ON THE HODGE-NEWTON FILTRATION FOR p-DIVISIBLE $\mathcal{O}$-MODULES

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Abstract. The notions Hodge-Newton decomposition and Hodge-Newton filtration for F-crystals are due to Katz and generalize Messing's result on the existence of the local-étale filtration for p-divisible groups. Recently, some of Katz's classical results have been generalized by Kottwitz to the context of F-crystals with additional structures and by Moonen to μ-ordinary p-divisible groups.

In this paper, we discuss further generalizations to the situation of crystals in characteristic p and of p-divisible groups with additional structure by endomorphisms.

1. Introduction

This paper is concerned with crystals in characteristic p and with p-divisible groups with given endomorphisms. We begin by defining these notions. Let p be a prime. Let B be a split unramified finite-dimensional semi-simple algebra over $\mathbb{Q}_p$ and let $\mathcal{O}_B$ be a maximal order of B. Let A be a noetherian ring and a formally smooth $\mathbb{F}_p$-algebra. Assume that there is an ideal $\mathfrak{I}$ of A such that A is $\mathfrak{I}$-adically complete and such that $A/\mathfrak{I}$ is a finitely generated algebra over a field with a finite p-basis. In [2], 2.2.3, de Jong shows that there is an equivalence of categories between crystals over a formal scheme $S = \mathrm{Spf}(A)$ as above and the category of locally free $\tilde{A}$-modules together with a connection. Here $\tilde{A}$ denotes a lift of A over $\mathbb{Z}_p$ in the sense of [2], 1.2.2 and 1.3.3, i.e. a p-adically complete flat $\mathbb{Z}_p$-algebra endowed with an isomorphism $\tilde{A}/p\tilde{A} \cong A$. We write σ for a lift of Frobenius on $\tilde{A}$. In the following definition we implicitly use this equivalence but consider an additional $\mathcal{O}_B$-action.

Definition 1. *Let S be as above and $a \in \mathbb{N} \setminus \{0\}$. A σ^a-F-crystal of $\mathcal{O}_B$-modules is a $4-$tuple (M, ∇, F, ι) consisting of*

- *a locally free $\tilde{A}$-module M of finite rank,*
- *an integrable nilpotent $W(k)$-connection ∇,*
- *a horizontal morphism $F : (\sigma^a)^*(M, \nabla) \to (M, \nabla)$ which induces an isomorphism after inverting p and*
- *a faithful action $\iota : \mathcal{O}_B \to \mathrm{End}(M)$ commuting with ∇ and σ^a-commuting with F.*

Note that when A is a perfect field, the connection can be reconstructed from (M, F), and thus will often be omitted from the notation. An important class of examples of σ-F-crystals of $\mathcal{O}_B$-modules is provided by the crystals associated via (contravariant) Dieudonné theory to p-divisible groups with endomorphisms. Let S be a formal scheme which is locally of the form $\mathrm{Spf}(A)$ for A as above. Then by [2], Main Theorem 1, the crystalline Dieudonné functor is an equivalence of categories between p-divisible groups over S and their Dieudonné crystals. For later use we define p-divisible $\mathcal{O}_B$-modules in a slightly more general context.

Definition 2. *For any $\mathbb{Z}_p$-scheme S, by a p-divisible $\mathcal{O}_B$-module (H,ι) over S we mean a p-divisible group H over S, together with a faithful action $\iota:\mathcal{O}_B\hookrightarrow \mathrm{End}(H)$.*

Remark 3. A standard reduction argument shows that to prove any of our results on σ-F-crystals of $\mathcal{O}_B$-modules or on p-divisible $\mathcal{O}_B$-modules, we may assume that B is simple, i.e. a matrix algebra over an unramified extension D of $\mathbb{Q}_p$ (compare [14], Section 3.23(b)). Then by Morita equivalence we may further assume $B=D$.

We now define Newton and Hodge polygons of σ^a-F-crystals of $\mathcal{O}_B$-modules in the case that $A=k$ is an algebraically closed field of positive characteristic p. In this case $\tilde{A}=W(k)$. Let (M,∇,F,ι) be a σ^a-F-crystal of $\mathcal{O}_B$-modules over $\mathrm{Spec}(k)$ where M is a $W(k)$-module of rank h. The Newton polygon ν of (M,∇,F,ι) is defined as the Newton polygon of the map F. It is an element of $(\mathbb{Q}_{\geq 0})^h_+$, i.e. an h-tuple of nonnegative rational numbers that are ordered increasingly by size. The notion polygon is used as one often considers the polygon associated to $\nu=(\nu_i)$ which is the graph of the continuous, piecewise linear function $[0,h]\to\mathbb{R}$ mapping 0 to 0 and with slope ν_i on $[i-1,i]$. We will refer to this polygon by the same letter ν, for example when we talk about points lying on ν, or about the slopes of ν. If (M,∇,F,ι) is the σ-F-crystal of $\mathcal{O}_B$-modules associated to a p-divisible $\mathcal{O}_B$-module (H,ι) over k then $\nu_i\in[0,1]$ for all i. Note that the isogeny class of H is uniquely determined by ν.

A second invariant of (M,∇,F,ι) is its Hodge polygon $\mu\in\mathbb{N}^h_+$. It is given by the condition that the relative position of M and $F(M)$ is μ. Then $\nu\preceq\mu$ with respect to the usual order, i.e. if we denote the entries of ν and μ by ν_i and μ_i, then $\sum_{i=1}^{l}(\nu_i-\mu_i)\geq 0$ for all $l\leq h$ with equality for $l=h$. If (M,∇,F,ι) is the Dieudonné module of some p-divisible $\mathcal{O}_B$-module, then the entries of μ are all 0 or 1.

For a σ^a-F-crystal of $\mathcal{O}_B$-modules (M,∇,F,ι) over an algebraically closed field k, we analyse how the Galois group $\mathrm{Gal}(\mathbb{F}_q\mid\mathbb{F}_p)$ (where $\mathbb{F}_q$ with $q=p^r$ is the residue field of B) permutes the entries of ν and μ. In Section 2 we show that the entries of the Newton polygon are fixed by this action, however in general the entries of the Hodge polygon are not. We call

$$\overline{\mu}=\frac{1}{r}\sum_{i=0}^{r-1}\sigma^i(\mu)$$

the *σ-invariant Hodge polygon* of (M,∇,F,ι). Here σ is the Frobenius. Then we also have $\nu\preceq\overline{\mu}$.

If (M,∇,F,ι) is a σ^a-F-crystal of $\mathcal{O}_B$-modules over a scheme S, the Newton polygon (resp. Hodge polygon, σ-invariant Hodge polygon) of (M,∇,F,ι) is defined to be the function assigning to each geometric point s of S the Newton polygon (resp. Hodge polygon, σ-invariant Hodge polygon) of the fiber of (M,∇,F,ι) at s.

In the literature the σ-invariant Hodge polygon of a p-divisible $\mathcal{O}_B$-module is often called the μ-ordinary polygon, because of its analogy with the ordinary polygon in the classical context (e.g. see [16], 2.3). Indeed, for p-divisible $\mathcal{O}_B$-modules $\overline{\mu}$ can be defined as follows. For a given $\mu\in\{0,1\}^h_+$, one can consider the set X of all isogeny classes of p-divisible $\mathcal{O}_B$-modules with Hodge polygon equal to μ. Their

Newton polygons all lie above μ and share the same start and end points. Thus, there is a partial order on X given by the natural partial order $\preceq$ on the set of Newton polygons. The μ-ordinary polygon $\overline{\mu}$ is the unique maximal element in X. In the classical case without endomorphisms the μ-ordinary polygon coincides with the Hodge polygon μ.

Definition 4. *Let (M, ∇, F, ι) be a σ^a-F-crystal of $\mathcal{O}_B$-modules over S. Let $(x_1, x_2) = x \in \mathbb{Z}^2$ be a point which lies on the Newton polygon ν of (M, ∇, F, ι) at every geometric point of S. Let ν_1 denote the polygon consisting of the first x_1 slopes of ν and ν_2 the polygon consisting of the remaining ones. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be the analogous parts of the σ-invariant Hodge polygon.*

1. *x is called a breakpoint of ν if the slopes of ν_1 are strictly smaller than the slopes of ν_2.*
2. *We say that (M, ∇, F, ι) has a Hodge-Newton decomposition at x if there are σ^a-F-subcrystals of $\mathcal{O}_B$-modules $(M_1, \nabla|_{M_1}, F|_{M_1}, \iota|_{M_1})$ and $(M_2, \nabla|_{M_2}, F|_{M_2}, \iota|_{M_2})$ of (M, ∇, F, ι) with $M_1 \oplus M_2 = M$ and such that the Newton polygon of M_i is ν_i and its σ-invariant Hodge polygon is $\overline{\mu}_i$.*
3. *We say that (M, ∇, F, ι) has a Hodge-Newton filtration in x if there is a σ^a-F-subcrystal of $\mathcal{O}_B$-modules $(M_1, \nabla|_{M_1}, F|_{M_1}, \iota|_{M_1})$ of (M, ∇, F, ι) with Newton polygon ν_1 and σ-invariant Hodge polygon $\overline{\mu}_1$ and such that M/M_1 is a σ^a-F-crystal of $\mathcal{O}_B$-modules with Newton polygon ν_2 and σ-invariant Hodge polygon $\overline{\mu}_2$.*

We will use the analogous notions for p-divisible $\mathcal{O}_B$-modules, also more generally over $\mathbb{Z}_p$-schemes.

Remark 5. Note that the existence of a Hodge-Newton decomposition or filtration in some x as in the previous definition immediately implies that x lies on $\overline{\mu}$ at every geometric point of S.

In this paper we give conditions for the existence of a Hodge-Newton decomposition over a perfect field of characteristic p, a Hodge-Newton filtration for families in characteristic p, and a Hodge-Newton filtration for deformations of p-divisible $\mathcal{O}_B$-modules to characteristic 0. First instances of these results can be found in Messing's thesis [11] in the classical context of p-divisible groups and in Katz's paper [5] for F-crystals in positive characteristics. Let us now explain the results in more detail.

1.1. **The Hodge-Newton decomposition for $\mathcal{O}_B$-modules.** Our result on the Hodge-Newton decomposition for p-divisible $\mathcal{O}_B$-modules over a perfect field of characteristic p is closely related to a more general result for F-crystals with additional structures. The generalization of Katz's result (which deals with the cases of GL_n and GSp_{2n}) to the context of unramified reductive groups is due to Kottwitz, in [9]. (To be precise the condition required in [9] is actually stronger than the one we give, on the other hand Kottwitz's argument for its sufficiency applies almost unchanged to our settings.)

The following notations are the same as in loc. cit.

Let F be a finite unramified extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ and $\varepsilon \in \{p, t\}$ a uniformizer. Let L be the completion of the maximal unramified extension of F in some algebraic closure. We write $\mathcal{O}_F$ and $\mathcal{O}_L$ for the valuation rings, and σ for the Frobenius of L over F.

Let G be an unramified reductive group over $\mathcal{O}_F$ and T a maximal torus of G over $\mathcal{O}_F$. Let P_0 be a Borel subgroup containing T with unipotent radical U.

Let $P = MN$ be a parabolic subgroup of G containing P_0 with Levi component M containing T and assume that all of these groups are defined over $\mathcal{O}_F$. Let A_P be the maximal split torus in the center of M and write $\mathfrak{a}_P = X_*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$. For $P = P_0$ we skip the index P. Let X_M be the quotient of $X_*(T)$ by the coroot lattice for M. Then the Frobenius σ acts on X_M and we denote by Y_M the σ-coinvariants. We identify $Y_M \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathfrak{a}_P$. Let Y_M^+ be the subset of Y_M of elements identified with elements $x \in \mathfrak{a}_P$ with $\langle x, \alpha \rangle > 0$ for each root α of A_P in N.

In [8], Kottwitz defines a morphism $w_G : G(L) \to X_G$, which induces a map κ_G assigning to each σ-conjugacy class of elements of $G(L)$ an element of Y_G. The classification of σ-conjugacy classes in [7] and [8] shows that a σ-conjugacy class of some $b \in G(L)$ is determined by $\kappa_G(b)$ and the Newton point ν of b, a dominant element of $X_*(A)_{\mathbb{Q}} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. For two elements $\nu_1, \nu_2 \in X_*(A)_{\mathbb{Q}}$ we write $\nu_1 \preceq_G \nu_2$ if $\nu_2 - \nu_1$ is a non-negative rational linear combination of positive coroots of G.

Let $K = G(\mathcal{O}_L)$. Let $b \in G(L)$ and $\mu \in X_*(T)$ dominant. The *affine Deligne-Lusztig set associated to b and μ* is the set

$$X_\mu(b) = \{g \in G(L)/K \mid g^{-1} b\sigma(g) \in K\varepsilon^\mu K\}.$$

In the function field case, the affine Deligne-Lusztig set is the set of $\overline{\mathbb{F}}_p$-valued points of a subscheme of the affine Grassmannian. In the case of mixed characteristic, a scheme structure is not known in general. However, for G the restriction of scalars of some GL_n or GSp_{2n} and μ minuscule, $X_\mu(b)$ is in bijection with the $\overline{\mathbb{F}}_p$-valued points of a moduli space of p-divisible $\mathcal{O}_B$-modules constructed by Rapoport and Zink, [14]. In the case of mixed characteristic, for G the restriction of scalars of some GL_n or GSp_{2n} (and any choice of b, μ), to every element $g \in X_\mu(b)$ we can associate a crystal with given endomorphisms and/or polarization. Let G be the restriction of scalars of $GL_B(V)$ or $GSp_B(V)$, for B an unramified finite extension of F and V a finite dimensional B-vector space. Let $\mathcal{O}_B$ be the valuation ring of B. We choose an $\mathcal{O}_B$-lattice Λ in V. In the symplectic case, we assume B is endowed with an involution $*$ of the first kind which preserves $\mathcal{O}_B$, and V with a non-degenerate alternating $*$-hermitian form with respect to which Λ is selfdual. Then, to any element $g \in G(L)$ we associate the σ-F-crystal of $\mathcal{O}_B$-modules $(\Lambda_{\mathcal{O}_L}, \Phi)$, where $\Lambda_{\mathcal{O}_L} = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_L$ and $\Phi = g^{-1}b\sigma(g) \circ (id_\Lambda \otimes \sigma)$. It is naturally endowed with a faithful action of $\mathcal{O}_B$ and in the case of $G = GSp_B(V)$ also a polarization. The isomorphism class of $(\Lambda_{\mathcal{O}_L}, \Phi)$ is determined by the image of g in $G(L)/K$, while its isogeny class (i.e. the associated isocrystal) only depends on b. For $g \in X_\mu(b)$, the corresponding module has Hodge polygon μ, and for μ minuscule it is the crystal of a p-divisible $\mathcal{O}_B$-module.

Theorem 6. *Let $\mu_M \in X_*(T)$ be M-dominant and let $b \in M(L)$ such that $\kappa_M(b) = \mu_M$ and $\nu \preceq_M \mu_M$. Then $X^M_{\mu_M}(b) \neq \emptyset$. Let μ_G be the G-dominant element in the Weyl group orbit of μ_M. If $\mu_M = \mu_G$ and $\kappa_M(b) \in Y_M^+$, then the natural inclusion $X^M_{\mu_M}(b) \hookrightarrow X^G_{\mu_G}(b)$ is a bijection.*

For F-crystals without additional structures, i.e. for GL_n and GSp_{2n}, this is Katz's Theorem 1.6.1. in [5]. If F is an extension of $\mathbb{Q}_p$ and if b is basic in M, this is shown by Kottwitz in [9]. If G is split, Theorem 6 is the same as [15], Thm. 1.

Applying it to the contravariant Dieudonné module of p-divisible $\mathcal{O}_B$-modules, this theorem yields the existence statement of the following corollary, see Section 3. The general assertion of the corollary is a special case of Corollary 9.

Corollary 7. *Let (H, ι) be a p-divisible $\mathcal{O}_B$-module over a perfect field of characteristic p, with Newton polygon ν and σ-invariant Hodge polygon $\overline{\mu}$. Let $(x_1, x_2) = x \in \mathbb{Z}^2$ be on ν. If x is a breakpoint of ν and x lies on $\overline{\mu}$, then (H, ι) has a unique Hodge-Newton decomposition associated to x.*

In the classical context of p-divisible groups without additional structures, the Hodge-Newton decomposition coincides with the multiplicative-bilocal-étale decomposition. Indeed, this is a simple consequence of the fact that the slopes of the Newton polygon of a p-divisible group are bounded by 0 and 1, while the slopes of its Hodge polygon are always either 0 or 1. Its existence and uniqueness are shown by Messing in [11].

1.2. **The Hodge-Newton filtration for families in characteristic p.** This result on the existence of a Hodge-Newton filtration for families of $\mathcal{O}_B$-modules over a smooth scheme of positive characteristic is a generalization of Katz's Theorem 2.4.2 in [5].

Theorem 8. *Let $S = \mathrm{Spf}(A)$ be a formal scheme as in Definition 1. Let (M, ∇, F, ι) be a σ-F-crystal of $\mathcal{O}_B$-modules over S with Newton polygon ν and σ-invariant Hodge polygon $\overline{\mu}$. Let $(x_1, x_2) = x \in \mathbb{Z}^2$ be a breakpoint of ν lying on $\overline{\mu}$ at every geometric point of S. Then there is a Hodge-Newton filtration of (M, ∇, F, ι) in x and it is unique.*

Furthermore, if $S = Spec(A)$ for A a perfect $\mathbb{F}_p$-algebra, then the Hodge-Newton filtration admits a unique splitting.

Theorem 8 applied to the crystal of a p-divisible $\mathcal{O}_B$-module yields the following corollary.

Corollary 9. *Let (H, ι) be a p-divisible $\mathcal{O}_B$-module over a formal scheme S which is locally of the form $\mathrm{Spf}(A)$ for some A as in Definition 1. Let ν be the Newton polygon of (H, ι) and $\overline{\mu}$ its σ-invariant Hodge polygon . Assume that there is a breakpoint $(x_1, x_2) = x \in \mathbb{Z}^2$ of ν lying on $\overline{\mu}$ for every geometric point of S. Then there is a unique Hodge-Newton filtration of (H, ι) in x. If $S = \mathrm{Spec}(A)$ for A a perfect $\mathbb{F}_p$-algebra, then the Hodge-Newton filtration admits a unique splitting.*

For p-divisible groups (without additional structures) the Hodge-Newton filtration coincides with the multiplicative-bilocal-étale filtration. Therefore, in this context, Katz's result was also known at the time due to the work of Messing in [11].

1.3. **The Hodge-Newton filtration for deformations of p-divisible $\mathcal{O}_B$-modules to characteristic 0.** Let k be a perfect field of positive characteristic p and R an artinian local ring with residue field k. By *a deformation* of a p-divisible $\mathcal{O}_B$-module (H, ι) over k to R we mean a p-divisible $\mathcal{O}_B$-module $(\mathcal{H}, \iota)$ over R together with an isomorphism $j : (H, \iota) \to (\mathcal{H}, \iota) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$.

Theorem 10. *Let R be an artinian local $\mathbb{Z}_p$-algebra of residue field k of characteristic p. Let $(\mathcal{H}, \iota)$ be a p-divisible $\mathcal{O}_B$-module over R and let H be its reduction over k. Let ν and $\overline{\mu}$ be the Newton polygon and the σ-invariant Hodge polygon of*

(H,ι). Assume that there is a Hodge-Newton decomposition of H associated to a breakpoint $x \in \mathbb{Z}^2$ of ν. Let $H_2 \subset H$ be the induced filtration. If $\nu_1 = \overline{\mu}_1$, then the filtration of (H,ι) lifts (in a unique way) to a Hodge-Newton filtration of $\mathcal{H}$ in x.

For p-divisible groups without $\mathcal{O}_B$-module structure, if a breakpoint of the Newton polygon lies on the Hodge polygon, then the two polygons necessarily share a side (of slope either 0 or 1). Therefore, in that case, the condition in this theorem coincides with the one in Theorem 8.

In [11] Messing establishes that the infinitesimal deformations of a p-divisible group over a perfect field of characteristic p are naturally endowed with a unique filtration lifting the multiplicative-bilocale-étale decomposition. In the case of p-divisible groups with additional structures, the existence of a Hodge-Newton filtration for deformations was previously observed in a special case in the work of Moonen. More precisely, in [12], Moonen establishes the existence of the slope decomposition for μ-ordinary p-divisible $\mathcal{O}_B$-modules over perfect fields and of the slope filtration for their deformations. Since, by definition, in these cases the slope decomposition agrees with the (finest) Hodge-Newton decomposition, Moonen's results can be regarded as special cases of the existence of the Hodge-Newton decomposition and filtration for p-divisible $\mathcal{O}_B$-modules. For the proof of Theorem 10, we use a generalization of Moonen's argument via crystalline Dieudonné theory.

Remark 11. There is also a variant of a Hodge-Newton decomposition and filtration for polarized p-divisible $\mathcal{O}_B$-modules. For this case we assume that p is odd, and that B is endowed with an involution $*$ that preserves $\mathcal{O}_B$ (see [6], Section 2). A polarized p-divisible $\mathcal{O}_B$-module is a p-divisible $\mathcal{O}_B$-module (H,ι) together with a polarization $\lambda : H \to H^\vee$ satisfying the condition

$$\lambda \circ b^* = b^\vee \circ \lambda \text{ for all } b \in \mathcal{O}_B.$$

Let (H,ι) be a polarized p-divisible $\mathcal{O}_B$-module of dimension n. From the existence of the polarization and the uniqueness of the Hodge-Newton decomposition and filtration one obtains the following results.

(1) If $x = (x_1, x_2)$ is a breakpoint of the Newton point of H, then $x' = (2n - x_1, n - x_2 + x_1)$ also is.
(2) If (H,ι) has a Hodge-Newton decomposition $H = H_1 \oplus H_2$ associated to x as in Theorem 6, then it also has a Hodge-Newton decomposition in x'. We may assume $x_1 \leq n$. Then there is a decomposition $H \cong H^\vee = H_1 \oplus H' \oplus H_1^\vee$ of $\mathcal{O}_B$-modules such that $H_1^\vee \oplus H' = H_2$ and that $(H_1 \oplus H') \oplus H_1^\vee$ is the Hodge-Newton decomposition in x'. Here H' is trivial if $x_1 = n$.
(3) If (H,ι) has a Hodge-Newton filtration $H_2 \subseteq H$ associated to x as in Theorem 8 or 10, then it also has a Hodge-Newton filtration $H_2' \subseteq H$ in x'. Again we assume $x_1 \leq n$. If $x_1 = n$, then the two filtrations coincide. In general there is a joint filtration $H_2' \subseteq H_2 \subseteq H$ of $\mathcal{O}_B$-modules such that $H_2^\vee \cong H/H_2'$ and $(H_2')^\vee \cong H/H_2$ where the isomorphisms are given by the principal polarization on H.

A detailed discussion of the notion of Hodge-Newton filtration for polarized F-crystals without endomorphisms can be found in [1].

The paper is organized as follows. In Section 2 we consider σ-F-crystals of $\mathcal{O}_B$-modules with additional structure in more detail. In Section 3 we prove Theorem

6 and explain the relation between the general result and the special case of p-divisible $\mathcal{O}_B$-modules. The last two sections concern the Hodge-Newton filtration for families of σ-F-crystals of $\mathcal{O}_B$-modules in characteristic p and for deformations (in the p-divisible case) to characteristic 0, respectively.

Our results on the existence of Hodge-Newton filtrations are used by the first author in [10]. There, the existence of a Hodge-Newton filtration allows to compare the corresponding Rapoport-Zink space with a similar moduli space of filtered p-divisible groups with endomorphisms. This leads to a proof of some cases of a conjecture of Harris ([4], Conjecture 5.2), namely that in the cases when a Hodge-Newton filtration exists, the l-adic cohomology of the corresponding moduli space is parabolically induced from that of a lower-dimensional one.

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2. σ-F-crystals with additional structures

Let (M, ∇, F, ι) be a σ-F-crystal of $\mathcal{O}_B$-modules over a scheme S over an algebraically closed field k as in Definition 1. Recall that we assume that B is an unramified field extension of $\mathbb{Q}_p$ of degree r. We consider (M, ∇, F^r, ι). One easily sees that the connection is again compatible with F^r and ι, thus it is a σ^r-F-crystal of $\mathcal{O}_B$-modules. Let $I = \text{ Hom } (\mathcal{O}_B, W(\overline{\mathbb{F}}_p))$. Note that M is a module over $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \tilde{A}$ which (as k contains $\overline{\mathbb{F}}_p$) is a product of r copies of $\tilde{A}$. Thus the module M decomposes naturally into a direct sum associated to the characters,

$$M = \oplus_{i \in I} M^{(i)}.$$

We identify the set I with $\mathbb{Z}/r\mathbb{Z}$, with r being the degree of the extension of k given by the residue field of $\mathcal{O}_B$. To do so we fix $j \in I$ and consider the bijection that to each $s \in \mathbb{Z}/r\mathbb{Z}$ associates the element $\sigma^s \circ j$ in I. As ∇ and F^r commute with the $\mathcal{O}_B$-action, they decompose into a direct sum of connections resp. σ^r-linear morphisms on the different summands. We obtain a decomposition of the σ^r-F-crystal of $\mathcal{O}_B$-modules (M, ∇, F^r, ι) into r summands $(M^{(i)}, \nabla|_{M^{(i)}}, F^r|_{M^{(i)}})$.

To compute Newton polygons and Hodge polygons, we assume for the remainder of this section that (M, ∇, F^r, ι) is defined over an algebraically closed field of characteristic p. For crystals over a scheme, one computes the polygons at each geometric point separately.

The following notations are the same as in [12], Section 1.

For each i, the Frobenius $F : M \to M$ restricts to a σ-linear map $F_i : M^{(i)} \to M^{(i+1)}$. In particular, $F^r|_{M^{(i)}} = \phi_i$ is given by $\phi_i = F_{i+r-1} \circ F_{i+r-2} \circ \cdots \circ F_i$. Moreover, for all $i \in I$, the morphism $F_i : M^{(i)} \to M^{(i+1)}$ is a semilinear morphism of σ^r-F-crystals, thus the $M^{(i)}$ are all isogenous. This implies, in particular, that they have the same Newton polygon.

The Newton polygon of (M,F) and the Newton polygon of $(M^{(i)},\phi_i)$ can easily be calculated from one another. Indeed, the $M^{(i)}$ are mutually isogenous summands of (M,F^r), thus a slope λ appears in the Newton polygon of (M,F) with multiplicity m if and only if the slope $r\lambda$ appears in the Newton polygon of each $(M^{(i)},\phi_i)$ with multiplicity m/r. We denote by ν' the Newton polygon of $(M^{(i)},\phi_i)$ and call it the r-reduction of ν. These calculations also show that ν is already invariant under the Galois group (as mentioned in the introduction).

The next step is to compute the σ-invariant Hodge polygon $\overline{\mu}$ more explicitly, see also [16], 2.3. It is defined as $\frac{1}{r}\sum_{i\in\mathbb{Z}/r\mathbb{Z}}\sigma^i(\mu)$, where μ is the Hodge polygon. We consider the decomposition

$$M/FM=\oplus_i M^{(i)}/F_{i-1}M^{(i-1)}$$

and let

$$d=h/r=\dim_k M^{(i)}/pM^{(i)}.$$

Let $m_i\in\mathbb{Z}_+^d$ be the relative position of $M^{(i)}$ and $F_{i-1}M^{(i-1)}$. Recall that the Newton polygon of $(M^{(i)},\phi_i)$ is ν' for all i. Let $\mu^{(i)}$ be the Hodge polygon of $(M^{(i)},\phi_i)$.

Remark 12. We can rearrange the entries of μ into r blocks of length d and such that the ith block is m_i and corresponds to the contribution by $M^{(i)}$. Then $\sigma(\mu)$ is obtained from μ by replacing each block by the following one. This implies that each entry of $\overline{\mu}$ occurs with a multiplicity divisible by r. If we use the same formal reduction procedure to define the r-reduction $\overline{\mu}'$ of $\overline{\mu}$ we obtain $\overline{\mu}'=\sum_i m_i$. However, $\overline{\mu}'$ is in general not equal to the Hodge polygons of the $(M^{(i)},\phi_i)$ (which also do not need to be equal), compare Lemma 15.

Note that in the special case where μ is minuscule (for example for the crystal associated to a p-divisible group), m_i can be written as $(1,\dots,1,0,\dots,0)$ with multiplicities $f(i)$ and $d-f(i)$. Here

$$f(i)=\dim_k M^{(i)}/F_{i-1}M^{(i-1)}$$

with $0\leq f(i)\leq d, i=1,\dots,m$. Note that f need not be constant. The data (d,f) is called the *type* of (M,F). This description together with Remark 12 implies the following result.

Corollary 13. *(*[12]*, Lemma 1.3.4) Let (M,F) be the crystal associated to a p-divisible group over an algebraically closed field of characteristic p. The entries of $\overline{\mu}'$ are $0\leq a_1\leq a_2\leq\dots\leq a_d$, where*

$$a_j=\#\{i\in I\mid f(i)>d-j\}.$$

In particular, the number of breakpoints of $\overline{\mu}$ is equal to the number of distinct values of the function f.

Remark 14. Let (M,F) be a (not necessarily minuscule) σ^r-F-crystal over a perfect field and let $F':M\rightarrow M$ be σ-linear. Let $\mu_F,\mu_{F'},\mu_{F\circ F'}$ denote the Hodge polygons of F,F', and $F\circ F'$. Then $\mu_{F\circ F'}\preceq\mu_F+\mu_{F'}$. Indeed, if we denote their slopes by $\mu_{F,i},\mu_{F',i},\mu_{F\circ F',i}$, we have to show that $\sum_{i=1}^l(\mu_{F,i}+\mu_{F',i}-\mu_{F\circ F',i})\leq 0$ for all $l\geq 1$. Considering the lth exterior power of M, this is equivalent to showing that the least Hodge slope of $(\wedge^l F)\circ(\wedge^l F')$ is greater than or equal to the sum of the least Hodge slopes of $\wedge^l F$ and $\wedge^l F'$. But this is obvious as the least Hodge slope of one of these morphisms is the minimal i such that the image of M under this morphism is not contained in $p^{i+1}M$.

Lemma 15. *In the situation above we have*

$$\nu' \preceq \mu^{(i)} \preceq \overline{\mu}'$$

for all $i \in \{1, \ldots, r\}$.

Proof. For all $i \in \mathbb{Z}/r\mathbb{Z}$, we choose an isomorphism between $M^{(i)}$ and $W(k)^d$. Then each restriction $F_i : M^{(i)} \to M^{(i+1)}$ of F induces a σ-linear morphism $W(k)^d \to W(k)^d$ which we also denote by F_i. Its Hodge polygon is m_i (where we use the notation from Remark 12). Then ϕ_i is identified with $F_{i+r-1} \circ \cdots \circ F_i$ where the indices are still taken modulo r. Its Newton polygon is ν' and its Hodge polygon $\mu^{(i)}$, thus we obtain the first of the two inequalities. The remark above shows that the Hodge polygon of $\phi_i = F_{i+r-1} \circ \cdots \circ F_i$ is less or equal to the sum of the Hodge polygons of $F_{i+r-1}, \ldots, F_i$. Hence $\mu^{(i)} \preceq \sum_{j \in I} m_j = \overline{\mu}'$. □

Remark 16. Lemma 15 implies that if ν' and $\overline{\mu}'$ have a point in common, then all the Hodge polygons $\mu^{(i)}$ also contain this point.

3. Hodge-Newton decompositions for p-divisible groups with additional structure

Proof of Theorem 6. Under the additional assumptions that F is an extension of $\mathbb{Q}_p$ and that b is basic in M, the theorem is [9], Theorem 4.1. But Kottwitz's proof still works without these additional assumptions. To leave out the assumption on the characteristic, the proof may remain unmodified. Instead of the assumption on b to be basic, Kottwitz's proof only uses in the proof of [9], Lemma 3.2 that M contains the centralizer M_b of ν. But this already follows from $\nu \in \mathfrak{a}_P^+$. □

To relate this theorem to [5], Theorem 1.6.1, we consider the special case $G = GL_n$. Let T be the diagonal torus and P_0 the Borel subgroup of lower triangular matrices. Then $X_*(T) \cong \mathbb{Z}^n$ and an element $\mu = (\mu_1, \ldots, \mu_n)$ is dominant if $\mu_1 \leq \cdots \leq \mu_n$. Let $b \in G(L)$. Then its Newton point $\nu = (\nu_i) \in X_*(T)_{\mathbb{Q}} \cong \mathbb{Q}^n$ is the same as the n-tuple of Newton slopes considered in Katz's paper. The map κ maps b to $\sum_i \nu_i = m = v_p(\det b) \in \mathbb{Z} \cong \pi_1(G)$. The Hodge polygon μ is of the form $(0, \ldots, 0, 1, \ldots, 1)$ with multiplicities $h - m$ and m. The ordering $\preceq$ takes the following explicit form. The simple coroots of G are the $e_i - e_{i-1}$ for $i = 2, \ldots, n$. Thus $\nu = (\nu_i) \preceq \mu = (\mu_i)$ if and only if

$$\sum_{i=1}^{l} (\nu_i - \mu_i) \geq 0$$

for all $l \in \{1, \ldots, n\}$ and with equality for $l = n$. As G is split, $X = Y$. Let $P \subseteq G$ be a maximal parabolic containing P_0 and let M be its Levi factor containing T. Then $M \cong GL_j \times GL_{n-j}$ for some j. The condition $\nu \in Y_M^+$ is equivalent to $\nu_a - \nu_b > 0$ for all $a > j$ and $b \leq j$. Thus j is a breakpoint of the associated Newton polygon considered by Katz. We have

$$\pi_1(M) = \mathbb{Z}^n / \{(x_i) \mid \sum_{i=1}^{j} x_i = 0 = \sum_{i=j+1}^{n} x_i\}.$$

Hence $\kappa_M(b) = \mu$ if and only if $\sum_{i=1}^{j} \nu_i = \sum_{i=1}^{j} \mu_i$ and $\sum_{i=j+1}^{n} \nu_i = \sum_{i=j+1}^{n} \mu_i$. This means that the polygons associated to μ and ν have the point $(j, \sum_{i=1}^{j} \nu_i)$ in common. From these considerations we see that in this special case Theorem 6 reduces to Katz's theorem on the Hodge-Newton decomposition.

A similar translation shows the existence statement of Corollary 7 over an algebraically closed field. Indeed, let $G = \mathrm{Res}_{B|\mathbb{Q}_p}(GL_d)$. Then $X_*(A)_{\mathbb{Q}}$ can be written as $\{(x_{ij}) \in \mathbb{Q}^h \mid 1 \leq i \leq r, 1 \leq j \leq d\}$ and the action of σ by mapping (x_{ij}) to (y_{ij}) with $y_{ij} = x_{i-1,j}$. We can identify the coinvariants under this action of the Galois group with the subset of $(x_{ij}) \in X_*(A)_{\mathbb{Q}}$ with $x_{ij} = x_{i+1,j}$ for all i and j, or, using the r-reduction, with $\mathbb{Q}^d$. Thus the Newton polygon of H, considered as an element of Y_G is identified with ν' and its Hodge polygon is identified with $\overline{\mu}' \in \mathbb{Q}^d$. As for the split case above one easily sees that the condition $b \in Y_M^+$ for $M = \mathrm{Res}_{B|\mathbb{Q}_p}(GL_j \times GL_{h-j})$ is equivalent to ν' having a breakpoint at j. Also, $\kappa_M(b) = \overline{\mu}'$ translates into the condition that the two polygons coincide in this point. Thus Theorem 6 implies the existence statement in the corollary for p-divisible $\mathcal{O}_B$-modules over an algebraically closed field.

Similarly, for $G = GSp_{2n}$ or $\mathrm{Res}_{B|\mathbb{Q}_p} GSp_{2d}$, one may show that in this case, Theorem 6 implies the polarized variant of Katz's Theorem or of Corollary 7. This is a second strategy to obtain the results explained in Remark 11.

4. Hodge-Newton filtrations in characteristic p

Proof of Theorem 8. Our proof consists in reducing to Katz's result by deducing the statement from the existence of a compatible system of Hodge-Newton decompositions for the σ^r-F-crystals $(M^{(i)}, \nabla|_{M^{(i)}}, \phi_i)$, for all i. Here the $M^{(i)}$ are the same as in Section 2. To be able to use them, we first consider the case that k is algebraically closed and that S is affine.

Let x be the break-point of ν' corresponding to the breakpoint of ν in the statement. Then it follows from Remark 16 that x also lies on $\mu^{(i)}$ for all i. Therefore, by [5], Theorem 1.6.1, for each $i \in I$, there is a unique filtration $(M_1^{(i)}, \nabla|_{M_1^{(i)}}, \phi_i|_{M_1^{(i)}}) \subset (M^{(i)}, \nabla|_{M^{(i)}}, \phi_i)$ associated to x as in Definition 4 and with the desired splitting property. Note that we have slightly less assumptions on S than Katz, but the proof of his theorem still holds in this situation.

By construction, for all i, the sets of Newton slopes of $M_1^{(i)}$ and $M^{(i)}/M_1^{(i)}$ are distinct, therefore the map $F_i : M^{(i)} \to M^{(i+1)}$ maps $M_1^{(i)}$ into $M_1^{(i+1)}$. This implies that $(\oplus_i M_1^{(i)}, \nabla|_{\oplus_i M_1^{(i)}}, F|_{\oplus_i M_1^{(i)}}, \iota|_{\oplus_i M_1^{(i)}})$ inherits a natural structure of sub-σ-F-crystal of $\mathcal{O}_B$-modules of (M, ∇, F, ι) associated to the breakpoint of ν as in the statement.

Finally, if k is not algebraically closed, we first get an analogous filtration over $S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ where $\overline{k}$ is an algebraic closure of k. The uniqueness of the filtration implies that it is stable by the Galois group $\mathrm{Gal}(\overline{k}|k)$, and thus it descends to a filtration over S itself. A similar argument extends the result to non-affine S. □

5. Lifting the Hodge-Newton filtration

5.1. **Explicit coordinates for μ-ordinary p-divisible groups.** Let (H,ι) be a p-divisible $\mathcal{O}_B$-module over an algebraically closed field k of characteristic p. Assume that $x = (d', n')$ is the first breakpoint of its Newton polygon ν', that x also lies on the Hodge polygon $\overline{\mu}'$, and that ν' and $\overline{\mu}'$ coincide up to this breakpoint. Then by Corollary 7, (H,ι) has a Hodge-Newton decomposition in x, so its Dieudonné module decomposes as $M = M_1 \oplus M_2$. Here M_1 corresponds to the first parts of ν' and $\overline{\mu}'$, thus it is μ-ordinary and its polygons are constant. Let $\lambda = l/r$ with $0 \leq l \leq r$ be their slope. From [12], 1.2.3 we obtain that M_1 has generators $e_{i,j}$ with $i \in \mathbb{Z}/r\mathbb{Z}$, $1 \leq j \leq d'$ and such that $F(e_{i,j}) = p^{a(i)}e_{i+1,j}$ where $a(i) \in \{0,1\}$ is equal to 1 on exactly l elements i. As the Newton polygon and the Hodge polygon of M both have slope l/r up to x, we have that $f(i) = d$ if $a(i) = 1$. (Of course, $f(i) \leq d - d'$ if $a(i) = 0$.) For the whole module M, we may complement the coordinates $e_{i,j}$ to obtain $e_{i,j}$ with $1 \leq j \leq d$ of each $M^{(i)}$ such that $e_{i,j}$ is in the image of F if and only if $j > d - f(i)$.

5.2. **Description of the universal deformation.** We recall the description of the universal deformation via crystalline Dieudonné theory from [12], 2.1., see also [3], 7. Let A be a formally smooth $W(k)$-algebra with k algebraically closed and φ_A a lift of the Frobenius of A/pA with $\varphi_A|_{W(k)} = \sigma$. Then p-divisible groups over A can be described by 4-tuples $(\mathcal{M}, \mathrm{Fil}^1(\mathcal{M}), \nabla, F_{\mathcal{M}})$ consisting of

- a free A-module $\mathcal{M}$ of finite rank
- a direct summand $\mathrm{Fil}^1(\mathcal{M}) \subset \mathcal{M}$
- an integrable, topologically quasi-nilpotent connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes \hat{\Omega}_{A/W(k)}$
- a φ_A-linear endomorphism $F_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$.

The 4-tuple corresponding to the universal deformation of a p-divisible $\mathcal{O}_B$-module (H,ι) over $\mathrm{Spec}(k)$ can be described explicitly. Let (M, F, ι) be its Dieudonné module. As in the previous section, one can find a basis $e_{i,j}$ with $1 \leq j \leq d$ of each $M^{(i)}$ such that $e_{i,j}$ is in the image of F if and only if $j > d - f(i)$. The Hodge filtration $\mathrm{Fil}^1(M)$ of M is given by the $W(k)$-module generated by all $e_{i,j}$ with $j > d - f(i)$. The submodule M^0 generated by the $e_{i,j}$ with $j \leq d - f(i)$ is a complement. Let A be the formal power series ring

$$W(k)[[u_{l,m}^{(i)} \mid i \in \mathbb{Z}/r\mathbb{Z}, 1 \leq m \leq d - f(i) < l \leq d]]$$

and let $g_{\mathrm{univ}} = \prod_i g^{(i)} \in GL_{\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(k)}(M)(A) = \prod_i GL_{d,W(k)}(A)$ be given by the r-tuple of matrices $g^{(i)} = (x_{a,b}^{(i)})_{1 \leq a,b \leq d}$ with

$$x_{a,b}^{(i)} = \begin{cases} \delta_{a,b} & \text{if } a \leq d - f(i) \text{ or } b > d - f(i) \\ u_{a,b}^{(i)} & \text{otherwise.} \end{cases}$$

Using this notation let $\mathcal{M} = M \otimes_{W(k)} A$, $\mathrm{Fil}^1(\mathcal{M}) = \mathrm{Fil}^1(M) \otimes_{W(k)} A$ and $F_{\mathcal{M}} = g_{\mathrm{univ}} \circ (F_M \otimes \varphi_A)$. Then (see loc. cit.) there is a unique integrable topologically quasi-nilpotent connection ∇ on $\mathcal{M}$ that is compatible with $F_{\mathcal{M}}$. Besides, $\mathcal{O}_B$ acts by endomorphisms on $(\mathcal{M}, \mathrm{Fil}^1(\mathcal{M}), \nabla, F_{\mathcal{M}})$ and this tuple corresponds to the universal deformation of the p-divisible $\mathcal{O}_B$-module.

Proof of Theorem 10. This proof is an adaptation of the corresponding proof for the μ-ordinary case in [12], Proposition 2.1.9. Therefore we will not repeat each calculation but rather explain the differences. Again, Grothendieck-Messing deformation theory implies the uniqueness of the lifting. Besides it is enough to prove existence in the case that k is algebraically closed, the generally case follows using descent and the uniqueness. Finally it is enough to prove the theorem for $\mathcal{H}$ the universal deformation of H.

We use the description of the universal deformation given above. Let $M_1 \subset M$ be the submodule corresponding to the quotient H_1 of H. Then we have to show that M_1 lifts to a sub-4-tuple of $\mathcal{M}$ with $\mathcal{M}_1 = M_1 \otimes_{W(k)} A$. Let $d' = \min\{d - f(i) \mid i \leq r, f(i) < d\}$. The rank of M_1 is rd' and $M_1 = \oplus_i \mathrm{Span}\,(e_{i,j}, 1 \leq j \leq d')$. Denote by $M_1^{(i)}$ the parts of M_1 in the different $M^{(i)}$. From the explicit description of M_1 in Section 5.1 we obtain that for each i either $M_1^{(i)} \subseteq (M^0)^{(i)}$ or $(M^0)^{(i)}$ is trivial (the second case is equivalent to $f(i) = d$). In both cases g^{univ} maps $\mathcal{M}_1$ to itself and hence $F_{\mathcal{M}}$ restricts to $F_{\mathcal{M}_1} = F_{M_1} \otimes \varphi_A$ on $\mathcal{M}_1$.

It remains to prove that $\nabla(\mathcal{M}_1) \subseteq \mathcal{M}_1 \otimes \hat{\Omega}_{A/W(k)}$, i.e. that the connection restricts to a connection on $\mathcal{M}_1$. This calculation is the same as in Moonen's paper, see [12],(2.1.9.1), at least under the condition $j < d'$. But this is enough as one only needs this equality for those j. $\square$

Using duality, one easily obtains an analogous theorem in the case that ν and $\overline{\mu}$ have the last breakpoint of ν in common and coincide from that point on.

Finally, we remark that the further assumption in this theorem, that the Hodge polygon and Newton polygon coincide up to the considered breakpoint or from this breakpoint on, is necessary. Indeed, a first argument for this is already visible in the proof: if this stronger condition is not satisfied, then the key property that either $M_1^{(i)} \subseteq (M^0)^{(i)}$ or that $(M^0)^{(i)}$ is trivial for each i does not hold. Thus there is some i such that g^{univ} no longer maps M_1 to itself, which makes a lifting of the filtration impossible. Besides, the necessity of this condition can also be seen independently of the proof by comparing the dimensions of the corresponding period spaces.

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