} = \tilde{\Sigma}^g_{K,g'}$. The group scheme $\mathcal{H}[\mathcal{P}_K^m]/\mathcal{H}[F^{mg}]$ fits in the short exact sequence

$$(*) \qquad 0 \to \tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}] \to \mathcal{H}[\mathcal{P}_K^m]/\mathcal{H}[F^{mg}] \to \mathcal{H}^{et}[\mathcal{P}_K^m] \to 0.$$

We consider the standard splitting $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^h = (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'-g} \oplus (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{h'}$. Then the datum of a level structure on $\mathcal{H}[\mathcal{P}^m]/\mathcal{H}[F^{mg}]$ over a local R-algebra A is equivalent to the following data (see proposition 3):

(1) a splitting isomorphism

$$\mathcal{H}[\mathcal{P}_K^m]/\mathcal{H}[F^{mg}] \simeq \tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}] \times \mathcal{H}^{et}[\mathcal{P}_K^m];$$

(2) a level structure

$$\alpha^0 : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'-g} \to \left(\tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}]\right)(A)$$

We denote by $\tilde{\alpha}$ the universal Igusa structure defined over B_m and by $\tilde{\alpha}^0$ its local component (defined as in condition (2) above).

We explicitly compute B_m/R as follows. We first construct the extension B/R over which the group scheme $\tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}]$ is endowed with the universal Drinfeld structure, as in condition (2). Then, we define a finite extension B'/B over which the short exact sequence $(*)$ splits, as in condition (1). The above data give rise to a morphism of R-algebras $B_m \to B'$, which is by construction an isomorphism.

We choose a parameter X on $\mathcal{H}$ and identify $R \simeq k[[X_1, \dots X_{h'}; T_g, \dots T_{g'-1}]]$.

Claim (1): The universal Drinfeld level structure $\tilde{\alpha}^0$ on $\tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}]$ is defined over the finite extension

$$B = k[[X_1, \dots X_{h'}; \varphi_g^{(m)}, \dots \varphi_{g'-1}^{(m)}]] \supset R = k[[X_1, \dots X_{h'}; T_g, \dots T_{g'-1}]],$$

where the parameters $\varphi_i^{(m)}$ are defined as follows.

Let X be a parameter on $\tilde{\Sigma}$ and f_π the power series corresponding to the multiplication by π on $\tilde{\Sigma}$. There exists a unique power series $\phi_\pi \in R[[X]]$ such that

$$f_\pi(X) = \phi_\pi(X^{q^{mg}}).$$

Furthermore, for any $m \geq 1$, the power series $\phi_{\pi^m} = \phi_\pi^{\circ m}$ satisfies the condition

$$\phi_{\pi^m}(X^{q^{mg}}) = f_{\pi^m}(X).$$

For $e_g^{(m)}, \dots e_{g'-1}^{(m)}$ the canonical basis for $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^{g'-g}$, we define the parameters $\varphi_i^{(m)}$ as the solutions to the following recursive equations:

- for $m = 1$: $h_r(\varphi_r^{(1)}) = 0$ where

$$h_r(X) = \frac{\phi_\pi(X)}{\prod_{x \in M_{r-1}}(X - \tilde{\alpha}(x))}$$

  and $M_{r-1} = \langle e_g^{(1)}, \dots, e_{r-1}^{(1)} \rangle$ $(g \leq r \leq g' - 1)$;
- for any $m > 1$: $\phi_\pi(\varphi_i^{(m)}) = \varphi_i^{(m-1)}$ $(g \leq i \leq g' - 1)$.

To prove our claim we consider the following exact sequence over R

$$0 \to \tilde{\Sigma}[F^{mg}] \to \tilde{\Sigma}[\mathcal{P}_K^m] \to \frac{\tilde{\Sigma}[\mathcal{P}_K^m]}{\tilde{\Sigma}[F^{mg}]} \to 0.$$

Then $\tilde{\Sigma}[F^{mg}] = \operatorname{Spec} R[[X]]/(X^{q^{mg}})$ and $\tilde{\Sigma}[\mathcal{P}_K^m] = \operatorname{Spec} R[[X]]/(f_{\pi^m}(X))$.

Thus, the previous short exact sequence gives rise to the identification

$$\frac{\tilde{\Sigma}[\mathcal{P}_K^m]}{\tilde{\Sigma}[F^{mg}]} = \operatorname{Spec} R[[X]]/(\phi_{\pi^m}(X)),$$

where the projection $l : \tilde{\Sigma}[\mathcal{P}_K^m] \to \tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}]$ corresponds to the morphism of Hopf R-algebras

$$l^* : R[[X]]/(\phi_{\pi^m}(X)) \to R[[X]]/(f_{\pi^m}(X))$$

mapping X to $X^{q^{mg}}$.

To conclude that $B = k[[X_1, \dots X_{h'}; \varphi_g^{(m)}, \dots \varphi_{g'-1}^{(m)}]]/R$ has the required universal properties, as well as being regular, finite and flat, we argue as in [2] (Lemma following Proposition 4.3, pp. 572–573). In order to apply those arguments to our case, it suffices to remark that the power series $h_r(X)$ satisfy the congruences

$$h_r(X) \equiv T_r \qquad \mathrm{mod}\ (X, T_g, \dots T_{r-1}),$$

for all $g \leq r \leq g-1$, and that the above construction lies uniquely over the subring $k[[T_g, \dots T_{g'-1}]] \subset R$.

Claim (2): The finite extension B'/B defined by the existence of a splitting of the short exact sequence $(*)$ is

$$B' = k[[Z_1^{(m)}, \dots Z_{h'}^{(m)}]][[\varphi_g^{(m)}, \dots \varphi_{g'-1}^{(m)}]] \supset B = k[[X_1, \dots X_{h'}; \varphi_g^{(m)}, \dots \varphi_{g'-1}^{(m)}]],$$

where the new parameters $Z_i^{(m)}$ $(0 \leq i \leq h')$ satisfy the recursive equations:

$$\phi_\pi(Z_i^{(m)}) = Z_i^{(m-1)},\ Z_i^{(0)} = X_i \qquad \forall i.$$

We consider the following diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{H}^{et}[\mathcal{P}_K^m], \tilde{\Sigma}[\mathcal{P}_K^m]) & \xrightarrow{\ l_*\ } & \mathrm{Ext}^1(\mathcal{H}^{et}[\mathcal{P}_K^m], \frac{\tilde{\Sigma}[\mathcal{P}_K^m]}{\tilde{\Sigma}[F^b]}) \\ \downarrow{\scriptstyle F^{mg}} & \swarrow{\scriptstyle \iota_*} & \\ \mathrm{Ext}^1(\mathcal{H}^{et\,(q^{mg})}[\mathcal{P}_K^m], \tilde{\Sigma}^{(q^{mg})}[\mathcal{P}_K^m]) & & \end{array}$$

where the morphism l_* (resp. ι_*) is induced by the projection $l : \tilde{\Sigma}[\mathcal{P}_K^m] \to \tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}]$ (resp. by inclusion $\iota : \tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}] \to \tilde{\Sigma}^{(q^{mg})}[\mathcal{P}_K^m]$) via push-forward on extensions, and F^{mg} maps to an extension its mg-th q-Frobenius twist.

We observe that the map i_* is injective. In fact, the inclusion ι fits in the exact sequence

$$0 \to \frac{\tilde{\Sigma}[\mathcal{P}_K^m]}{\tilde{\Sigma}[F^{mg}]} \to \tilde{\Sigma}^{(q^{mg})}[\mathcal{P}_K^m] \to \frac{\tilde{\Sigma}[\pi^m F^{mg}]}{\tilde{\Sigma}[\mathcal{P}_K^m]} \to 0,$$

thus the corresponding morphism i_* fits in the long exact sequence

$$\mathrm{Hom}\,(\mathcal{H}^{et}[\mathcal{P}_K^m], \tfrac{\tilde{\Sigma}[\pi^m F^{mg}]}{\tilde{\Sigma}[\mathcal{P}_K^m]}) \longrightarrow \mathrm{Ext}^1(\mathcal{H}^{et}[\mathcal{P}_K^m], \tfrac{\tilde{\Sigma}[\mathcal{P}_K^m]}{\tilde{\Sigma}[F^{mg}]}) \xrightarrow{\ \iota_*\ }$$

$$\xrightarrow{\ \iota_*\ } \mathrm{Ext}^1(\mathcal{H}^{et\,(q^{mg})}[\mathcal{P}_K^m], \tilde{\Sigma}^{(q^{mg})}[\mathcal{P}_K^m]).$$

On the other hand, multiplication by π^m on $\tilde{\Sigma}[\pi^m F^{mg}]$ induces an isomorphism

$$Q = \frac{\tilde{\Sigma}[\pi^m F^{mg}]}{\tilde{\Sigma}[\mathcal{P}_K^m]} \simeq \tilde{\Sigma}[F^{mg}].$$

It follows that the group scheme Q is connected and $\mathrm{Hom}\,(\mathcal{H}^{et}[\mathcal{P}_K^m], Q) = 0$, which implies that the morphism ι_* is injective. From the injectiveness of the map ι_*, we deduce that the existence of a splitting of $(*)$ is equivalent to the existence of splitting of the short exact sequence

$$0 \to \tilde{\Sigma}^{(q^{mg})}[\mathcal{P}_K^m] \to \mathcal{H}^{(q^{mg})}[\mathcal{P}_K^m] \to \mathcal{H}^{et(q^{mg})}[\mathcal{P}_K^m] \to 0.$$

We move our focus from $\mathcal{H}$ to $\mathcal{H}^{(q^{mg})}$, and regard $\mathcal{H}^{(q^{mg})}$ as a deformation of the Barsotti-Tate $\mathcal{O}_K$-module $H_{g',h'}^{(q^{mg})}/k$. We denote by $k[[X'_1,\dots,X'_{h'};T'_g,\dots,T'_{g'-1}]]$, the universal deformation ring for $H_{g',h'}^{(q^{mg})}/k$, where the parameters X'_j and T'_i are defined in the usual way. Then $\mathcal{H}^{(q^{mg})}$ corresponds to the morphism

$$(F^{mg})^* : k[[X'_1,\dots,X'_{h'};T'_g,\dots,T'_{g'-1}]] \to k[[X_1,\dots,X_{h'};T_g,\dots,T_{g'-1}]]$$

which maps $X'_j \mapsto X_j^{q^{mg}}$ and $T'_i \mapsto T_i^{q^{mg}}$. Arguing as we did for proposition 11, we deduce that defining a splitting isomorphism of the extension structure underlying $\mathcal{H}^{(q^{mg})}[\mathcal{P}_K^m]$, corresponds to defining an extension

$$k[[Y'_1,\dots,Y'_{h'};T'_g,\dots,T'_{g'-1}]] \supset k[[X'_1,\dots,X'_{h'};T'_g,\dots,T'_{g'-1}]],$$

where the new parameters Y'_j satisfy the equations $f'_{\pi^m}(Y'_j) = X'_j$, for f'_π the power series of the multiplication by π on $\mathcal{H}^{(q^{mg})}$.

Therefore, we deduce that constructing a splitting as in condition (1) over $k[[X_1,\dots,X_{h'};T_g,\dots,T_{g'-1}]]$ is equivalent to constructing the unique extension B'' of $k[[X_1,\dots,X_{h'};T_g,\dots,T_{g'-1}]]$ which completes the following cartesian diagram.

$$\begin{array}{ccc}
B'' & \longleftarrow & k[[X_1,\dots,X_{h'};T_g,\dots,T_{g'-1}]] \\
\uparrow & & \uparrow{\scriptstyle (F^{mg})^*} \\
k[[Y'_1,\dots,Y'_{h'};T'_g,\dots,T'_{g'-1}]] & \longleftarrow & k[[X'_1,\dots,X'_{h'};T_g,\dots,T_{g'-1}]]
\end{array}$$

Equivalently, $B'' = k[[Z_1,\dots,Z_{h'};T_g,\dots,T_{g'-1}]]$, where the new parameters Z_j are defined by the equations

$$(F^{mg})^* f_{\pi^m}(Z_j) = X_j^{q^{mg}} \text{ for all } j.$$

Since $(F^{mg})^* f_{\pi^m} = (\phi_{\pi^m})^{q^{mg}}$, we deduce that the parameters of B'' satisfy the equations

$$(F^{mg})^*(f_{\pi^m}(Z_j)) - X_j^{q^{mg}} = (\phi_{\pi^m}(Z_j) - X_j)^{q^{mg}} = 0,$$

for all $j = 1,\dots,h'$.

Finally, since the ring B'' is assumed reduced, this implies that $\phi_{\pi^m}(Z_i) = X_i$ (for all i), and thus B'' is regular, finite and flat over $k[[X_1,\dots,X_{h'};T_g,\dots,T_{g'-1}]]$.

It follows that $B' = B_m = k[[Z_1,\dots,Z_{h'};\varphi_g^{(m)},\dots\varphi_{g'-1}^{(m)}]]$ is regular, finite and flat over $R \simeq k[[X_1,\dots X'_h;T_g,\dots T_{g'-1}]]$. □

The above proof also implies the following proposition.

Proposition 15. *Maintaining the above notations. Suppose that $\mathcal{G}/X$ satisfy "the versality condition":*

- *at each point $x \in X$ the natural map from the formal completion of X at x to the formal space of deformations in equal characteristic of $\mathcal{G}(x)$ with p-rank bounded by h is an isomorphism (thus, in particular X is smooth).*

Then, the Igusa cover I_m is a smooth k-scheme.

3.3.2. We compare Igusa and Drinfeld covers over X. The following proposition extends the result of Harris and Taylor in the case of constant p-rank (see proposition 6) to the general case.

Let X be a reduced k-scheme and $\mathcal{G}$ a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over X, of constant height n and maximal p-rank h. We denote by $X^{(h)}$ the open subscheme of X where the p-rank is equal to h.

For any $m \geq 1$, we consider the Igusa and Drinfeld covers of level m of X, which we denote respectively by I_m and X_m, and the closed subschemes $X_{m,M} \subset X_m$, associated with the direct summands M of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$ over $\mathcal{O}_K/\mathcal{P}_K^m$.

Proposition 16. *Let M be a direct summand of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$. We choose an epimorphism of kernel M*

$$p_M : (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \to (\mathcal{P}_K^{-m}/\mathcal{O}_K)^h.$$

Then, associated with p_M, there exists an isomorphism $j_M^* : I_m \to X_{m,M}$ which makes the following diagram commute.

$$\begin{array}{ccc} I_m & \xrightarrow{j_M^*} & X_{m,M} \\ \downarrow & & \downarrow \\ X & \xrightarrow{F^{mg}} & X \end{array}$$

Proof. We define the morphism j_M^* in a manner which generalizes the construction of proposition 6. By definition, a morphism j_M^* which makes the above diagram commute is equivalent to the datum of a Drinfeld structure on the group scheme $\mathcal{G}^{(q^{m(n-h)})}[\mathcal{P}_K^m]$ defined over I_m vanishing on the submodule $M \subset (\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$.

Let β be the morphism:

$$(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n \xrightarrow{p_M} (\mathcal{P}_K^{-m}/\mathcal{O}_K)^h \xrightarrow{\alpha} \frac{\mathcal{G}[\mathcal{P}_K^m]}{\mathcal{G}[F^{m(n-h)}]}(I_m) \xrightarrow{\iota} \mathcal{G}^{(q^{m(n-h)})}[\mathcal{P}_K^m](I_m)$$

where α is the universal Igusa structure defined over I_m, and ι the natural inclusion induced by the $(m(n-h))$-th power of the q-Frobenius. We claim that β is a Drinfeld structure.

To check this statement it suffices to prove that β induces a Drinfeld structure over the completions of I_m at a point x, for all geometric closed points $x \in I_m$. For all point x of I_m which lies above a point $\bar{x} \in X^{(h)} \subset X$, this follows from proposition 6, thus it suffices to consider the points x of p-rank $h(x) < h$.

Let x be a geometric closed point of I_m with $h(x) = h' < h$, and denote by $\bar{x}$ its image in X. We choose an isomorphism $\mathcal{G}^0(x) \simeq \Sigma_{K,g'}$, where $g' = n - h'$. and consider the natural map $\phi(x)$ from the completion of X at $\bar{x}$ to the space of deformation in equal characteristic of $\mathcal{G}(\bar{x})$ with p-rank bounded by h. Then, it suffices to prove our statement in the case when $\phi(x)$ is an isomorphism, in which case, after choosing a parameter X on $\mathcal{G}_x = \mathcal{G}/\mathcal{O}^{\wedge}_{I_m,x}$, we can identify

$$\mathcal{O}^{\wedge}_{I_m,x} = k[[Z_1, \ldots, Z_{h'}; \varphi_g^{(m)}, \ldots, \varphi_{g'-1}^{(m)}]]$$

where the parameters Z_i's and $\varphi_j^{(m)}$'s are defined as in proposition 14, and regard $\mathcal{G}_x$ as the universal extension of $\mathcal{G}_x^{et}$ by $\tilde{\Sigma} = \tilde{\Sigma}^g_{K,g'}$.

We consider the following commutative diagram (we write $N = \mathcal{P}_K^{-m}/\mathcal{O}_K$)

$$\begin{array}{ccccc} N^h & \xrightarrow{\alpha} & \frac{\mathcal{G}_x[\mathcal{P}_K^m]}{\mathcal{G}_x[F^{mg}]}(\mathcal{O}^\wedge_{I_m,x}) & \xrightarrow{\iota} & \mathcal{G}_x^{(q^{mg})}[\mathcal{P}_K^m](\mathcal{O}^\wedge_{I_m,x}) \\ \downarrow{\scriptstyle (f_1,f_2)} & & \downarrow{\scriptstyle s(\mathcal{O}^\wedge_{I_m,x})} & & \downarrow{\scriptstyle \sigma(\mathcal{O}^\wedge_{I_m,x})} \\ N^{h-h'} \times N^{h'} & \longrightarrow & \left(\frac{\tilde{\Sigma}[\mathcal{P}_K^m]}{\tilde{\Sigma}[F^{mg}]} \times \mathcal{G}_x^{et}[\mathcal{P}_K^m]\right)(\mathcal{O}^\wedge_{I_m,x}) & \xrightarrow{(\iota^0,\iota^{et})} & \left(\tilde{\Sigma}^{(q^{mg})} \times \mathcal{G}_x^{et}\right)[\mathcal{P}_K^m](\mathcal{O}^\wedge_{I_m,x}) \end{array}$$

where:

- (f_1, f_2) is the canonical splitting of N^h,
- s is the splitting isomorphism induced by the datum of a level structure on $\mathcal{G}_x[w^m]/\mathcal{G}_x[F^{mg}]$ (see proposition 3),
- σ is the splitting isomorphism on $\mathcal{G}_x^{(q^{mg})}[w^m]$ induced by s (see the proof of proposition 14).

We remark that, in particular, all the vertical maps are isomorphisms.

We denote by (α^0, α^{et}) the morphism in the diagram

$$N^{h-h'} \times N^{h'} \to (\tilde{\Sigma}[\mathcal{P}_K^m]/\tilde{\Sigma}[F^{mg}] \times \mathcal{G}_x^{et}[\mathcal{P}_K^m])(\mathcal{O}^\wedge_{I_m,x}),$$

and write $(\beta^0, \beta^{et}) = \sigma \circ \beta_x$.

Proposition 3 implies that in order to check that β_x is a level structure it suffices to check that both β^0 and β^{et} are level structures. Since β^{et} clearly is a level structure, we are left to consider the morphism

$$\beta^0 : \ker \mathrm{p} \to \tilde{\Sigma}(q^{mg})[\mathcal{P}_K^m](\mathcal{O}^\wedge_{I_m,x})$$

where $\mathrm{p} = f_2 \circ p_M : N^n \twoheadrightarrow N^{h'}$ (and thus $\ker \mathrm{p} \supset M$).

We choose a basis $\{e_0, \dots, e_{g'-1}\}$ of $\ker \mathrm{p}$ such that $M = \langle e_0, \dots, e_{g-1}\rangle$. Then, the morphism β^0 maps the vectors e_i to 0 for $0 \le i \le g-1$, and for $g \le i \le g'-1$ to $v_i = \iota(\alpha^0(e_i))$, where $X(v_i) = (F^{mg})^*(\varphi_i^{(m)})$ for X the chosen parameter on $\tilde{\Sigma}$.

In order to conclude it suffices to prove that, for all i, $(F^{mg})^*(\varphi_i^{(m)}) = \theta_i'^{(m)}$ (where we use the $'$ to extend the notations established in section 2.2.3 for the Barsotti-Tate $\mathcal{O}_K$-module $\tilde{\Sigma}$ to its Frobenius twist $\tilde{\Sigma}^{(q^{mg})}$). Equivalently, we need to verify that the $(F^{mg})^*(\varphi_i^{(m)})$ satisfy the following equations defining the local parameters $\theta_i'^{(m)}$ (see [2], p. 572–574)

- $g_r'(\theta_r'^{(1)}) = 0$, for $g_r'(X) = \frac{f_\pi'(X)}{\prod_{x \in M_{r-1}}(X - \tilde{\alpha}(x))}$ and $M_{r-1} = \langle e_0^{(1)}, \dots, e_{r-1}^{(1)}\rangle$, for $m = 1$ and all r;
- $f_\pi'(\theta_i'^{(m)}) = \theta_i'^{(m-1)}$ for $m > 1$ and all i.

By definition (see claim (1) in the proof of proposition 14), the parameters $\varphi_i^{(m)}$ satisfy the equations

- $h_r(\varphi_r^{(1)}) = 0$ for $m = 1$ and all r,
- $\phi_\pi(\varphi_i^{(m)}) = \varphi_i^{(m-1)}$ for $m > 1$ and all i,

where $(F^{mg})^*(\phi_\pi) = f_\pi'$ and $(F^{mg})^*(h_r) = g_r'$), for all r.

Thus, β^0 is a level structure on $\tilde{\Sigma}^{(q^b)}$. In fact, more precisely, it is the universal level structure on it. This not only proves that the morphism j_M^* is well defined, but also that it is an isomorphism since it induces an isomorphism on complete local rings. $\square$

3.4. **The branched locus of a Drinfeld cover.** So far, we have described how the Drinfeld covers X_m of X decompose as unions of proper subschemes $X_{m,M}$, each of them isomorphic, up to an inseparable morphism, to the corresponding Igusa cover. We conclude this section by shedding some light on how the subschemes $X_{m,M}$ intersect. In the case when the Barsotti-Tate $\mathcal{O}_K$-module $\mathcal{G}/X$ satisfies the appropriate "versality condition", this question amounts to investigating the intersection of the smooth components of the singular fiber of the Drinfeld covers. (The following results are the analogue in this context of Section 13.2 in [9].)

3.4.1. Let X be a reduced k-scheme and $\mathcal{G}$ a one-dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module over X, of height n and maximal p-rank h. Let $X^{(h)}$ denote the open subscheme of X where the p-rank is maximal. We already established that the subschemes $X_{m,M}$ are disjoint when restricted over the open $X^{(h)} \subset X$, thus we focus on their restrictions over the proper closed subscheme $X^{[h-1]} = X - X^{(h)}$.

It is an easy consequence of the definitions that for any positive integer m and any M direct summand of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$, we have

$$(X_{m,M})^{\mathrm{red}}_{|X^{[h-1]}} = \bigcup_{M' \supset M} X^{[h-1]}_{m,M'}$$

where M' varies among the direct summands of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h+1$, containing M. Thus, for any two direct summand M_1, M_2 of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, of rank $n-h$, we have

$$(X_{m,M_1} \cap X_{m,M_2})^{\mathrm{red}} = X_{m,L},$$

where L is the minimal direct summand of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, containing $M_1 + M_2$. By definition the closed subscheme $X_{m,L}$ of X_m/X is supported only over $X^{[h_L]} \subset X$, for h_L the corank of L.

In the cases when X satisfies the "versality condition", using the explicit computations of local parameters as they appear in the proof of proposition 11, it is possible to describe the complete local ring of the intersection $X_{m,M_1} \cap X_{m,M_2}$ at a point x in terms of the linear equations describing the transition matrix between two basis of M_1 and M_2 inside $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$.

It is an easy observation that these intersections are in general non transversal, not even reduced. On the other hand, they are reduced (resp. transversal) when the module M_1+M_2 is a direct summand (resp. direct summand of rank $2n-h_1-h_2$). In fact, if we assume $L = M_1 + M_2$, then we can choose a base of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$ of the form $\{\underline{e}, \underline{u}, \underline{v}, \underline{t}\}$ such that $M_1 = \langle \underline{e}, \underline{u} \rangle$, $M_2 = \langle \underline{e}, \underline{v} \rangle$ and $L = \langle \underline{e}, \underline{u}, \underline{v} \rangle$. Let us choose local parameters θ's as in proposition 11 corresponding to the above basis, i.e. such that $\mathcal{O}^\wedge_{x,X_{m,M_1}} \simeq k[[Y, \underline{\theta}_v, \underline{\theta}_t]]$ and $\mathcal{O}^\wedge_{x,X_{m,M_2}} \simeq k[[Y, \underline{\theta}_u, \underline{\theta}_t]]$. Then, the complete local ring at a point $x \in X_{m,M_1} \cap X_{m,M_2} \subset X_m$ is

$$\mathcal{O}^\wedge_{x,X_{m,M_1} \cap X_{m,M_2}} \simeq k[[Y, \underline{\theta}_v, \underline{\theta}_t]]/(\underline{\theta}_v) = k[[Y, \underline{\theta}_u, \underline{\theta}_t]]/(\underline{\theta}_u).$$

We give an example in the case when M_1+M_2 is not a direct summand, for $n=2$ and $h=1$. Let e, f denote the canonical basis of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^2$, we consider $M_1 = \langle e \rangle$ and $M_2 = \langle e + \pi f \rangle$. Then $X_{m,M_1} \cap X_{m,M_2}$ is zero dimensional non-reduced. For a point x of this intersection we have

$$\mathcal{O}^\wedge_{x,X_{m,M_1} \cap X_{m,M_2}} = k[[\theta_f]]/(f_\pi(\theta_f)) \simeq k[[\theta_f]]/(\theta_f^q).$$

4. On the number of connected components

We conclude this paper with an application of the above description of the geometry in positive characteristic to counting number of connected components of Drinfeld covers. Our results will apply in particular to the class of Shimura varieties studied by Harris and Taylor in [3].

4.1. **On Drinfeld covers.** Let X be a reduced k-scheme and $\mathcal{G}/X$ be a one dimensional compatible Barsotti-Tate $\mathcal{O}_K$-module , of height n. We assume that $\mathcal{G}/X$ satisfies the "versality condition":

- at each point x the map from the formal completion of X at x to the formal space of deformation in equal characteristic of $\mathcal{G}(x)$ is an isomorphism.

Thus X is smooth of pure dimension $n-1$.

Then, the results in the previous section imply the following remark.

Remark 17. *For any $m \geq 1$, the Drinfeld cover X_m of X decomposes as a union of $q^{(m-1)(n-1)}(q^{n-1}+q^{n-2}+\cdots q^2+q+1)$ smooth closed subvarieties of pure dimension $n-1$.*

Indeed, $X_m = \bigcup_M X_{m,M}$ where each $X_{m,M}$ is a smooth closed subvariety of pure dimension $n-1$, and M varies among the direct summands of $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$ which are free of rank 1. Thus it suffices to check that there are exactly $q^{(m-1)(n-1)}(q^{n-1}+q^{n-2}+\cdots q^2+q+1)$ such M's.

For $m=1$, the number of all direct summands of $(\mathcal{P}_K^{-1}/\mathcal{O}_K)^n$, of rank 1, is simply the number of all k-subspaces of k^n of dimension 1, i.e.

$$\#\mathbb{P}^{n-1}(k(w)) = q^{n-1}+q^{n-2}+\cdots q^2+q+1.$$

For $m \geq 2$, we count the number of direct summands of $(\mathcal{P}_K^{-2}/\mathcal{O}_K)^n$ by grouping them accordingly to their reduction modulo $\mathcal{P}_K$. Given $M_0 = \langle v_0 \rangle \subset (\mathcal{P}_K^{-1}/\mathcal{O}_K)^n$, there are $q^{(m-1)n}$ possible lifts of the vector v_0 in $(\mathcal{P}_K^{-m}/\mathcal{O}_K)^n$, and any two of them span the same submodule if and only if they are multiple of each other by a 1-unit. Thus, for any M_0, there are exactly $q^{(m-1)(n-1)}$ direct summands which reduce to M_0 modulo $\mathcal{P}_K$.

4.1.1. Let us further assume that the locus $X^{[0]} \subset X$ is not empty, i.e. that there exists a point $x \in X$ such that $\mathcal{G}(x)$ has p-rank equal to 0 (such a point is called *supersingular*). Then

(1) for any $m \geq 1$, the Drinfeld cover X_m is the union of smooth closed subvarieties crossing at all supersingular points $x \in X_m^{[0]}$;
(2) the map $X_m \to X$ is totally ramified at all supersingular points $x \in \bar{X}_m^{[0]}$.

These two remarks together directly imply the following proposition.

Proposition 18. *Let X be a connected k-scheme and $\mathcal{G}/X$ a one dimensional Barsotti-Tate $\mathcal{O}_K$-module of height n. Suppose*

(1) $\mathcal{G}/X$ satisfy the versality condition given above;
(2) X has at least one supersingular point.

Then X_m is connected and the subschemes $X_{m,M}$ are its irreducible components.

Further more, the above proposition extends to the analogous result in characteristic zero.

Proposition 19. *Let $\mathcal{X}$ be a connected smooth $\mathcal{O}_{\hat{K}^{nr}}$-scheme and $\mathcal{G}/\mathcal{X}$ a one dimensional Barsotti-Tate $\mathcal{O}_K$-module of height n.*

Suppose

(1) *$\mathcal{G}/\mathcal{X}$ satisfy the "versality condition" at each point $x \in \mathcal{X} \times_{\mathcal{O}_{\hat{K}^{nr}}} k$ the natural morphism from the formal completion of $\mathcal{X}$ at x to the formal space of deformations of $\mathcal{G}(x)$ is an isomorphism;*

(2) *$\mathcal{X} \times_{\mathcal{O}_{\hat{K}^{nr}}} k$ has at least one supersingular point.*

Then, for any $m \geq 1$, the generic fiber of Drinfeld cover $\mathcal{X}_m$ of $\mathcal{X}$ of level m is irreducible.

Proof. Under the above assumptions we know that the scheme $\mathcal{X}_m$ is regular, its generic fiber smooth and its special fiber connected. In fact, more precisely, the special fiber $\mathcal{X}_m \times_{\mathcal{O}_{\hat{K}^{nr}}} k$ consists of smooth irreducible components crossing at the supersingular points. The regularity of $\mathcal{X}_m$ at the supersingular points implies that the generic fiber $\mathcal{X}_m$ is also connected and thus irreducible since smooth. □

4.2. **On some simple Shimura varieties.** We conclude by applying the above results to the Shimura varieties studied in [3]. We refer to [3] (I.7 and III.1) for the technical definition of this class of Shimura varieties, and limit ourself to outline how they fit in the context of this paper.

The Shimura varieties in [3] belong to a certain subclass of the class of PEL-type Shimura varieties. In particular, these Shimura varieties arise as moduli spaces of abelian varieties together with certain additional structures (namely, a polarization, the action of a simple division algebra $B/\mathbb{Q}$ and a level structure). In some cases, when the Hasse principle fails, these PEL-type moduli spaces give the Shimura varieties only after passing to connected components. In the following, we temporarily ignore this problem and refer to these moduli spaces as "Shimura varieties", postponing the discussion of this issue to the very end. In particular, we shall see that even in those cases, our result on the number of the connected components of the moduli spaces implies the analogous result for the Shimura varieties.

4.2.1. Let F be the ground field. We assume $F = E.F^+$, for E a quadratic imaginary extension of $\mathbb{Q}$ where p splits completely and F^+ a totaly real field. Then F is a CM-field with maximal totally real subfield F^+. Let u be a place of E above p and $w = w_1, \ldots, w_r$ the places of F above u. The completion F_w of F at our chosen prime $w|p$ plays the role of the local field K in this context. In the following, we write $\mathcal{O}_{F_w}$ for the ring of integers of F_w, and k_w for its residue field. We also choose $\hat{F}_w^{nr}$ the completion of a maximal unramified extension of F_w, and write $\mathcal{O}_{\hat{F}_w^{nr}}$ for its ring of integers and k for its residue field.

4.2.2. Let $G/\mathbb{Q}$ be the algebraic groups associated with one of these Shimura varieties and B the central division algebra over F appearing in the moduli data. We write B^{op} for the opposite algebra of B. Under the hypotheses in [3], we have

$$G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \prod_{i=1}^{r} B_{w_i}^{op}.$$

For $i = 1, \ldots, r$, let $\mathcal{O}_{B_{w_i}}^{\mathrm{op}}$ be a maximal ideal of $B_{w_i}^{\mathrm{op}}$. We assume B is split at w, and choose an isomorphism $\mathcal{O}_{B_w}^{\mathrm{op}} \cong M_n(\mathcal{O}_{F_w})$.

4.2.3. Let U be a level, i.e. a sufficiently small open compact subgroup U of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the finite adeles of $\mathbb{Q}$. We always assume U has the form

$$U = U^p(\underline{m}) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^{r} \ker((\mathcal{O}_{B_{w_i}}^{\mathrm{op}})^\times \to (\mathcal{O}_{B_{w_i}}^{\mathrm{op}}/w_i^{m_i})^\times)$$

inside $G(\mathbb{A}_f) = G(\mathbb{A}_f^p) \times \mathbb{Q}_p^\times \times \prod_{i=1}^r B_{w_i}^{op}$ (where $\mathbb{A}_f^p$ denotes the finite adeles away from p), for some positive integers $m_1, \dots, m_r$.

We call the integer $m = m_1$ the *level at* w and we say that w *does not divide the level* (resp. p *does not divide the level*) if $m_1 = 0$ (resp. if $m_1 = m_2 = \cdots = m_r = 0$). Further more, for any given level $U = U^p(m_1, m_2 \dots, m_r)$, we define $U_1 = U^p(0, m_2, \dots m_r)$ and $U_0 = U^p(0, 0, \dots, 0)$. Thus $U \subset U_1 \subset U_0$.

4.2.4. Let X_U denote the Shimura variety of level U associated with G. It is a smooth projective scheme defined over F. Moreover, it admits a integral model $\mathcal{X}_U$ over $\mathcal{O}_{F_w}$ with the following properties ([3], Lemma III.4.1, p.111-112):

(1) $\mathcal{X}_U$ is regular and flat over $\mathcal{O}_{F_w}$ (for any U);
(2) if w does not divide U ($m_1 = 0$), then $\mathcal{X}_U/\mathcal{O}_{F_w}$ is smooth and the natural morphism $\mathcal{X}_U \to \mathcal{X}_{U_0}$ is finite étale and Galois;
(3) if w divides U ($m_1 \neq 0$), then $\mathcal{X}_U \to \mathcal{X}_{U_1}$ is finite flat.

These results are based on the following important observation ([3], III.4 p. 108).

Remark 20. *Let $V \subset G(\mathbb{A}_f)$ be a level not divisible by w, $\mathcal{X} = \mathcal{X}_V$ the associated integral Shimura variety over $\mathcal{O}_{F_w}$ and $\mathcal{A}$ the universal abelian scheme over $\mathcal{X}$.*

There exists a canonical compatible one-dimensional Barsotti-Tate $\mathcal{O}_{F_w}$-module $\mathcal{G} \subset \mathcal{A}[p^\infty]$ such that

- *$\forall x \in \mathcal{X} \times_{\mathcal{O}_{F_w}} k$: the natural morphism from the completion of $\mathcal{X} \times_{\mathcal{O}_{F_w}} \mathcal{O}_{\hat{F}_w^{nr}}$ at x to the formal space of deformations of $\mathcal{G}(x)$ is an isomorphism.*

Moreover, for a general level U, the scheme $\mathcal{X}_U \to \mathcal{X}_{U_1}$ is the Drinfeld cover of level $m = m_1$ associated with $\mathcal{G}$.

The above remark is the key for translating the results of section 3 into the context of the Shimura varieties studied in [3]. In particular, the following result can be obtained as corollary to Proposition 19.

Corollary 21. *The number of connected components of the Shimura varieties is independent of the level at w.*

Proof. In order to deduce this corollary, we need to show that all the connected components of the reduction of the Shimura varieties, of level not divisible by w, contain at least one supersingular point. In fact, it suffices to check this for levels U_0 not divisible by p, since the general case of levels U_1 not divisible by w will then follow using the fact that the morphism $\mathcal{X}_{U_1} \to \mathcal{X}_{U_0}$ is finite étale and Galois.

In [3] (Lemma III.4.3, p. 114) Harris and Taylor prove the existence of at least one supersingular point. In the lemma below we prove that, for any level U not divisible by p, the group of automorphisms of the integral model $\mathcal{X}_U$, preserving the p-rank stratification (associated with $\mathcal{G}$) of the special fiber $\bar{X}$ of $\mathcal{X}$ and thus mapping supersingular points to supersingular points, acts transitively on the connected components. These two results combined allow us to conclude. □

In the following lemma, we consider the Shimura varieties X_U as varieties over the complex numbers $\mathbb{C}$, via an embedding of the CM-field F in $\mathbb{C}$ which extends a fixed embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ characterized by the conditions $G(F^+_\tau) \cong U(1, n-1)$, and $G(F^+_\sigma) \cong U(0, n)$ for all other real places σ of F^+ (see Lemma I.7.1 pp.52–55).

Lemma 22. *Maintaining the above notations. Let Z_1, Z_2 be any two connected components of the Shimura variety $X_U/\mathbb{C}$, for any level U not divisible by p.*

Then, there exists an automorphism of its integral model $\mathcal{X}_U/\mathrm{Spec}\,\mathcal{O}_{F_w}$ such that:

- *its restriction to the generic fiber maps Z_1 isomorphically to Z_2;*
- *its restriction to the special fiber preserves the p-rank stratification.*

Proof. In [6] (Section 8, pp. 398-400) Kottwitz describes the generic fiber X_U of the integral models $\mathcal{X}_U$ as

$$X_U(\mathbb{C}) = \coprod_{i\in I} S_U^{(i)}, \qquad \forall i : S_U^{(i)} = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U \times G(\mathbb{R})/U_\infty,$$

for U_∞ a subgroup of $G(\mathbb{R})$ determined by the data defining the moduli problems of the Shimura varieties ([6], p. 386) and I the (finite) set of locally trivial elements in $H^1(\mathbb{Q}, G)$ (see [6], section 7, p. 393 and ff., where $I = \ker^1(\mathbb{Q}, G)$). The definitions of U_∞ and I play no role in our proof. Using this decomposition, we deduce the statement of the lemma from three known facts.

(1) For each $i \in I$, there exists an automorphism σ_i of the integral model $\mathcal{X}_U/\mathrm{Spec}\,\mathcal{O}_{F_w}$ which restricted to the generic fiber X_U maps $S_U^{(1)}$ isomorphically to $S_U^{(i)}$, and which restricted to the special fiber $\bar{X}_U$ preserves the p-rank stratification (see [6], p. 400).
(2) The action of $G(\mathbb{A}_f^p)$ on $\mathcal{X}_U/\mathrm{Spec}\,\mathcal{O}_{F_w}$ when restricted to the generic fiber X_U stabilizes each $S_U^{(i)}$ and induces a transitive action on $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$. Moreover, when restricted to the special fiber $\bar{X}_U$ it preserves the p-rank stratification (see [3], Section III.4, pp. 109-110).
(3) the space $G(\mathbb{R})/U_\infty$ is connected (in fact, $U(1, n-1) \twoheadrightarrow G(\mathbb{R})/U_\infty$ and $U(1, n-1)$ is connected).

□

4.2.5. Finally, let us consider the cases when the PEL moduli spaces X_U give the Shimura varieties only after passing to connected components. More precisely, the PEL moduli spaces $X_U/\mathbb{C}$ decompose as $X_U(\mathbb{C}) = \coprod_{i\in I} S_U^{(i)}$, where each $S_U^{(i)}$ is a canonical model for the Shimura variety of level U associated with G, and $\#I \neq 1$ when the Hasse principle for $H^1(\mathbb{Q}, G)$ fails. On the other hand, since the set I is independent of the level U, our result on the number of connected components of the moduli spaces X_U implies the analogous result for the canonical models S_U even in cases when $\#I \neq 1$. Thus, corollary 21 still holds.

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