THE NEWTON STRATIFICATION

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1. Introduction

The Newton stratification of moduli spaces for abelian varieties in positive characteristic naturally arise in the study of Shimura varieties of PEL type. This stratification is defined by the loci where the equivalence class of the F-isocrystals attached to the abelian varieties is constant. In the following, we recall the notion of F-isocrystals with G-structure and the definition of the Newton stratification of the reduction modulo p of Shimura varieties. We discuss the geometry of the Newton strata and of the associated Oort's foliations and their relation to the local models constructed by Rapoport and Zink, and to some moduli spaces for abelian varieties in positive characteristic, called Igusa varieties. In particular, we discuss a formula, in the appropriate Grothendieck group, computing the l-adic cohomology of the Newton strata in terms of those of the Igusa varieties and the Rapoport-Zink spaces. Finally, in the last section, we focus on a special class of Shimura varieties, which is closely related to and includes that studied by Harris and Taylor in [8]. For these Shimura varieties, the associated local models are easily related to Lubin-Tate spaces and Drinfeld modular varieties. As the geometry and cohomology of these latter spaces are well understood by the work of Boyer ([3]), we deduce many interesting results on the geometry and cohomology of these Shimura varieties.

2. F-isocrystals with G-structure

The notion of F-isocrystal with G-structure is due to Kottwitz. In this section, we recall some of his definitions and results from [12] and [13], as well as results from the work of Rapoport and Richartz in [26].

2.1. Let k be a finite field of characteristic p (e.g. $k = k(\nu)$), $\overline{k}$ an algebraic closure of k, and L the fraction field of the Witt ring $W(\overline{k})$, $L = W(\overline{k})[\frac{1}{p}]$. The Frobenius automorphism on $\overline{k}$, $x \mapsto x^p$, induces the Frobenius automorphism of L over $\mathbb{Q}_p$ which we denote by σ. For any connected reductive group G over $\mathbb{Q}_p$, we also denote by σ the corresponding automorphism of $G(L)$. We say that two elements b, b' of $G(L)$ are *σ-conjugate* if there exists $g \in G(L)$ such that $b' = gb\sigma(g)^{-1}$. We define $B(G)$ to be the set of σ-conjugacy classes of $G(L)$ and write $[b] \in B(G)$ for the class of an element $b \in G(L)$.

2.2. An *F-isocrystal* is a finite dimensional vector space V over L together with a σ-linear automorphism $\Phi : V \to V$ (called the Frobenius morphism). An *F-isocrystal with G-structure*, for G a connected reductive group over $\mathbb{Q}_p$, is an exact faithful tensor functor from the category of finite dimensional p-adic representations of G to that of F-isocrystals. In the cases of interest to us (e.g. $G = GL(V)$ or $G = GSp(V)$), any such functor N is uniquely determined by its value on the natural representation of G. Therefore, in these cases, an F-isocrystal with G-structure can be identified with a classical F-isocrystal endowed with a G-structure, i.e. an F-isocrystal of the form (V_L, Φ), where $V_L = V \otimes_{\mathbb{Q}_p} L$, for V a finite dimensional $\mathbb{Q}_p$-vector space together with a G-structure, $G \subset GL(V)$, and where the Frobenius morphism Φ commutes with the G-structure on V_L.

To each element $b \in G(L)$ we associate an F-isocrystal with G-structure N_b, via $N_b(W, \rho) = (W_L, \rho(b)(\mathrm{id}_W \otimes \sigma))$. It follows from the definition that any such functor N is defined by a unique $b \in G(L)$, and that if b, b' are σ-conjugate in $G(L)$ then the corresponding functors $N_b, N_{b'}$ are isomorphic. Thus, the set $B(G)$ can be identified with the set of isomorphism classes of F-isocrystals with G-structure.

E.g., for $G = GL(V)$, $B(G)$ can be regarded as the set of isomorphism classes of h-dimensional F-isocrystals, where $h = \dim_{\mathbb{Q}_p} V$.

2.3. The Dieudonné-Manin description of the category of F-isocrystals makes it possible to give a simple classification of the σ-conjugacy classes in $G(L)$. In particular, we recall that the category of F-isocrystals is semisimple, and that the simple objects are parameterized by rational numbers (called the *slopes*). Let $\mathbb{D}$ be the diagonalizable pro-algebraic group over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. For any finite dimensional p-adic representation (W, ρ) of G, and $b \in G(L)$, the slope decomposition of the associated F-isocrystal $N_b(W, \rho)$ gives a homomorphism $\nu_{\rho,b} : \mathbb{D} \to GL(W)$ defined over L. Thus, for each $\rho : G \to GL(W)$, we get a map $\nu_\rho : G(L) \to \mathrm{Hom}_L(\mathbb{D}, GL(W))$, via $b \mapsto \nu_{\rho,b}$. We fix $b \in G(L)$, and consider the elements $\nu_{\rho,b}$ as ρ varies. The functoriality of the construction implies that there exists a unique element $\nu_b \in \mathrm{Hom}_L(\mathbb{D}, G)$ (called the *slope homomorphism* of b) such that $\rho \circ \nu_b = \nu_{\rho,b}$ for all ρ. This defines a map

$$\nu : G(L) \to \mathrm{Hom}_L(\mathbb{D}, G)$$

by $b \mapsto \nu_b$ ([12], Section 4).

For each $b \in G(L)$, the corresponding slope homomorphism ν_b can be characterized as follows ([12], Section 4). It is the unique element in $\mathrm{Hom}_L(\mathbb{D}, G)$ for which there exists an integer $n \geq 1$, an element $c \in G(L)$ and a unit $u \in \mathbb{Z}_p^*$ such that $n\nu_b \in \mathrm{Hom}_L(\mathbb{G}_m, G)$, $Int(c) \circ (n\nu_b)$ is defined over the fixed field of σ^n on L, and $c(b\sigma)^n c^{-1} = c((n\nu_b)(up))c^{-1}\sigma^n$ as elements in $G(L) \rtimes \langle\sigma\rangle$ (here the inclusion $\mathrm{Hom}_L(\mathbb{G}_m, G) \subset \mathrm{Hom}_L(\mathbb{D}, G)$ is the one induced by the $\mathbb{Q}_p$-homomorphism

$\mathbb{D} \to \mathbb{G}_m$ corresponding to the usual inclusion $\mathbb{Z} \subset \mathbb{Q}$, and $Int(c)$ denotes the inner automorphism $x \to cxc^{-1}$).

These properties imply that the mapping $b \mapsto \nu_b$ satisfies the conditions $\sigma(b) \mapsto \sigma(\nu_b)$ and $gb\sigma(g)^{-1} \mapsto Int(g) \circ \nu_b$, for all $g \in G(L)$. In particular, the $G(L)$-conjugacy class of the homomorphism ν_b (which we denote by $[\nu_b]$) depends only on the σ-conjugacy class of b, and is fixed by σ (the latter property follows from the equality $\nu_b = Int(b) \circ \sigma(\nu_b)$). Thus, the map $b \mapsto \nu_b$ induces a map

$$\bar{\nu} : B(G) \to \mathcal{N}(G)$$

by $[b] \mapsto \bar{\nu}_{[b]} = [\nu_b]$, where $\mathcal{N}(G) = (Int(G(L))\backslash \mathrm{Hom}_L(\mathbb{D}, G))^{\langle\sigma\rangle}$ is the set of the σ-invariant $G(L)$-conjugacy classes of homomorphisms $\mathbb{D}_L \to G_L$. We call $\mathcal{N}(G)$ the *set of Newton points of* G. The map $\bar{\nu}$ (the definition of which is due to Rapoport and Richartz in [26], Section 1) is called the *Newton map of the group* G, and defines a natural transformation of set-valued functors on the category of connected reductive groups. We remark that, in the cases of interest to us, the Dieudonné-Manin classification of F-isocrystals implies that the Newton map is injective.

E.g., for $G = GL(V)$, the slope decomposition of the F-isocrystal $N_b(V)$ associated to an element $b \in G(L)$ is a decomposition $V_L = \oplus_1^r V_i$ into Φ-stable subspaces of dimension $h_i = \dim_L V_i$, for which there exist $\mathcal{O}_L$-lattices $M_i \subset V_i$ with $\Phi^{h_i} M_i = \pi^{d_i} M_i$, where $d_i = \lambda_i h_i \in \mathbb{Z}$, for some uniquely determined rational numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ (each V_i is called the *isotypical component* of slope λ_i in V). Then, the associated slope homomorphism $\nu_b : \mathbb{D}_L \to G_L$ is equal to $\nu_b = \oplus_i \lambda_i \cdot \mathrm{id}_{V_i}$. Classically, the data of the slopes $\lambda_1, \ldots, \lambda_r$ and their multiplicities $h_1, \ldots, h_r$ are used to form a convex polygon with integral vertices, starting at $(0,0)$ and ending at (h, d), where $h = \dim_L V$ and $d = \sum_i d_i$ (called respectively the *height* and *dimension* of the F-isocrystal $N_b(V)$). It follows by the construction that this polygon, which is called *Newton polygon* of the F-isocrystal $N_b(V)$, uniquely determines the isomorphism class of the corresponding F-isocrystal.

2.4. To each $b \in G(L)$ (or equivalently, to each pair (G, b), $b \in G(L)$), we associate a connected reductive group J_b over $\mathbb{Q}_p$,

$$J_b(R) = \{g \in G(L \otimes_{\mathbb{Q}_p} R) | g = b\sigma(g)b^{-1}\}$$

for any $\mathbb{Q}_p$-algebra R. Then, $J_b(\mathbb{Q}_p)$ is the group of automorphism of the F-isocrystal with G-structure N_b associated with b. If $b' = hb\sigma(h)^{-1}$ is a σ-conjugate of b in $G(L)$, then the inner automorphism $Int(h)$ on $G(L)$ induces a $\mathbb{Q}_p$-isomorphism $J_b \to J_{b'}$. Thus the isomorphism class of J_b depends only on the σ-conjugacy class of b, $[b] \in B(G)$.

We say that an element $b \in G(L)$ is *basic* if its slope homomorphism $\nu_b \in \mathrm{Hom}_L(\mathbb{D}_m, G)$ factors through the center $Z = Z(G)$ of G. We remark that if b is basic, then ν_b is necessarily defined over $\mathbb{Q}_p$. We say that a σ-conjugacy class $[b] \in B(L)$ is *basic* if it consists of basic elements, and write $B(G)_b \subset B(G)$ for the set of basic classes in $B(G)$. We recall that an element $b \in G(L)$ is basic if and only if the corresponding connected reductive group J_b is an inner twist of G ([12], Section 5).

Let us assume for simplicity that G is a quasi-split connected reductive group (a similar description of $B(G)$ for G any connected reductive group is also possible). Then, any element $b \in G(L)$ is σ-conjugate to a basic element of some Levi subgroup of G. Equivalently, each $[b] \in B(G)$ is in the image of $B(M)_b$, for some Levi subgroup M of G, under the map $B(M) \to B(G)$ induced by the inclusion $M(L) \subset G(L)$. Furthermore, for any Levi subgroup M of G and $b \in B(M)_b$, the centralizer $\mathrm{Cent}_G(\nu_b)$ of ν_b in G is a Levi subgroup containing M. We say that b is *G-regular* if $\mathrm{Cent}_G(\nu_b)$ is equal to M, and write $B(M)_{br}$ for the set of G-regular basic elements of $B(M)$. Then, each $[b] \in B(G)$ is in the image of $B(M)_{br}$, for some Levi subgroup M of G. Moreover, given any two Levi subgroups M_1, M_2 of G and two basic G-regular elements $b_i \in M_i(L)$, $i = 1, 2$, b_1, b_2 have the same image in $B(G)$ if and only if there exists $g \in G(\mathbb{Q}_p)$ such that $Int(g)(M_1) = M_2$ and $Int(g)(b_1) = b_2$. (We remark that since $g \in G(\mathbb{Q}_p)$ the isomorphism $Int(g) : M_1 \to M_2$ induces a map $B(M_1) \to B(M_2)$ which extends to be the identity on $B(G)$.)

Let M be a Levi subgroup of G and b is a basic G-regular element of $M(L)$. It is a consequence of the definitions that the connected reductive group associated with the pair (M, b) is the same as the group J_b associated with the pair (G, b), $b \in G(L)$. Thus, in particular, J_b is an inner twist of M. Then, it follows from the previous results that, for each $b \in G(L)$, the corresponding connected reductive group J_b is an inner twist of a Levi subgroup of G ([12], Section 6).

In the case when G is quasi-split, we can be more explicit. Indeed, for each $[b] \in B(G)$, $b \in G(L)$, since the $G(L)$-conjugacy class of the slope morphism ν_b is fixed by σ, there exists $g \in G(L)$ such that $Int(g) \circ \nu_b$ is defined over $\mathbb{Q}_p$ ([12], Section 6). Thus, after replacing b by $gb\sigma(g)^{-1}$, we may assume that ν_b is defined over $\mathbb{Q}_p$. We write $M_b = \mathrm{Cent}_G(\nu_b)$. It is immediate that $b \in M_b(L)$, and b is basic and G-regular in $M_b(L)$. Thus, in particular, the group J_b is an inner form of M_b.

E.g., for $G = GL(V)$, the Levi subgroup M_b associated to an element $b \in G(L)$ is the stabilizer in G of the decomposition $V_L = \oplus_i V_i$ underlying the slope decomposition of the F-isocrystal $N_b(V)$. I.e. it is the Levi subgroup corresponding to the partition $(h_1, \ldots, h_r)$ of $h = \dim_{\mathbb{Q}_p} V$, determined by the L-dimensions of the isotypical components of $N_b(V)$, $h_i = \dim_L V_i$. In particular, $b \in G(L)$ is basic if the corresponding F-isocrystal $N_b(V)$ has a unique isotypical component (in which case the F-isocrystal is called *isoclinic*).

2.5. Let G be a reductive connected group, and consider $\mathcal{N}(G)$ the set of Newton points of G. If $A \subset G$ is a maximal torus with Weyl group Ω, then $\mathcal{N}(G) = (X_*(A)_{\mathbb{Q}}/\Omega)^\Gamma$, for Γ the absolute Galois group of $\mathbb{Q}_p$ ([26], Section 1). We define a partial order $\preceq$ on $\mathcal{N}(G)$, via $\bar{\nu} \preceq \bar{\nu}'$ if the orbit $\Omega \cdot \bar{\nu}$ under the Weyl group in $X_*(A)_{\mathbb{R}}$ lies in the convex hull of the orbit $\Omega \cdot \bar{\nu}'$.

Under the Newton map $\bar{\nu} : B(G) \to \mathcal{N}(G)$, the partial order on $\mathcal{N}(G)$ defines a partial order on the set $B(G)$, via $[b] \preceq [b']$ if and only if $\bar{\nu}_{[b]} \preceq \bar{\nu}_{[b']}$. We write $[b] \prec [b']$ if $[b] \preceq [b']$ and $[b] \neq [b']$. The order $\preceq$ on $B(G)$ is called the *Bruhat-Tit order*. Under the Bruhat-Tit order, the set of basic elements $B(G)_b$ is equal to the set of minimal elements. Moreover, for all $[b] \in B(G)$ the set $X_{[b]} = \{[b'] \in B(G) | [b'] \preceq [b]\}$ is finite ([26], Section 2). We remark that, for any representation $\rho : G \to GL(W)$, the corresponding map $B(G) \to B(GL(W))$ (resp. $\mathcal{N}(G) \to \mathcal{N}(GL(W))$) preserves the order $\preceq$, i.e. if $[b] \preceq [b']$ in $B(G)$ then $\rho([b]) \preceq \rho([b'])$ in $B(GL(W)$ (resp. if $\bar{\nu} \preceq \bar{\nu}'$ in $\mathcal{N}(G)$ then $\rho(\bar{\nu}) \preceq \rho(\bar{\nu}')$ in $\mathcal{N}(GL(W)))$ ([26], Section 2).

E.g., for $G = GL(V)$, the Bruhat-Tit order on $B(G)$ can be described as follows. For any $[b], [b'] \in B(G)$, let $\bar{\nu}_{[b]}, \bar{\nu}_{[b]}$ be the corresponding Newton points in $\mathcal{N}(G)$. As described above, we regard $\bar{\nu}_{[b]}$ and $\bar{\nu}_{[b']}$ as two convex polygons with integral vertices starting at $(0,0)$. Then, $[b] \preceq [b']$ (resp. $\bar{\nu}_{[b]} \preceq \bar{\nu}_{[b']}$) if and only if the Newton polygon $\bar{\nu}_{[b]}$ lies above the Newton polygon $\bar{\nu}_{[b']}$ and has the same end-point. (We warn that the opposite partial order on the set of Newton polygons is also sometimes used in the literature.)

2.6. Let (G,μ) be a pair consisting of a reductive connected group over $\mathbb{Q}_p$ and a conjugacy class of cocharacters of G. (Such pairs arise naturally in the context of Shimura varieties, with μ minuscule). We assume for simplicity that G is quasi-split and μ unramified (i.e. μ defined over L).

To μ we associate an element $\bar{\mu} \in \mathcal{N}(G)$ as follows. The class μ determines an element in $Int(G(L))\backslash \mathrm{Hom}_L(\mathbb{D}, G)$, via pullback under the natural morphism $\mathbb{D} \to \mathbb{G}_m$. For each μ, there exists an integer $r \geq 1$ such that this element (which we still denote by μ) is fixed under the action of σ^r (e.g. r equal to the degree over $\mathbb{Q}_p$ of the field of definition of μ). We define

$$\bar{\mu} = \frac{1}{r}\sum_{1}^{r} \sigma^i(\mu).$$

Its definition is independent of the choice of r, and it implies that $\bar{\mu}$ is σ-invariant. Thus, $\bar{\mu} \in \mathcal{N}(G)$ and is called the *Hodge point of* μ.

We say that an element $[b] \in B(G)$ is μ*-admissible* if the corresponding Newton point $\bar{\nu}_{[b]} \in \mathcal{N}(G)$ satisfies the condition $\bar{\nu}_{[b]} \preceq \bar{\mu}$. We write $B(G,\mu) \subset B(G)$ for the set of μ-admissible elements ([13], Section 4). Then, it follows from the above discussion that the set $B(G,\mu)$ is finite, and that it contains a unique basic element. If we regard $B(G,\mu)$ as a partially ordered set under the restriction of the Bruhat-Tit order of $B(G)$, then $B(G,\mu)$ has a unique minimum (which is its unique basic element), and in the cases of interest to us, a unique maximum (called the μ*-ordinary* element), which is the unique element mapping to the Hodge point of μ under the Newton map.

E.g., for $G = GL(V)$, a conjugacy class of cocharacters μ of G is uniquely determine by its weights and their multiplicities, i.e. by integral numbers $w_1 < w_2 \cdots < w_s$ and multiplicities $t_1, \ldots t_s$, for some $s \geq 1$. Similarly to the construction of Newton polygons, these data can be used to form a convex polygon with integral vertices, starting at $(0,0)$. This polygon is called the *Hodge polygon* of μ and the set $B(G,\mu)$ can be identified with the set of all Newton polygons which lie above the Hodge polygon of μ and have the same end-point. If the class of cocharacters μ is minuscule (i.e. if μ has weights in $\{0,1\}$), then it is uniquely determined by the multiplicities $(h-d,d)$ of the weights $(0,1)$ (for $h = \dim_{\mathbb{Q}_p} V \geq d \geq 0$). It follows that the Hodge polygon of μ is the convex polygon of slopes in $\{0,1\}$, starting at $(0,0)$ and ending at (h,d), with breakpoint $(h-d,0)$. Thus, an element $[b] \in B(G)$ is μ-admissible if and only if the corresponding F-isocrystals N_b of height h has dimension d, and slopes in the closed interval $[0,1]$. Then, $B(G,\mu)$ is equal to the set of F-isocrystals of Barsotti-Tate groups of height h and dimension d. Equivalently, it can be identified with the set of isogeny classes of Barsotti-Tate groups over $\bar{k}$, of height h and dimension d. The unique basic element of $B(G,\mu)$ corresponds to the isoclinic F-isocrystal of slope d/h. The μ-ordinary element of $B(G,\mu)$ corresponds to the ordinary F-isocrystal, i.e. the F-isocrystal of slopes 0 and 1.

3. Newton strata and Oort's foliations

In this section, we explain how the geometry in characteristic p of PEL type Shimura varieties can be discussed in terms of the F-isocrystals associated to the p-divisible part of the corresponding abelian varieties.

A first study of the behavior in families of the Newton polygons of F-crystals associated with Barsotti-Tate groups is due to Grothendieck. His results were later improved by Katz and are known as Grothendieck's Specialization Theorem ([11], Theorem 2.3.1). A converse to this theorem and its application to the study of the Newton polygon stratification of moduli spaces of abelian varieties in positive characteristic (which is defined by the loci where the isogeny class of the p-divisible part of the abelian varieties is constant) are due to Oort in [22]. Subsequently, in the work of Rapoport and Richartz in [26], these ideas have been reformulated in and applied to the study of Shimura varieties. More recently, in [23], Oort has obtained new results on the geometry of the Newton polygon strata of moduli spaces of abelian varieties in positive characteristic p by introducing two foliations on the strata. The first one (called the *central foliation*) is defined by the loci where the isomorphism class of the p-divisible part of the abelian varieties is constant. The second one (called the *isogeny foliation*) is defined by the loci where the p-isogeny class of the abelian varieties is constant. (The latter, as we will explain in the next section, is closely related to the construction of moduli spaces of p-divisible group due to Rapoport and Zink in [27]). In [16] and [17], these constructions of Oort have been generalized in and applied to the study of Shimura varieties. In the following, we recall these notions and results following respectively the work in [26] and [17].

3.1. For $K^p \subset G(\mathbb{A}_f^p)$ a sufficiently small open compact subgroup, let S_{K^p} over $\mathcal{O}_{E_\nu}$ be the canonical integral model of the Shimura variety $Sh_K \otimes_E E_\nu$, of level $K = K^p K_{p,0} \subset G(\mathbb{A}_f)$, and $\bar{S}_{K^p}$ over $k(\nu)$ its reduction modulo p. We write A for the universal abelian variety over $\bar{S}_{K^p}$, and $H = A[p^\infty]$ for its p-divisible group. Then, H is a Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure, namely in this case a Barsotti-Tate group endowed with a quasi-polarization and an action of $\mathcal{O}_B \otimes \mathbb{Z}_p$, the p-adic completion of $\mathcal{O}_B$ in $B_{\mathbb{Q}_p}$.

Let h denote the conjugacy class of cocharacters $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$ in the Shimura datum. After choosing a local embedding at p of the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, $\nu : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, h determines a conjugacy class of cocharacters of $G_{\mathbb{Q}_p}$, $\mu_{\bar{\mathbb{Q}}_p} = \nu \circ \mu_h : \mathbb{G}_m \to G_{\bar{\mathbb{Q}}_p}$. The conditions satisfied by the Shimura datum imply that the corresponding conjugacy class $\mu_{\bar{\mathbb{Q}}_p}$ is minuscule. We recall that a Barsotti-Tate group H with $G_{\mathbb{Q}_p}$-structure is said to be *compatible* (with $\mu_{\bar{\mathbb{Q}}_p}$) if the induced $G_{\mathbb{Q}_p}$-structure on its Lie algebra, $\mathrm{Lie}(H)$, satisfies the *determinant condition*, i.e. for all $g \in \mathcal{O}_B \otimes \mathbb{Z}_p$: $\det(g, \mathrm{Lie}(H)) = \det(g, V_1)$ (here, V_1 denotes the subspace of V_L of weight 1 with respect to $\mu_{\bar{\mathbb{Q}}_p}$). It follows that the universal Barsotti-Tate group H over $\bar{S}_{K^p}$ is a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure.

3.2. To each geometric closed point x of $\bar{S}_{K^p}$, we associate H_x the fiber of H at x, $\mathbb{D}H_x$ the (covariant) Dieudonné module of H_x, and $N_x = \mathbb{D}H_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the F-isocrystal of H_x. We write $k_x \supset \overline{k(\nu)}$ for the field of definition of x. For any point x in $\bar{S}_{K^p}$, the corresponding F-isocrystal N_x determines a unique element $b_x \in B(G_{\mathbb{Q}_p})$ (here, we simply write b in place of $[b]$ for an element in $B(G)$). It is important to remark that, since the $G_{\mathbb{Q}_p}$-structure on H_x is compatible with $\mu_{\bar{\mathbb{Q}}_p}$, the corresponding element $b_x \in B(G_{\mathbb{Q}_p})$ is $\mu_{\bar{\mathbb{Q}}_p}$-admissible, i.e. $b_x \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$.

The generalization of Katz-Grothendieck's Specialization Theorem to this context (which is due to Rapoport-Richartz in [26], Theorem 3.6) implies that for each $b \in B(G_{\mathbb{Q}_p})$ the subset

$$\bar{S}_{K^p}(\preceq b) = \{x \in \bar{S}_{K^p} | b_x \preceq b\}$$

is a Zariski-closed subspace of $\bar{S}_{K^p}$. We define $\bar{S}_{K^p}(\preceq b)$ as a closed subscheme of $\bar{S}_{K^p}$, endowed with the induced reduced structure. It follows from the definition that if $b \preceq b'$ then $\bar{S}_{K^p}(\preceq b)$ is contained in $\bar{S}_{K^p}(\preceq b')$. Thus, they form a stratification by closed subschemes of $\bar{S}_{K^p}$, indexed by the elements in $B(G_{\mathbb{Q}_p})$. We call this the *Newton stratification of* $\bar{S}_{K^p}$ and, for each $b \in B(G)$, we call $\bar{S}_{K^p}(\preceq b)$ the *closed Newton stratum attached to* b. For each $b \in B(G)$, we define the *Newton stratum attached to* b as

$$\bar{S}_{K^p}(b) = \bar{S}_{K^p}(\preceq b) - \bigcup_{b' \prec b} \bar{S}_{K^p}(\preceq b').$$

Then, $\bar{S}_{K^p}(b)$ is a locally closed reduced subscheme of $\bar{S}_{K^p}$, which is open inside the corresponding closed Newton stratum $\bar{S}_{K^p}(\preceq b)$.

We remark that since H is a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure then the strata $\bar{S}_{K^p}(b)$ are empty for all $b \in B(G_{\mathbb{Q}_p}) - B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. Thus, the Newton stratification of $\bar{S}_{K^p}$ is actually indexed by the elements $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. The converse of this statement, i.e. that the strata $\bar{S}_{K^p}(b)$ are non-empty for all $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, is also expected to hold, but it is currently known only in some special cases. E.g., in the classical case of $G = GL_n$ or $G = GSp_{2n}$, this was conjectured by Manin in [15] and proved first by Honda in [9] and later by Oort in [21]. Furthermore, the converse of the generalization of Katz-Grothendieck's Specialization Theorem to this context would imply that, for each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, the closed Newton stratum attached to b is equal to the Zariski-closure of the corresponding Newton stratum, or equivalently, that for each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ the stratum $\bar{S}_{K^p}(b)$ is open and dense in $\bar{S}_{K^p}(\preceq b)$ (and thus, in particular, non-empty). In the case of $G = GL_n$ or $G = GSp_{2n}$, this was conjectured by Grothendieck in [6] and proved by Oort in [21].

It follows from the property of the Bruhat-Tit order on $B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ that there is a unique Newton stratum which is closed in $\bar{S}_{K^p}$, the stratum corresponding to the basic element in $B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, and that there is a unique Newton stratum which is open in $\bar{S}_{K^p}$, the stratum corresponding to the $\mu_{\bar{\mathbb{Q}}_p}$-ordinary element in $B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. We call them respectively the *basic Newton stratum* and the *$\mu_{\bar{\mathbb{Q}}_p}$-ordinary Newton stratum*. We recall that in [28] Wedhorn proved that the $\mu_{\bar{\mathbb{Q}}_p}$-ordinary Newton stratum is dense in $\bar{S}_{K^p}$ (and thus, in particular, non-empty).

If we consider the *classical* Newton stratification of $\bar{S}_{K^p}$ (a.k.a. the Newton polygon stratification), which is defined by looking at the isogeny class of the Barsotti-Tate groups underlying the H_x's (i.e., with our notations, $\bar{S}_{K^p}(\preceq \rho(b)) = \{x \in \bar{S}_{K^p} | \rho(b_x) \preceq \rho(b)\}$, for $\rho : G \to GL(V)$ the standard representation), then it follows from the functorial property of the Bruhat-Tit order on $B(G)$ that the Newton polygon stratification is coarser than or equal to the Newton stratification . While in general the corresponding inclusions are strict, in the cases of interest to us, the two stratifications are actually the same (see [26], Theorem 3.8).

It is an immediate consequence of the definition that, as the level K^p varies, the action of $G(\mathbb{A}_f^p)$ on the Shimura varieties respects the Newton stratifications. Furthermore, for each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ and any $K_1^p \subset K^p$, the Newton stratum of $\bar{S}_{K_1^p}$ attached to b is equal to the pullback of the corresponding Newton stratum of $\bar{S}_{K^p}$, under the natural projection between Shimura varieties $\bar{S}_{K_1^p} \to \bar{S}_{K^p}$.

3.3. Let l be a prime number, $l \neq p$, and $\mathcal{L}_\rho$ the étale l-adic local system on the Shimura varieties, associated with a representation $\rho \in \mathrm{Rep}_{\mathbb{C}}(G)$. For all (sufficiently small) open compact subgroups $K^p \subset G(\mathbb{A}_f^p)$ and $K_p \subset K_{p,0}$, we write $f_{K_p} = f_{K^pK_p} : Sh_{K^pK_p} \to Sh_{K^pK_{p,0}}$ for the natural projection between the Shimura varieties, and $R^q\psi(f_{K_p*}(\mathcal{L}_\rho))$ for the nearby cycle sheaves over $\bar{S}_{K^p}$ of $f_{K_p*}(\mathcal{L}_\rho)$, for all $q \geq 0$. With abuse of notation, we also write $R^q\psi(f_{K_p*}(\mathcal{L}_\rho))$ for their restriction to the Newton strata $\bar{S}_{K^p}(b)$, for all $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. For all $t, q \geq 0$, we define

$$H_c^t(\bar{S}(b), R^q\psi(\mathcal{L}_\rho)) = \varinjlim_{K^p, K_p} H_c^t(\bar{S}_{K^p}(b), R^q\psi(f_{K_p*}(\mathcal{L}_\rho)).$$

These cohomology spaces naturally inherit an action of $G(\mathbb{A}_f) \times W_{E_\nu}$, and as l-adic representations of $G(\mathbb{A}_f) \times W_{E_\nu}$, they are admissible. We define

$$H_c(\bar{S}(b), R\psi(\mathcal{L}_\rho)) = \sum_{t,q} (-1)^{t+q} H_c^t(\bar{S}(b), R^q\psi(\mathcal{L}_\rho)) \in \mathrm{Groth}(G(\mathbb{A}_f) \times W_{E_\nu}).$$

We remark that, if the Shimura varieties considered are proper, then it follows from the theory of nearby cycles and the above properties of the Newton stratification that the following equality holds in $\mathrm{Groth}(G(\mathbb{A}_f) \times W_{E_\nu})$:

$$H_{\text{ét}}(Sh, \mathcal{L}_\rho) = \sum_{b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})} H_c(\bar{S}(b), R\psi(\mathcal{L}_\rho)).$$

3.4. Let $\mathbb{X}$ over $\overline{k(\nu)}$ be a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure, and write $b = b_{\mathbb{X}}$ for the corresponding element in $B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. We say that a ($\mu_{\bar{\mathbb{Q}}_p}$-compatible) Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure Y, defined over a field $k \supset \overline{k(\nu)}$, is *geometrically isomorphic* to $\mathbb{X}$ (and write $Y \cong_g \mathbb{X}$) if there exists a field extension k' of k over which the two become isomorphic. We define the *central leaf* associated to $\mathbb{X}$ to be

$$C_{\mathbb{X}} = \{x \in \bar{S}_{K^p} | H_x \cong_g \mathbb{X}\}.$$

It follows from the definition that $C_{\mathbb{X}} \subset \bar{S}_{K^p}(b)$. Furthermore, it is a closed subspace of $\bar{S}_{K^p}(b)$, and if endowed with the induced reduced subscheme structure then it is smooth ([22], Sections 2 and Section 3; [17], Proposition 1). We remark that the latter property is tautological. (Indeed, given such scheme structure, Serre-Tate theory implies that the complete local ring of $C_{\mathbb{X}}$ at any closed geometric point depends only on the geometric isomorphism class of $\mathbb{X}$, and in particular it is independent of the point. Since the smooth locus of a non-empty reduced scheme is non-empty, we conclude that $C_{\mathbb{X}}$ is smooth.)

It follows from the definition that any two central leaves $C_{\mathbb{X}}, C_{\mathbb{X}'}$ of $\bar{S}_{K^p}(b)$ either are disjoint or coincide, and that any closed geometric point x of $\bar{S}_{K^p}(b)$ is contained in exactly one central leaf (which we denote by C_x). Moreover, if $\mathbb{X}, \mathbb{X}'$ are two $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate groups with $G_{\mathbb{Q}_p}$-structure, with $b = b_{\mathbb{X}} = b_{\mathbb{X}'}$ (thus $\mathbb{X}$ and $\mathbb{X}'$ are isogenous), then there exists a scheme T and two finite surjective morphisms $C_{\mathbb{X}} \twoheadleftarrow T \twoheadrightarrow C_{\mathbb{X}'}$. In particular, $\dim C_{\mathbb{X}} = \dim C_{\mathbb{X}'}$ ([22], Section 2 and Section 3;

[17], Section 5). Because of these properties, the central leaves are said to form a foliation (called the *central foliation*) of $\bar{S}_{K^p}(b)$.

We remark that, for any $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, if the Newton stratum $\bar{S}_{K^p}(b)$ is not empty then there exists a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure $\mathbb{X}_0$ over $\overline{k(\nu)}$, with $b_{\mathbb{X}_0} = b$, such that the corresponding central leaf $C_{\mathbb{X}_0}$ is not empty, and in such case, all central leaves $C_{\mathbb{X}}$ are not empty for all $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate groups with $G_{\mathbb{Q}_p}$-structure $\mathbb{X}$ over $\overline{k(\nu)}$, with $b_{\mathbb{X}} = b$.

We also remark that if we consider the *original* central foliations of the Newton strata, which is defined by fixing the isomorphism class of the Barsotti-Tate groups underlying the H_x's, then it follows from the definition that it is coarser than or equal to the corresponding above central foliations. In particular, the central leaves (as defined above) are unions of connected components of the corresponding original ones ([17], Proposition 1).

Finally, it is an immediate consequence of the definition that, as the level K^p varies, the action of $G(\mathbb{A}_f^p)$ on the Shimura varieties respects the central foliations. Furthermore, for each a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure $\mathbb{X}$ and any $K_1^p \subset K^p$, the associated central leaf of $\bar{S}_{K_1^p}(b)$ ($b = b_{\mathbb{X}}$) is equal to the pullback of the corresponding central leaf of $\bar{S}_{K^p}$, under the natural projection $\bar{S}_{K_1^p}(b) \to \bar{S}_{K^p}(b)$.

3.5. Let x be a closed geometric point of $\bar{S}_{K^p}(b)$ defined over $\overline{k(\nu)}$. We write A_x for the associated abelian variety with G-structure, the fiber of A at x. We say that an abelian variety with G-structure B, defined over a field $k \supset \overline{k(\nu)}$, is *geometrically p-isogenous* to A_x (and write $B \sim_g A_x$) if there exist a field extension k' of k and a p-power isogeny $A_x \otimes_{\overline{k(\nu)}} k' \to B \otimes_k k'$ of abelian varieties with the G-structures. We define the *isogeny leaf* of x to be

$$I_x = \{y \in \bar{S}_{K^p} | A_y \sim_g A_x\}.$$

It follows from the definition that $I_x \subset \bar{S}_{K^p}(b)$. Furthermore, it is a closed subspace of $\bar{S}_{K^p}(b)$. We regard I_x as a subscheme of $\bar{S}_{K^p}(b)$ via the induced reduced scheme structure ([22], Section 4).

It follows from the definition that any two isogeny leaves $I_x, I_{x'}$ of $\bar{S}_{K^p}(b)$ either are disjoint or coincide, and that any closed geometric point x of $\bar{S}_{K^p}(b)$ is contained in exactly one isogeny leaf. Moreover, if x, x' are two closed geometric point of $\bar{S}_{K^p}(b)$, then there exists a scheme J and two finite surjective morphisms $I_x \twoheadleftarrow J \twoheadrightarrow I_{x'}$. In particular, $\dim I_x = \dim I_{x'}$ ([22], Section 4; [17], Section 5). Thus, the isogeny leaves form a foliation (called the *isogeny foliation*) of $\bar{S}_{K^p}(b)$.

Finally, it is an immediate consequence of the definition that, as the level K^p varies, the isogeny foliation is respected by the action of $G(\mathbb{A}_f^p)$ on the Shimura varieties, i.e. for all $g \in G(\mathbb{A}_f^p) : g(I_x) = I_{g(x)}$, for any geometric closed point x of $\bar{S}_{K^p}$.

4. Igusa varieties and Rapoport-Zink spaces

In this section, we introduce the notions of Igusa variety and Rapoport-Zink space, and explain their relation to the central and isogeny foliations of the Newton strata of Shimura varieties of PEL type.

A crucial ingredient in the study of the bad reduction of modular curves is due to Igusa. In [10], Igusa introduced some new moduli spaces for elliptic curves (now called *Igusa curves*), which exist exclusively in positive characteristic, and proved that up to a finite inseparable morphism they can be identified with the smooth components of the reduction of the corresponding modular curves. Many years later, in [8], Harris and Taylor generalized Igusa's ideas in the context of some simple Shimura varieties. They defined some new varieties, which they called *Igusa varieties*, as smooth covering spaces of the Newton strata of Shimura varieties with good reduction, and applied their new construction to the study of the cases of bad reduction. In particular, similarly to the case of modular curves, the smooth components of the bad reduction of a simple Shimura variety can be identified, up to a finite inseparable morphism, to a *compactification* of the Igusa variety defined over the $\mu_{\bar{\mathbb{Q}}_p}$-ordinary Newton stratum of the corresponding Shimura variety with good reduction ([18], Proposition 16). In [16] and [17], Harris-Taylor's new notion of Igusa variety is extended to the context of Shimura varieties of PEL type and applied to the study of their bad reductions. In this generality, the Igusa varieties arise as smooth covering spaces of the central leaves of the Newton strata of the Shimura varieties with good reduction. In particular, even in the case corresponding to the $\mu_{\bar{\mathbb{Q}}_p}$-ordinary Newton stratum, their dimension is in general strictly smaller than the dimension of the Shimura varieties. (We will see in the last section that, in the case of the simple Shimura varieties considered in [8], each central foliation consists of a unique leaf, equal to the whole Newton stratum.) This difference in dimension is explained by the existence of the isogeny foliation (which in the case of simple Shimura varieties is zero dimensional). As it was already mentioned, the isogeny foliation is closely related to certain moduli spaces of p-divisible group with additional structure constructed by Rapoport and Zink in [27] as local analogues of Shimura varieties. More precisely, Rapoport-Zink's p-adic uniformization theorem completely describes the geometry of the isogeny leaves in terms of these new spaces.

4.1. Let $\mathbb{X}$ be a Barsotti-Tate group and $N = N_{\mathbb{X}}$ the associated F-isocrystal. We write $N = \oplus_i N^i$ for the slope decomposition of N, each N^i isoclinic of slope λ_i. We also order the slope of N in decreasing order: $1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$. We call a *slope filtration* of $\mathbb{X}$ a filtration $0 = \mathbb{X}_0 \subset \mathbb{X}_1 \subset \mathbb{X}_2 \cdots \subset \mathbb{X}_r = \mathbb{X}$ of $\mathbb{X}$ by Barsotti-Tate subgroups satisfying the condition that for each $i \in \{0, \dots, r\}$ the corresponding subquotient $\mathbb{X}^i = \mathbb{X}_i/\mathbb{X}_{i-1}$ is an isoclinic Barsotti-Tate group of slope λ_i (thus, for each i, N^i can be identified with the F-isocrystal associated to $\mathbb{X}^i$). It follows from the definition that if a slope filtration exists then it is unique. The notion of slope filtration for Barsotti-Tate groups is due to Grothendieck who in [6] proved that any Barsotti-Tate group defined over a field of positive characteristic admits a slope filtration and that this filtration is split in the case when the field is perfect. Grothendieck's result was later extended by Katz ([11]) to families of Barsotti-Tate groups over a smooth curve in characteristic p, and more recently by Zink ([29]) over regular schemes and by Oort and Zink ([24]) over normal schemes. In [24], Oort and Zink also discuss counterexamples to more general statements.

Let $\mathbb{X}$ over $\overline{k(\nu)}$ be a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure, $N = N_{\mathbb{X}}$ the associated F-isocrystal with $G_{\mathbb{Q}_p}$-structure, $b = b_{\mathbb{X}}$ the corresponding element in $B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, and $1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$ the slopes of $\mathbb{X}$. We write $\mathbb{X} = \oplus_i \mathbb{X}^i$ for the slope decomposition of $\mathbb{X}$, each $\mathbb{X}^i$ an isoclinic Barsotti-Tate group of slope λ_i defined over $\overline{k(\nu)}$. Then, it follows from the definition that $\mathbb{X}$ has a natural M_b-structure (for M_b the centralizer of the slope morphism ν_b of

$b = b_{\mathbb{X}}$). Indeed, the action of the maximal order $\mathcal{O}_B \otimes \mathbb{Z}_p$ of $B_{\mathbb{Q}_p}$ on $\mathbb{X}$ respects the slope decomposition of $\mathbb{X}$, i.e. each isoclinic component $\mathbb{X}^i$ inherits a structure of $\mathcal{O}_B \otimes \mathbb{Z}_p$-module. Moreover, the quasi-polarization of $\mathbb{X}$, $\ell : \mathbb{X} \to \mathbb{X}^\vee$, naturally decomposes as $\ell = \oplus_i \ell^i$, the direct sum of $\mathcal{O}_B \otimes \mathbb{Z}_p$-equivariant isomorphisms $\ell^i : \mathbb{X}^i \to (\mathbb{X}^{r+1-i})^\vee$ satisfying the conditions $\ell^i = c \cdot (\ell^{r+1-i})^\vee$ for all $i \in \{0, \dots, r\}$ and a constant $c \in \mathbb{Z}_{(p)}^\times$ independent of i.

In the following we will assume that $\mathbb{X}$ is *completely slope divisible*. The notion of complete slope divisibility is due to Zink in [29]; we recall this definition. Let $s > 0$ and $t_1, \dots t_r$ be integers such that $s \geq t_1 > t_2 > \cdots > t_r \geq 0$. A Barsotti-Tate group $\mathbb{X}$ is said to be completely slope divisible with respect to these integers if $\mathbb{X}$ has a filtration by Barsotti-Tate subgroups $0 = \mathbb{X}_0 \subset \mathbb{X}_1 \subset \mathbb{X}_2 \cdots \subset \mathbb{X}_r = \mathbb{X}$ such that for each $i \in \{1, \dots r\}$ the quasi isogenies $p^{-t_i} F^s : \mathbb{X}_i \to \mathbb{X}_i^{(p^s)}$ are isogenies, and the induced morphisms $p^{-t_i} F^s : \mathbb{X}_i / \mathbb{X}_{i-1} \to (\mathbb{X}_i / \mathbb{X}_{i-1})^{(p^s)}$ are isomorphisms (here, $F : \mathbb{X} \to \mathbb{X}^{(p)}$ denotes the relative Frobenius of $\mathbb{X}$). Note that the last condition implies that for each i the Barsotti-Tate subquotient $\mathbb{X}^i = \mathbb{X}_i / \mathbb{X}_{i-1}$ is isoclinic of slope $\lambda_i = t_i / s$, and that the filtration is a slope filtration. In [24], Oort and Zink proved that if $\mathbb{X}$ is a Barsotti-Tate group defined over an algebraically closed field, then $\mathbb{X}$ is completely slope divisible if and only if it is isomorphic to the direct sum of isoclinic Barsotti-Tate groups defined over finite fields. On the other hand, we recall that an element $b \in B(G_{\mathbb{Q}_p})$ is said to be *decent* if it contains an element of $G(L)$ defined over an unramified finite field extension of $\mathbb{Q}_p$. In [12] Kottwitz proved that if $G_{\mathbb{Q}_p}$ is connected then every element in $B(G_{\mathbb{Q}_p})$ is decent. The above result of Oort and Zink implies that if $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ then b is decent if and only if there exists a completely slope divisible $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure defined over $\overline{k(\nu)}$ associated with b.

4.2. Let $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, and $\mathbb{X}$ a completely slope divisible $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure associated with b (the above observations imply that under our assumptions such a Barsotti-Tate group always exists). For $K^p \subset G(\mathbb{A}_f^p)$ a sufficiently small open compact subgroup, let $C = C_{\mathbb{X}} \subset \bar{S}_{K^p}(b)$ denote the central leaf and Newton stratum associated respectively with $\mathbb{X}$ and b inside the reduction of the Shimura variety of level K^p. It is an easy application of Zink's work to our context that the restriction of the universal family of Barsotti-Tate groups H over C admits a slope filtration ([16], Section 3.2.3). Moreover, if we write $H^1, \dots, H^r$ for the Barsotti-Tate subquotients associated with the slope filtration of H over C, then it follows from the definition that the Barsotti-Tate group $H^{\rm sp} = \oplus_i H^i$ has a natural M_b-structure and that for each point x of C the Barsotti-Tate group with M_b-structure $H_x^{\rm sp}$ is geometrically isomorphic to $\mathbb{X}$. In particular, for each $i \in \{0, \dots, r\}$, the Barsotti-Tate group H_x^i is geometrically isomorphic to $\mathbb{X}^i$.

For any integer $m \geq 1$, we define $\mathrm{Ig}_m = \mathrm{Ig}_{\mathbb{X},m}$ the *Igusa variety of level* m *attached to* $\mathbb{X}$ as the universal space over $C = C_{\mathbb{X}}$ for the existence of an isomorphism of truncated Barsotti-Tate groups with M_b-structure $j_m : \mathbb{X}[p^m]_C \cong H^{\rm sp}[p^m]$. (By a homomorphism of truncated Barsotti-Tate groups with M_b-structure we mean a homomorphism of the corresponding finite flat group schemes which commutes with the induced M_b-structures and which extends étale locally to any depth $m' \geq m$, for m the depth of the truncation.) It follows from the definition that for each $m \geq 1$ the corresponding Igusa variety $\mathrm{Ig}_m \to C$ is naturally endowed with an action of the group of isomorphisms of the m-th truncation of $\mathbb{X}$. We denote this group by $\Gamma_m = \Gamma_{\mathbb{X},m}$ and regard it as the quotient of $\Gamma = \Gamma_{\mathbb{X}}$, the group of automorphisms of $\mathbb{X}$, by the subgroup of those automorphisms which induce the identity on $\mathbb{X}[p^m]$. For each $m \geq 1$, the Igusa variety $\mathrm{Ig}_m \to C$ is finite étale and Galois over C, with Galois group Γ_m. In particular, it is a smooth scheme over $\overline{k(\nu)}$ ([17], Proposition 4). In the following, we sometimes write $\mathrm{Ig}_0 = C$ and $\mathrm{Ig}_{m,K^p} = \mathrm{Ig}_{\mathbb{X},m,K^p}$ for the Igusa variety of level m, when we wish to include the level of the corresponding Shimura variety in the notation.

It follows from the moduli interpretation of the Igusa varieties that, as the level m varies, they form a projective system. Furthermore, if we regard the Igusa varieties as endowed with an action of Γ (via the natural projections $\Gamma \twoheadrightarrow \Gamma_m$, for all $m \geq 1$), then the projective system naturally inherits an action of Γ. We regard Γ as a subgroup of the group of the quasi-self-isogenies of $\mathbb{X}$, which we identify with $J_b(\mathbb{Q}_p)$. Let $\rho \in J_b(\mathbb{Q}_p)$. If ρ^{-1} is an isogeny of $\mathbb{X}$, we define $f_i = f_i(\rho)$ and $e_i = e_i(\rho)$ to be respectively the maximal and minimal positive integers such that $\oplus_i \mathbb{X}^i[p^{f_i}] \subset \ker(\rho^{-1}) \subset \oplus_i \mathbb{X}^i[p^{e_i}]$ (thus $f_i \leq e_i$, for all i). We define $S = S_{\mathbb{X}}$ as the set of elements in $J_b(\mathbb{Q}_p)$ whose inverse is an isogeny of $\mathbb{X}$ satisfying the conditions $f_{i-1} \geq e_i$, for all $i \geq 2$. It follows that S is a submonoid of $J_b(\mathbb{Q}_p)$ which contains Γ. Moreover, the action of Γ on the tower of Igusa varieties can be extended to the monoid S, each $\rho \in S$ defining a compatible system of morphisms $\rho : \mathrm{Ig}_m \to \mathrm{Ig}_{m-e}$, for $e = e_1(\rho)$ and any $m \geq e$ ([17], Lemma 5).

As we also allow the level K^p to vary, then the corresponding Igusa varieties Ig_{m,K^p} form a projective system indexed by both the levels m and K^p. Moreover, for each $g \in G(\mathbb{A}_f^p)$ and $K_1^p \subset g^{-1}K^p g$, the morphism $g : C_{K_1^p} \to C_{K^p}$ canonically lifts to morphisms $g : \mathrm{Ig}_{m,K_1^p} \to \mathrm{Ig}_{m,K^p}$ for all $m \geq 1$, defining an action of $G(\mathbb{A}_f^p)$ on the projective system of Igusa varieties. It is easy to see that the action of $G(\mathbb{A}_f^p)$ commutes with the previously defined action of $S \subset J_b(\mathbb{Q}_p)$.

4.3. Let l be prime number, $l \neq p$, and $\mathcal{L}_\rho$ the étale l-adic local system over the tower of Shimura varieties associated with a representation $\rho \in \mathrm{Rep}_{\mathbb{C}}(G)$. With abuse of notations, we also denote by $\mathcal{L} = \mathcal{L}_\rho$ the pullback of $\mathcal{L}_\rho$ to the tower of Igusa varieties, under the natural morphisms $\mathrm{Ig}_{\mathbb{X},m,K^p} \to C_{\mathbb{X},K^p} \subset \bar{S}_{K^p}(b) \subset \bar{S}_{K^p}$. For all $t \geq 0$, we define

$$H_c^t(\mathrm{Ig}_{\mathbb{X}}, \mathcal{L}) = \varinjlim_{m,K^p} H_c^t(\mathrm{Ig}_{\mathbb{X},m,K^p}, \mathcal{L}).$$

These cohomology groups naturally inherit an action of $S_{\mathbb{X}} \times G(\mathbb{A}_f^p)$ and it is not hard to see that this action uniquely extends to an action of $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$. Moreover, as l-adic representations of $J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$, they are admissible ([17], Proposition 7). We define

$$H_c(\mathrm{Ig}_{\mathbb{X}}, \mathcal{L}) = \sum_t (-1)^t H_c^t(\mathrm{Ig}_{\mathbb{X}}, \mathcal{L}) \in \mathrm{Groth}(J_b(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)).$$

4.4. In [27], Rapoport and Zink associate to each class $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ a tower of moduli spaces of $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate groups with $G_{\mathbb{Q}_p}$-structure in the category of rigid analytic spaces. For each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, the associated tower of Rapoport-Zink spaces is a projective system of rigid analytic spaces defined over L, indexed by the set of open compact subgroups K_p of $G(\mathbb{Q}_p)$. This projective system is naturally endowed with an action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ and with a non-effective decent datum to E_ν. Moreover, similarly to the case of the corresponding Shimura varieties, in the case of $K_p = K_{p,0}$ a maximal open compact of $G(\mathbb{Q}_p)$, the corresponding Rapoport-Zink space admits a smooth integral model in the category of formal schemes over $\mathcal{O}_L$. We recall Rapoport's and Zink's construction in more detail.

Let $\mathbb{X}$ over $\overline{k(\nu)}$ be a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure associated with b. For $K_p = K_{p,0}$, we write $\mathcal{M}_b = \mathcal{M}_{b,\mathbb{X}}$ for the corresponding Rapoport-Zink formal scheme over $\mathcal{O}_L$. The set-valued functor on the category of formal $\mathcal{O}_L$-schemes associated to $\mathcal{M}_b$ is defined as follows. For any $\mathcal{O}_L$-scheme S where p is locally nilpotent, $\mathcal{M}_b(S)$ is the set of isomorphism classes of pairs (H, β), where H is a $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure defined over S, and β is a quasi-isogeny $\beta : \mathbb{X} \times_{\overline{k(\nu)}} \bar{S} \to H \times_S \bar{S}$, defined over $\bar{S} = Z(p) \subset S$. After identifying $J_b(\mathbb{Q}_p)$ with the group of quasi-self-isogenies of $\mathbb{X}$, we define an action of $J_b(\mathbb{Q}_p)$ on $\mathcal{M}_b$ by right translations, i.e. for all $\rho \in J_b(\mathbb{Q}_p) : (H, \beta) \mapsto (H, \beta \circ \rho)$. Similarly, we define a non-effective descent datum on $\mathcal{M}_b$, via the σ-semi-linear isomorphism of $\mathcal{M}_b$ arising from the Frobenius of $\mathbb{X}$, $(H, \beta) \mapsto (H, F^{-1} \circ \beta)$. We remark that the descent datum and the action of $J_b(\mathbb{Q}_p)$ commute with each other.

Let H denote the universal family of Barsotti-Tate groups over $\mathcal{M}_b$, we define $\mathcal{M}_{b,K_{p,0}}$ to be the rigid analytic fiber over L of $\mathcal{M}_b$ and T_pH the Tate module of H over $\mathcal{M}_{b,K_{p,0}}$, regarded as a locally constant étale $\mathbb{Z}_p$-sheaf over $\mathcal{M}_{b,K_{p,0}}$. For each $K_p \subset K_{p,0}$, the *Rapoport-Zink space of level* K_p *attached to* b, $\mathcal{M}_{b,K_p}$, is defined as the universal space over $\mathcal{M}_{b,K_{p,0}}$ for the existence of a K_p-level structure on T_pH (i.e. the covering space parameterizing the classes modulo K_p of trivialization of T_pH). For each K_p, the space $\mathcal{M}_{b,K_p}$ is finite étale and Galois over $\mathcal{M}_{b,K_{p,0}}$. In particular, it is a smooth rigid analytic space of dimension $D_b = \dim \mathcal{M}_{b,K_{p,0}}$ ([27], Section 5.34). It also follows from the definition that the action of $J_b(\mathbb{Q}_p)$ and the descent datum on $\mathcal{M}_{b,K_{p,0}}$ canonically lift to $\mathcal{M}_{b,K_p}$. Moreover, as the level K_p varies, these covers naturally form a projective system endowed with the obvious action of $G(\mathbb{Q}_p)$.

We remark that although the definition depends on the choice of a Barsotti-Tate group $\mathbb{X}$ associated with b, the isomorphism class of the corresponding Rapoport-Zink spaces depends only on the class $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. We also point out that the Rapoport-Zink spaces are only locally of finite type. In fact, they naturally arise as a direct limit of subspaces of finite type. More precisely, we may describe $\mathcal{M}_b$ as $\mathcal{M}_b = \varinjlim_{n,d} \mathcal{M}_{b,\mathbb{X}}^{n,d}$, where for each pair of positive integers n, d, we denote by $\mathcal{M}_{b,\mathbb{X}}^{n,d}$ the formal subscheme of $\mathcal{M}_b$ where the universal quasi-isogeny $\beta : \mathbb{X} \times_{\overline{k(\nu)}} \mathcal{M}_b^{\mathrm{red}} \to H \times_{\mathcal{M}_b} \mathcal{M}_b^{\mathrm{red}}$, defined over the reduced fiber of $\mathcal{M}_b$, satisfies the condition that $p^n\beta$ is a well-defined isogeny of degree less than or equal to p^d (the latter is a closed condition in $\mathcal{M}_b^{\mathrm{red}}$). We remark that the truncated Rapoport-Zink spaces $\mathcal{M}_{b,\mathbb{X}}^{n,d}$ are not stable under either the action of $J_b(\mathbb{Q}_p)$ or the descent datum. Moreover, their definition depends on the choice of a Barsotti-Tate group $\mathbb{X}$ associated with b.

4.5. Let l be a prime number, $l \neq p$. For all $t \geq 0$, we define

$$H_c^t(\mathcal{M}_b, \bar{\mathbb{Q}}_l) = \varinjlim_{K_p} H_c^t(\mathcal{M}_{b,K_p} \times_L \hat{\bar{E}}_\nu, \bar{\mathbb{Q}}_l).$$

It follows from the definition that the above cohomology groups are naturally endowed with commuting actions of $G(\mathbb{Q}_p)$, $J_b(\mathbb{Q}_p)$ and of the Weil group W_{E_ν}. As representations of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_{E\nu}$, they are smooth but it is not known whether they are also admissible. On the other hand, for any admissible representation ρ of $J_b(\mathbb{Q}_p)$, the groups $\mathrm{Ext}^i_{J_b(\mathbb{Q})}(H^t_c(\mathcal{M}_b, \bar{\mathbb{Q}}_l), \rho)$, regarded as smooth representations of $G(\mathbb{Q}_p) \times W_{E_\nu}$, are admissible and vanish for almost all $i, t \geq 0$ ([5], Corollary 4.4.14; [16], Section 8.2). (Here, $\mathrm{Ext}^i_{J_b(\mathbb{Q}_p)}(\cdot, \cdot)$ are the derived functors on the category of smooth $J(\mathbb{Q}_p)$-representations.) It follows that to each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$ we can associate a functor $\mathcal{E}_b : \mathrm{Groth}(J_b(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_\nu})$ defined as

$$\mathcal{E}_b(\rho) = \sum_{i,t \geq 0} (-1)^{i+t} \mathrm{Ext}^i_{J_b(\mathbb{Q})}(H^t_c(\mathcal{M}_b, \bar{\mathbb{Q}}_l), \rho)(-D_b).$$

In the following we will also denote by $\mathcal{E}_b$ the functor from $\mathrm{Groth}(J_b(\mathbb{Q}_p) \times G(\mathbb{A}^p_f))$ to $\mathrm{Groth}(G(\mathbb{A}_f) \times W_{E_\nu})$ obtained by extending the above functor by the identity on $\mathrm{Groth}(G(\mathbb{A}^p_f))$.

Alternatively, using Berkovich's theory of nearby cycle sheaves for formal schemes ([1],[2]), the l-adic cohomology of the Rapoport-Zink spaces can be computed in terms of the cohomology of the reduced fiber $\mathcal{M}^{\mathrm{red}}_b$ with coefficient in the appropriate nearby cycle sheaves. For each $K_p \subset K_{p,0}$, we write $g_{K_p} : \mathcal{M}_{b,K_p} \to \mathcal{M}_{b,K_{p,0}}$ for the natural projection among Rapoport-Zink spaces, and $R^q\psi(g_{K_p*}(\bar{\mathbb{Q}}_l))$ for the nearby cycle sheaves over $\mathcal{M}^{\mathrm{red}}_b$ of $g_{K_p*}(\bar{\mathbb{Q}}_l)$, for all $q \geq 0$. Then, the following equality holds in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_\nu})$, for each $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p)$,

$$\mathcal{E}_b(\rho) = \sum_{i,p,q \geq 0} (-1)^{i+p+q} \varinjlim_{K_p} \mathrm{Tor}^i_{J_b(\mathbb{Q}_p)}(H^p_c(\mathcal{M}^{\mathrm{red}}_b, R^q\psi(g_{K_p*}(\bar{\mathbb{Q}}_l))), \rho)$$

([16], Theorem 8.7). (Here, the interchange between Tor-groups and Ext-groups reflects the fact that we are considering cohomology with compact support.)

4.6. Finally, we recall Rapoport's and Zink's p-adic uniformization Theorem ([27], Chapter 6). Let K^p be a level away from p, $\bar{S}_{K^p}$ the reduction in positive characteristic of the corresponding Shimura variety, and $\bar{S}_{K^p}(b)$ the Newton stratum associated to an element $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$. For any closed geometric point x of $\bar{S}_{K^p}(b)$, we write I_x and $(A_x, \lambda_x, \iota_x; \bar{\eta}_x)$ respectively for the corresponding isogeny leaf in $\bar{S}_{K^p}(b)$ and polarized abelian variety with G-structure and K^p-level structure over $\overline{k(\nu)}$. For simplicity, we choose $\mathcal{M}_b = \mathcal{M}_{b,\mathbb{X}}$ for $\mathbb{X} = A_x[p^\infty]$. Let $\mathcal{M}^{\mathrm{red}}_b$ denote the reduced fiber of $\mathcal{M}_b$, and for any closed geometric points y of $\mathcal{M}^{\mathrm{red}}_b$ write (H_y, β_y) for the fiber at y of the universal family (H, β) over $\mathcal{M}_b$. We define a morphism of $\overline{k(\nu)}$-schemes

$$\Phi_x : \mathcal{M}^{\mathrm{red}}_b \to I_x \subset \bar{S}_{K^p}$$

as $y \mapsto (A_x/(\ker(p^n\beta_y)), \lambda', \iota'; \bar{\eta}')$, where the additional structures on the abelian variety $A_x/(\ker(p^n\beta_y))$ are induced by the those of A_x, via pushforward under the isogeny $A_x \twoheadrightarrow A_x/(\ker(p^n\beta_y))$ (here, n is a sufficiently large positive integer depending on y). The morphism Φ_x is well-defined and surjective onto I_x. Moreover, using Serre-Tate theory, it is easy to see that Φ_x can be canonically extended to a formally étale morphism of formal $\mathcal{O}_L$-schemes $\Phi^{/x}$, from $\mathcal{M}_b$ to the formal completion of S_{K^p} along the isogeny leaf I_x, $(S_{K^p})^{/I_x}$. Finally, it follows from the definition of the Rapoport-Zink covers that the rigid analytic fiber of $\Phi^{/x}$ canonically lifts to a

$G(\mathbb{Q}_p)$-equivariant projective system of morphisms $\Phi_{K_p}^{/x}$, from $\mathcal{M}_{b,K_p}$ to the rigid covers of $(S_{K^p})^{/I_x}$ corresponding to the Shimura varieties $Sh_{K^pK_p} \to Sh_{K^pK_{p,0}}$, for all $K_p \subset K_{p,0}$.

5. The product structure of the Newton strata

In this section, we explain how the geometry and cohomology of the Newton strata in the reduction of the Shimura varieties can be described in terms of those of the corresponding Rapoport-Zink spaces and Igusa varieties.

In [23] (Theorem 5.3) Oort proved that the central and isogeny foliations define an almost product structure on the Newton strata of Siegel varieties. More precisely, he proved that, for each Newton stratum W in the reduction of a Siegel variety, there exist two schemes T and J of finite type over $\bar{\mathbb{F}}_p$ and a finite surjective $\bar{\mathbb{F}}_p$-morphism $\Phi : T \times J \to W$ such that for each closed geometric point x of J, $\Phi(T \times \{x\})$ is a central leaf in W and every central leaf in W can be obtained in this way, and for each closed geometric point y of T, $\Phi(\{y\} \times J)$ is an isogeny leaf in W and every isogeny leaf in W can be obtained in this way. In [16], [17], these ideas are adapted to the context of Shimura varieties. In particular, in the cases of interest to us, for each Newton stratum $\bar{S}_{K^p}(b)$ in the reduction of the Shimura variety $\bar{S}_{K^p}$, for K^p a sufficiently small open compact subgroup of $G(\mathbb{A}_f^p)$ and $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, the schemes T and J in Oort's theorem can be chosen equal respectively to an Igusa variety $\mathrm{Ig}_{m,\mathbb{X}}$ and the reduced fiber of a truncated Rapoport-Zink space $\mathcal{M}_{b,\mathbb{X}}^{n,d}$, for some sufficiently large integers m, n, d and any choice of a completely slope divisible Barsotti-Tate group $\mathbb{X}$ associated with b. Further more, in these cases, for each closed geometric point x' in $\mathrm{Ig}_{m,\mathbb{X}}$ the restriction of the morphism Φ to the space $\mathcal{M}_{b,\mathbb{X}}^{n,d} \times \{x'\}$ agrees (up to a purely inseparable morphism) with the morphism Φ_x in Rapoport's and Zink's p-adic uniformization theorem, for x the image of x' in $\bar{S}_{K^p}(b)$. These observations lead to a group-equivariant description of the geometry and cohomology of the Newton strata in terms of those of the corresponding products of Igusa varieties and Rapoport-Zink spaces. We recall the main results of [16],[17].

5.1. Let K^p a sufficiently small open compact subgroup of $G(\mathbb{A}_f^p)$, and $\bar{S}_{K^p}$ the corresponding Shimura variety modulo p. For each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, let $\mathbb{X}$ be a complete slope divisible $\mu_{\bar{\mathbb{Q}}_p}$-compatible Barsotti-Tate group with $G_{\mathbb{Q}_p}$-structure associated with b, and for all m, n, d denote by $\mathrm{Ig}_{m,\mathbb{X}}$ and $\bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d}$ the corresponding Igusa varieties and the reduced fibers of the truncated Rapoport-Zink spaces. There exists a system of finite (and almost all surjective) $\overline{k(\nu)}$-morphisms

$$\pi_N : \mathrm{Ig}_{m,\mathbb{X}} \times \bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d} \to \bar{S}_{K^p}(b)$$

indexed by quadruples of positive integers m, n, d, N satisfying the conditions $m \geq d$ and $N \geq \frac{d}{\delta \log_p q}$, for $\delta = \delta_b \in \mathbb{Q}$ a constant depending only on $b \in B(G)$. As the indexes m, n, d, N vary, these morphisms are compatible under the projections among the Igusa varieties and the inclusion among the reduced truncated Rapoport-Zink spaces, up to a purely inseparable morphism of $\bar{S}_{K^p}(b)$ (namely a power of the q-Frobenius morphism $F_{\bar{S}} = F_{\bar{S}_{K^p}(b)}$ of $\bar{S}_{K^p}(b)$, the actual power depending on the variation of the index N). Furthermore, the resulting compatible system of morphism is also invariant under the action of $S_{\mathbb{X}} \subset J_b(\mathbb{Q}_p)$ on the product of the Igusa varieties by the Rapoport-Zink spaces, and equivariant for the descent data to E_ν. Finally, as we also allow the level $K^p \subset G(\mathbb{A}_f^p)$ to vary, the resulting morphisms form a $G(\mathbb{A}_f^p)$-equivariant projective system. We refer to [16] (Section 4) and [17] (Section 5) for the details of the above construction. Here, we simply recall its main properties and its application to the computation of the cohomology of the Newton strata.

5.2. Let x be a closed geometric point of $\bar{S}_{K^p}$. For each triple of integers m, n, d, with $m \geq d$, we consider the morphisms $\pi_N : \mathrm{Ig}_{m,\mathbb{X}} \times \bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d} \to \bar{S}_{K^p}(b)$ as N varies. It follows from the compatibility among the various π_N's that, for each $N \geq N_0 = [\frac{d}{\delta \log_p q}] + 1$, there is an equality of finite sets $\pi_N^{-1}(F_{\bar{S}}^N(x)) = \pi_{N_0}^{-1}(F_{\bar{S}}^{N_0}(x))$. Moreover, as m varies, the sets $\pi_N^{-1}(F_{\bar{S}}^N(x))$ form an inverse system under the projection maps among the Igusa varieties, and the corresponding limits form a direct system under the inclusions among the reduced truncated Rapoport-Zink spaces, as n, d vary. We call *the fiber at* x the resulting set

$$\Pi^{-1}(x) = \varinjlim_{n,d} \left(\varprojlim_{m} \pi_N^{-1}(Fr^{fN} x) \right),$$

endowed with the topology of direct limit of inverse limits of discrete sets. The topological space $\Pi^{-1}(x)$ inherits a continuous action of the monoid $S_{\mathbb{X}}$, which it is easy to see extends uniquely to a continuous action of the group $J_b(\mathbb{Q}_p)$.

Alternatively, we can describe the fiber at x, $\Pi^{-1}(x)$, as follows. We define the topological space $\mathrm{Ig}_{\mathbb{X}}(\overline{k(\nu)}) = \varprojlim_m \mathrm{Ig}_{m,\mathbb{X}}(\overline{k(\nu)})$, or equivalently

$$\mathrm{Ig}_{\mathbb{X}}(\overline{k(\nu)}) = \{(B, \lambda, i, \bar{\mu}^p; j) | (B, \lambda, i, \bar{\mu}^p) \in \bar{S}_{K^p}(b)(\overline{k(\nu)}) \text{ and } j : \mathbb{X} \cong B[p^\infty]\},$$

endowed with the topology of inverse limit of discrete sets. In the following, we simply write (B, j) (resp. B) in place of $(B, \lambda, i, \bar{\mu}^p; j)$ (resp. $(B, \lambda, i, \bar{\mu}^p)$) for a point of $\mathrm{Ig}_{\mathbb{X}}(\overline{k(\nu)})$ (resp. of $\bar{S}_{K^p}(b)(\overline{k(\nu)})$). We regard $\bar{\mathcal{M}}_b(\overline{k(\nu)})$ as endowed with the discrete topology and the product $\mathrm{Ig}_{\mathbb{X}}(\overline{k(\nu)}) \times \bar{\mathcal{M}}_b(\overline{k(\nu)})$ with the product topology. The action of the monoid $S_{\mathbb{X}}$ on the product of Igusa varieties and Rapoport-Zink spaces induces a continuous action of $S_{\mathbb{X}}$ on the topological space $\mathrm{Ig}_{\mathbb{X}}(\overline{k(\nu)}) \times \bar{\mathcal{M}}_b(\overline{k(\nu)})$, namely for all $\rho \in S_{\mathbb{X}} : ((B, j), (H, \beta)) \mapsto \big((B/j(\ker \rho^{-1}), j \circ \rho), (H, \beta \circ \rho)\big)$. It is easy to see that this action extends to a continuous action of $J_b(\mathbb{Q}_p)$. We define a map

$$\Pi : \mathrm{Ig}_{\mathbb{X}}(\overline{k(\nu)}) \times \bar{\mathcal{M}}_b(\overline{k(\nu)}) \to \bar{S}_{K^p}(b)(\overline{k(\nu)})$$

as $((B, j), (H, \beta)) \mapsto B/j(\ker(p^n \beta)$, where the abelian variety $B/j(\ker(p^n \beta))$ is endowed with the additional structures induced by those of B (for n a sufficiently large integer depending on β). It is easy to see that the map Π is continuous for the discrete topology on $\bar{S}_{K^p}(b)(\overline{k(\nu)})$, and $J_b(\mathbb{Q}_p)$-invariant. In particular, for each $x \in \bar{S}_{K^p}(b)(\overline{k(\nu)})$, the preimage under Π of x is naturally endowed with a continuous action of $J_b(\mathbb{Q}_p)$. Moreover, it follows from the construction that it is a principle homogeneous $J_b(\mathbb{Q}_p)$-space and that it can be identified (as a topological $J_b(\mathbb{Q}_p)$-space) with the fiber at x, $\Pi^{-1}(x)$.

5.3. Let l be a prime number, $l \neq p$. The existence of the morphisms π_N enable us to compare the l-adic nearby cycles sheaves of the Shimura varieties, when restricted to the Newton strata, with those of the corresponding Rapoport-Zink spaces, when restricted to the truncated subspaces, after pulling back to the common covers

$\mathrm{Ig}_{m,\mathbb{X}} \times \bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d}$. Indeed, it follows from the above construction, that, for each quadruple of sufficiently large integers m, n, d, N, the pullbacks over $\mathrm{Ig}_{m,\mathbb{X}} \times \bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d}$ of the l-adic nearby cycles sheaves defined over $\bar{S}_{K^p}(b)$ and $\bar{\mathcal{M}}_b$, pulled back respectively via the morphism π_N and the second projection $\mathrm{Ig}_{m,\mathbb{X}} \times \bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d} \to \bar{\mathcal{M}}_{b,\mathbb{X}}^{n,d} \subset \bar{\mathcal{M}}_b$, are isomorphic ([17], Proposition 21). By combining this observation with a $J_b(\mathbb{Q}_p)$-equivariant version of the Künneth formula ([16], Section 5), we compute the l-adic cohomology of the Newton strata in terms of the l-adic cohomology of the corresponding Igusa varieties and Rapoport-Zink spaces. More precisely, let $\mathcal{L}_\rho$ be the étale l-adic local system on the Shimura varieties, associated with an representation $\rho \in \mathrm{Rep}_{\mathbb{C}}(G)$. Then, for each $b \in B(G_{\mathbb{Q}_p}, \mu_{\bar{\mathbb{Q}}_p})$, the following equality holds in $\mathrm{Groth}(G(\mathbb{A}_f) \times W_{E_\nu})$:

$$H_c(\bar{S}(b), R\psi(\mathcal{L}_\rho)) = \mathcal{E}_b(H_c(\mathrm{Ig}_{\mathbb{X}}, \mathcal{L}_\rho)).$$

We recall that, in the case of b basic, an analogous formula computing the cohomology of the basic Newton stratum in terms of that of the corresponding basic Rapoport-Zink space was obtained by Fargues in [5] (Corollary 4.6.3), as an application of Rapoport's and Zink's p-adic uniformization theorem.

6. The signature $(1, n-1)$ case and Lubin-Tate spaces

We discuss in further details the geometry of the reduction modulo p of a special class of PEL type Shimura varieties, which contains the class of simple Shimura varieties considered in [8]. The Shimura varieties in this class parameterize abelian varieties whose local geometry is completely controlled by a one-dimensional Barsotti-Tate $\mathcal{O}_K$-module, for K a local p-adic field. In this case, we allow p to be ramified in the reflex field E, and K to be ramified over $\mathbb{Q}_p$. The Rapoport-Zink spaces associated with this class of Shimura varieties are closely related to the formal moduli spaces for one-dimensional formal Lie groups constructed by Lubin and Tate in [14], and to the modular varieties constructed by Drinfeld in [4]. We recall that the cohomology of these varieties is completely understood, due to the work of Boyer in [3].

6.1. Let $\mathcal{O}_K$ be the ring of integers of a p-adic field K. A *one-dimensional Barsotti-Tate $\mathcal{O}_K$-module* is a one-dimensional Barsotti-Tate group together with a faithful action of $\mathcal{O}_K$. If H is a one-dimensional Barsotti-Tate $\mathcal{O}_K$-module defined over a $\mathcal{O}_K$-scheme S where p is locally nilpotent, we say that H is *compatible* if the two action of $\mathcal{O}_K$ on the Lie algebra of H (one from the action of $\mathcal{O}_K$ on H and one from the structure morphism $\mathcal{O}_K \to \mathcal{O}_S$) coincide. In the following we assume all one-dimensional Barsotti-Tate $\mathcal{O}_K$-modules to be compatible. We say that a one-dimensional Barsotti-Tate $\mathcal{O}_K$-module is *ind-étale* (resp. *formal*) if the underlying Barsotti-Tate group is ind-étale (resp. formal).

Let k be an algebraically closed field of characteristic p containing the residue field of K. It follows from the definition that if H is a one-dimensional Barsotti-Tate $\mathcal{O}_K$-module, then the height h of the underlying Barsotti-Tate group is divisible by the degree $[K : \mathbb{Q}_p]$. We define the height of H as $h(H) = \frac{h}{[K:\mathbb{Q}_p]}$. For each integer $g \geq 1$, there is a unique up to isomorphism formal one-dimensional Barsotti-Tate $\mathcal{O}_K$-module $\Sigma_g = \Sigma_{K,g}$ over k of height g. Furthermore, every one-dimensional Barsotti-Tate $\mathcal{O}_K$-module H over k is of the form $\Sigma_g \times (K/\mathcal{O}_K)^r$, for some non-negative integers g and r, with $g + r$ equal to the height of H. We call r the *p-rank* of H, and denote it by $r = \mathrm{rk}(H)$. It follows from the definitions that if H is a one-dimensional Barsotti-Tate $\mathcal{O}_K$-modules over k, of height n and p-rank r, then the Newton polygon of the Barsotti-Tate group underlying H is the convex polygon which starts at $(0,0)$, ends at $(n[K:\mathbb{Q}_p],1)$, and has a unique break-point at $(r[K:\mathbb{Q}_p],0)$. In particular, any two one-dimensional Barsotti-Tate $\mathcal{O}_K$-modules over k are isogenous if and only if they are isomorphic, i.e. if and only if they have the same height and p-rank.

6.2. We recall the specific assumptions on the moduli data. Let $(B,*,v,\langle,\rangle,h)$ be a Shimura datum of PEL type. We assume B to be central over a number field F. The positivity condition on $*$ implies that F is either a CM field or a totally real field. We assume F is a CM field and fix $\Phi \subset \mathrm{Hom}(F,\mathbb{C})$ a CM type of F. Let G be the unitary similitude group defined over $\mathbb{Q}$ associated with the Shimura datum. The similitude factor c defines a character $c: G \to \mathbb{G}_m$, and we write $G_1 = \ker(c)$. For each $\tau \in \Phi$, we write (p_τ, q_τ) for the signature of G_1 at τ, $p_\tau + q_\tau = n$, i.e.

$$G_{1/\mathbb{R}} \cong \prod_{\tau\in\Phi} U(p_\tau, q_\tau).$$

In the following we assume that there exists a place $\tau_0 \in \Phi$ such that $(p_{\tau_0}, q_{\tau_0}) = (1, n-1)$, and $(p_\tau, q_\tau) = (0,n)$ for all $\tau \in \Phi - \{\tau_0\}$.

Let F^+ be the maximal totally real subfield of F. For simplicity, we assume that F is of the form $F = F^+\mathcal{K}$, for $\mathcal{K}$ a quadratic imaginary extension of $\mathbb{Q}$, and that the CM type Φ is induced by a CM type of $\mathcal{K}$. Then, under all these assumptions, the reflex field of the Shimura datum is simply $E = \tau_0(F)$.

Let p be a prime number which is split in $\mathcal{K}$ (but not necessarily unramified in F). After choosing a local embedding $\nu : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, to each $\tau \in \Phi$, $\tau : F \to \bar{\mathbb{Q}}$, we associate the embedding $\nu \circ \tau : F \to \bar{\mathbb{Q}}_p$. We write $v_1, \dots, v_r$ for the places of F corresponding to these embeddings as τ varies in Φ, and $v = v_1$ for the place corresponding to τ_0. Then, the set of places of F dividing p is exactly $\{v_1, \dots, v_r, v_1^c, \dots, v_r^c\}$. We assume B is split at p. Then $B_{\mathbb{Q}_p}$ is of the form $B_{\mathbb{Q}_p} \cong \prod_{i=1}^r (M_n(F_{v_i}) \times M_n(F_{v_i^c})^{\mathrm{opp}})$, and $G(\mathbb{Q}_p) \cong \prod_{i=1}^r GL_n(F_{v_i}) \times \mathbb{Q}_p^\times$. We denote by ϵ the idempotent in $\mathcal{O}_B \otimes \mathbb{Z}_p$ corresponding to the matrix of $M_n(F_v)$ with entry 1 in $(1,1)$ and 0 elsewhere.

6.3. Let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup, and S_{K^p} the integral model of the Shimura variety Sh_K, for $K = K^pK_{p,0}$. We write A for the universal abelian variety over S_{K^p}, and $A[p^\infty]$ for its p-divisible group. Then it follows from our assumptions that the Barsotti-Tate group $A[p^\infty]$ decomposes as

$$A[p^\infty] = \prod_{i=1}^r (A[v_i^\infty] \times A[(v_i^c)^\infty]),$$

where the polarization of A induces an isomorphism of Barsotti-Tate $\mathcal{O}_{F_{v_i}}$-modules between $A[v_i^\infty]$ and $A[(v_i^c)^\infty]$, for all $i \geq 1$, the Barsotti-Tate $\mathcal{O}_{F_{v_i}}$-modules $A[v_i^\infty]$ are ind-étale, for all $i > 1$, and for $i = 1$ the Barsotti-Tate group $H = \epsilon A[v^\infty]$ is a compatible one-dimensional $\mathcal{O}_{F_v}$-module.

6.4. For each closed geometric point x of $\bar{S}_{K^p}$, we write H_x (resp. A_x) for the fiber at x of H (resp. A). We regard the height and p-rank of H_x, as x varies, as

$\mathbb{Z}$-valued functions on $\bar{S}_{K^p}$. Then, the height is constant and equal to n, while the p-rank is lower semicontinuous. I.e., for each integer $i \geq 0$, the subspace

$$\bar{S}_{K^p}^{\leq i} = \{x \in \bar{S}_{K^p} | \mathrm{rk}(H_x) \leq i\}$$

is a closed subspace of $\bar{S}_{K^p}$. (This result is due to Messing in [20], Proposition II.4.9.) We call the associated stratification of $\bar{S}_{K^p}$ by closed reduced subschemes the *p-rank stratification* of $\bar{S}_{K^p}$, and for each integer i, we define *the i-th p-rank stratum* of $\bar{S}_{K^p}$ as the locally closed reduced subscheme

$$\bar{S}_{K^p}^{(i)} = \bar{S}_{K^p}^{\leq i} - \bar{S}_{K^p}^{\leq i-1} = \{x \in \bar{S}_{K^p} | \mathrm{rk}(H_x) = i\}.$$

We remark that the stratum $\bar{S}_{K^p}^{(i)}$ is empty unless $i \in \{0, \dots, n-1\}$, in which case it is smooth of pure dimension i ([8], Corollary III.4.4). It also follows from the work of Messing ([20], Proposition II.4.9) that, for each i, $0 \leq i \leq n-1$, the restriction of H over the stratum $\bar{S}_{K^p}^{(i)}$ admits a unique filtration $0 \subset H_0 \subset H$, where H_0 is a formal one-dimensional Barsotti-Tate $\mathcal{O}_{F_v}$-module of height $n-i$ and $H_1 = H/H_0$ is an ind-étale Barsotti-Tate $\mathcal{O}_{F_v}$-module of height i. This filtration is called the *formal-ind-étale* filtration of H.

We remark that, for each closed geometric point x of $\bar{S}_{K^p}$, the isogeny and isomorphism classes of A_x only depend on H_x, namely on its p-rank. Thus, in particular, the Newton stratification of $\bar{S}_{K^p}$ coincides with the p-rank stratification, the basic (resp. μ-ordinary) stratum equal to the 0-th (resp. $(n-1)$-th) p-rank stratum. Moreover, each Newton stratum has a unique central leaf, which is equal to the whole stratum, and the slope filtration of H defined over it agrees with the formal-ind-étale filtration.

6.5. It follows from the above discussion, that the theory of local models for this class of Shimura varieties can also be described in terms of the appropriate one-dimensional Barsotti-Tate groups. More precisely, for each integer i, $0 \leq i \leq n-1$, we write $\mathcal{M}_{i,n}$ for the formal Rapoport-Zink space parameterizing one-dimensional Barsotti-Tate $\mathcal{O}_{F_v}$-modules, of height n and p-rank i. We write $\breve{E}$ and $\mathcal{O}_{\breve{E}}$ for the maximal unramified extension of the local reflex field E_ν and its ring of integers (in the unramified case $\breve{E} = L$). Then, $\mathcal{M}_{i,n}$ is by definition a formal $\mathcal{O}_{\breve{E}}$-scheme, and it follows from the above considerations that its reduced fiber is a zero-dimensional k-scheme. These are the local models associated with the group $G' = \mathrm{Res}_{F_v/\mathbb{Q}_p}(GL_n)$ (under our current assumptions, $G(\mathbb{Q}_p) \cong \prod_{i=1}^r GL_n(F_{v_i}) \times \mathbb{Q}_p^\times$.)

The geometry and cohomology of these Rapoport-Zink spaces and their rigid analytic covers is easily described in terms of those of the Lubin-Tate spaces ([14]) and of Drinfeld varieties ([4]). To make the connection, we recall the following two results of Messing. Let S be an $\mathcal{O}_{F_v}$-scheme where p is locally nilpotent. There is an equivalence of categories between the category of formal Barsotti-Tate $\mathcal{O}_{F_v}$-modules over S and that of formal Lie groups over S, together with an action of $\mathcal{O}_{F_v}$, which satisfy the condition that multiplication by p on the Lie group is an epimorphism with finite and locally free kernel ([20], Corollary II.4.5.). Let H be a Barsotti-Tate $\mathcal{O}_{F_v}$-module over S. If H has constant height n and constant p-rank r, then it admits a unique filtration $0 \subset H_0 \subset H$, where H_0 is a formal Barsotti-Tate $\mathcal{O}_{F_v}$-module over S of constant height $n-r$, and $H_1 = H/H_0$ is an ind-étale Barsotti-Tate $\mathcal{O}_{F_v}$-module over S of constant height r ([20], Proposition II.4.9).

For $i = 0$, the Rapoport-Zink space $\mathcal{M}_{0,n}$ parameterizes formal one-dimensional Barsotti-Tate $\mathcal{O}_{F_v}$-modules, of height n. It follows from the fact that any quasi-isogeny of height 0 between formal one-dimensional Barsotti-Tate $\mathcal{O}_{F_v}$-modules over an algebraically closed field is an isomorphism, that $\mathcal{M}(\overline{k(\nu)}) = \mathbb{Z}$ (the bijection given by the height of the quasi-isogeny). Furthermore, the infinitesimal deformation functor for formal one-dimensional Barsotti-Tate $\mathcal{O}_{F_v}$-modules of height n, $\mathcal{D}_{0,n}$ (which was studied by Lubin and Tate in [14], and by Drinfeld in [4]), is represented by the formal $\mathcal{O}_{\breve{E}}$-scheme $Spf(\mathcal{O}_{\breve{E}})[[T_1, \dots, T_{n-1}]]$. We conclude that the Rapoport-Zink space $\mathcal{M}_{0,n}$ is (non-canonically) isomorphic to a disjoint union of copies of $Spf(\mathcal{O}_{\breve{E}})[[T_1, \dots, T_{n-1}]]$, indexed by $\mathbb{Z}$ ([27], Proposition 3.79).

For $i > 0$, it follows from the fact that over an algebraically closed field any quasi-isogeny between one-dimensional $\mathcal{O}_{F_v}$-modules splits as the product of a quasi-isogeny between their ind-étale parts by a quasi-isogeny between their formal parts, that $\mathcal{M}_{i,n}(\overline{k(\nu)}) = GL_i(F_v)/GL_i(\mathcal{O}_{F_v}) \times \mathbb{Z}$. Furthermore, the infinitesimal deformation functor for one-dimensional $\mathcal{O}_{F_v}$-modules of height n and p-rank i, $\mathcal{D}_{i,n}$ (which was studied by Drinfeld in [4]), is also represented by $Spf(\mathcal{O}_{\breve{E}})[[T_1, \dots, T_{n-1}]]$. We conclude that the Rapoport-Zink space $\mathcal{M}_{i,n}$ is (non-canonically) isomorphic to a disjoint union of copies of $Spf(\mathcal{O}_{\breve{E}})[[T_1, \dots, T_{n-1}]]$, indexed by $GL_i(F_v)/GL_i(\mathcal{O}_{F_v}) \times \mathbb{Z}$. Moreover, it follows from the existence and uniqueness of the formal-ind-étale filtration of Barsotti-Tate $\mathcal{O}_{F_v}$-modules, that the Rapoport-Zink space $\mathcal{M}_{i,n}$ admits a surjective homomorphism θ onto a disjoint union of copies of $\mathcal{M}_{0,n-i}$, indexed by $GL_i(F_v)/GL_i(\mathcal{O}_{F_v})$. In the above coordinates, the corresponding homomorphism of $\mathcal{O}_{\breve{E}}$-algebras $\theta^* : \mathcal{O}_{\breve{E}}[[T_1, \dots, T_{n-i-1}]] \to \mathcal{O}_{\breve{E}}[[T_1, \dots, T_{n-1}]]$ is defined by $T_s \mapsto T_s$, for all $s \in \{1, \dots, n-i-1\}$. In particular, the map θ is formally smooth of relative dimension i.

6.6. Let $K_p \subset K_{p,0} = GL_n(\mathcal{O}_{F_v})$ be an open compact subgroup of $GL_n(F_v)$.

For each i, $0 \leq i \leq n-1$, we write $\mathcal{M}_{i,n,K_{p,0}}$ for the rigid analytic fiber of $\mathcal{M}_{i,n}$, and $\mathcal{M}_{i,n,K_p} \to \mathcal{M}_{i,n,K_{p,0}}$ for the associated finite étale cover, of level K_p. We also denote by $\mathcal{D}_{i,n,K_p} \to \mathcal{D}_{i,n,K_{p,0}}$ the rigid analytic Drinfeld cover of level K_p (a.k.a. the rigid analytic Lubin-Tate cover in the case of $i = 0$). It follows from the universal properties of the covers that, for each $i \geq 0$, the above description in the case of level $K_{p,0}$ extends to the cases of higher levels $K_p \subset K_{p,0}$. I.e., there exists a $GL_n(F_v)$-equivariant system of isomorphisms between $\mathcal{M}_{i,n,K_p}$ and disjoints union of $GL_i(F_v)/GL_i(\mathcal{O}_{F_v}) \times \mathbb{Z}$-copies of $\mathcal{D}_{i,n,K_p}$, indexed by the open compact subgroups K_p.

For simplicity, we now assume K_p is of the form

$$K_{p,m} = \{A \in GL_n(\mathcal{O}_{F_v}) | A \equiv \mathbb{I} \mod v^m\},$$

for some $m \geq 0$. For each $m \geq 0$, let $\mathcal{D}_{i,n,m}$ be the formal Drinfeld space of level m (a.k.a. the formal Lubin-Tate space of level m in the case of $n = 0$). The space $\mathcal{D}_{i,n,m}$ is defined as the formal $\mathcal{O}_{\breve{E}}$-scheme representing the infinitesimal deformation functor for one-dimensional $\mathcal{O}_{F_v}$-modules of height n and p-rank i, endowed with a Drinfeld structure of level m (its construction is due to Drinfeld in [4]). It follows from the definition that the rigid analytic fiber of $\mathcal{D}_{i,n,m}$ can be canonically identified with $\mathcal{D}_{i,n,K_{p,m}}$. Moreover, for all $m' \geq m \geq 0$, the projections $\mathcal{D}_{i,n,K_{p,m'}} \to \mathcal{D}_{i,n,K_{p,m}}$ canonically extend to homomorphisms of formal $\mathcal{O}_{\breve{E}}$-schemes $\mathcal{D}_{i,n,m'} \to \mathcal{D}_{i,n,m}$. In [4] (Proposition 4.3) Drinfeld proved that the formal schemes $\mathcal{D}_{i,n,m}$ are regular and flat and that the homomorphisms $\mathcal{D}_{i,n,m'} \to \mathcal{D}_{i,n,m}$ are finite and flat, of degree $\#GL_n(\mathcal{O}_{F_v}/v^{m'})/\#GL_n(\mathcal{O}_{F_v}/v^m)$.

For $i > 0$, we write $\mathcal{D}_{0,n-i,m}$ for the corresponding formal Lubin-Tate space of level m, for each $m \geq 0$. Let H be the universal one-dimensional Barsotti-Tate $\mathcal{O}_{F_v}$-module over $\mathcal{D}_{i,n,0}$, $0 \subset H_0 \subset H$ denote the formal-ind-étale filtration of H, and $H_1 = H/H_0$. Étale locally on $\mathcal{D}_{i,n,m}$, the datum of a Drinfeld structure of level m on H determines a direct summand N of $\mathcal{O}_{F_v}^n$ modulo $K_{p,m}$ (each N of rank $n-i$), and two classes of isomorphisms $N \cong T_pH_0$ and $\mathcal{O}_{F_v}^n/N \cong T_pH_1$. After choosing an isomorphism between the 2-flag $0 \subset N \subset \mathcal{O}_{F_v}^n$ and the standard 2-flag of ranks $(n-i, n)$, the latter data can be identified with structures of level m on H_0 and H_1, respectively. Thus, for each $m \geq 1$, the restriction of θ to $\mathcal{D}_{i,n,0}$ (which we denote by $\theta_0 : \mathcal{D}_{i,n,0} \to \mathcal{D}_{0,n-i,0}$) canonically lifts to a $GL_n(\mathcal{O}_{F_v})$-equivariant homomorphism of formal $\mathcal{O}_{\breve{E}}$-schemes

$$\theta_m : \mathcal{D}_{i,n,m} \to \coprod_{P_{n-i,n}(F_v)\backslash GL_n(F_v)/K_{p,m}} \left(\coprod_{GL_i(\mathcal{O}_{F_v}/v^m)} \mathcal{D}_{0,n-i,m} \right),$$

where $P_{n-i,n}$ denotes the parabolic subgroup of GL_n which stabilizes the standard 2-flag of dimensions $(n-i, n)$. For all $m \geq 1$, the homomorphisms θ_m are surjective, finite and flat. Moreover, as the level m varies, they form a $GL_n(F_v)$-equivariant projective system, under the natural projections among the Drinfeld and Lubin-Tate spaces and epimorphisms among the indexing sets.

6.7. It is an immediate consequence of the latter construction that, for each $i > 0$, the l-adic cohomology groups with compact supports of the tower of Drinfeld spaces $\mathcal{D}_{i,n,K_p}$, and thus also that of the associated tower of Rapoport-Zink spaces $\mathcal{M}_{i,n,K_p}$, regarded as representations of $GL_n(F_v)$, are parabolically induced by $P_{n-i,n}(F_v)$, and thus in particular contain no supercuspidal representations of $GL_n(F_v)$. (This simple but important observation is due to Boyer in [3], and reappeared in the work of Harris and Taylor in [8].)

This observation, together with the formula in section 5.3, implies that, for any étale l-adic local system $\mathcal{L}_\rho$ associated with a representation $\rho \in \mathrm{Rep}_{\mathbb{C}}(G)$, and for each $i > 0$, the cohomology groups of the i-th p-rank strata $\bar{S}^{(i)}_{K_p}$, with compact supports and coefficients in the vanishing cycles sheaves of the pushforwards of $\mathcal{L}_\rho$ (as defined in section 3.3), contain no supercuspidal representations of $G(\mathbb{Q}_p)$. Thus, in particular, in the case when the Shimura varieties are proper (i.e. in the case when $[F^+ : \mathbb{Q}_p] \geq 2$), for any étale l-adic local system $\mathcal{L}_\rho$, only the cohomology groups of the basic strata (i.e. the 0-th p-rank strata) contribute, in the sense of the formula in section 3.3, to the supercuspidal part of the cohomology of the Shimura varieties.

We conclude remarking that the above phenomena are expected to occur in much larger generality, e.g. for all Shimura varieties of PEL type which are unramified at p. Indeed, a conjecture of Harris ([7], Conjecture 5.2) predicts that, given an unramified local datum (G, μ) of EL or PEL type, and any element $b \in B(G, \mu)$, if b is non-basic then the l-adic cohomology with compact supports of the Rapoport-Zink spaces attached to the triple (b, G, μ) is parabolically induced from that of the Rapoport-Zink spaces attached to (b, M_b, μ_b), for a specified choice of a class of cocharacters μ_b of M_b associated with μ, and of a parabolic subgroup P_b of G associated with M_b (we recall that $b \in B(M_b, \mu_b)$ is basic). More precisely, for each

$b \in B(G,\mu)$, Harris's conjecture predicts, for all $\rho \in \mathrm{Groth}(J_b(\mathbb{Q}_p))$, the following equalities in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{E_v})$:

$$\mathcal{E}_{(b,G,\mu)}(\rho) = \mathrm{Ind}_{P_b(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathcal{E}_{(b,M_b,\mu_b)}(\rho).$$

In particular, via the formula of section 5.3, Harris' conjecture (and also Kottwitz's conjecture in [25]) implies that, for any Shimura datum (G,h) of PEL type which is unramified at p, and any étale l-adic local system $\mathcal{L}_\rho$ over the corresponding Shimura varieties, associated with a representation $\rho \in \mathrm{Rep}_{\mathbb{C}}(G)$, the corresponding cohomology groups of the Newton strata (as defined in section 3.3) contain no supercuspidal representations of $G(\mathbb{Q}_p)$ except in the case of the basic Newton strata. In [3] Boyer proved Harris' conjecture for Drinfeld modular varieties, e.g. some instances of $G = GL_n$. In [19], Boyer's method was extended to establish new cases of Harris' conjecture.

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