ON CERTAIN UNITARY GROUP SHIMURA VARIETIES

by

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Abstract. —

In this paper, we study the local geometry at a prime p of a certain class of (PEL) type Shimura varieties. We begin by studing the Newton polygon stratification of the special fiber of a Shimura variety with good reduction at p. Each stratum can be described in terms of the products of the reduced fiber of the corresponding Rapoport-Zink space with some smooth varieties (we call the Igusa varieties), and of the action on them of a certain p-adic group T_α, which depends on the stratum. (The definition of the Igusa varieties in this context is based upon a result of Zink on the slope filtration of a Barsotti-Tate group and on the notion of Oort's foliation.) In particular, we show that it is possible to compute the étale cohomology with compact supports of the Newton polygon strata, in terms of the étale cohomology with compact supports of the Igusa varieties and the Rapoport-Zink spaces, and of the group homology of T_α. Further more, we are able to extend Zariski locally the above constructions to characteristic zero and obtain an analoguous description for the étale cohomology of the Shimura varieties in both the cases of good and bad reduction at p. As a result of this analysis, we obtain a description of the l-adic cohomology of the Shimura varieties, in terms of the l-adic cohomology with compact supports of the Igusa varieties and of the Rapoport-Zink spaces.

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***Résumé* (Sur certaines variétés de Shimura associoées à des groupes unitaires)**

Dans cet article, nous étudions la géométrie locale, en un premier p, d'une certaine classe de variétés de Shimura de type PEL. Nous commençons par étudier la stratification par le polygone de Newton de la fibre spéciale des variétés de Shimura ayant bonne réduction en p. Chaque strate peut être décrite en termes de produits des fibres réduites des espaces de Rapoport-Zink correspondants avec certaines variétés lisses, les variétés d'Igusa, et de l'action sur ces objets d'un certain groupe p-adique T_α, qui dépend de la strate. Nous montrons en particulier qu'il est possible de calculer la cohomologie étale à support compact des strates du polygone de Newton, en termes de la cohomologie étale à support compact des variétés d'Igusa et des espaces de Rapoport-Zink, et de l'homologie des groupes de T_α. De plus, nous parvenons à étendre localement (au sens de la topologie de Zariski) les constructions précédentes à la caractéristique nulle et à obtenir une description analogue de la cohomologie étale des variétés de Shimura, dans les cas de bonne comme de mauvaise réduction en p. Comme conséquence de cette étude, nous obtenons une description de la cohomologie l-adique des variétés de Shimura, en termes de la cohomologie l-adique à support compact des variétés d'Igusa et des espaces de Rapoport-Zink.

Contents

1. Introduction

In this paper, we study a certain class of (PEL) type Shimura varieties. These varieties arise as moduli spaces of polarized abelian varieties, endowed with an action of a division algebra and a level structure. Their l-adic cohomology is the object of a conjecture of Langlands.

In [**29**] Rapoport and Zink introduce local analogues of the Shimura varieties, which are (PEL) type moduli spaces for Barsotti-Tate groups, in the category of rigid analytic spaces. These spaces can be used to give rigid analytic uniformizations of isogeny classes of abelian varieties inside the corresponding Shimura varieties. In [**28**] Rapoport reports a conjecture of Kottwitz for the l-adic cohomology groups

with compact supports of the Rapoport-Zink spaces. This conjecture is "heuristically compatible" (in the sense of the p-adic uniformization given in [**29**]) with the corresponding global conjecture on Shimura varieties.

In [**14**] Harris and Taylor prove the local Langlands conjecture by studying a particular class of (PEL) type Shimura varieties. In their work, they analyse the reduction mod p of the Shimura varieties via the notion of Igusa varieties. These varieties arise as finite étale covers of the locus, inside the reduction of the Shimura varieties with no level structure at p, where the Barsotti-Tate group associated to the abelian variety lies in a fixed isomorphism class. Their analysis strongly relies on the fact that, for the class of Shimura varieties they consider, the pertinent Barsotti-Tate groups are one dimensional, and thus Drinfeld's theory of elliptic modules applies.

For general (PEL) type Shimura varieties such an assumption on the dimension of the Barsotti-Tate groups which control the deformation of the abelian varieties does not hold. On the other hand, it might be possible to describe the geometry and the cohomology of general (PEL) type Shimura varieties by combining together Harris-Taylor's and Rapoport-Zink's techniques. We consider the Newton polygon stratification of the reduction of the Shimura varieties, which is defined by the loci where the Barsotti-Tate group associated to the abelian variety lies in a fixed isogeny class. The idea is to analyse each Newton polygon stratum along two main "directions": one corresponding to deforming the abelian varieties without altering the isomorphism class of the associated Barsotti-Tate group, the other corresponding to varying the abelian varieties inside one isogeny class.

In this paper, we carry out this plan for a simple class of (PEL) type Shimura varieties and, as a result, we obtain a description of the l-adic cohomology groups of the Shimura varieties, in terms of the l-adic cohomology with compact supports groups of the Igusa varieties and of the Rapoport-Zink spaces, in the appropriate Grothendieck group, for any prime number $l \neq p$.

More precisely, the class of (PEL) type Shimura varieties we are interested in arises as the class of moduli spaces of polarized abelian varieties endowed with the action of a division algebra and with a level structure associated to the data $(E, B, *, V, <,>)$ where:

- E is an imaginary quadratic extension of $\mathbb{Q}$ in which the prime p splits (we write $(p) = u \cdot u^c$);
- B is a central division algebra over E of dimension h^2 which splits at u;
- $*$ is a positive involution of the second kind on B;
- $V = B$ viewed as a B-module;
- $<,>: V \times V \to \mathbb{Q}$ is a non degenerate alternating $*$-hermitian pairing.

We denote by G the algebraic group over $\mathbb{Q}$ of the automorphisms of V which preserves $<,>$ up to scalar multiple, and by G_1 the algebraic subgroup of G of the automorphisms which preserves $<,>$, i.e. $0 \to G_1 \to G \to \mathbb{G}_m \to 0$. Finally, we also

assume

$$G_1(\mathbb{R}) = U(q, h-q),$$

for some integer q, $1 \leq q \leq h-1$. (For $q = 0, h$ the corresponding class of Shimura varieties has good reduction at the prime p.)

We remark that when $q = 1$ the above class of Shimura varieties is a subclass of the one studied by Harris and Taylor in [**14**] (namely, the case when the totally real part of the ground field is trivial).

For any sufficiently small open compact subgroup $U \subset G(\mathbb{A}^\infty)$, we call the Shimura variety of level U the smooth projective scheme X_U over $\operatorname{Spec} E$, of dimension $q(h-q)$, which arises as the moduli space of polarized abelian varieties endowed with a *compatible* action of B and with a structure of level U, classified up to isogeny (see [**24**]).

Our goal is to study the virtual representation of the group $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$:

$$H(X, \mathbb{Q}_l) = \sum_{i \geq 0} (-1)^i \varinjlim_U H^i_{et}(X_U \times_E (\hat{E}^{nr}_u)^{ac}, \mathbb{Q}_l).$$

We obtain the following theorem.

Theorem 1 (Main Theorem). — *There is an equality of virtual representations of the group $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$:*

$$H(X, \mathbb{Q}_l)^{\mathbb{Z}_p^\times} = \sum_{\alpha, k, i, j} (-1)^{k+i+j} \varinjlim_{V_p} \operatorname{Ext}^k_{T_\alpha - smooth} \left(H^i_c(\mathcal{M}^{\mathrm{rig}}_{\alpha, V_p}, \mathbb{Q}_l(-D)), H^j_c(J_\alpha, \mathbb{Q}_l) \right)$$

where:

- *$D = q(h-q)$ is the dimension of the Shimura varieties;*
- *the action of $\mathbb{Z}_p^\times$ on $H(X, \mathbb{Z}/l^r\mathbb{Z})$ is defined via the embedding*

$$\mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times \times (B_u^{op})^\times = G(\mathbb{Q}_p) \subset G(\mathbb{A}^\infty);$$

- *α varies among all the Newton polygons of height h and dimension q;*
- *for each α, T_α is a p-adic group of the form $T_\alpha = \prod_i GL_{r_i}(D_i)$ for some finite dimensional division algebras $D_i/\mathbb{Q}_p$;*
- *$H^j_c(J_\alpha, \mathbb{Q}_l)$ are representations of $T_\alpha \times G(\mathbb{A}^{\infty,p}) \times (\mathbb{Q}_p^\times/\mathbb{Z}_p^\times) \times (W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p})$ associated to the l-adic cohomology with compact support groups of the Igusa varieties, for all $j \geq 0$;*
- *$H^k_c(\mathcal{M}^{\mathrm{rig}}_{\alpha, V_p}, \mathbb{Q}_l)$ are the l-adic cohomology with compact supports groups of the rigid analytic Rapoport-Zink space of level V_p, for all $k \geq 0$ and any open compact subgroup $V_p \subset G(\mathbb{Q}_p)/\mathbb{Q}_p^\times$; as the level V_p varies, they form a direct limit of representations of $T_\alpha \times W_{\mathbb{Q}_p}$, endowed with an action of $G(\mathbb{Q}_p)/\mathbb{Q}_p^\times$.*

In the following, we outline in more detail the content of this paper.

In [**24**] Kottwitz proves that the Shimura varieties without level structure at p, i.e. associated to a subgroup U of the form

$$U = U^p(0) = U^p \times \mathbb{Z}_p^\times \times \mathcal{O}_{B_u^{op}}^\times,$$

admit smooth integral models over $\operatorname{Spec} \mathcal{O}_{E_u}$ ($\mathcal{O}_{E_u} = \mathbb{Z}_p$). We denote by $\bar{X} = \bar{X}_{U^p(0)}$ the reduction $X_{U^p(0)} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$ of a Shimura variety with no level structure at p and by $\mathcal{A}$ the universal abelian variety over $\bar{X}$. It follows from the definition of the moduli space and Serre-Tate's theorem that the deformation theory of the abelian variety $\mathcal{A}$ over $\bar{X}$ is controlled by a Barsotti-Tate group of height h and dimension q over $\bar{X}$, which we denote by $\mathcal{G}/\bar{X}$ ($\mathcal{G} \subset \mathcal{A}[p^\infty]$).

In [**25**] Oort studies the Newton polygon stratification of a moduli space of abelian varieties in positive characteristic. This is a stratification by locally closed subschemes which are defined in terms of the Newton polygons of the Barsotti-Tate groups associated to the abelian varieties. (Newton polygons associated to Barsotti-Tate groups were first introduced and studied by Grothendieck in [**13**] and Katz in [**21**]). For any Newton polygon α of dimension q and height h, the associated stratum $\bar{X}^{(\alpha)}$ is the locus where the Barsotti-Tate group $\mathcal{G}$ has constant Newton polygon equal to α, i.e. constant isogeny class.

The first step towards the main theorem is to notice that the decomposition of $\bar{X}$ as the disjoint union of the open Newton polygon strata $\bar{X}^{(\alpha)}$ induces an equality of virtual representations of the group $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$:

$$\sum_{i\geq 0}(-1)^i H^i(X, \mathbb{Q}_l) = \sum_{i\geq 0}(-1)^i H^i(\bar{X}, \mathbb{Q}_l) = \sum_\alpha \sum_{j\geq 0}(-1)^j H_c^j(\bar{X}^{(\alpha)}, \mathbb{Q}_l).$$

Thus, we may restrict ourself to study each Newton polygon stratum separately. Since we are assuming $q \geq 1$, to each isogeny class of Barsotti-Tate groups of dimension q and height h correspond possibly many distinct isomorphism classes. It is a result of Oort that, for any given Barsotti-Tate group H over $\bar{\mathbb{F}}_p$, with Newton polygon equal to α, the set of geometric points x of $\bar{X}^{(\alpha)}$ such that $\mathcal{G}_x \simeq H$ is a closed subset of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. Moreover, the corresponding reduced subscheme of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is a smooth scheme over $\operatorname{Spec} \bar{\mathbb{F}}_p$, which is called the leaf associated to H and is denoted by C_H.

In this work, we focus our attention on a distinguished leaf $C_\alpha = C_{\Sigma_\alpha}$ inside each Newton polygon stratum, which we call the central leaf, and define the Igusa varieties as covering spaces of the central leaf C_α. Before introducing the definition of Igusa variety, we recall a result of Zink (see [**30**]). This result extends the classical result in Dieudonné's theory of p-divisible groups which states that any p-divisible group defined over a perfect field is isogenous to a completely slope divisible one, i.e. to a direct product of isoclinic slope divisible p-divisible groups. In [**30**] Zink shows that over a regular scheme of characteric p any p-divisible group with constant Newton polygon of slopes $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ is isogenous to a p-divisible group $\mathcal{G}$ which

admits a filtration (called the slope filtration)

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G}$$

whose factors $\mathcal{G}^i = \mathcal{G}_i/\mathcal{G}_{i-1}$ are isoclinic slope divisible p-divisible groups of slope λ_i. In particular, it follows from Zink's work that the Barsotti-Tate group $\mathcal{G}$ over C_α admits a slope filtration (see remark 2.14).

Definition 2. — *For any positive integer m, we define the Igusa variety of level m, $J_{\alpha,m}$, over C_α to be the universal space for the existence of isomorphisms*

$$j^i_m : \Sigma^i[p^m] \to \mathcal{G}^i[p^m]$$

which extend étale locally to any higher level $m' \geq m$ (we denote by Σ^i the isoclinic piece of Σ_α of slope λ_i, for each i).

The notion of Igusa varieties was first introduced by Igusa in [**16**] in the theory of elliptic curves and used to describe the reduction at a bad prime p of modular curves (see [**22**]). In [**14**], Harris and Taylor introduce and study a higher dimensional analogue of the Igusa curves which they use to describe the reduction at a bad prime p of Shimura varieties. We remark that, both in the classical theory of modular curves and in the case of the Shimura varieties considered by Harris and Taylor in [**14**], the Igusa varieties are finite étale covers of the whole open Newton polygon stratum, and not of the central leaf. This is because, in the case when $\mathcal{G}$ is one dimensional, there is a unique leaf inside each open Newton polygon stratum, namely the whole stratum itself. We prove that, for any integer $m > 0$, the morphism $J_{\alpha,m} \to C_\alpha$ is finite, étale and Galois. (The definition of the Igusa varieties can be easily given over other leaves of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, namely over any leaf associated to a completely slope divisible p-divisible group.)

As mentioned, our idea is to study the Newton polygon stratum along two "directions": one corresponding to deforming the abelian varieties without altering the isomorphism class of the associated Barsotti-Tate group, the other to varying the Barsotti-Tate group (and the associated abelian variety) inside its isogeny class. The first one is related to the leaves of Oort's foliation (and thus to the Igusa varieties), the latter corresponds to the Rapoport-Zink spaces.

Following the work of Rapoport and Zink ([**29**]), to the Barsotti-Tate group Σ_α we associate a formal scheme $\mathcal{M}_\alpha = \mathcal{M}_{\Sigma_\alpha}$ over $\hat{\mathbb{Z}}_p^{nr} = W(\bar{\mathbb{F}}_p)$, which is formally locally of finite type. It arises as a moduli space for Barsotti-Tate groups H/S together with a quasi-isogeny $\beta : \Sigma_\alpha \times \bar{S} \to H \times \bar{S}$ over the reduction modulo p $\bar{S}$ of S, classified up to isomorphisms of Barsotti-Tate group H/S (S is a scheme over $\hat{\mathbb{Z}}_p^{nr}$ on which p is locally nilpotent). We call $\mathcal{M}_\alpha$ the Rapoport-Zink space with no level structure, associated to α. For any pair of positive integers (n, d), we denote by $\mathcal{M}_\alpha^{n,d}$ the closed formal subscheme of $\mathcal{M}_\alpha$ over which $p^n\beta$ is an isogeny with kernel contained in $\Sigma_\alpha[p^d]$, and by $\bar{\mathcal{M}}_\alpha^{n,d}$ its reduced fiber over $\bar{\mathbb{F}}_p$.

For any set of positive integers m, n, d, with $m \geq d$, and all integers $N \geq d/\delta B$ (where $\delta \in \mathbb{Q}$ and $B \in \mathbb{N}$ are two numbers which depend on the Newton polygon α), we define some morphisms $\pi_N : J_{\alpha,m} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}_\alpha^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. (The definition of the map π_N is based on the observation that the iterated action of Frobenius on a Barsotti-Tate group makes the slope filtration more and more split.) When $n = d = 0$, the morphism $\pi_m^{0,0}$ is simply the structure morphism $\mathrm{q}_m : J_{\alpha,m} \to C_\alpha \hookrightarrow \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (the space $\bar{\mathcal{M}}_\alpha^{0,0}$ is just a point, namely the point corresponding to the pair $(\Sigma_\alpha, \mathrm{id})$ over $\bar{\mathbb{F}}_p$). We prove that the morphisms $\pi_N : J_{\alpha,m} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}_\alpha^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ are finite, and surjective on geometric points for m, n, d sufficiently large. Moreover, they are compatible with the projections among the Igusa varieties and with the inclusions among the Rapoport-Zink spaces, and also $\pi_{N+1} = (Fr^B \times 1)\pi_N$, for all N (we denote by Fr the absolute Frobenius on $\bar{X}$). Further more, there is a natural way of defining a Galois action on the Igusa varieties and on the Rapoport-Zink spaces, which is compatible under the morphisms π_N with the Galois action on the Newton polygon strata.

One may hope to also endow the system of covers $J_{\alpha,m} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}_\alpha^{n,d}$ of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ with an action of the group T_α of quasi-isogenies of Σ_α over $\bar{\mathbb{F}}_p$, leaving the morphisms π_N invariant (such an action exists for example on geometric points). From the definitions we have an action of subgroup $\Gamma_\alpha = \mathrm{Aut}\,(\Sigma_\alpha)$ of T_α on the tower of Igusa varieties, and a natural action of T_α on the space $\bar{\mathcal{M}}_\alpha$. We show that the diagonal action of Γ_α on the system of covers $J_{\alpha,m} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}_\alpha^{n,d}$ extends to an action of a certain submonoid $\Gamma_\alpha \subset S_\alpha \subset T_\alpha$ which leaves the morphisms π_N invariant. Moreover, the action of S_α on the systems of covers $J_{\alpha,m} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}_\alpha^{n,d}$ induces an action on the étale cohomology with compact supports groups, and this action extends uniquely to a smooth action of the entire group T_α. As a result of the analysis of the action of T_α on the cohomology groups, we prove the existence of a spectral sequence of Galois representations

$$\oplus_{i+j+k=n} Tor^i_{\mathcal{H}_r(T_\alpha)} \left(H_c^k(\bar{\mathcal{M}}_\alpha, \mathbb{Z}/l^r\mathbb{Z}), H_c^j(J_\alpha, \mathbb{Z}/l^r\mathbb{Z})\right) \Rightarrow H_c^n(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \mathbb{Z}/l^r\mathbb{Z}),$$

where $H_c^j(J_\alpha, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_m H_c^j(J_{\alpha,m}, \mathbb{Z}/l^r\mathbb{Z})$.

As we let the level structure away from p vary, the action of $G(\mathbb{A}^{\infty,p})$ on the Shimura varieties with no level structure at p preserves the above constructions. Therefore, the above spectral sequences give rise to an equality of representations of $G(\mathbb{A}^{\infty,p}) \times W_{\mathbb{Q}_p}$

$$\sum_{i+j+k=n} \varinjlim_{U^p} Tor^i_{\mathcal{H}(T_\alpha)} \left(H_c^k(\bar{\mathcal{M}}_\alpha, \mathbb{Q}_l), H_c^j(J_{\alpha,U^p}, \mathbb{Q}_l)\right) =$$

$$= \varinjlim_{U^p} H_c^n(\bar{X}^{(\alpha)}_{U^p(0)} \times \bar{\mathbb{F}}_p, \mathbb{Q}_l).$$

We are left to study the case of Shimura varieties with level structure at p. Our idea is to extend the above construction to characteristic zero and to compare the Shimura varieties with level structure to the product of some smooth lifts of the Igusa varieties with the Rapoport-Zink spaces of the same level. More precisely, we

are interested in studying the associated vanishing cycles sheaves as we can then use them to compute the l-adic cohomology of these spaces in characteristic zero in terms of the cohomology of their reduction in positive characteristic.

We denote by $\mathcal{X}$ (resp. $\mathfrak{C}_\alpha$) the formal completion of X along $\bar{X} \times \bar{\mathbb{F}}_p$ (resp. $C_\alpha \times \bar{\mathbb{F}}_p$), and by $\mathcal{J}_{\alpha,m}$ the finite étale cover of $\mathfrak{C}_\alpha$ corresponding to $J_{\alpha,m}/C_\alpha$. We denote by $\mathfrak{X}_M^{\mathrm{rig}}$ over $\mathfrak{X}^{\mathrm{rig}}$ the rigid analytic space associated to the Shimura variety with structure of level M at p and by $\mathcal{M}_{\alpha,M}^{\mathrm{rig}}$ over $\mathcal{M}_\alpha^{\mathrm{rig}}$ the Rapoport-Zink space of level M.

For $m \geq d+t$, we construct some morphisms

$$\pi_N(t) : \left(\mathcal{J}_{\alpha,m} \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}_\alpha^{n,d}\right)(t) \to \mathcal{X}(t),$$

such that $(\pi_N(t))^{\mathrm{red}} \circ (1 \times Fr^{NB}) = \pi_N$, for all $N \geq (d+t)/\delta B$, which are compatible with the projections among the lifts of the Igusa varieties and with the inclusions among the Rapoport-Zink spaces, and which extend Zariski locally to the formal schemes in characteristic zero. (For any formal scheme $\mathcal{Y}$ over $\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}$, we write $\mathcal{Y}(t)$ for the subscheme defined by the t-th power of an ideal of definition $\mathcal{I}$ of $\mathcal{Y}$, $p \in \mathcal{I}$.) For $m \geq d+t+M$, we prove that for any affine open V of $\mathcal{J}_{\alpha,m} \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}_\alpha^{n,d}$ there exists a formally smooth morphism π_V over V lifting $\pi_N(t)_{|V(t)}$ with the property that

$$\mathfrak{X}_M^{\mathrm{rig}} \times_{\mathcal{X}^{\mathrm{rig}},\pi_V} V^{\mathrm{rig}} \simeq \mathcal{M}_{\alpha,M}^{\mathrm{rig}} \times_{\mathcal{M}_\alpha^{\mathrm{rig}},pr_{2|V}} V^{\mathrm{rig}}.$$

The existence of the morphisms $\pi_N(t)$, and of the corresponding local liftings to characteristic zero, enable us to compare the vanishing cycles sheaves (in the sense of Berkovich's [**2**] and [**3**]) of the Shimura varieties and of the Rapoport-Zink spaces when level structure at p is considered. More precisely, as the level at p varies, we obtain a system of compatible equalities of representations of $G(\mathbb{A}^{\infty,p}) \times W_{\mathbb{Q}_p}$

$$\sum_{i,j,k,q} (-1)^{i+j+k+q} \varinjlim_{U^p} Tor^i_{\mathcal{H}(T_\alpha)} \left(H_c^k(\bar{\mathcal{M}}_{\alpha,M}, R^q\Psi_\eta(\mathbb{Q}_l)), H_c^j(J_{\alpha,U^p}, \mathbb{Q}_l)\right) =$$

$$= \sum_{n,q} (-1)^{n+q} \varinjlim_{U^p} H_c^n(\bar{X}^{(\alpha)}_{U^p(M)} \times \bar{\mathbb{F}}_p, R^q\Psi_\eta(\mathbb{Q}_l)),$$

It is easy to see that, as α varies among the Newton polygons of dimension q and height h, the right hand side computes the l-adic cohomology groups of the Shimura variety of level M. It is more subtle to realise that the cohomology groups of the special fibers of the Rapoport-Zink spaces, with coefficients in the vanishing cycles sheaves, compute not the cohomology of the Rapoport-Zink spaces but its contragredient dual, up to Tate twist.

As the level M varies, the above equalities piece together in the statement of the main theorem.

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2. Preliminaries

In this section, we shall introduce the definitions and results which are the starting point of our work. In particular, we shall define the class of (PEL) type Shimura varieties we study. These are some smooth projective varieties defined over an imaginary quadratic extension of $\mathbb{Q}$, which arise as moduli spaces of polarized abelian varieties and which admit smooth integral models at p, in the cases when no level structure at p is considered. (We shall follow the definitions given by Harris and Taylor in [**14**]).

In section 2.2, we shall focus our attention on the reduction in positive characteristic of the Shimura varieties with no level structure at p and introduce in this context the Newton polygon stratification. This stratification was defined by Grothendieck in [**13**] and extensively studied by Oort in the general context of moduli spaces of abelian varieties (see [**25**]). Inside each Newton polygon stratum, we shall distinguish some smooth closed subschemes which are defined by fixing the isomorphism class of the p-divisible group associated to the abelian variety. These are the leaves of Oort's foliation (see [**26**]). We shall also point out a certain isomorphism class of p-divisible groups for any given Newton polygon, whose corresponding leaf (the central leaf) will play an important role in our analysis.

From the theory of Barsotti-Tate groups, we shall recall the definition of some (PEL) type moduli spaces introduced by Rapoport and Zink (in [**29**]) and also the notion of slope filtration. The slope filtration of a Barsotti-Tate group over a perfect field of characteristc p was first studied by Grothendieck (see [**21**]). Here, we shall recall some recent results of Zink which study the case of a Barsotti-Tate group over a smooth scheme of characteristic p (see [**30**]). We shall also introduce the notion of level structure on a Barsotti-Tate group, as a specification of Katz's and Mazur's notion of full set of sections on finite flat group schemes (see [**22**]).

Finally, we shall recall some results of Berkovich on the theory of vanishing cycles in the context of rigid-analytic spaces associated to some special formal schemes (see [**2**] and [**3**]).

2.1. Shimura varieties. — In this section we shall introduce the simple unitary group Shimura varieties which are studied in this paper. This class is a sub-class of the Shimura varieties introduced by Kottwitz in [**23**] and contains a sub-class of the class studied by Harris and Taylor in [**14**] (see section 2.1.8).

We shall follow the exposition and notations of Chapter IV in [**14**].

2.1.1. Let E be an imaginary quadratic extension of $\mathbb{Q}$ in which the prime p splits. We denote by c the complex conjugation in $Gal(E/\mathbb{Q})$ and by u, u^c the two primes of E above p.

Let B over E be a division algebra of dimension h^2 such that:

- E is the center of B;
- B splits at u;
- there is a positive involution of the second kind $*$ on B.

(We recall that an involution on B, $* : B \to B^{op}$, is said of the second kind if $z^* = z^c$ for all $z \in E$, and is positive if $tr_{B/\mathbb{Q}}(xx^*) > 0$ for all $x \in B^\times$.)

The above conditions on B imply that $B_u = B \otimes_E E_u$ is isomorphic to $M_n(\mathbb{Q}_p)$. We fix such an isomorphism $B_u \simeq M_n(\mathbb{Q}_p)$ and denote by $\mathcal{O}_{B_u}$ the maximal order of B_u corresponding to $M_n(\mathbb{Z}_p) \subset M_n(\mathbb{Q}_p)$. Then, there is a a unique maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ in B which is stable under the involution $*$ and such that $(\mathcal{O}_B)_u = \mathcal{O}_{B_u}$. We also define $\epsilon \in \mathcal{O}_{B_u}$ to be the idempotent element which maps, under the above isomorphism, to the matrix with entries equal to 1 in position $(1,1)$ and 0 every where else.

2.1.2. Let V denote the $B \otimes B^{op}$-module underlying B and choose a non-degenerate $*$-hermitian alternanting pairing $<,>$ on V. This choice defines an involution of the second kind $\#$ on B^{op} by

$$\langle (b_1 \otimes b_2)x, y\rangle = \langle x, (b_1^* \otimes b_2^\#)y\rangle$$

for all $x, y \in V$, $b_1 \in B$ and $b_2 \in B^{op}$.

Let us denote by q $(0 \leq q \leq h)$ the positive integer such that the pairing $<,>$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ has invariant $(q, h-q)$. We assume $q \neq 0, h$. This is equivalent to assuming that the corresponding class of Shimura varieties has bad reduction at p.

2.1.3. Let G be the algebraic group over $\mathbb{Q}$ (resp. $\mathbb{A}^\infty$) defined by

$$G(R) = \{(\lambda, g) \in R^\times \times (B^{op} \otimes R)^\times \mid gg^\# = \lambda\},$$

for any $\mathbb{Q}$-algebra (resp. $\mathbb{A}^\infty$-algebra) R.

There is a distinguished normal subgroup G_1 of G, namely the subgroup of the automorphisms of V which preserves the pairing $<,>$, i.e.

$$0 \to G_1 \to G \to \mathbb{G}_m \to 0,$$

where the morphism $\nu : G \to \mathbb{G}_m$ is defined by $(\lambda, g) \mapsto \lambda$. Then,

$$G_1(\mathbb{R}) \simeq U(q, h-q).$$

Let $\mathbb{A}^\infty$ denote the finite adels of $\mathbb{Q}$. We observe that $G(\mathbb{A}^\infty)$ naturally decomposes as $G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)$. Moreover, since $(p) = u \cdot u^c$ in E, we can identify

$$B^{op} \otimes_{\mathbb{Q}} \mathbb{Q}_p = B_u^{op} \times B_{u^c}^{op},$$

and thus any $(\lambda, g) \in G(\mathbb{Q}_p)$ can be written uniquely as $(\lambda, g_1, g_2) \in \mathbb{Q}_p \times B_u^{op} \times B_{u^c}^{op}$ with $(g_1 g_2^\#, g_2 g_1^\#) = (\lambda, \lambda)$. There is a natural isomorphism

$$(\mathbb{Q}_p)^\times \times (B_u^{op})^\times \to G(\mathbb{Q}_p)$$

which is defined by $(\lambda, g_1) \mapsto (\lambda, g_1, \lambda(g_1^\#)^{-1})$.

2.1.4. Let us recall the following definitions about abelian schemes (see [**14**], Lemma IV.1.1, p. 93, and Section IV.4, p. 112).

Let S be a E-scheme and A/S an abelian scheme of dimension h^2. Suppose that there is a morphism $i : B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $Lie(A)$ is a locally free $\mathcal{O}_S$-module of rank h^2 with an action of B and it decomposes as

$$Lie(A) = Lie^+(A) \oplus Lie^-(A),$$

where $Lie^+(A)$ (resp. $Lie^-(A)$) is the module $Lie(A) \otimes_{\mathcal{O}_S \otimes E} \mathcal{O}_S$ and the map $E \to \mathcal{O}_S$ is the natural map (resp. the complex conjugate of the natural map). Both $Lie^+(A)$ and $Lie^-(A)$ are locally free $\mathcal{O}_S$-modules.

Definition 2.1. — We call the pair (A, i) compatible if $Lie^+(A)$ has rank qh.

Suppose now that S is a $\mathcal{O}_{E_u}$-scheme, A an abelian scheme over S and $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Definition 2.2. — We call the pair (A, i) compatible if $Lie(A) \otimes_{\mathbb{Z}_p \otimes \mathcal{O}_E} \mathcal{O}_{E_u}$ is locally free of rank qh.

We remark that, if p is locally nilpotent on S, then the pair (A, i) is compatible if and only if the Barsotti-Tate group $G = \epsilon A[u^\infty]$ is a q-dimensional Barsotti-Tate group, ($\epsilon \in \mathcal{O}_{B_u}$ is the idempotent defined in section 2.1.1). In fact, in this case, we can identify $Lie(A[u^\infty]) = Lie(A) \otimes_{\mathbb{Z}_p \otimes \mathcal{O}_E} \mathcal{O}_{E_u}$ as $\mathcal{O}_{B_u}$-modules, and thus we have $Lie(G) = \epsilon Lie(A[u^\infty])$ as $\mathcal{O}_{E_u} = \mathbb{Z}_p$-modules. It follows that saying that the pair (A, i) is compatible is equivalent to say that $Lie(G)$ is locally free of rank q, since we also have the equalities

$$\mathcal{O}_{B_u} \otimes_{\mathbb{Z}_p} Lie(G) = \mathcal{O}_{B_u} \otimes_{\mathbb{Z}_p} \epsilon Lie(A[u^\infty]) = Lie(A[u^\infty]) = Lie(A).$$

On the other hand, saying that $Lie(G)$ is a locally free $\mathbb{Z}_p$-module of rank q is equivalent to saying that the Barsotti-Tate group G has dimension q.

It follows from the fact that A/S has dimension h^2 that $A[u^\infty]$ has height h^2 (half the height of $A[p^\infty]$) and thus that G has height h.

2.1.5. Let U be an open compact subgroup of $G(\mathbb{A}^\infty)$. We shall define a functor X_U on the category of pairs (S, s), where S is a connected locally noetherian E-scheme and s is a geometric point on S, to sets. We define $X_U(S, s)$ to be the set of equivalence classes of quadruples $(A, \lambda, i, \bar{\mu})$ where:

- A is an abelian scheme over S of dimension h^2;
- $\lambda : A \to A^\vee$ is a polarization;
- $i : B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that (A, i) is compatible and $\lambda \circ i(b^*) = i(b)^\vee \circ \lambda$ for all $b \in B$;
- $\bar{\mu}$ is a $\pi_1(S, s)$-invariant U-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\mu : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$ which takes the pairing $<,>$ on $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty$ to a $(\mathbb{A}^\infty)^\times$-scalar multiple of the λ-Weil pairing.

Two quadruples $(A, \lambda, i, \bar{\mu})$ and $(A', \lambda', i', \bar{\mu}')$ are equivalent if there exists an isogeny $\beta : A \to A'$ which takes λ to a $\mathbb{Q}^\times$-multiple of λ', i to i' and $\bar{\mu}$ to $\bar{\mu}'$ (see [**24**], p. 390).

If s' is a second geometric point on S then $X_U(S, s)$ is canonically in bijection with $X_U(S, s')$ (see [**24**], p. 391). Therefore X_U can be viewed as a functor on the category of connected locally noetherian E-schemes to sets. Moreover, we can extend X_U on the category of locally noetherian E-schemes by setting $X_U(S) = \prod_i X_U(S_i)$ for any $S = \coprod_i S_i$ with S_i connected for all i.

2.1.6. We say that an open compact subgroup U of $G(\mathbb{A}^\infty)$ is sufficiently small if there exists a prime x in $\mathbb{Q}$ such that the projection of U in $G(\mathbb{Q}_x)$ contains no elements of finite order other than 1. If U is sufficiently small then the functor X_U on the category of locally noetherian E-schemes to sets is represented by a smooth projective scheme X_U/E (see [**24**], p. 391).

If $V \subset U$ then there is a natural finite étale morphism $X_V \twoheadrightarrow X_U$ and if V is normal in U this map is Galois with Galois group U/V.

2.1.7. Moroever, there is a natural action of the group $G(\mathbb{A}^\infty)$ on the system of Shimura varieties defined by composition on the right with the isomorphism $\mu : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$. More precisely, for any $g \in G(\mathbb{A}^\infty)$ and any open compact sufficiently small subgroup U of $G(\mathbb{A}^\infty)$, there exists a natural finite étale morphism

$$g : X_U \to X_{g^{-1}Ug}$$

defined by setting $(A, \lambda, i, \bar{\mu}) \mapsto (A, \lambda, i, \overline{\mu g})$.

2.1.8. We remark that when $q = 1$ the class of Shimura varieties we have introduced is a sub-class of the class of Shimura varieties studied by Harris and Taylor in [**14**] (and thus the geometry and the cohomology of these Shimura varieties is already known). More precisely, for $q = 1$, we recover the sub-class corresponding to the case when the totally real extension of $\mathbb{Q}$ inside the ground field is trivial.

2.1.9. Let U^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$ (i.e. there exists a prime $x \neq p$ in $\mathbb{Q}$ such that the projection of U^p in $G(\mathbb{Q}_x)$ contains no elements of finite order other than 1). For any non-negative integer m we define

$$U^p(m) = U^p \times \mathbb{Z}_p^\times \times \ker\left(\left(\mathcal{O}_{B_u^{op}}\right)^\times \to \left(\mathcal{O}_{B_u^{op}}/u^m\right)^\times\right).$$

It is a sufficiently small open compact subgroup of $G(\mathbb{A}^\infty)$.

We call $X_{U^p(m)}$ a Shimura variety with structure of level m at u (or with no level structure at u if $m = 0$).

2.1.10. The Shimura varieties with no level structure at u admit smooth integral models over $\operatorname{Spec} \mathcal{O}_{E_u}$ (see [**24**], chapter 5, pp.389–392). For completeness, we recall Kottwitz's construction in this context.

We define a functor $\mathcal{X}_{U^p}$ on the category of pairs (S, s), where S is a connected locally noetherian $\mathcal{O}_{E_u}$-schemes and s is a geometric point on S, to sets. We set $\mathcal{X}_{U^p}(S, s)$ to be the set of equivalence classes of quadruples $(A, \lambda, i, \bar{\mu}^p)$ where:

- A is an abelian scheme over S of dimension h^2;
- $\lambda : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that (A, i) is compatible and $\lambda \circ i(b^*) = i(b)^\vee \circ \lambda$ for all $b \in \mathcal{O}_B$;
- $\bar{\mu}^p$ is a $\pi_1(S, s)$-invariant U^p-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\mu^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ which takes the pairing $<,>$ on $V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ to a $(\mathbb{A}^{\infty,p})^\times$-scalar multiple of the λ-Weil pairing (we denote by $V^p A_s$ the Tate space of A_s away from p).

Two quadruples $(A, \lambda, i, \bar{\mu}^p)$ and $(A', \lambda', i', (\bar{\mu}^p)')$ are equivalent if there exists a prime-to-p isogeny $\beta : A \to A'$ which takes λ to a $\mathbb{Z}_{(p)}^\times$-multiple of λ', i to i' and $\bar{\mu}$ to $(\bar{\mu}^p)'$.

As in 2.1.5 $\mathcal{X}_{U^p}(S, s)$ is canonically independent on s. We denote again by $\mathcal{X}_{U^p}$ the induced functor on the category of connected locally noetherian $\mathcal{O}_{E_u}$-schemes. We also extend $\mathcal{X}_{U^p}$ to all locally noetherian $\mathcal{O}_{E_u}$-schemes as in 2.1.5.

The functor $\mathcal{X}_{U^p}$ on the category of locally noetherian $\mathcal{O}_{E_u}$-schemes to sets is represented by a projective scheme $\mathcal{X}_{U^p}$ over $\mathcal{O}_{E_u}$ and there is a canonical isomorphism

$$\mathcal{X}_{U^p} \times_{\mathrm{Spec}\, \mathcal{O}_{E_u}} \mathrm{Spec}\, E_u = X_{U^p(0)} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, E_u$$

(see [**14**], p. 113).

Proposition 2.3. — *Let U^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Let x be a closed point of $\mathcal{X}_{U^p} \times_{\mathcal{O}_{E_u}} k(u)^{ac}$, associated to a quadruple $(A, \lambda, i, \bar{\mu}^p)$.*

The formal completion $(\mathcal{X}_{U^p} \times_{\mathcal{O}_{E_u}} \mathcal{O}_{\hat{E}_u^{nr}})_x^\wedge$ is the universal formal deformation space of the Barsotti-Tate group $G = \epsilon A[u^\infty]/k(u)^{ac} = \bar{\mathbb{F}}_p$, thus

$$(\mathcal{X}_{U^p} \times \mathcal{O}_{\hat{E}_u^{nr}})_x^\wedge \simeq \mathrm{Spf}\, W(\bar{\mathbb{F}}_p)[[T_1, \ldots, T_{q(n-q)}]].$$

Proof. — By definition, the completion $(\mathcal{X}_{U^p} \times \mathcal{O}_{\hat{E}_u^{nr}})_x^\wedge$ is the formal deformation space for deforming (A, λ, i). By Serre-Tate Theorem, this is the same as deforming $(A[p^\infty], \lambda, i)$, and since $\lambda : A[u^\infty] \to A[(u^c)^\infty]$ is an isomorphism, this is also the same as deforming $(A[u^\infty], i)$. Finally, we observe that deforming the $\mathcal{O}_{B_u}$-module $A[u^\infty]$ is equivalent to deforming the $\mathbb{Z}_p = \epsilon\mathcal{O}_{B_u}$-module $G = \epsilon A[u^\infty]$.

Since the Barsotti-Tate group G has dimension q and height h, its formal deformation space is isomorphic to $\mathrm{Spf}\, W(\bar{\mathbb{F}}_p)[[T_1, \ldots, T_{q(n-q)}]]$ (see [**17**]). □

Corollary 2.4. — *For any sufficiently small open compact subgroup U^p of $G(\mathbb{A}^{\infty,p})$, the Shimura variety $\mathcal{X}_{U^p}$ is a smooth projective scheme over $\mathcal{O}_{E_u}$.*

2.1.11. If $V^p \subset U^p$ then there is a natural finite étale morphism $\mathcal{X}_{V^p} \twoheadrightarrow \mathcal{X}_{U^p}$ which is compatible with the map $X_{V^p(0)} \twoheadrightarrow X_{U^p(0)}$ defined in 2.1.5. Moreover, if V^p is normal in U^p then the map is Galois with Galois group U^p/V^p (see [**14**], Lemma IV.4.1, part (6)).

There is a natural action of the group $G(\mathbb{A}^{\infty,p})$ on the integral models of the Shimura varieties, which is compatible with the action we previously defined on the Shimura varieties. More precisely, for any $g \in G(\mathbb{A}^{\infty,p})$ and any open compact sufficiently small subgroup U^p of $G(\mathbb{A}^{\infty,p})$, there exists a natural finite étale morphism

$$g : \mathcal{X}_{U^p} \to \mathcal{X}_{g^{-1}U^p g}$$

defined by setting $(A, \lambda, i, \bar{\mu}) \mapsto (A, \lambda, i, \overline{\mu g})$, and whose restriction to the generic fibers is the morphism

$$g : X_{U^p(0)} \to X_{g^{-1}U^p(0)g},$$

we defined in section 2.1.7.

2.1.12. We remark that the above action of $G(\mathbb{A}^{\infty,p})$ on the integral models of the Shimura varieties extends to an action of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \subset G(\mathbb{A}^\infty)$, also compatible with the previously defined action on the generic fibers.

In fact, let $g \in \mathbb{Q}_p^\times$ and U^p be an open compact sufficiently small subgroup of $G(\mathbb{A}^{\infty,p})$. Then, we have $g^{-1}U^p(0)g = U^p(0)$. Let us assume $\mathrm{val}_p(g) \leq 0$ and define a morphism

$$g : \mathcal{X}_{U^p} \to \mathcal{X}_{U^p}$$

by setting $(A, \lambda, i, \bar{\mu}) \mapsto (A/A[(u^c)^{-val_p(g)}], \lambda', i', \bar{\mu}')$, where the structures on the abelian variety $A/A[(u^c)^{-val_p(g)}]$ are induced by the ones on A via the isogeny $g : A \twoheadrightarrow A/A[(u^c)^{-val_p(g)}]$, i.e.

- λ' is the unique prime-to-p polarization such that $g^\vee \circ \lambda' \circ g = p^{-\mathrm{val}_p(g)}\lambda$;
- for all $b \in \mathcal{O}_B$, we have $g^{-1} \circ i'(b) \circ g = i(b) \in \mathrm{End}(A) \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$;
- $\bar{\mu}' = \overline{\mu \circ g}$.

Let us choose $v \in \mathcal{O}_E$ such that $\mathrm{val}_u(v) = 0$ and $\mathrm{val}_{u^c}(v) = -val_p(g)$. Then the isogeny $v : A \to A$ gives rise to an equivalence between the quadruples $(A/A[(u^c)^{-val_p(g)}], \lambda', i', \overline{\mu}')$ and $(A, \lambda, i, \overline{\mu \circ v})$. It follows that the morphism $g : \mathcal{X}_{U^p} \to \mathcal{X}_{U^p}$ is indeed an isomorphism and also that on the generic fibers it restricts to the morphism

$$g : X_{U^p(0)} \to X_{g^{-1}U^p(0)g},$$

defined in section 2.1.7.

2.1.13. Let us reinterpret some of the above definitions and results in terms of the cohomology of the Shimura varieties.

Let l be a prime number, $l \neq p$, and consider the constant abelian torsion étale sheaf $\mathbb{Z}/l^r\mathbb{Z}$, for any integer $r \geq 1$.

For any level $U \subset G(\mathbb{A}^\infty)$, we consider the étale cohomology of the Shimura varieties over E_u with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$, $H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Z}/l^r\mathbb{Z})$, for any integer $i \geq 0$. They form an A-R l-adic system, we write

$$H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Q}_l) = \varprojlim_r H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Z}/l^r\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

For any $U' \subset U$, the natural morphisms $X_{U'} \to X_U$ give rise to some morphisms

$$H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Q}_l) \to H^i_{et}(X_{U'} \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Q}_l).$$

The groups $H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Q}_l)$, together with the above morphisms, form an inductive system. For all $i \geq 0$, we write

$$H^i(X, \mathbb{Q}_l) = \varinjlim_U H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Q}_l).$$

For any $g \in G(\mathbb{A}^\infty)$, the morphisms $g : X_U \to X_{g^{-1}Ug}$ also give rise to some morphism among the l-adic cohomology groups of the Shimura varieties, and morever the induced morphisms piece together in an isomorphism of the direct limit $H^i(X, \mathbb{Q}_l)$. These isomorphisms define an action of $G(\mathbb{A}^\infty)$ on the groups $H^i(X, \mathbb{Q}_l)$, for all i, and moreover it is easy to see that this action commutes with the natural action of the Weil group $W_{E_u} = W_{\mathbb{Q}_p}$ $(E_u = \mathbb{Q}_p)$. Further more, the $H^i(X, \mathbb{Q}_l)$ are admissible representations of the product $W_{\mathbb{Q}_p} \times G(\mathbb{A}^\infty)$.

In the following, we are interested in studing the virtual representation of $W_{\mathbb{Q}_p} \times G(\mathbb{A}^\infty)$

$$H^\bullet(X, \mathbb{Q}_l) = \sum_i (-1)^i H^i(X, \mathbb{Q}_l).$$

We will also work with torsion coefficients and, for any integer $r \geq 1$, consider the $\mathbb{Z}/l^r\mathbb{Z}$-representations of $W_{\mathbb{Q}_p} \times G(\mathbb{A}^\infty)$

$$H^i(X, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_U H^i_{et}(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Z}/l^r\mathbb{Z}),$$

for all $i \geq 0$. Let us remark that these representations are smooth, but not *a priori* admissible.

2.2. Newton polygon stratification. — In [**25**] Oort introduces a statification on the reduction in characteristic $p > 0$ of a moduli space of abelian varieties. This stratification is called the Newton polygon stratification and is defined in terms of the Newton polygons of the Barsotti-Tate groups associated to the p-torsion of the abelian varieties. In particular, each Newton polygon stratum is characterized by the property that the Barsotti-Tate group over it has constant Newton polygon, i.e. costant isogeny class. When Barsotti-Tate groups considered have dimension greater than one, to each isogeny class correspond many isomorphism classes. In [**26**], Oort defines some closed subvarieties inside each Newton polygon stratum, which are the loci where the Barsotti-Tate groups associated to the abelian varieties have constant isomorphism class. We refer to these varieties as the leaves of Oort's foliation.

In this section we shall recall the definitions of the Newton polygon stratification and of the leaves, in the context of the Shimura varieties introduced in 2.1. Moreover, we shall give an alternative proof of the fact that the leaves are closed and smooth subvarieties of the Newton polygon strata.

2.2.1. Let U^p be as in 2.1.10 and denote by $\bar{X}$ the reduction $\mathcal{X}_{U^p} \times_{\mathrm{Spec}\mathcal{O}_{E_u}} \mathrm{Spec}k(u)$, where $k(u)$ is the residue field of $\mathcal{O}_{E_u}$ ($k(u) = \mathbb{F}_p$).

Let $\mathcal{A}$ be the universal abelian variety over $\bar{X}$. It follows from the definition of the moduli space that there is a natural action of $\mathcal{O}_B$ on $\mathcal{A}$ and therefore an action of $\mathcal{O}_{B_u}$ on $\mathcal{A}[p^\infty]$. Let $\epsilon \in \mathcal{O}_{B_u}$ be the idempotent element defined in 2.1.1 and write $\mathcal{G} = \epsilon\mathcal{A}[p^\infty]$. $\mathcal{G}$ is a Barsotti-Tate group of height h and dimesion q over $\bar{X}$ (see section 2.1.4). For any point x of $\bar{X}$ we denote by $\alpha(x)$ the Newton polygon of $\mathcal{G}_x$.

Proposition 2.5. — *(see* [**25**]*, section 2.3, p. 387) Let α be a Newton polygon of height h and dimension q. The set of the points x of $\bar{X}$ such that $\alpha(x) \leq \alpha$ (i.e. such that no point of $\alpha(x)$ is stricly below α) is a closed subset of $\bar{X}$.*

We denote by $\bar{X}^{[\alpha]}$ the corresponding reduced subscheme of $\bar{X}$ and call it the closed Newton polygon stratum determined by α.

We also define the open Newton polygon stratum $\bar{X}^{(\alpha)} = \bar{X}^{[\alpha]} - \cup_{\beta<\alpha}\bar{X}^{[\beta]}$.

2.2.2. The following definition of a leaf is due to Oort, who also proved that the leaves are closed smooth subvarieties of the Newton polygon strata (see [**26**]). Here, we provide an alternative proof of these facts in our context. More precisely, we now show that the leaves are locally closed smooth subvarieties of the Newton polygon strata. Later, in proposition 4.7, we will prove that the leaves are indeed closed. We thank A. Vasiu for suggesting to us the proof of proposition 2.7.

Lemma 2.6. — *Let H be a Barsotti-Tate group defined over a finite extension k_0 of $\mathbb{F}_p$. Let X be a scheme over $\bar{\mathbb{F}}_p \supset k_0$ and G/X a Barsotti-Tate group.*

Then, the set

$$C_H = \{x \in X |\, G_x \times_{k(x)} k(x)^{ac} \simeq H \times_{k_0} k(x)^{ac}\}$$

is a constructible subset of X.

Moreover, if we further assume that $X = X_0 \times_{k_0} \bar{\mathbb{F}}_p$, for some k_0-scheme X_0, and G is the pullback of a Barsotti-Tate group over X_0, then the set C_H is a constructible subset of X_0.

Proof. — In [**31**], Zink proves that, for any positive integer N, the functor $Y_N = Y_{H,N}$, defined as

$$Y_N(T/X) = \{ \text{ isomorphisms } \mathcal{G}_T[p^N] \simeq H_T[p^N]\},$$

is represented by a scheme of finite type over X.

Thus, in particular, $C_{H,N} = \mathrm{im}(Y_N \to X)$ is a constructible subset of X,

$$C_{H,N} = \{x \in X |\, G_x[p^N] \times k(x)^{ac} \simeq H[p^N] \times k(x)^{ac}\}.$$

Moreover, Zink also proves that, for any algebraically closed field $K/\bar{\mathbb{F}}_p$, there exists a positive integer N_K such that

$$C_{H,N}(K) = C_{H,N_K}(K) = \{x \in X(K) |\, G_x \simeq H \times K\},$$

for all $N \geq N_K$.

In particular, for $K = \bar{\mathbb{F}}_p$ and $N_0 = N_{\bar{\mathbb{F}}_p}$, we have that for all $N \geq N_0$

$$C_{H,N}(\bar{\mathbb{F}}_p) = C_{H,N_0}(\bar{\mathbb{F}}_p) = \{x \in X(\bar{\mathbb{F}}_p) |\, G_x \simeq H \times \bar{\mathbb{F}}_p\}.$$

Since the subsets $C_{H,N}$ are constructible, the above equalities imply that $C_{H,N} = C_{H,N_0}$, for all $N \geq N_0$, and thus

$$C_H = \{x \in X |\, G_x \times k(x)^{ac} \simeq H \times k(x)^{ac}\} = C_{H,N_0}$$

and is a contructible subset of X.

Finally, in order to show that the subscheme C_H is a constructible subset of X_0, it suffices to observe that the set of closed points of C_H is stabilized by the action of the Galois group $Gal(\bar{\mathbb{F}}_p/k_0)$ (which follows from the fact that the Barsotti-Tate groups G_0 and H are defined over X_0/k_0 and k_0, respectively). □

Proposition 2.7. — *Let H be a Barsotti-Tate group over a finite extension $k_0/\mathbb{F}_p$, of height h and dimension q. We denote by α the Newton polygon of H, by $\bar{X}^{(\alpha)}$ the associated open Newton polygon stratum and by $\mathcal{G}$ the universal Barsotti-Tate group over $\bar{X}^{(\alpha)}$.*

Then there exists a unique reduced locally closed subscheme C_H of $\bar{X}^{(\alpha)} \times k_0$ such that for any geometric point $x \in \bar{X}^{(\alpha)} \times k_0$ we have $G_x \simeq H$ if and only if $x \in C_H$.

Moreover, the scheme C_H is smooth.

Proof. — By the previous lemma, we know that the set

$$C_H = \{x \in \bar{X}^{(\alpha)} \times k_0 |\, \mathcal{G}_x \times k(x)^{ac} \simeq H \times k(x)^{ac}\}$$

is constructible. Thus, it remains to prove that the set C_H is locally closed (in which case it inherits a unique structure of reduced locally closed subscheme), and moreover that, as a subscheme of $\bar{X}^{(\alpha)} \times k_0$, C_H is smooth.

Let us fix an algebraically closed field $\Omega/\bar{\mathbb{F}}_p$ with trascendence degree of the continuum (thus all the point of $\bar{X}^{(\alpha)}$ can be viewed as Ω-points).

Since $C = C_H \times \Omega$ is constructible, C is a finite union of locally closed sets C_i with irreducible closure. Moreover, for each closed point x of C, the ring $\mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x}$ and the restriction of $\mathcal{G}$ over it are independent of x. For simplicity, we denote this ring by A and its Barsotti-Tate group by G/A. For any prime ideal $\mathcal{P}$ of A (neccessarly closed), we choose a morphism $f_{\mathcal{P}} : A \to \Omega$ with $\ker f_{\mathcal{P}} = \mathcal{P}$ (such a morphism exists since Ω

has degree of the continuum). We denote by J the intersection of all primes $\mathcal{P}$ of A such that $f_{\mathcal{P}\,*}\mathcal{G} \simeq H$.

Let x_0 be a smooth closed point of C, i.e. x_0 is in the closure of only one C_i, say C_0, and it is a smooth point of C_0. Let I_0 denote the ideal in $\mathcal{O}_{\bar{X}\times\Omega,x_0}$ defining $\mathcal{O}_{C_0,x_0}$. Then, a prime ideal $\mathcal{P}$ of $\mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x_0}$ contains $I_0^{\wedge}$ if and only if it contains I_0, or equivalently if and only if $f_{\mathcal{P}\,*}\mathcal{G} \simeq H$. As C_0 is by definition reduced we have

$$I_0^{\wedge} = \cap_{\mathcal{P}\supset I_0}\mathcal{P}.$$

It follows that, under an isomorphism $(\mathcal{G}, \mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x_0}) \simeq (G, A)$, the ideal $I_0^{\wedge}$ corresponds to the ideal J. In particular, the ring A/J is formally smooth and a prime ideal $\mathcal{P}$ of A contains J if and only if $f_{\mathcal{P}\,*}\mathcal{G} \simeq H$.

Now let $x \in C_i - C_j$ be a closed point such that $x \in \bar{C}_j$ (the closure of C_j). Let $\mathcal{P}'$ be the prime ideal of $\mathcal{O}_{\bar{X}\times\Omega,x}$ defining $\bar{C}_j$ and let J_x be the ideal of $\mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x}$ which corresponds to J under an isomorphism $(\mathcal{G}, \mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x}) \simeq (G, A)$. Since $\mathcal{P}'$ is a prime in C we have $\mathcal{P}'^{\wedge} \supset J_x$. Thus, for any other prime $\mathcal{P} \supset \mathcal{P}'$, we have $\mathcal{P}^{\wedge} \supset J_x$ and so $f_{\mathcal{P}^{\wedge}\,*}\mathcal{G} \simeq H$, i.e. $\mathcal{P} \in C$. Hence, there exists a neighbourhood U_x of x in $\bar{X}$ such that $U_x \cap \bar{C}_j$ is contained in C. We conclude that for any point x in C, there exists a neighbourhood U_x in $\bar{X}$ such that $U_x \cap \bar{C} = C$. Thus C is a locally closed subset of $\bar{X}$, and by definition is contained in $\bar{X}^{(\alpha)}$ (thus, it is also a locally closed subset of $\bar{X}^{(\alpha)}$).

Moreover, if x is any point of C and $I \subset \mathcal{O}_{\bar{X}\times\Omega,x}$ is the ideal defining C, then (as C is reduced) $I^{\wedge}$ is the intersection of the prime ideals of $\mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x}$ which contain I, i.e. the intersection of prime ideals $\mathcal{P}$ such that $f_{\mathcal{P}\,*}\mathcal{G} \simeq H$. Thus, under an isomorphism $(\mathcal{G}, \mathcal{O}^{\wedge}_{\bar{X}\times\Omega,x}) \simeq (G, A)$, $I^{\wedge}$ corresponds to J and we see that C is formally smooth at x and hence smooth.

□

2.2.3. We remark that both the definition of the Newton polygon stratification and the definition of Oort's foliation of the Newton polygon strata are independent of the level structure away from p. Thus, as the level U^p of the structure away from p varies, the corresponding morphisms among the reductions of the Shimura varieties preserve the Newton polygon startification and also Oort's foliation. Analogously, the action of the group $G(\mathbb{A}^{\infty,p})$ on the Shimura varieties with no level structure at p also respects the Newton polygon strata and the leaves inside them. We thus obtain, by restriction, an action of the group $G(\mathbb{A}^{\infty,p})$ on the Newton polygon strata (resp. on the leaves) via finite étale morphisms (see section 2.1.11).

2.3. Some distinguished Barsotti-Tate groups. — In this section we shall define a distinguished p-divisible group Σ_α over $\mathbb{F}_p$, for each Newton polygon α of dimension q and height h. The associated leaf $C_\alpha = C_{\Sigma_\alpha}$ inside the corrisponding Newton polygon stratum $\bar{X}^{(\alpha)}$ is called the central leaf and it will play an important

role in our work. In this section, we shall also discuss some of the properties of the group of the quasi-selfisogenies of Σ_α (for all α).

2.3.1. We start by recalling the definition of a certain simple isoclinic p-divisible group Σ_λ over $\mathbb{F}_p$, for any given slope $\lambda \in \mathbb{Q}$ ($0 \leq \lambda \leq 1$). The definition of Σ_λ is introduced by de Jong and Oort in [**19**] (Paragraph 5.3, p. 227).

Let λ be a rational number, $0 \leq \lambda \leq 1$ and write $\lambda = \frac{n}{m+n}$ (with n, m relatively prime). We describe the p-divisible group Σ_λ over $\mathbb{F}_p$ by its covariant Dieudonné module $M(\Sigma_\lambda)$. This is the free $\mathbb{Z}_p$-module with basis $e_0, e_1, \ldots, e_{m+n-1}$ on which the actions of F and V are given by $F(e_i) = e_{i+n}$ and $V(e_i) = e_{i+m}$ for all i (for any non-negative integer j we write $e_j = p^l e_i$ if $j = i + l(m+n)$ with $0 \leq i \leq m+n-1$ and $l \geq 0$). The p-divisible group Σ_λ has height $m+n$ and is isoclinic of slope λ.

We denote by t the endomorphism of $M(\Sigma_\lambda)$ (and also the corresponding isogeny on Σ_λ) which maps e_i to e_{i+1} for all i. It is an isogeny of degree p and $t^{m+n} = p$.

Proposition 2.8. — *(see* [**19**]*, Lemma 5.4, p. 227) We choose $r, s \in \mathbb{Z}$ such that $rm + sn = 1$. For every algebraically closed field k of characteristic p we have*

$$\mathrm{End}\left((\Sigma_\lambda)_k\right) = W(\mathbb{F}_{p^{m+n}})[t]$$

where $xt = t\sigma^{s-r}(x)$ for all $x \in W(\mathbb{F}_{p^{m+n}})$ (we denote by σ the Frobenius map). The ring $\mathrm{End}\left((\Sigma_\lambda)_k\right)$ *is a non-commutative discrete valuation ring with uniformizer t and valuation define by* $\log_p \deg(\cdot)$

We write $\mathcal{O}_\lambda = W(\mathbb{F}_{p^{m+n}})[t]$ and $D_\lambda = \mathcal{O}_\lambda[1/p]$. D_λ is a central simple algebra over $\mathbb{Q}_p$ of rank $(n+m)^2$ and invariant λ, and $\mathcal{O}_\lambda$ is a maximal order inside D_λ.

2.3.2. For convenience, we also recall de Jong's and Oort's Isogeny theorem. This result illustrates one of the properties which make isoclinic p-divisible groups easier to understand than general p-divisible groups.

Theorem 2.9. — *(see* [**19**]*, Corollary 2.17, p. 217-218) Let A be a noetherian complete local domain with algebraically closed residue field k, normal and with field of fractions K of characteristic p. Let G be an isoclinic p-divisible group over* $\mathrm{Spec}\, A$.

Then there exists a p-divisible group G_0 over $\mathrm{Spec}\, k$ *and an isogeny over A*

$$G_0 \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, A \to G.$$

2.3.3. An isoclinic p-divisible group G of slope λ is called slope divisible if there exists an integer $b > 0$ such that $p^{-\lambda b} F^b$ is an isogeny, or equivalently an isomorphism between G and $G^{(p^b)}$.

It follows from the definition that the p-divisible groups Σ_λ are slope divisible, for all λ.

2.3.4. Let α be a Newton polygon of dimension q and height h, and denote by $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ its slopes. For each i, we denote by r_i the multiplicity of the slope λ_i in α and define the Barsotti-Tate groups

$$\Sigma_\alpha^i = \Sigma_{\lambda_i}^{\oplus r_i} \qquad \text{and} \qquad \Sigma_\alpha = \oplus_i \Sigma_\alpha^i.$$

Thus, for all i, Σ_α^i are slope divisible isoclinic p-divisible groups of slope λ_i defined over $\mathbb{F}_p$, and Σ_α is a p-divisible group over $\mathbb{F}_p$ with Newton polygon equal to α. Moreover, the p-divisible group Σ_α^i may be identified with the isoclinic pieces of Σ_α (see section 2.4).

2.3.5. For every algebraically closed field k of characteristic p we have

$$(\mathrm{End}_k(\Sigma_\alpha) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times = \prod_i \mathrm{GL}_{r_i}(D_{\lambda_i}).$$

We write T_α for the group of the quasi-selfisogenies of Σ_α, i.e. $T_\alpha = \prod_i \mathrm{GL}_{r_i}(D_{\lambda_i})$, and Γ_α for the group of the automorphism of Σ_α, i.e. $\Gamma_\alpha = \prod_i \mathrm{GL}_{r_i}(\mathcal{O}_{\lambda_i})$ ($\Gamma_\alpha \subset T_\alpha$).

2.3.6. We now fix a Newton polygon α of dimension q and height h (and write $\Sigma = \Sigma_\alpha$). In the following, we analyse in more detail the group $T = T_\alpha$ and, in particular, we introduce a certain submonoid $S = S_\alpha$ of T, where $S \supset \Gamma = \Gamma_\alpha$. (The role of the submonoid S will be explained in section 3.4.)

Let $\rho \in T$. For any $i = 1, \dots k$ we denote by $\rho_i : \Sigma^i \to \Sigma^i$ the quasi-selfisogeny of Σ^i induced by ρ and define:

$$e_i = e_i(\rho) = e(\rho_i) = \min\{m \in \mathbb{Z} \mid p^m \rho_i \text{ is an isogeny}\}$$

and

$$f_i = f_i(\rho) = f(\rho_i) = \max\{m \in \mathbb{Z} \mid (p^m \rho_i)^{-1} \text{ is an isogeny}\}.$$

We also write $e = e(\rho) = \max_i e_i(\rho)$ and $f = f(\rho) = \min_i f_i(\rho)$. It follows from the definitions that e and f are respectively the minimal and maximal integers such that $p^e \rho$ and $(p^f \rho)^{-1}$ are isogenies.

It is easy to see that if ρ^{-1} is an isogeny then f_i, e_i are respectively the maximal and the minimal integers such that

$$\Sigma^i[p^{f_i}] \subset \ker(\rho_i^{-1}) \subset \Sigma^i[p^{e_i}].$$

In particular, $e_i \geq f_i$ for all i.

Definition 2.10. — We define $S \subset T$ to be the subgroup

$$S = \{\rho \in T \mid \rho^{-1} \text{ is an isogeny}, e_i \leq f_{i-1}, 1 < i \leq k\}.$$

2.3.7. For all $i = 1, \dots k$, we denote by $t_i \in \mathrm{End}(\Sigma_{\lambda_i})$ the uniformizer we defined in section 2.3, and write $\tau_i = t_i^{\oplus r_i} \in \mathrm{End}(\Sigma^i = \Sigma_{\lambda_i}^{\oplus r_i})$. We define fr to be the isogeny of $\Sigma = \oplus_i \Sigma^i$

$$fr = \oplus_i \tau_i^{a_i},$$

where the a_i denote the numerators of the slopes λ_i (written in minal form), for all i. Equivalently, fr is the unique isogeny which fits in the following commutative diagram (see section 2.3.7)

$$\begin{tikzcd} & \Sigma \arrow[dl, "F"'] \arrow[dr, "fr"] & \\ \Sigma^{(p)} & & \Sigma. \arrow[ll, "\simeq", "\nu"'] \end{tikzcd}$$

where we may identify Σ and $\Sigma^{(p)}$, via ν, since the Barsotti-Tate group Σ is defined over $\mathbb{F}_p$.

2.3.8. If $B = \mathrm{lcm}(b_1, \dots, b_k)$, where the b_i are the denominators of the rational numbers λ_i (written in minimal form), then it follows from the definition of the morphism fr that $fr^B = \oplus_i p^{\lambda_i B}$ and, in particular, that fr^B is in the center of T (for all i, $\lambda_i B \in \mathbb{Z}$).

Lemma 2.11. — *Maintaing the above notations.*

The set S is a submonoid of T and has the properties that the quasi-selfisogenies $p^{-1}, fr^{-B} \in S$ and also that $T =< S, p, fr^B >$.

Proof. — To see that S is a submonoid of T, it suffices to remark that

$$e_i(\omega\rho) \leq e_i(\rho) + e_i(\omega) \text{ and } f_i(\omega\rho) \geq f_i(\rho) + f_i(\omega),$$

which follow easily for the definitions.

It is also clear from the definition of S that $p^{-1}, fr^{-B} \in S$. In fact, we have

$$e_i(p^{-1}) = 1 = f_{i-1}(p^{-1}) \text{ and } e_i(fr^{-B}) = \lambda_i B > f_{i-1}(fr^{-B}) = \lambda_{i-1} B,$$

for all $i = 2, \dots k$.

We now show that $T =< S, p, fr^B >$.

For any $\rho \in T$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $p^m \rho^{-1}$ is an isogeny, i.e. such that $\rho = p^m \omega$ with ω^{-1} an isogeny ($\omega = p^{-m}\rho$). Moreover, for any isogeny ρ we have

$$e_i(p^n \rho) = e_i(\rho) - n \text{ and } f_i(p^n \rho) = f_i(\rho) - n,$$

and thus ρ satisfies the conditions $e_i(\rho) \geq f_{i-1}(\rho)$, for all $1 < i \leq k$, if and only if $p^n \rho$ does for some integer n.

Then, it suffices to prove that for any isogeny ρ there exists a positive integer m sucht that $fr^{mB}\rho$ satisfies the above inequalities. This fact follows directly from the following inequalities:

$$e_i(fr^{mB}\rho) \leq e_i(\rho) + m\lambda_i B \text{ and } f_{i-1}(fr^{mB}\rho) \geq f_{i-1}(\rho) + m\lambda_{i-1} B,$$

where $\lambda_i < \lambda_{i-1}$, for all $i = 2, \dots k$. □

2.4. Slope filtration. — Dieudonné's classification of p-divisible groups implies that any p-divisible group defined over a perfect field is isogenous to a direct product of slope divisible isoclinic p-divisible groups. In [**30**] Zink investigates what remains true if the perfect field is replaced by a ring of characteric p. In particular, Zink shows that over a regular scheme of characteric p any p-divisible group is isogenous to a p-divisible group which admits a filtration by p-divisible subgroups which factors are slope divisible isoclinic p-divisible groups ordered with the decreasing order of the slopes.

In this section we shall recall some of the definitions and results from [**30**]. For completeness, let us mention that this result over a smooth curve was previously established by Katz in [**21**], and also that, more recently, it was extended by Oort and Zink to the case over a normal scheme (see [**27**]).

2.4.1. Let H be a p-divisible group over a scheme S of characteristic p and $\lambda \in \mathbb{Q}$. We say that H is slope divisible with respect to λ if there are positive integers a, b such that $\lambda = \frac{a}{b}$ and the quasi-isogeny $p^{-a}F^b : H \to H^{(p^b)}$ is an isogeny. If H isoclinic and slope divisible of slope λ then the above isogeny is in fact an isomorphism.

Theorem 2.12. — *(see* [**30**]*, Theorem 7, p. 9) Let S be a regular scheme over $\mathbb{F}_p$. Let H be a p-divisible group over S with constant Newton polygon. We denote by $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ the slopes of H.*

Then there is a p-divisible group G over S which is isogenous to H, and which has a filtration by closed immersions of p-divisible groups:

$$0 = G_0 \subset G_1 \subset \cdots \subset G_k = G$$

such that for each i the factor G_i/G_{i-1} is isoclinic of slope λ_i and G_i is slope divisible with respect to λ_i.

We say a p-divisible group with the properties described for G completely slope divisible and call its filtration the slope filtration.

We remark that any isogeny among two p-divisible groups endowed with slope filtrations respects the filtrations.

In fact, suppose $\phi : G \to H$ is an isogeny among p-divisible group endowed with a slope filtration over a reduced scheme S of characteristic p. When $k = 1$, the statement is trivial, so we may assume $k \geq 2$. Moreover, using induction on k, it suffices to prove the statement for $k = 2$.

First, we assume $S = \operatorname{Spec} K$ and we consider the morphism

$$\phi_1 : G_1 \hookrightarrow G \to H \twoheadrightarrow H_2.$$

By Diedonné's theory, there is no non zero morphism between two isoclinic Barsotti-Tate groups with different slopes (see [**30**], p.13). Thus, the morphism ϕ_1 is identically zero, or equivalently the isogeny ϕ maps G_1 to H_1.

For obtaining the same result in the general case, it suffices to remark that the above considerations imply that $(\phi_1)_x$ vanishes for any point x in S. Thus, the morphism ϕ_1 over S is zero.

Finally, we remark that from the existence of an isogeny ψ such that $\phi\psi = p^d$, for some d, we deduce that the induced morphisms ϕ_i are isogenies, for all i.

2.4.2. We conclude this section with the following two remarks which will play a key role in our definition of Igusa varieties (see section3).

Remark 2.13. — (see [**27**]) Let G be a Barsotti-Tate group over a field K of positive characteristic p. Then, G is completely slope divisible if and only if $G \times_K L$ is completely slope divisible, for some $L \supset K$.

Remark 2.14. — (see [**30**], proof of Theorem 7, p. 15) Let $\mathcal{G}$ be a Barsotti-Tate group over a connected regular scheme S over $\mathbb{F}_p$. Let η be the generic point of S and assume that $\mathcal{G}_\eta$ is completely slope divisible (i.e. admits a slope filtration as in theorem 2.12).

Then, the Barsotti-Tate group $\mathcal{G}$ over S is also completely slope divisible.

2.5. Rapoport-Zink spaces. — In [**29**] Rapoport and Zink formulate moduli problems of (PEL) type for Barsotti-Tate groups, associated to any decent Barsotti-Tate group over a perfect field k of characteristic p. They prove that the corresponding moduli spaces exist in the category of rigid analytic spaces over the fraction field of the Witt vectors of k and, in the cases when the moduli problems impose no level structures, they admit integral models in the category of formal schemes over $W(k)$.

In this section we shall recall their constructions together with some of the main results in the case which is our interest.

2.5.1. Let $\mathbb{X}$ be a decent Barsotti-Tate group over a perfect field k of characteristic p. (We recall that a Barsotti-Tate group is called decent if it arises by base change from a Barsotti-Tate defined over a perfect field.) To $\mathbb{X}$ we associate a functor $\mathcal{M} = \mathcal{M}_{\mathbb{X}}$ on the category of schemes S over $W = W(k)$ such that p is locally nilpotent on S to sets. For such a scheme S, we denote by $\bar{S}$ the closed subscheme defined by the sheaf of ideals $p\mathcal{O}_S$, and we view it as a scheme over $\operatorname{Spec} k$.

We define $\mathcal{M}(S)$ to be the set of equivalence classes of pairs (H, β) where:

- H is a Barsotti-Tate group over S;
- $\beta : \mathbb{X}_{\bar{S}} \to H_{\bar{S}}$ is a quasi-isogeny.

Two pairs (H, β) and (H', β') are equivalent if the quasi-isogeny $\beta' \circ \beta^{-1} : H_{\bar{S}} \to H'_{\bar{S}}$ lifts to an isomorphism between H and H' over S.

Theorem 2.15. — *(see* [**29**]*, Theorem 2.16, p. 54) The functor $\mathcal{M}$ is represented by a formal scheme over* $\operatorname{Spf} W$*, which is formally locally of finite type.*

For any other Barsotti-Tate group $\mathbb{X}'$ and isogeny $\phi : \mathbb{X}' \to \mathbb{X}$, there is a canonical isomorphism between the corresponding Rapoport-Zink spaces

$$\phi^* : \mathcal{M}_{\mathbb{X}} \to \mathcal{M}_{\mathbb{X}'}$$

$$(H, \beta) \mapsto (H, \beta \circ \phi).$$

2.5.2. For any pair of positive integers n, d, we denote by $\mathcal{M}^{n,d}$ the subfunctor of $\mathcal{M}$ defined by the condition that $p^n\beta$ is an isogeny and its kernel is contained inside $\mathbb{X}[p^d]$ (or equivalently, both $p^n\beta$ and $p^{d-n}\beta^{-1}$ are isogenies). The functor $\mathcal{M}^{n,d}$ is represented by the p-adic completion of a scheme of finite type over $\operatorname{Spf} W$ and can be identified to a closed formal subscheme of $\mathcal{M}$, of finite type over $\operatorname{Spf} W$. Moreover, it follows from the definitions that, in the sense of Zariski sheaves, we have

$$\mathcal{M} = \varinjlim_{n,d} \mathcal{M}^{n,d}.$$

We call the spaces $\mathcal{M}^{n,d}$ the truncated Rapoport-Zink spaces. (Let us remark that the truncated Rapoport-Zink spaces we use are not exactly the ones introduced in [**29**], paragraph 2.22, p. 58. For any pair of positive integers n, d, Rapoport and Zink consider the closed subfunctor $\mathcal{M}_{n,d}$ of $\mathcal{M}$ defined by the condition that $p^n\beta$ is an isogeny of degree less than or equal to p^d. Thus, there are natural inclusions $\mathcal{M}_{n,d} \subset \mathcal{M}^{n,d} \subset \mathcal{M}_{n,dh}$.)

We observe that, for any other Barsotti-Tate group $\mathbb{X}'$ and isogeny $\phi : \mathbb{X}' \to \mathbb{X}$, the corresponding isomorphism $\phi^* : \mathcal{M}_{\mathbb{X}} \to \mathcal{M}_{\mathbb{X}'}$ does not preserve the truncated Rapoport-Zink spaces (nor the open subspaces we define below).

2.5.3. For any pair of positive integers n, d, we define:

$$U^{n,d} = \{t \in \mathcal{M}^{n,d} \mid \exists V \subset \mathcal{M} \text{ open}, t \in V \subset \mathcal{M}^{n,d}\}.$$

It follows from the fact that $\mathcal{M}$ is formally locally of finite type that the $U^{n,d}$ form an open cover of $\mathcal{M}$, i.e.

$$\mathcal{M} = \cup_{n,d} U^{n,d},$$

where the $U^{n,d}$ are open formal subschemes of $\mathcal{M}$, of finite type over $\operatorname{Spf} W$.

2.5.4. Let us assume from now on that the field k is algebraically closed and write K for the fraction field of W. We denote by $\mathcal{M}^{\mathrm{rig}}$ the rigid analytic space associated to the formal scheme $\mathcal{M}$. In [**29**] Rapoport and Zink introduce a tower of rigid analytic coverings of $\mathcal{M}^{\mathrm{rig}}$ over $\operatorname{Spm} K$. We now recall their construction (see [**29**], paragraph 5.34, pp. 254–256).

We first recall the following result.

Proposition 2.16. — *(see* [**29**]*, Proposition 5.17, p. 237). The rigid analytic space* $\mathcal{M}^{\mathrm{rig}}$ *over* $\operatorname{Spm} K$ *is smooth.*

Let $x \in \mathcal{M}^{\mathrm{rig}}(L)$ be represented by the pair $(H, \beta) \in \mathcal{M}(\mathcal{O}_L)$. The p-adic Tate modules $T_p(H)$, for the points $x \in \mathcal{M}^{\mathrm{rig}}$, piece together in a locally constant $\mathbb{Z}_p$-sheaf

for the étale topology on $\mathcal{M}^{\mathrm{rig}}$. We denote this sheaf by $\mathcal{T}$. Namely, for all integers $n \geq 0$, $T_p(H) \otimes \mathbb{Z}_p/p^n$ is the generic fiber of the finite flat group scheme $H[p^n]$, which is étale.

For any open compact subgroup $U \subset GL_n(\mathbb{Z}_p)$, we define $\mathcal{M}_U^{\mathrm{rig}}$ to be the finite étale covering of $\mathcal{M}^{\mathrm{rig}}$ parametrizing the classes modulo U of trivializations of $\mathcal{T}/\mathcal{M}^{\mathrm{rig}}$,

$$\alpha : \mathcal{T} \to \mathbb{Z}_p^n \pmod U.$$

We remark that if

$$U = U(M) = \{A \in GL_n(\mathbb{Z}_p) | A \equiv \mathbf{1}_n (\mathrm{mod}\ p^M)\}$$

the space $\mathcal{M}_{U(M)}^{\mathrm{rig}}$ parametrizes the classes of trivializations of the generic fiber of the p^M-torsion of the universal Barsotti-Tate group over $\mathcal{M}$.

2.5.5. If $U' \subset U$, then there is a natural morphism $\mathcal{M}_{U'}^{\mathrm{rig}} \to \mathcal{M}_U^{\mathrm{rig}}$. More generally, for any two subgroups U', U, an element $g \in GL_n(\mathbb{Q}_p)$ such that $g^{-1}U'g \subset U$ gives rise to a morphism

$$g : \mathcal{M}_{U'}^{\mathrm{rig}} \to \mathcal{M}_U^{\mathrm{rig}}.$$

2.5.6. Let T denote the group of quasi-selfisogenies of $\mathbb{X}$ over k and σ the Frobenius automorphism of W (we also denote by σ its extension to K and the Frobenius automorphism of k).

There is a natural action of the group

$$GL_n(\mathbb{Q}_p) \times T \times Frob^{\mathbb{Z}}$$

on the system of Rapoport-Zink spaces, where the action of $GL_n(\mathbb{Q}_p) \times T$ is linear and the action of $Frob^{\mathbb{Z}}$ is σ-semilinear.

2.5.7. We first define the action of T on the formal scheme $\mathcal{M}$ defined by

$$(H, \beta) \mapsto (H, \beta \circ \rho), \qquad \rho \in T$$

(see [**29**], section 2.33, p. 64). We remark that, for any other Barsotti-Tate group $\mathbb{X}'$ and isogeny ϕ, the isomorphism $\phi^* : \mathcal{M}_{\mathbb{X}} \to \mathcal{M}_{\mathbb{X}'}$ commutes with the action of the group of the quasi-selfisogenies, where we identify $T_{\mathbb{X}}$ with $T_{\mathbb{X}'}$ via the isomorphism $\rho \mapsto \phi^{-1}\rho\phi$, for all $\rho \in T_{\mathbb{X}}$.

The action of T on $\mathcal{M}$ induces an action of T on $\mathcal{M}^{\mathrm{rig}}$, which extends canonically to an action of T on the covers $\mathcal{M}_U^{\mathrm{rig}}$, for all level $U \subset GL_n(\mathbb{Z}_p)$. In fact, for any level U, let us denote a point $t \in \mathcal{M}_U^{\mathrm{rig}}(L)$, for some extension L of K, by a triple $(H, \beta, [\alpha])$, where H is a Barsotti-Tate group defined over the ring of integers $\mathcal{O}_L$ of L, $\beta : \mathbb{X} \to H \times_{\mathcal{O}_L} k$ a quasi-isogeny, and $[\alpha]$ the U-orbit of an isomorphism $\alpha : T_p(H) \to \mathbb{Z}_p^n$. Then, for any $\rho \in T$, we define

$$\rho : \mathcal{M}_U^{\mathrm{rig}} \to \mathcal{M}_U^{\mathrm{rig}}$$

$$(H, \beta, [\alpha]) \mapsto (H, \beta \circ \rho, [\alpha]).$$

It is clear that the above action of T on the system of the Rapoport-Zink spaces $\mathcal{M}_U^{\mathrm{rig}}$ commutes with the previously defined action of $GL_n(\mathbb{Q}_p)$.

2.5.8. We now introduce the σ-semilinear action of Frobenius. Let us recall that defining a σ-semilinear automorphism of the formal scheme $\mathcal{M}$ is equivalent to defining an isomorphism of formal schemes over W

$$Frob : \mathcal{M} \to \mathcal{M}^{(p)},$$

where $\mathcal{M}^{(p)}$ denotes the pullback under σ of $\mathcal{M}$.

If we identify the space $\mathcal{M}^{(p)}$ with the Rapoport-Zink space associated to the Barsotti-Tate group $\mathbb{X}^{(p)}/k$, then we can describe the morphism $Frob$ in terms of its universal property. The morphism $Frob$ is defined by

$$(H, \beta) \mapsto (H, \beta \circ F^{-1}),$$

where $F : \mathbb{X} \to \mathbb{X}^{(p)}$ is the Frobenius morphism of the Barsotti-Tate group $\mathbb{X}$ (see [**29**], section 3.48, pp. 100-101).

We observe that $Frob$ is indeed an isomorphism, since we can define its inverse by setting

$$(G, \rho) \mapsto (G, \rho \circ F).$$

Moreover, it is easy to see that this action commutes with the action of T. In fact, for any quasi-selfisogeny $\rho \in T$, the equality $F \circ \rho = \rho^{(p)} \circ F$ implies

$$Frob \circ \rho = \rho^{(p)} \circ Frob : \mathcal{M} \to \mathcal{M}^{(p)}.$$

Finally, we remark that the automorphism $Frob$ of $\mathcal{M}$ gives rise to an automorphism of $\mathcal{M}^{\mathrm{rig}}$, which is also σ-semilinear and which extends canonically to σ-semilinear automorphisms of the covers $\mathcal{M}_U^{\mathrm{rig}}$, for all level $U \subset GL_n(\mathbb{Z}_p)$. If we denote a point on $\mathcal{M}_U^{\mathrm{rig}}$ by a triple $(H, \beta, [\alpha])$ as before, then, for all level U, we define

$$Frob : \mathcal{M}_U^{\mathrm{rig}} \to (\mathcal{M}_U^{\mathrm{rig}})^{(p)}$$

$$(H, \beta, [\alpha]) \mapsto (H, \beta \circ F^{-1}, [\alpha]).$$

It is clear that the action of $Frob$ on the system of the Rapoport-Zink spaces $\mathcal{M}_U^{\mathrm{rig}}$ commutes with the previously defined action of $GL_n(\mathbb{Q}_p)$, and thus gives rise to an action of the group $GL_n(\mathbb{Q}_p) \times T \times Frob^{\mathbb{Z}}$.

2.5.9. Let us remark that, if we restrict our attention to the reduced fiber $\bar{\mathcal{M}}$ of $\mathcal{M}$ over k, then the action of $T \times Frob^{\mathbb{Z}}$ extends to an action of the product

$$T \times Frob^{\mathbb{Z}} \times Fr^{\mathbb{N}},$$

where Fr is also a σ-semilinear endomorphism of $\bar{\mathcal{M}}$, namely the relative Frobenius.

As before, we describe Fr as a k-linear morphism

$$Fr : \bar{\mathcal{M}} \to \bar{\mathcal{M}}^{(p)},$$

where $\bar{\mathcal{M}}^{(p)}$ is the pullback under $\sigma : k \to k$ of $\bar{\mathcal{M}}$. If we identify the space $\bar{\mathcal{M}}^{(p)}$ with the reduced fiber of the Rapoport-Zink space associated to the Barsotti-Tate group $\mathbb{X}^{(p)}/k$, then the morphism Fr is defined by setting

$$(H, \beta) \mapsto (H^{(p)}, \beta^{(p)}),$$

where $H^{(p)}$ and $\beta^{(p)} : \mathbb{X}^{(p)} \to H^{(p)}$ are the pullbacks under σ of H and $\beta : \mathbb{X} \to H$, respectively.

We observe that indeed the relative Frobenius Fr commutes with the action of $T \times Frob^{\mathbb{Z}}$. For any $\rho \in T_\alpha$,

$$(\beta\rho)^{(p)} = \beta^{(p)}\rho^{(p)},$$

and also

$$(\beta F_{\mathbb{X}}^{-1})^{(p)} = \beta^{(p)}(F_{\mathbb{X}}^{-1})^{(p)} = \beta^{(p)}F_{\mathbb{X}^{(p)}}^{-1},$$

where $F_{\mathbb{X}}$ and $F_{\mathbb{X}^{(p)}}$ are the Frobenius morphism on $\mathbb{X}$ and $\mathbb{X}^{(p)}$, respectively.

2.5.10. We now focus our attention of the Rapoport-Zink space $\mathcal{M}_\alpha = \mathcal{M}_{\Sigma_\alpha}$ over $W(\bar{\mathbb{F}}_p)$, associated to the Barsotti-Tate $\Sigma_\alpha/\bar{\mathbb{F}}_p$, for any given Newton polygon α of dimension q and height h (see section 2.3.4).

Let T_α be the group of the quasi-selfisogenies of $\Sigma_\alpha/\bar{\mathbb{F}}_p$. In section 2.5.7 and 2.5.8, we defined the action of $T_\alpha \times Frob^{\mathbb{Z}}$ on $\mathcal{M}_\alpha$. It follows from the definition that this action does not preserve the truncated Rapoport-Zink spaces $\mathcal{M}_\alpha^{n,d} \subset \mathcal{M}_\alpha$. In the following, we analyze how this action moves the truncated Rapoport-Zink spaces. (We remark that, in particular, the action of the subgroup $\Gamma = \mathrm{Aut}\,(\Sigma_\alpha)$ preserves the truncated Rapoport-Zink spaces.)

2.5.11. Let $\rho \in T_\alpha$ and write $e = e(\rho)$ and $f = f(\rho)$ (see section 2.3.6). It is not hard to see that if $(H, \beta) \in \mathcal{M}^{n,d}$ then $(H, \beta\rho) \in \mathcal{M}^{n+e,d+e-f}$. In fact, it follows from the definitions that both $p^{n+e}\beta\rho = (p^n\beta)(p^e\rho)$ and $p^{d-f-n}(\beta\rho)^{-1} = (p^f\rho)^{-1}(p^{d-n}\beta^{-1})$ are isogenies.

Thus, for each $\rho \in T_\alpha$ and any pair of integers n, d, the action of T_α on $\mathcal{M}_\alpha$ give rise to morphisms

$$\rho : \mathcal{M}_\alpha^{n,d} \to \mathcal{M}_\alpha^{n+e,d+e-f}$$

such that for any positive integers n', d', with $n' \geq n$ and $d' \geq d$,

$$i_{n'+e,d'+e-f}^{n+e,d+e-f} \circ \rho = \rho \circ i_{n',d'}^{n,d}$$

where we denote by $i_{n',d'}^{n,d}$ the natural inclusion of $\mathcal{M}_\alpha^{n,d}$ in $\mathcal{M}_\alpha^{n',d'}$.

Moreover, the restrictions of the above morphisms associated to $\rho \in T_\alpha$ give rise to morphisms

$$\rho : U^{n,d} \to U^{n+e,d+e-f},$$

which are open embeddings of formal schemes over $W(\bar{\mathbb{F}}_p)$.

On the other hand, from the equality $p = FV = VF$ we deduce that, for all positive integers n, d,

$$Frob : \mathcal{M}_\alpha^{n,d} \to (\mathcal{M}_\alpha^{n+1,d+1})^{(p)} = (\mathcal{M}_\alpha^{(p)})^{n+1,d+1},$$

and also that

$$Frob : U^{n,d} \to (U^{n+1,d+1})^{(p)}.$$

In fact, if $p^n\beta$ and $p^{d-n}\beta^{-1}$ are isogenies, then $p^{n+1}\beta F^{-1} = p^n\beta pF^{-1}$ and $p^{d-n}F\beta^{-1}$ are also an isogenies.

2.5.12. Let us now consider the reduced fibers $\bar{\mathcal{M}}_\alpha$ and $\bar{\mathcal{M}}_\alpha^{n,d}$ of $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^{n,d}$ respectively, for all n, d. Then, $\bar{\mathcal{M}}_\alpha$ and $\bar{\mathcal{M}}_\alpha^{n,d}$ are reduced schemes over $\bar{\mathbb{F}}_p$ (the latter of finite type over $\bar{\mathbb{F}}_p$, for all n, d), and there is an action of $T \times Frob^{\mathbb{Z}} \times Fr^{\mathbb{N}}$ on $\bar{\mathcal{M}}_\alpha$. From the above discussion, we already know how the action of $T \times Frob^{\mathbb{Z}}$ moves the spaces $\bar{\mathcal{M}}_\alpha^{n,d}$, let us now remark that the action of Fr respects the truncated Rapoport-Zink spaces.

In fact, if a quasi-isogeny $\beta : \Sigma \to H$ is such that $p^n\beta$ with kernel contained in $\Sigma[p^d]$, then the same holds for the quasi-isogeny $\beta^{(p)} : \Sigma^{(p)} \to H^{(p)}$. Equivalently, for all positive integers n, d, the relative Frobenius morphism maps the scheme $\bar{\mathcal{M}}_\alpha^{n,d}$ to $\bar{\mathcal{M}}_\alpha^{n,d\,(p)}$.

Let us also remark that the action of Γ on the reduced fiber $\bar{\mathcal{M}}_\alpha^{n,d}$ of the truncated Rapoport-Zink spaces is particularly simple. More presecisely, for any n, d the subgroup $\Gamma_d \subset \Gamma$ of the automorphisms of Σ_α which induce the identity on $\Sigma_\alpha[p^d]$ acts trivially on $\bar{\mathcal{M}}_\alpha^{n,d}$. Indeed, for any $\gamma \in \Gamma_d$ and any $(H, \beta) \in \bar{\mathcal{M}}^{n,d}$, there exists $\bar{\gamma} \in \mathrm{Aut}\,(H)$ such that $(p^n\beta) \circ \gamma = \bar{\gamma} \circ (p^n\beta)$, thus the pair (H, β) is equivalent to the pair $(H, \beta \circ \gamma)$.

2.5.13. We observe that, depending of the choice of the Barsotti-Tate group $\Sigma/\mathbb{F}_p$ in its isogeny class, there is another natural isomorphism $frob$ between $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^{(p)}$, namely the isomorphism defined as

$$frob : \mathcal{M}_\alpha \to \mathcal{M}_\alpha^{(p)}$$

$$(H, \beta) \mapsto (H, \beta \circ \nu^{-1}),$$

where $\nu : \Sigma_\alpha \to \Sigma_\alpha^{(p)}$ is the natural identification over $\mathbb{F}_p$ (see section 2.3.7).

Differently from $Frob$, the action of $frob$ preserves the truncated Rapoport-Zink spaces $\mathcal{M}_\alpha^{n,d}$ (and thus also the open $U_\alpha^{n,d}$) inside $\mathcal{M}_\alpha$, but is not compatible with the action of T_α. In fact, for any $\rho \in T_\alpha$, we have

$$\nu^{-1} \circ \rho^{(p)} \circ \nu = fr \circ \rho \circ fr^{-1},$$

which is equivalent to

$$\rho^{(p)} \circ frob = frob \circ (fr \circ \rho \circ fr^{-1}) : \mathcal{M}_\alpha \to \mathcal{M}_\alpha^{(p)}.$$

On the other hand, since fr^B is in the center of T, the same equality shows that the morphism $frob^B$ does commute with the action of T.

Finally, it follows from the definitions that

$$Frob = frob \circ fr^{-1},$$

since the equality $\nu \circ fr = F$ implies that

$$\beta \circ F^{-1} = \beta \circ fr^{-1} \circ \nu^{-1},$$

for any Barsotti-Tate group H and any quasi-isogeny $\beta : \Sigma \to H$.

2.5.14. We reinterpret the above definitions in terms of the l-adic cohomology with compact supports of the Rapoport-Zink rigid analytic spaces associated to the Barsotti-Tate group Σ_α.

Let l be a prime, $l \neq p$, and consider the constant abelian torsion étale sheaf $\mathbb{Z}/l^r\mathbb{Z}$, for some integer $r \geq 1$. For any open compact subgroup U of $GL_n(\mathbb{Z}_p)$ and any integer $i \geq 0$, we consider the i-th étale cohology group with compact supports of the rigid analytic space $\mathcal{M}_U^{\mathrm{rig}}$ with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$,

$$H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}).$$

For any $U' \subset U$, the natural projection $\mathcal{M}_{U'}^{\mathrm{rig}} \to \mathcal{M}_U^{\mathrm{rig}}$ give rise to a morphism between the corresponding cohomology groups and therefore the cohomogoly groups of the Rapoport-Zink spaces piece together in a direct limit

$$\varinjlim_U H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}).$$

We remark that, since the open subgroups $U(M)$ (for all integers $M \geq 0$) form a cofinal system of compact opens of $GL_n(\mathbb{Z}_p)$, the above limit can be also computed as a direct limit over the open subgroups U of the form $U = U(M)$, for some positive integer M, i.e.

$$\varinjlim_U H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_M H_c^i(\mathcal{M}_{U(M)}^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}).$$

The action of $GL_n(\mathbb{Q}_p) \times T_\alpha \times Frob^{\mathbb{Z}}$ on the system of Rapoport-Zink spaces gives rise an action of the $GL_n(\mathbb{Q}_p) \times T_\alpha \times Frob^{\mathbb{Z}}$ on étale cohomology groups, which naturally extends to an action of $GL_n(\mathbb{Q}_p) \times T_\alpha \times W_{\mathbb{Q}_p}$, where $W_{\mathbb{Q}_p}$ is the Weil group of $\mathbb{Q}_p$ ($K = \mathbb{Q}_p^{nr}$).

Proposition 2.17. — *For all integers $i \geq 0$, the $\mathbb{Z}/l^r\mathbb{Z}$-representation*

$$\varinjlim_U H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z})$$

of $GL_n(\mathbb{Q}_p) \times T_\alpha$ is smooth.

Proof. — Let us write $H^i = \varinjlim_U H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z})$. Then, it follows from the definitions that, for any open subgroup U of $GL_n(\mathbb{Q}_p)$, we have

$$(H^i)^U = H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}).$$

Moreover, let us consider the opens $V = U^{n,d} \subset \mathcal{M}_\alpha$ (for all integers $n, d \geq 0$), which form a cover of opens of finite type of $\mathcal{M}$. Then, for any level U, the associated open cover of $\mathcal{M}_U^{\mathrm{rig}}$ (whose opens V_U^{rig} are the pullbacks under the natural projection

$\mathcal{M}_U^{\mathrm{rig}} \to \mathcal{M}^{\mathrm{rig}}$ of the rigid analytic spaces associated to the opens $V$of $\mathcal{M}_\alpha$) is also formed of opens of finite type and we have

$$H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_{V_U^{\mathrm{rig}}} H_c^i(V_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}),$$

a direct limit of finite modules. FInally, we remark that the action of the subgroup $\Gamma_\alpha \subset T_\alpha$ on H^i preserves the subspaces $H_c^i(V_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z})$, for all opens V and all levels U, and thus

$$(H^i)^{U\times\Gamma'} = \varinjlim_V H_c^i(V_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z})^{\Gamma'},$$

for any level U and any open compact subgroup $\Gamma' \subset \Gamma_\alpha$. □

Maintaining the notations introduced in the above proof, for any integer $i \geq 0$ and open compact subgroup $U \subset GL_n(\mathbb{Q}_p)$, we defined the i-th l-adic cohomology group of $\mathcal{M}_U^{\mathrm{rig}}$ as

$$H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Q}_l) = \varinjlim_{V_U^{\mathrm{rig}}} (\varprojlim_r H_c^i(V_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Z}/l^r\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l).$$

As the level U varies, the l-adic cohomology groups of the Rapoport-Zink spaces form a direct system, endowed with an action of $GL_n(\mathbb{Q}_p) \times T_\alpha$. Further more, the corresponding l-adic representations of $GL_n(\mathbb{Q}_p) \times T_\alpha$

$$\varinjlim_U H_c^i(\mathcal{M}_U^{\mathrm{rig}} \times_K \hat{K}^{ac}, \mathbb{Q}_l)$$

are smooth. (This fact is a direct consequence of the definition.)

2.6. Full set of sections. — In [**9**] Drinfeld introduces the notion of full level structure in the context of his theory of elliptic modules. In [**22**] Katz and Mazur develop this notion in the context of finite locally free commutative group-schemes.

In this section we shall recall the definition and some basic properties of their notion of a full set of sections of a finite flat scheme of finite presentation.

2.6.1. Let S be a scheme and Z a finite flat S-scheme of finite presentation and rank $N > 1$ (or equivalently Z is finite locally free over S of rank $N > 1$). For every affine S-scheme $\operatorname{Spec} R$, the R-scheme $Z_R = Z \times_S \operatorname{Spec} R$ is of the form $\operatorname{Spec} B$ where B is an R-algebra which is as an R-module locally free of rank N. Any $f \in B$ defines an R-linear endomorphism of B. We denote by $\operatorname{Norm}(f)$ its determinant and by $\det(T - f)$ its characteristic polynomial, which is a monic polynomial in $R[T]$ of degree N.

Definition 2.18. — (see [**22**], section 1.8.2, p.33) A set of N points (not necesserally distinct) $P_1, \dots P_N \in Z(S)$ is a full set of sections of Z/S if either of the following equivalent conditions is satisfied:

1. for every affine S-scheme $\operatorname{Spec} R$ and for every $f \in B = H^0(Z_R, \mathcal{O})$ we have $\det(T - f) = \prod_{i=1}^N (T - f(P_i))$ in $R[T]$;
2. for every affine S-scheme $\operatorname{Spec} R$ and for every $f \in B = H^0(Z_R, \mathcal{O})$ we have $\operatorname{Norm}(f) = \prod_{i=1}^N f(P_i)$ in R.

2.6.2. If Z is a finite étale S-scheme of rank N the above conditions are equivalent to the following ones:

1. the morphism $\coprod_i S \to Z$ defined by the N-sections $P_1, \dots P_N$ is an isomorphism of S-scheme;
2. for every geometric point $\operatorname{Spec} k \to S$ the N points $(P_i)_k \in Z(k)$ are all distinct (see [**22**], Lemma 1.8.3, pp. 33-34).

Proposition 2.19. — *(see* [**22**]*, Proposition 1.9.1, p.38) Let Z be a finite flat S-scheme of finite presentation and rank $N > 1$. Let $P_1, \dots P_N \in Z(S)$ (not necesserally distinct).*

Then there exists a unique closed subscheme W of S which is universal for the relation "$P_1, \dots, P_N$ is a full set of sections of Z/S", i.e. such that for any S-scheme T the induced points $P_{1,T}, \dots, P_{N,T} \in Z(T)$ are a full set of sections for Z_T/T if and only if the structure morphism $T \to S$ factors through W.

2.6.3. Suppose now that Z is a finite flat S-group scheme of finite presentation and rank $N > 1$ and let A be a finite abelian abstract group of order N (e.g. Z is the p^m-torsion of a Barsotti-Tate group over S of height h and $A = (\mathbb{Z}/p^m)^h$).

Definition 2.20. — (see [**22**], section 1.10.5, p.44) A group morphism $\phi : A \to Z(S)$ is an A-generator of Z/S if the set of N points $\{\phi(a) \mid a \in A\}$ is a full set of sections of Z/S.

It follows directly from proposition 2.19 that the functor on S-schemes to sets which maps T/S to the sets of A-generators of Z_T/T is represented by a closed subscheme $W(A, Z/S)$ of $Z \times_S \cdots \times_S Z$. In particular, $W(A, Z/S)$ is finite over S.

Moreover, if Z, Z' are two isomorphic finite flat S-group schemes of finite presentation and rank $N > 1$ then $W(A, Z) \simeq W(A, Z')$.

We remark that there is a natural action of the group $\operatorname{Aut}(A)$ on the S-scheme $W(A, Z/S)$, namely the one defined by $\phi \mapsto \phi \circ g$, for any $g \in \operatorname{Aut}(A)$.

2.7. Vanishing cycles. — In [**2**] and [**3**] Berkovich constructs and studies the vanishing cycles functor from the category of étale sheaves on the generic fiber $\mathcal{X}_\eta$ of a formal scheme $\mathcal{X}$ to the category of étale sheaves on the closed fiber $\mathcal{X}_s$ of $\mathcal{X}$, when the formal scheme $\mathcal{X}$ is locally finitely presented over the ring of integers W of a non archimedean field K with residue field k. In this case, the generic fiber $\mathcal{X}_\eta$ is an analytic space over K and the closed fiber $\mathcal{X}_s$ is a scheme over k. Berkovich proves that for each pair of formal schemes $\mathcal{X}, \mathcal{Y}$ over W there exists an ideal of definition $\mathcal{I}$ of $\mathcal{Y}$ such that, if two morphisms $\varphi, \psi : \mathcal{Y} \to \mathcal{X}$ coincide modulo $\mathcal{I}$, then the morphisms between the vanishing cycles sheaves induced by φ and ψ coincide.

In this section we shall recall the definition of the functor that associates to a formal scheme $\mathcal{X}$ locally finitely presented over W a K-analytic space $\mathcal{X}_\eta$, together with the definition and some of the relevant properties of the vanishing cycles functor.

Finally, we shall focus on the construction and properties of the vanishing cycles of $\mathbb{Z}/l^r\mathbb{Z}$, for a prime number $l \neq p$ and an integer $r \geq 1$ (see [**14**], pp. 46-47).

2.7.1. We now describe the functor from the category of formal schemes locally finitely presented over W to the category of K-analyitc spaces which associates to a formal scheme $\mathcal{X}$ its generic fiber $\mathcal{X}_\eta$ over K.

If $\mathcal{X} = \operatorname{Spf} A$ where A is topologically finitely presented over W, then $\mathcal{X}_\eta = \operatorname{Spm} A_K$ where $A_K = A \otimes_W K$ is naturally a stricly K-affinoid algebra.

We define the reduction map $\pi : \mathcal{X}_\eta \to \mathcal{X}_s$ by sending a seminorm $|\cdot|_x$ on A_K to the prime ideal $\ker |\cdot|_x$ of $A_k = A \otimes_W k$. If $\mathcal{Y}$ is a open subset of $\mathcal{X}_s$ then $\pi^{-1}(\mathcal{Y})$ is a closed analytic domain of $\mathcal{X}_\eta$ and in particular, when $\mathcal{Y}$ is a open formal subscheme, we have $\pi^{-1}(\mathcal{Y}) = \mathcal{Y}_\eta$.

2.7.2. Let $\mathcal{X}$ be a formal scheme locally finitely presented over W, we recall the following two facts.

Proposition 2.21. — *(see* [**2**]*, Lemma 2.1, pp. 542-543, and Proposition 2.3, p. 543)*

1. *The correspondence $\mathcal{Y} \mapsto \mathcal{Y}_s$ induces an equivalence between the category of étale formal schemes over $\mathcal{X}$ and the category of étale schemes over $\mathcal{X}_s$.*
2. *If $\phi : \mathcal{Y} \to \mathcal{X}$ is an étale morphism of formal schemes, then the induced morphism $\phi_\eta : \mathcal{Y} \to \mathcal{X}$ between the generic fibers is quasi-étale.*

2.7.3. For simplicity, we suppose the field K separably closed. The functor $\mathcal{Y}_s \mapsto \mathcal{Y}_\eta$, which we obtain by composing the functors $\mathcal{Y}_s \mapsto \mathcal{Y}$ and $\mathcal{Y} \mapsto \mathcal{Y}_\eta$, induces a left exact functor Ψ_η from the category of étale sheaves over $\mathcal{X}_\eta$ to the category of étale sheaves over $\mathcal{X}_s$ (we recall that any sheaf on the étale site of $\mathcal{X}_\eta$ extend uniquely to a sheaf on the quasi-étale site of $\mathcal{X}_\eta$). Ψ_η is called the vanishing cycles functor of $\mathcal{X}$, and we denote by $R^q\Psi_\eta$ its right derived functors on the category of étale abelian sheaf on $\mathcal{X}_\eta$.

Proposition 2.22. — *(see* [**2**]*, Corollary 4.5, p. 549 and Corollary 5.4, p. 555; and* [**3**]*, Corollary 2.5, p. 373 and Theorem 3.1, p. 374) Let $\mathcal{X}$ be a formal scheme locally finitely represented over W and $\mathcal{F}$ an étale abelian sheaf on $\mathcal{X}_\eta$.*

1. *For any étale morphism $\mathcal{Y} \to \mathcal{X}$, $R^q\Psi_\eta(\mathcal{F})_{|\mathcal{Y}_s} \simeq R^q\Psi_\eta(\mathcal{F}_{|\mathcal{Y}_\eta})$, for all $q \geq 0$.*
2. *For any morphism $\varphi : \mathcal{Z} \to \mathcal{X}$ of formal schemes locally finitely represented over W, we have*
$$R^{\cdot}\Psi_\eta(R^{\cdot}\varphi_{\eta *}\mathcal{F}) \simeq R^{\cdot}\varphi_{s*}(R^{\cdot}\Psi_\eta(\mathcal{F})).$$
3. *If $\mathcal{X}$ is a smooth formal scheme and n is relatively prime to* $\operatorname{char} k$*, then $\Psi_\eta(\mathbb{Z}/n\mathbb{Z})_{\mathcal{X}_\eta} = (\mathbb{Z}/n\mathbb{Z})_{\mathcal{X}_s}$ and $R^q\Psi_\eta(\mathbb{Z}/n\mathbb{Z})_{\mathcal{X}_\eta} = 0$ for $q \geq 1$.*
4. *For any subscheme $\mathcal{Y} \subset \mathcal{X}_s$, we denote by $\mathcal{X}^\wedge_{|\mathcal{Y}}$ the formal completion of $\mathcal{X}$ along $\mathcal{Y}$ and by $\mathcal{F}^\wedge_{|\mathcal{Y}}$ the pullback of $\mathcal{F}$ over $(\mathcal{X}^\wedge_{|\mathcal{Y}})_\eta$, then $(\mathcal{X}^\wedge_{|\mathcal{Y}})_\eta$is canonically*

isomorphic to $\pi^{-1}(\mathcal{Y})$*. If* $\mathcal{F}$ *is constructible with torsion orders prime to* $\operatorname{char} k$*, then there are canonical isomorphisms*

$$R^q\Psi_\eta(\mathcal{F})_{|\mathcal{Y}} \simeq R^q\Psi_\eta(\mathcal{F}^\wedge_{|\mathcal{Y}})$$

for all $q \geq 0$*.*

5. *If* $\mathcal{X}$ *is locally of finite type and all the irreducible components of* $\mathcal{X}_s$ *are proper, then there is a spectral sequence*

$$E_2^{p,q} = H_c^p(\mathcal{X}_s, R^q\Psi_\eta(\mathcal{F})) \Longrightarrow H_c^{p+q}(\mathcal{X}_\eta, \mathcal{F}).$$

2.7.4. Let $\mathcal{T}$ be a scheme of finite type over W and $\mathcal{F}$ be an étale abelian torsion sheaf on $\mathcal{T}_\eta$. Suppose $\mathcal{X}$ and $\mathcal{X}'$ are schemes of finite type over $\mathcal{T}$, and let $\mathcal{Y} \subset \mathcal{X}_s$ and $\mathcal{Y}' \subset \mathcal{X}'_s$ be subschemes. Then, any morphism of formal schemes $\varphi : \mathcal{X}'^\wedge_{/\mathcal{Y}'} \to \mathcal{X}^\wedge_{/\mathcal{Y}}$ over $\mathcal{T}^\wedge$ induces some morphisms of sheaves on $\mathcal{Y}'_s$

$$\psi_\eta(\varphi, \mathcal{F}) : \varphi_s^*(R^q\Psi_\eta(\mathcal{F}_{|\mathcal{X}_\eta}))|_{\mathcal{Y}_s} \to R^q\Psi_\eta(\mathcal{F}_{|\mathcal{X}'_\eta})|_{\mathcal{Y}'_s},$$

for all integer $q \geq 0$.

Proposition 2.23. — *(see* [**3**]*, Theorem 4.1, p. 382) Let* $\mathcal{F}$ *be an abelian étale constructible sheaf on* $\mathcal{T}_\eta$ *with torsion orders prime to* $\operatorname{char} k$*.*

Given the schemes $\mathfrak{X} = \mathcal{X}'^\wedge_{/\mathcal{Y}'}$ *and* $\mathfrak{X}' = \mathcal{X}^\wedge_{/\mathcal{Y}}$ *over* $\mathcal{T}^\wedge$*, there exists an ideal of definition* $\mathcal{I}'$ *of* $\mathfrak{X}'$ *such that for any pair of morphisms* $\varphi, \phi : \mathfrak{X}' \to \mathfrak{X}$ *over* $\mathcal{T}^\wedge$ *that coincide modulo* $\mathcal{I}'$*, we have*

$$\psi_\eta(\varphi, \mathcal{F}) = \psi_\eta(\phi, \mathcal{F})$$

for all $q \geq 0$*.*

Further more, let $f : \mathcal{Y}' \to \mathcal{Y}$. Then, any finite étale formal scheme $\mathfrak{q} : \mathcal{Z} \to \mathcal{X}'^\wedge_{/\mathcal{Y}'}$, with degree relatevely prime to the torsion orders of $\mathcal{F}$, and any morphism $\varphi : \mathcal{Z} \to \mathcal{X}^\wedge_{/\mathcal{Y}}$ such that $\varphi_s = f \circ \mathfrak{q}_s$ induce a morphism

$$\theta(\varphi, \mathcal{F}) = \frac{1}{\deg(q)} Tr \circ q_{s\,*}\psi_\eta(\varphi, \mathcal{F}) \circ i : f^* R^q\Psi_\eta(\mathcal{F}_{|\mathcal{X}_\eta})|_{\mathcal{Y}_s} \to R^q\Psi_\eta(\mathcal{F}_{|\mathcal{X}'_\eta})|_{\mathcal{Y}'_s},$$

where $Tr : q_{s\,*}R^q\Psi_\eta(\mathcal{F}_{/\mathcal{Z}}) = q_{s\,*}q_s^*R^q\Psi_\eta(\mathcal{F}_{|\mathcal{X}_\eta})|_{\mathcal{Y}_s} \to R^q\Psi_\eta(\mathcal{F}_{|\mathcal{X}'_\eta})|_{\mathcal{Y}'_s}$ is the trace map.

By closely following the argument in [**3**], one can prove the following mild generalization of proposition 2.23.

Proposition 2.24. — *Maintaining the notations of proposition 2.23. Given* $\mathcal{T}$*,* $\mathcal{F}$*,* $\mathfrak{X} = \mathcal{X}'^\wedge_{/\mathcal{Y}'}$ *and* $\mathfrak{X}' = \mathcal{X}^\wedge_{/\mathcal{Y}}$ *over* $\mathcal{T}^\wedge$*, there exists an ideal of definition* $\mathcal{I}'$ *of* $\mathfrak{X}'$ *such that for any finite étale formal scheme* $\mathfrak{q} : \mathcal{Z} \to \mathfrak{X}'$ *and any pair of morphisms* $\varphi, \phi : \mathcal{Z} \to \mathfrak{X}$ *over* $\mathcal{T}^\wedge$ *that coincide modulo* $\mathcal{I}'$*, we have*

$$Tr \circ \mathfrak{q}_{s\,*}\psi_\eta(\varphi, \mathcal{F}) = Tr \circ \mathfrak{q}_{s\,*}\psi_\eta(\phi, \mathcal{F})$$

for all $q \geq 0$*, where* $Tr : \mathfrak{q}_{s\,*}R^q\Psi_\eta(\mathcal{F}_{/\mathcal{Z}}) = \mathfrak{q}_{s\,*}\mathfrak{q}_s^*R^q\Psi_\eta(\mathcal{F}_{/\mathfrak{X}'}) \to R^q\Psi_\eta(\mathcal{F}_{/\mathfrak{X}'})$ *is the trace map.*

In particular, if there exists a morphism $f : \mathfrak{X}'_s = \mathcal{Y}' \to \mathfrak{X}_s = \mathcal{Y}$ *such that* $\varphi_s = \phi_s = f \circ \mathfrak{q}_s$ *and* $\deg(\mathfrak{q})$ *is relatively prime to the torsion orders of* $\mathcal{F}$*, then*

$$\theta(\varphi, \mathcal{F}) = \theta(\phi, \mathcal{F})$$

for all $q \geq 0$*.*

Proof. — Let us first remark that it suffices to find such an ideal $\mathcal{I}'$ separately for each q, since all the sheaves are constructible and equal to zero for $q \geq 1 + 2\dim(\mathfrak{X}_\eta)$.

For any étale morphism $\mathcal{U}_s \to \mathfrak{X}_s$ (resp. $\mathcal{U}'_s \to \mathfrak{X}'_s$, $\mathcal{V}_s \to \mathcal{Z}_s$), we denote by $\mathcal{U} \to \mathfrak{X}$ (resp. $\mathcal{U}' \to \mathfrak{X}'$, $\mathcal{V} \to \mathcal{Z}$) the corresponding étale morphism of formal schemes under the equivalence of category stated in proposition 2.21.

Our first step is to remark that there exists a finite étale covering $\{u_\alpha : \mathcal{U}_{s,\alpha} \to \mathfrak{X}_s\}$ such that the canonical morphisms

$$\oplus_\alpha u_{s\,!} H^q(\mathcal{U}_\eta, \mathcal{F}) \to R^q\Psi_\eta(\mathcal{F}_{/\mathfrak{X}})$$

is surjective. For any $\mathcal{U}_s = \mathcal{U}_{s,\alpha} \to \mathfrak{X}_s$, we write $\mathcal{U}'_s = f_s^*\mathcal{U}_s$ and $\mathcal{V}_s = q_s^*\mathcal{U}'_s$. Then, the two morphisms $\varphi, \phi : \mathcal{Z} \to \mathfrak{X}$ extend to two morphisms from $\mathcal{V}$ to $\mathcal{U}$, and $\mathcal{V} \to \mathcal{U}'$ is a finite étale morphism with degree equal to $\deg(q)$. Moreover, in order to prove the statement, it is sufficient to show that the associated morphisms $\varphi^*, \phi^* : H^q(\mathcal{U}_\eta, \mathcal{F}) \to H^q(\mathcal{V}_\eta, \mathcal{F})$ satisfy the condition $Tr \circ \varphi^* = Tr \circ \phi^*$, where $Tr : H^q(\mathcal{V}_\eta, \mathcal{F}) \to H^q(\mathcal{U}'_\eta, \mathcal{F})$ denote the trace map. Further more, we can assume $\mathcal{U} = \mathrm{Spf}\,(A)$, $\mathcal{U}' = \mathrm{Spf}\,(B)$ and $\mathcal{V} = \mathrm{Spf}\,(C)$.

Let $\mathfrak{a}$ (resp. $\mathfrak{b}$) be the maximal ideal of definition of A (resp. B). We observe that $\mathfrak{b}C$ is an ideal of definition of C. For any $0 < r < 1$, we set

$$U(r) = \{x \in \mathcal{U}_\eta \,|\, |f(x)| \geq r \,\forall f \in \mathfrak{a}\},$$

and analogously define $U'(r) \subset \mathcal{U}'_\eta$ and $V'(r) \subset \mathcal{V}_\eta$. Then, the $U(r)$ (resp. $U'(r)$, $V(r)$) are some affinoid domains which exhaust $\mathcal{U}_\eta$ (resp. $\mathcal{U}'_\eta$, $\mathcal{V}_\eta$). For each r, the morphisms $\varphi, \phi : \mathcal{V} \to \mathcal{U}$ and $q : \mathcal{V} \to \mathcal{U}'$ induce some morphisms $\varphi_r, \phi_r : V(r) \to U(r)$ and $q_r : V(r) \to U'(r)$. Moreover, it follows from Lemma 6.3.12 in [**4**], that there exists r such that the canonical morphism $j : H^q(\mathcal{U}'_\eta, \mathcal{F}) \to H^q(U'(r), \mathcal{F})$ is an injection. We fix such a number r, $0 < r < 1$, and consider the following commutative diagram.

$$\begin{tikzcd}
H^q(\mathcal{U}_\eta, \mathcal{F}) \arrow[r, shift left, "\varphi^*"] \arrow[r, shift right, "\phi^*"'] \arrow[d] & H^q(\mathcal{V}_\eta, \mathcal{F}) \arrow[r, "Tr"] \arrow[d] & H^q(\mathcal{U}'_\eta, \mathcal{F}) \arrow[d, hook, "j"] \\
H^q(U(r), \mathcal{F}) \arrow[r, shift left, "\varphi_r^*"] \arrow[r, shift right, "\phi_r^*"'] & H^q(V(r), \mathcal{F}) \arrow[r, "Tr"] & H^q(U'(r), \mathcal{F})
\end{tikzcd}$$

By Theorem 7.1 in [**2**], there exists $\epsilon \in \mathfrak{S}(U(r))$ such that $\varphi_r^* = \phi_r^*$ on $H^q(U(r), \mathcal{F})$ if $d(\varphi_r, \phi_r) < \epsilon$. Moreover, without loss of generality we may assume that ϵ is defined by triple $(U(r), \{f_i\}, \{t_i\})$, for some elements $f_i \in A$ and some $t_i > 0$, $1 \leq i \leq m$, i.e. $d(\varphi_r, \phi_r) < \epsilon$ if $\max_{y \in V(r)} |(\varphi_r^* f_i - \phi_r^* f_i)(y)| \leq t_i$, for all i. Then, the ideal of

definition $\mathfrak{b}^n$, for any $n \geq 1$ such that $r^n \leq t_i$ (for all i), possesses the property that, for any pair of morphisms $\varphi_r, \phi_r : V(r) \to U(r)$ which coincide modulo $\mathfrak{b}^n$, one has $d(\varphi_r, \phi_r) < \epsilon$ (see Lemma 8.4 in [**2**]).

It follows that for $\mathfrak{I} = \mathfrak{b}^n$ we have $\varphi_r^* = \phi_r^*$ and thus also $Tr \circ \varphi_r^* = Tr \circ \phi_r^*$. Since the morphism j is injective, we deduce that $Tr \circ \varphi^* = Tr \circ \phi^*$. □

Finally, let us remark that if the morphism $\psi_\eta(\varphi, \mathcal{F})$ is an isomorphism then such is also the induced morphism $\theta(\varphi, \mathcal{F})$ on the vanishing cycles sheaves over $\mathcal{Y}'$ (one can define its inverse as $\frac{1}{\deg(q)} Tr \circ \psi_\eta(\varphi, \mathcal{F})^{-1} \circ i$).

2.7.5. Let l be a prime number, $l \neq p$, and $r \geq 1$ some integer.

We recall the following result on the vanishing cycles $R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})$.

Lemma 2.25. — *(see* [**14**]*, Lemma II.5.6, p. 47) Suppose that R is a complete noetherian local W-algebra. The natural map*

$$R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathrm{Spf}\, R} \to R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathrm{Spf}\, R[[T_1,\ldots,T_r]]}$$

is an isomorphism.

The above lemma can be reformulated as follows

Proposition 2.26. — *Let $\mathcal{X}, \mathcal{Y}$ be two formal schemes, locally finitely represented over W, and $\pi : \mathcal{Y} \to \mathcal{X}$ be a smooth morphism.*

Then the map:

$$\psi_\eta(\pi, \mathbb{Z}/l^r\mathbb{Z}) : \pi^* R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{X}_\eta} \to R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{Y}_\eta}$$

is an isomorphism.

Proof. — It suffices to check that $\psi_\eta(\pi, \mathbb{Z}/l^r\mathbb{Z})$ induces an isomorphism on fibers. Thus, let y be a point of $\mathcal{Y}_s$ and consider the map

$$\psi_\eta(\pi, \mathbb{Z}/l^r\mathbb{Z})_y : (R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{X}_\eta})_{\pi(y)} \to (R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{Y}_\eta})_y$$

It follows from part 5 of proposition 2.22 that

$$(R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{X}_\eta})_{\pi(y)} = R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{O}^\wedge_{\mathcal{X},\pi(y)}}$$

and

$$(R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{Y}_\eta})_y = R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})_{\mathcal{O}^\wedge_{\mathcal{Y},y}}.$$

To say that the morphism π is smooth at the point y is equivalent to say that there exists an isomorphism $\mathcal{O}^\wedge_{\mathcal{Y},y} \simeq \mathcal{O}^\wedge_{\mathcal{X},\pi(y)}[[T1,\ldots,T_r]]$, compatible with the morphism $\pi^* : \mathcal{O}^\wedge_{\mathcal{X},\pi(y)} \to \mathcal{O}^\wedge_{\mathcal{Y},y}$. Therefore the previous lemma suffices to conclude. □

3. Igusa varieties

In [**14**] Harris and Taylor introduce natural analogues of the Igusa curves in the theory of elliptic modular curves and call them Igusa varieties. These form towers of finite étale coverings of the open Newton polygon strata in the reduction of the Shimura variety.

In this section we shall define some varieties we shall also call Igusa varieties which are the natural generalization of the ones introduced in [**14**]. These form a tower of finite étale coverings not of an open Newton polygon stratum but only of the central leaf inside it. (In the case considered by Harris and Taylor in [**14**], i.e. when the dimension of the pertinent Barsotti-Tate group is one, there is a unique leaf inside each open Newton polygon stratum, which is the stratum itself.)

3.1. The general case. — We shall recall the general construction underlying the notion of Igusa variety which was introduced by Harris and Taylor in the context of [**14**] (see section III.1, pp.70-71).

3.1.1. Let $\mathbb{X}$ be a p-divisible groups defined over $\bar{\mathbb{F}}_p$ (e.g. $\mathbb{X} = \Sigma_\lambda$, $0 \leq \lambda \leq 1$, as in section 2.3).

The functor on $\bar{\mathbb{F}}_p$-schemes to groups which maps S to $\mathrm{Aut}\,(\mathbb{X}[p^m]/S)$ is represented by a scheme $\mathrm{Aut}\,(\mathbb{X}[p^m])$ of finite type over $\mathrm{Spec}\,\bar{\mathbb{F}}_p$ (for all m). If $m_1 \geq m_2$ then there is a natural map $\mathrm{Aut}\,(\mathbb{X}[p^{m_1}]) \to \mathrm{Aut}\,(\mathbb{X}[p^{m_2}])$.

For each m we define $\mathrm{Aut}^{\,1}(\mathbb{X}[p^m])$ to be the intersection of the scheme theoretic images of $\mathrm{Aut}\,(\mathbb{X}[p^{m'}])$ in $\mathrm{Aut}\,(\mathbb{X}[p^m])$ over all $m' \geq m$. Then the scheme thoeretic image of $\mathrm{Aut}^{\,1}(\mathbb{X}[p^{m_1}])$ in $\mathrm{Aut}\,(\mathbb{X}[p^{m_2}])$ for $m_1 > m_2$ is in fact $\mathrm{Aut}^{\,1}(\mathbb{X}[p^{m_2}])$.

3.1.2. Suppose now that S is a reduced $\bar{\mathbb{F}}_p$-scheme and H a p-divisible group over S. We now consider the functor on S-schemes to sets which maps T/S to the set of isomorphisms over T $j_m : \mathbb{X}[p^m] \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} T \to H[p^m] \times_S T$. This functor is represented by a scheme $X_m(\mathbb{X}, H/S)$ of finite type over S.

We define $Y_m(\mathbb{X}, H/S)$ to be the intersection of the scheme theorestic images of

$$X_{m'}(\mathbb{X}, H/S) \to X_m(\mathbb{X}, H/S)$$

over $m' \geq m$ and write $J_m(\mathbb{X}, H/S) = Y_m(\mathbb{X}, H/S)^{\mathrm{red}}$. We denote the universal isomorphism over $J_m(\mathbb{X}, H/S)$ by

$$j_m^{\mathrm{univ}} : \mathbb{X}[p^m] \to H[p^m].$$

It follows from the definitions that there is a natural action of the group of automorphism of $\mathbb{X}/\bar{\mathbb{F}}_p$ on the schemes $J_m(\mathbb{X}, H/S)$, for all m, which is defined via composition on the right of the restrictions to the p^m-torsion with the universal isomorphism j_m^{univ}.

3.2. Igusa varieties over the central leaves. — We maintain the notations established in sections 2.1, 2.2 and 2.3. Inside each open Newton polygon stratum of

the reduction in characteristic p of the Shimura variety, we consider the central leaf and define the Igusa varieties as covering spaces of the central leaves.

3.2.1. Let U^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. We denote by X the Shimura variety $X_{U^p(0)}$ over $\operatorname{Spec} E_u$, and write $\mathcal{X} = \mathcal{X}_{U^p}$. Then, $X = \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_{E_u}} \operatorname{Spec} E_u$. We denote by $\bar{X}$ the reduction $\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_{E_u}} \operatorname{Spec} k(u)$, where $k(u)$ is the residue field of $\mathcal{O}_{E_u}$ ($k(u) = \mathbb{F}_p$).

Let α be a Newton polygon of height h and dimension q and Σ_α be the Barsotti-Tate group over $\mathbb{F}_p$ definede in section 2.3.4. We denote by $\bar{X}^{(\alpha)}$ the open Newton polygon stratum inside $\bar{X}$ associated to α and by C_α the central leaf inside $\bar{X}^{(\alpha)}$, i.e. the leaf associated to Σ_α. We define the Igusa varieties as covering spaces of the central leaves C_α (for all α) as follows.

3.2.2. We breifly recall the notation introduced in 2.3.4. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be the slopes of the Newton polygon α. For each i, we denote by r_i the multiplicity of the slope λ_i in α. We define $\Sigma^i = \Sigma^i_\alpha = \Sigma_{\lambda_i}^{\oplus r_i}$ and $\Sigma = \Sigma_\alpha = \oplus_i \Sigma^i_\alpha$.

Lemma 3.1. — $\operatorname{Aut}^1(\Sigma^i[p^m])$ *is finite over* $\bar{\mathbb{F}}_p$ *and*

$$\operatorname{Aut}^1\left(\Sigma^i[p^m]\right)^{\mathrm{red}} \simeq \mathrm{GL}_{r_i}\left(\mathcal{O}_{\lambda_i}/p^m\mathcal{O}_{\lambda_i}\right)$$

Proof. — The same proof of Lemma III.1.5 of [**14**] (pp.70-71) applys to this lemma using the result in Proposition 2.8. □

We also see that

$$J_m\left(\Sigma^i, \Sigma^i/\operatorname{Spec}\bar{\mathbb{F}}_p\right) = \operatorname{Aut}^1\left(\Sigma^i[p^m]\right)^{\mathrm{red}} \simeq \mathrm{GL}_{r_i}\left(\mathcal{O}_{\lambda_i}/p^m\mathcal{O}_{\lambda_i}\right).$$

In fact if S is any reduced scheme over $\operatorname{Spec}\bar{\mathbb{F}}_p$ then

$$\begin{aligned} J_m\left(\Sigma^i, \Sigma^i/S\right) &= \left(J_m\left(\Sigma^i, \Sigma^i/\operatorname{Spec}\bar{\mathbb{F}}_p\right) \times_{\operatorname{Spec}\bar{\mathbb{F}}_p} S\right)^{\mathrm{red}} \\ &\simeq \left(\left(\mathrm{GL}_{r_i}\left(\mathcal{O}_{\lambda_i}/p^m\mathcal{O}_{\lambda_i}\right)\right)_S\right)^{\mathrm{red}} \\ &= \left(\mathrm{GL}_{r_i}\left(\mathcal{O}_{\lambda_i}/p^m\mathcal{O}_{\lambda_i}\right)\right)_S \end{aligned}$$

(see [**14**], section III.1, pp.70-71).

3.2.3. Let us consider the central leaf $C = C_\alpha$, i.e. the leaf of $\bar{X}^{(\alpha)}$ associated to the Barsotti-Tate $\Sigma = \Sigma_\alpha$.

We focus our attention on the Barsotti-Tate group $\mathcal{G} = \epsilon\mathcal{A}[u^\infty]$ over the central leaf $C = C_\alpha$. We denote by C_r the irreducible components of C, by $\eta = \eta_r$ the generic point of C_r and by $\bar{\eta} = \bar{\eta}_r$ the associated geometric point (for all r).

It follows from the definition of C that $\mathcal{G}_{\bar{\eta}} \simeq \Sigma \times k(\bar{\eta})$ and thus, in particular, that $\mathcal{G}_{\bar{\eta}}$ is completely slope divisible (for any $\bar{\eta} = \bar{\eta}_r$). We deduce, by remark 2.13, that the Barsotti-Tate groups $\mathcal{G}_{\eta_r}$ are completely slope divisible, and also, by remark 2.14, that $\mathcal{G}/C_r$ is complete slope divisible, for all r. Thus the same it is true for $\mathcal{G}$ over C (by definition $C = \cup_r C_r$, but indeed $C = \coprod_r C_r$, since it is smooth).

We denote by $(0) \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_k = \mathcal{G}$ the slope filtration of $\mathcal{G}$ over C and by $\mathcal{G}^i$ the subquotients $\mathcal{G}_i/\mathcal{G}_{i-1}$ $(i = 1, \ldots, k)$. The $\mathcal{G}^i$ are slope divisible isoclinic Barsotti-Tate group of slope λ_i, and for all geometric points $x \in C$ we have ${\mathcal{G}^i}_x \simeq \Sigma^i$. (This follows from the fact that, for any geometric point x of C, $\mathcal{G}_x \simeq \Sigma$ and any isogeny between Barsotti-Tate groups endowed with slope filtrations respects the filtrations.)

Definition 3.2. — For each positive integer m we define the Igusa variety of level m

$$J_m = J_m(\Sigma_1, \mathcal{G}_1/C \times \bar{\mathbb{F}}_p) \times_{C\times\bar{\mathbb{F}}_p} J_m(\Sigma_2, \mathcal{G}_2/C \times \bar{\mathbb{F}}_p) \times_{C\times\bar{\mathbb{F}}_p} \cdots J_m(\Sigma_k, \mathcal{G}_k/C \times \bar{\mathbb{F}}_p)$$

and denote by $j^{\mathrm{univ}}_{m,i} : \Sigma^i[p^m] \to \mathcal{G}^i[p^m]$ the universal isomorphisms over J_m.

If $m' \geq m$ then there is a natural surjection $\mathrm{q}_{m',m} : J_{m'} \twoheadrightarrow J_m$ over $C \times \bar{\mathbb{F}}_p$. The Igusa varieties J_m together with the morphisms $\mathrm{q}_{m',m}$ form a projective system of schemes over $C \times \bar{\mathbb{F}}_p$. Moreover, there is a natural action of the group Γ on the tower of Igusa varieties, which is defined by composition on the left. If we write $\Gamma_m = \prod_i \mathrm{GL}_{r_i}(\mathcal{O}_{\lambda_i}/p^m\mathcal{O}_{\lambda_i})$ then the action of Γ on J_m factors via the natural projection $\Gamma \twoheadrightarrow \Gamma_m$.

Proposition 3.3. — *The Igusa variety J_m over $C \times \bar{\mathbb{F}}_p$ is finite étale and Galois with Galois group Γ_m.*

Proof. — It is sufficient to show that for any closed point x of $C \times \bar{\mathbb{F}}_p$ the following two conditions hold

- the group Γ_m acts faithfully and transitively on the points of $(J_m)^\wedge_x$;
- if y is a point of $(J_m)^\wedge_x$ then $(J_m)^\wedge_y \simeq (C \times \bar{\mathbb{F}}_p)^\wedge_x$;

or equivalently that $J_m \times_{C\times\bar{\mathbb{F}}_p} \operatorname{Spec} \mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x} \simeq (\Gamma_m)_{\operatorname{Spec}\mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x}}$.

Following the argument of Proposition III.1.7 in [**14**] (pp.73-74) in order to conclude it suffices to prove the following lemma. □

Lemma 3.4. — *Maintaining the notations as above. For any closed point x in $C\times\bar{\mathbb{F}}_p$ and for all i*

$$\mathcal{G}^i \times_{C\times\bar{\mathbb{F}}_p} \operatorname{Spec} \mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x} \simeq \Sigma^i \times_{C\times\bar{\mathbb{F}}_p} \operatorname{Spec} \mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x}.$$

Proof. — By the Isogeny Theorem of de Jong and Oort (theorem 2.9), and the fact that over $\bar{\mathbb{F}}_p$ any two Barsotti-Tate groups with equal Newton polygons are isogenous, there exists an isogeny

$$\psi : \Sigma^i \times_{C\times\bar{\mathbb{F}}_p} \operatorname{Spec} \mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x} \twoheadrightarrow \mathcal{G}^i \times_{C\times\bar{\mathbb{F}}_p} \operatorname{Spec} \mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x}$$

over $\operatorname{Spec} \mathcal{O}^\wedge_{C\times\bar{\mathbb{F}}_p,\,x}$. We choose an integer $d > 0$ such that the kernel of the isogeny ψ is contained in the p^d-torsion subgroup.

Let $\bar{\mathcal{M}}^{0,d}_{\Sigma^i}$ be the reduced fiber of the Rapoport-Zink space $\mathcal{M}^{0,d}_{\Sigma^i}$ and denote by $\mathcal{H}$ the universal Barsotti-Tate group over $\bar{\mathcal{M}}^{0,d}_{\Sigma^i}$. We consider the subset

$$Y = \{t \in \bar{\mathcal{M}}^{0,d}_{\Sigma^i} \mid \mathcal{H}_t \times k(t)^{ac} \simeq \Sigma^i \times k(t)^{ac}\}.$$

It follows from lemma 2.6 that Y is a constructible subset of $\bar{\mathcal{M}}^{0,d}_{\Sigma^i}$. We now show that Y is finite. In fact, if $t \in Y$, then $\mathcal{H}_t \times k(t)^{ac} \simeq \Sigma^i \times k(t)^{ac}$ and thus there exists an isogeny $\phi_t \in \mathrm{End}\,(\Sigma^i \times k(t)^{ac}) = \mathrm{End}_{\bar{\mathbb{F}}_p}(\Sigma^i)$ with kernel contained in $\Sigma^i[p^d]$ such that

$$t = (\mathcal{H}_t, \beta_t) \simeq (\Sigma^i, \phi_t).$$

Thus the map which takes t to ϕ_t defines a bijection between Y and the set

$$\mathrm{End}_{\bar{\mathbb{F}}_p}(\Sigma^i) \cap p^d \mathrm{End}_{\bar{\mathbb{F}}_p}(\Sigma^i)^{-1} / \mathrm{Aut}_{\bar{\mathbb{F}}_p}(\Sigma^i) \simeq \mathrm{M}_{r_i}(\mathcal{O}_{\lambda_i}) \cap p^d \mathrm{M}_{r_i}(\mathcal{O}_{\lambda_i})^{-1} / \mathrm{GL}_{r_i}(\mathcal{O}_{\lambda_i}).$$

Since the set of matrixes

$$\begin{pmatrix} T^{a_1} & & & \\ & T^{a_2} & & x_{i,j} \\ & & \ddots & \\ & 0 & & T^{a_{r_i}} \end{pmatrix}$$

where $a_1 \geq a_2 \geq \cdots \geq a_{r_i} \geq 0$ are integers such that $a_i \leq d$ for all i, and $x_{i,j} \in \mathcal{O}_{\lambda_i}$ are such that $x_{i,j} = 0$ for $i < j$ and $\mathrm{val}_p(x_{i,j}) \leq a_j$ for $i > j$, is a system of representatives of the coset space $\mathrm{M}_{r_i}(\mathcal{O}_{\lambda_i}) \cap p^d \mathrm{M}_{r_i}(\mathcal{O}_{\lambda_i})^{-1} / \mathrm{GL}_{r_i}(\mathcal{O}_{\lambda_i})$, this set (and therefore Y) is finite. It follows that there is a reduced finite subscheme Y of $\bar{\mathcal{M}}^{0,d}_{\Sigma^i}$ such that, for any geometric point t of $\bar{\mathcal{M}}^{0,d}_{\Sigma^i}$, $t \in Y$ if and only if $\mathcal{H}_t \simeq \Sigma^i$.

By the universal property of the Rapoport-Zink space $\mathcal{M}^{0,d}_{\Sigma^i}$, we have that the pair $\left(\mathcal{G}^i \times_{C \times \bar{\mathbb{F}}_p} \mathrm{Spec}\, \mathcal{O}^{\wedge}_{C \times \bar{\mathbb{F}}_p, x}, \psi\right)$ defines a morphism of schemes

$$u : \mathrm{Spec}\, \mathcal{O}^{\wedge}_{C \times \bar{\mathbb{F}}_p, x} \to \bar{\mathcal{M}}^{0,d}_{\Sigma^i}.$$

Moreover, if we denote by η the generic point of $\mathrm{Spec}\, \mathcal{O}^{\wedge}_{C \times \bar{\mathbb{F}}_p, x}$, then $u(\eta) \in Y$ and thus, since Y is finite and $\mathcal{O}^{\wedge}_{C \times \bar{\mathbb{F}}_p, x}$ a domain, the map u has to factor via a (closed) point in Y. □

Corollary 3.5. — *The Igusa varieties are smooth schemes over* $\mathrm{Spec}\, \bar{\mathbb{F}}_p$.

Proof. — It follows directly from propositions 2.7, and 3.3. □

3.2.4. We remark that the definition of the Igusa varieties can be easily given over any leaf of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ which is associated to a completely slope divisible Barsotti-Tate group, and moreover the result of proposition 3.3 also holds for those Igusa varieties.

3.3. The action of $G(\mathbb{A}^{\infty,p})$ on the Igusa varieties. — We defined the Igusa varieties as covering spaces of the central leaf of an open Newton polygon stratum inside the reduction of a Shimura variety with no level structure at p. In this section, we shall investigates how the Igusa varieties vary as the level structure away from p on the Shimura variety varies.

3.3.1. Let α be a Newton polygon of dimension q and height h. For any open compact (sufficiently small) subgroup U^p of $G(\mathbb{A}^{\infty,p})$ and any positive integer m, we denote by

$$J_{U^p,m} = J_{\alpha,U^p,m}$$

the Igusa variety of level m over the central leaf $C_{U^p} = C_{\alpha,U^p}$ of the open Newton polygon stratum $\bar{X}^{(\alpha)}_{U^p}$ inside the reduction of a Shimura variety with no level structure at p and structure of level U^p away from p, $\bar{X}_{U^p(0)}$.

For all open compact subgroup $V^p \subset U^p$, the natural projections between the Shimura varieties $\bar{X}_{V^p(0)} \to \bar{X}_{U^p(0)}$ preserve both the Newton polygon stratification and Oort's foliation. Equivalently, they induces some morphisms $\bar{X}^{(\alpha)}_{V^p} \to \bar{X}^{(\alpha)}_{U^p}$ between the open Newton polygon strata and

$$\mathrm{q}_{V^p,U^p} : C_{V^p} \to C_{U^p}$$

between the corresponding central leaves. Moreover, these morphisms are finite and étale (see section 2.1.10).

It follows from the definition of the Igusa varieties that, for any level m, the morphisms $\mathrm{q}_{V^p,U^p} : C_{V^p} \to C_{U^p}$ give rise to some finite étale morphisms between the corresponding Igusa varieties of level m,

$$\mathrm{q}_{V^p,U^p} : J_{V^p,m} \to J_{U^p,m},$$

such that $\mathrm{q}_{m',m} \circ \mathrm{q}_{V^p,U^p} = \mathrm{q}_{V^p,U^p} \circ \mathrm{q}_{m',m}$, for any integers $m' \geq m$ and for any open compact subgroups $V^p \subset U^p$ of $G(\mathbb{A}^{\infty,p})$.

In fact, for all levels V^p, m, let us denote a point x on the Igusa variety $J_{V^p,m}$ by a $(4+k)$-tuple $(A, \lambda, i, \bar{\mu}; j_{m,1}, \ldots, j_{m,k})$, where $(A, \lambda, i, \bar{\mu})$ is the quadruple associated to the point $\mathrm{q}_m(x) \in C_{V^p}$ and $j_{m,i} : \Sigma^i[p^m] \to G^i[p^m]$ (for all $i = 1, \ldots, k$) are the isomorphisms defining the Igusa structures on the isoclinic subquotients G^i of the Barsotti-Tate group $G = \epsilon A[u^\infty]$. Then, the morphisms $\mathrm{q}_{V^p,U^p} : J_{V^p,m} \to J_{U^p,m}$ are defined by

$$(A, \lambda, i, \bar{\mu}; j_{m,1}, \ldots, j_{m,k}) \mapsto (A, \lambda, i, \bar{\mu}; j_{m,1}, \ldots, j_{m,k}),$$

where the V^p-orbit $\bar{\mu}$ of μ determines a unique U^p-orbit (which we still denote by $\bar{\mu}$).

3.3.2. Analogously, for any $g \in G(\mathbb{A}^{\infty,p})$, the corresponding morphisms between the reductions of the Shimura varieties

$$g : \bar{X}_{U^p} \to \bar{X}_{g^{-1}U^p g}$$

(see section 2.1.11) preserve the Newton polygon stratification and Oort's foliation, and induce some morphisms between Igusa varieties of the same level m,

$$g : J_{U^p,m} \to J_{g^{-1}U^p g,m},$$

for all $m \geq 0$ and $U^p \subset G(\mathbb{A}^{\infty,p})$, which commutes with the projections $\mathrm{q}_{m',m}$ and q_{V^p,U^p}. The morphisms $g : J_{U^p,m} \to J_{g^{-1}U^p g,m}$ are defined by

$$(A, \lambda, i, \bar{\mu}; j_{m,1}, \ldots, j_{m,k}) \mapsto (A, \lambda, i, \overline{\mu \circ g}; j_{m,1}, \ldots, j_{m,k}),$$

where U^p-orbit $\bar{\mu}$ of μ determines a unique $g^{-1}U^p g$-orbit of $\mu \circ g$, which we denote by $\overline{\mu \circ g}$.

3.4. The groups acting on the Igusa varieties. — In this section we investigates which abstract groups naturally act on the tower of Igusa varieties. More precisely, we shall show that there is a natural action of the submonoid

$$\mathbb{Q}_p^\times \times S_\alpha \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$$

of $\mathbb{Q}_p^\times \times T_\alpha \times Frob^{\mathbb{Z}} \times Fr^{\mathbb{Z}}$ on the Igusa varieties, where the action of $\mathbb{Q}_p^\times \times S$ is linear and the actions of $Frob$ and Fr are σ-semilinear (cfr. section 2.5). We also prove that this action commutes with the previously defined action of $G(\mathbb{A}^{\infty,p})$ on the Igusa varieties.

Let us remark that the action of the monoid $S = S_\alpha \subset T = T_\alpha$ (see definition 2.10) on the Igusa varieties $J_m/C \times \bar{\mathbb{F}}_p$ extends the action of the group $\Gamma = \Gamma_\alpha$, and also that the action of $\mathbb{Q}_p^\times$ on the Igusa varieties is compatible with the action of $\mathbb{Q}_p^\times \subset G(\mathbb{Q}_p)$ on the Shimura varieties (see section 2.1.12).

3.4.1. Let $(g, \rho, 1, 1) \in \mathbb{Q}_p^\times \times S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$ and write $e_i = e_i(\rho)$ and $f_i = f_i(\rho)$, for all i (see section 2.3.6), and $e = e_1$. We also assume that $-val_p(g) \geq e$.

For any positive integer m, such that $m \geq e$, we shall define a morphism

$$(g, \rho, 1, 1) : J_m \to J_{m-e_1}.$$

We recall that defining such a morphism $(g, \rho, 1, 1)$ on J_m is equivalent to give a $(4+k)$-tuple $(\mathcal{B}, \lambda', i', \bar{\mu}', j_{m-e,1}, \ldots, j_{m-e,k})$ over J_m which represents a point of J_{m-e}.

Let $(\mathcal{A}, \lambda, i, \bar{\mu}, j_1, \ldots, j_k)$ be the universal object over J_m, we write $\mathcal{G} = \epsilon\mathcal{A}[u^\infty]$ and denote by $\mathcal{G}^i$ its isoclinic subquotients.

Since $\rho \in S$, then ρ^{-1} is a well define isogeny and $\Sigma[p^{f_i}] \subset \ker(\rho_i^{-1}) \subset \Sigma[p^{e_i}] \subset \Sigma[p^e]$. We consider the following commutative diagram:

$$\begin{tikzcd}
\Sigma^i & \Sigma^i[p^{m-e}] \arrow[l, hook'] \arrow[r, "j_i\circ\rho_i"] & \frac{\mathcal{G}^i}{j_i(\ker(\rho_i^{-1}))}[p^{m-e}] \arrow[r, hook] & \frac{\mathcal{G}^i}{j_i(\ker(\rho_i^{-1}))} \\
\Sigma^i \arrow[u, two heads, "\rho_i{}^{-1}"] & \rho_i\Sigma^i[p^{m-e}] \arrow[l, hook'] \arrow[r] \arrow[u, two heads] \arrow[d, hook] & j_i(\rho_i\Sigma^i[p^{m-e}]) \arrow[r, hook] \arrow[u, two heads] \arrow[d, hook] & \mathcal{G}^i \arrow[u, two heads] \\
\ker(\rho_i^{-1}) \arrow[u, hook] \arrow[r, hook] & \Sigma^i[p^m] \arrow[r, "j_i"] & \mathcal{G}^i[p^m] & j_i(\ker(\rho_i^{-1})) \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

where $\rho_i : \Sigma^i \twoheadrightarrow \Sigma^i$ is the quasi-isogeny induced by ρ on the isoclinic subquotients and $j_i \circ \rho_i$ is defined as the isomorphism induced by j_i on the quotients, for each $i = 1, \dots k$. (The inclusion $\rho_i\Sigma^i[p^{m-e}] \subset \Sigma^i[p^m]$ follows from the inclusion $\ker(\rho_i^{-1}) \subset \Sigma^i[p^{e_i}] \subset \Sigma^i[p^e]$.) It is clear that since the isomorphism j_i are extendable to any higher leverl m', the same holds for the isomorphisms $j_i \circ \rho_i$.

For simplicity, we now write $\mathcal{K}_\rho^i = j_i(\ker(\rho_i^{-1})) \subset \mathcal{G}^i$.

Remark 3.6. — If $\rho \in S$, then there exists a unique subgrop $\mathcal{K}_\rho$ of $\mathcal{G}$ such that the corresponding subgroups inside the isoclinic subquotients of $\mathcal{G}$ are the $\mathcal{K}_\rho^i$ (for all i). Moreover, we have $\mathcal{G}[p^{f_k}] \subset \mathcal{K}_\rho \subset \mathcal{G}[p^{e_1}]$.

Let us argue by induction on k. If $k = 1$, then $\mathcal{K}_\rho = \mathcal{K}_\rho^1$ and thus there is nothing to prove.

If $k > 1$, we denote by $\mathcal{K}'_\rho$ the corresponding subgroup of $\mathcal{G}' = \mathcal{G}/\mathcal{G}^1$, then $\mathcal{G}'[p^{f_k}] \subset \mathcal{K}'_\rho \subset \mathcal{G}'[p^{e_2}]$. We define $\mathcal{K}_\rho = pr'|\, \mathcal{G}'[p^{e_2}]^{-1}(\mathcal{K}'_\rho) + i_1(\mathcal{K}_\rho^1)$, i.e. $\mathcal{K}_\rho$ is the unique subgroup which fits the following commutative diagram with exact rows. (We remark that we use the inequality $e_2 \leq f_1$ to deduce that $\mathcal{G}^1[p^{e_2}] \subset \mathcal{K}_\rho^1$.)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}^1[p^{e_2}] \arrow[r] \arrow[d, equal] & \mathcal{G}[p^{e_2}] \arrow[r] & \mathcal{G}'[p^{e_2}] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{G}^1[p^{e_2}] \arrow[r] \arrow[d, hook] & \mathcal{K}''_\rho \arrow[r] \arrow[d, hook] & \mathcal{K}'_\rho \arrow[r] \arrow[u, hook] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}_\rho^1 \arrow[r] & \mathcal{K}_\rho \arrow[r] & \mathcal{K}'_\rho \arrow[r] & 0
\end{tikzcd}$$

It is clear from the definition that $\mathcal{G}[p^{f_k}] \subset \mathcal{K}_\rho \subset \mathcal{G}[p^{e_1}]$, and thus we conclude.

3.4.2. We define the morphism

$$(g, \rho, 1, 1) : J_m \to J_{m-e}$$

to be associated to the $(4+k)$-tuple $(\mathcal{A}/\langle\mathcal{K}_\rho\rangle, \lambda', i', \bar{\mu}', j'_{m-e,1}, \dots, j'_{m-e,k})$ where:

1. $(\mathcal{A}/\langle\mathcal{K}_\rho\rangle, \lambda', i', \bar{\mu}')$ is the quadruple induced by the universal quadruple on J_m via the projection $\mathcal{A} \to \mathcal{A}/\langle\mathcal{K}_\rho\rangle$, where $\langle\mathcal{K}_\rho\rangle \subset \mathcal{A}[p^{-val_p(g)}]$ is a finite flat subgroup associated to $\mathcal{K}_\rho \subset \mathcal{G}[p^e] \subset \mathcal{G}[p^{-val_p(g)}]$;
2. $j'_{m-e,i}$ denotes the isomorphism $j_i \circ \rho_i : \Sigma^i[p^{m-e}] \to (\mathcal{G}^i/\mathcal{K}^i_\rho)[p^{m-e}]$.

The subgroup $\langle\mathcal{K}_\rho\rangle \subset \mathcal{A}[p^e]$ is defined as

$$(\mathcal{O}_{B_u} \otimes_{\mathbb{Z}_p} \mathcal{K}_\rho) \oplus (\mathcal{O}_{B_u} \otimes_{\mathbb{Z}_p} \mathcal{K}_\rho)^\perp \subset \mathcal{A}[u^{-val_p(g)}] \oplus \mathcal{A}[(u^c)^{-val_p(g)}],$$

and the structures on $\mathcal{A}/\langle\mathcal{K}_\rho\rangle$ are the ones induced by the structures on $\mathcal{A}$. More precisely, the polarization λ' on $\mathcal{A}/\langle\mathcal{K}_\rho\rangle$ is induced by $p^{-val_p(g)}\lambda : \mathcal{A} \to \mathcal{A}^\vee$, and the level structure is defined as

$$V \otimes \mathbb{A}^{\infty,p} \xrightarrow{\mu} V^p\mathcal{A} \longrightarrow V^p(\mathcal{A}/\langle\mathcal{K}_\rho\rangle).$$

3.4.3. It follows from the definition that for any $m \geq m' \geq e$

$$\mathrm{q}_{m-e,m'-e} \circ (g, \rho, 1, 1) = (g, \rho, 1, 1) \circ \mathrm{q}_{m,m'},$$

and that that the above definitions give rise to an action the submonoid

$$\{(g, \rho) \in \mathbb{Q}_p^\times \times S | -val_p(g) \geq e_1(\rho)\} \subset \mathbb{Q}_p^\times \times S$$

on the system of Igusa varieties.

We claim that the above action extends to an action of the monoid $\mathbb{Q}_p^\times \times S$. To prove it, it suffices to show that the element $(p^{-1}, 1) \in \mathbb{Q}_p^\times \times S$ acts invertibly on the Igusa varieties. More precisely, we claim that the element $(p^{-1}, 1)$ acts on J_m as the element $v^c \in E^\times \subset G(\mathbb{Q}) \subset G(\mathbb{A}^{\infty,p})$, for any $v \in E^\times$ such that $\mathrm{val}_u(v) = 1$, $\mathrm{val}_{u^c}(v) = 0$ and $v \equiv 1 \mod (u^c)^m$.

In fact, the pertinent subgroup $\mathcal{K}_1 \subset \mathcal{G}[p]$ is simply (0) and $\langle(0)\rangle = \mathcal{A}[u^c] \subset \mathcal{A}[p]$. Thus, the multiplication $v^c : \mathcal{A} \to \mathcal{A}$ gives rise to an isomorphism $\mathcal{A}/\langle(0)\rangle \simeq \mathcal{A}$, and under this identification the polarization λ' is simply λ, the level structure $\overline{\mu}' = \overline{v^c \circ \mu}$ and the isomorphisms $j'_{m,i} = v^c \circ j_{m,i} = j_{m,i}$.

3.4.4. We remark that the above argument also shows that the action of $\mathbb{Q}_p^\times$ on the Igusa varieties is compatible, under the projections $\mathrm{q}_{m',m}$, with the action of $\mathbb{Q}_p^\times \subset G(\mathbb{Q}_p)$ on the reductions of the Shimura varieties (see section 2.1.12).

3.4.5. Let us now define the action of Fr. As in section 2.5.9, we define the σ-semilinear action of Fr on the Igusa variaties as a linear morphisms

$$Fr : J_m \to J_m^{(p)},$$

where $J_m^{(p)}$ are the pullbacks under the Frobenius $\sigma : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ of the Igusa varieties $J_m/\bar{\mathbb{F}}_p$, for all m.

Let us denote by $C^{(p)}$ the pullback of the central leaf C under σ. Then, $C^{(p)}$ can be identified with the leaf $C_{\Sigma^{(p)}}$, and $J_m^{(p)}$ with the Igusa variety of level m over $C_{\Sigma^{(p)}}$.

Under the above identifications, the relative Frobenius on the central leaf

$$Fr : C \to C^{(p)}$$

is defined by setting

$$A = (A, \lambda, i, \bar{\mu}) \mapsto A^{(p)} = (A^{(p)}, \lambda^{(p)}, i^{(p)}, \bar{\mu}^{(p)}),$$

where $A^{(p)}$ denotes the pullbacks of A under σ, endowed with the structures induced from the ones of A.

If G denotes the Barsotti-Tate group associated to an abelian varieties A, then $G^{(p)}$ is the Barsotti-Tate group associated to $A^{(p)}$ and, for all i

$$(G^{(p)})^i = (G^i)^{(p)}.$$

Then, for any level m, the relative Frobenius on the Igusa variety of level m

$$Fr : J_m \to J_m^{(p)}$$

is defined by setting

$$(A, j_{m,1}, \ldots, j_{m,k}) \mapsto (A^{(p)}, j_{m,1}^{(p)}, \ldots, j_{m,k}^{(p)}),$$

and its action commutes with the action of the monoid $\mathbb{Q}_p^\times \times S$ on the Igusa varieties, i.e.

$$(j \circ (g, \rho, 1, 1))^{(p)} = j^{(p)} \circ (g, \rho, 1, 1)^{(p)},$$

for all $(g, \rho) \in \mathbb{Q}_p^\times \times S$.

3.4.6. We define the action of $Frob$ on the Igusa varieties, as an analogue of the action of $Frob$ on the Rapoport-Zink spaces (see section 2.5.8). As before, we define the σ-linear action of $Frob$ as a linear morphism

$$Frob : J_m \to J_{m-1}^{(p)},$$

for all $m \geq 0$.

Let us consider the following diagram:

$$\begin{tikzcd}
\Sigma^i \arrow[d, "F"] & \Sigma^i[p^m] \arrow[l, hook'] \arrow[r, "\simeq", "j_{m,i}"'] \arrow[d, dotted] & \mathcal{G}^i[p^m] \arrow[r, hook] \arrow[d, dotted] & \mathcal{G}^i \arrow[d, "F"] \\
\Sigma^{i\,(p)} & \Sigma^{i\,(p)}[p^{m-1}] \arrow[l, hook'] \arrow[r, "\simeq", "j_{m,i}F^{-1}"'] & \mathcal{G}^{i\,(p)}[p^{m-1}] \arrow[r, hook] & \mathcal{G}^{i\,(p)}
\end{tikzcd}$$

The isomorphisms $j_{m-1,i}F^{-1}$ are simply the restriction of the isomorphism $j_{m,i}^{(p)}$ to the p^{m-1}-torsion. Moreover, the subgroups $\mathcal{G}^i[F] = j_{m,i}(\Sigma^i[F])$ naturally piece together as the subquotients of the finite flat subgroup $\mathcal{G}[F]$.

Furthermore, we have $\mathcal{G}[F] \subset \mathcal{G}[p]$ and $\mathcal{A}[F] = \langle \mathcal{G}[F] \rangle \subset \mathcal{A}[p]$. Thus, analogously to the above definition of the action of $\mathbb{Q}_p^\times \times S$, we set the action of the element $(p^{-1}, 1, Frob, 1)$ to be defined by the morphism associated to the $(6+k)$-tuple

$$(\mathcal{A}^{(p)}, \lambda^{(p)}, i^{(p)}, \bar{\mu}^{(p)}, j_{m-1,1}^{(p)}, \ldots, j_{m-1,k}^{(p)}),$$

or equivalently $(p^{-1}, 1, Frob, 1) = \mathrm{q}_{m,m-1} \circ Fr$.

It follows that the action of $Frob$ on the Igusa varieties is defined as

$$Frob = (p,1,1,1) \circ \mathrm{q}_{m,m-1} \circ Fr,$$

i.e. by the morphism associated to the $(6+k)$-tuple

$$(\mathcal{A}^{(p)}, \lambda^{(p)}, i^{(p)}, \bar{\mu}^{(p)} \circ (v^c)^{-1}, j^{(p)}_{m-1,1}, \ldots, j^{(p)}_{m-1,k}),$$

where $v \in E^\times$ is an element such that $\mathrm{val}_u(v) = 1$, $\mathrm{val}_{u^c}(v) = 0$ and $v \equiv 1 \mod (u^c)^m$ (see section 3.4.3).

It is clear that the action of $Frob$ commutes with the previously defined action of $\mathbb{Q}_p^\times \times S \times Fr^{\mathbb{N}}$, and therefore there is an action of the monoid

$$\mathbb{Q}_p^\times \times S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$$

on the system of Igusa varieties. Moreover, it is easy to see that this action commutes with the previuosly defined action of $G(\mathbb{A}^{\infty,p})$.

3.4.7. Let us remark that, as in section 2.5.13, depending of the choice of the Barsotti-Tate group $\Sigma/\mathbb{F}_p$ in its isogeny class, there is a natural isomorphism between J_m and $J_m^{(p)}$ (for any $m \geq 0$), which arise from the fact that the Barsotti-Tate group Σ is defined over $\mathbb{F}_p$, namely

$$frob : J_m \to J_m^{(p)}$$
$$(A, j_{m,i}) \mapsto (A, j_{m,i} \circ \nu_{m,i}^{-1}),$$

where $\nu_{m,i}$ denotes the restriction to $\Sigma^i[p^m]$ of the identification $\nu : \Sigma \simeq \Sigma^{(p)}$ (see section 2.3.7).

As in the case of the Rapoport-Zink space, the action of $frob$ is invertible (and thus defines an effective descent datum on the Igusa varieties), but does not commute with the action of T_α (though $frob^B$ does, see section 2.3.8).

Proposition 3.7. — *Maintaining the above notations. We remark that $e(p^{-1}) = 1$ and we write $a = e(fr^{-B}) = \lambda_1 B$.*

1. *If $m \geq 1$, the element $(p^{-1}, p^{-1}, 1, 1) \in \mathbb{Q}_p^\times \times S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$ acts on J_m as*

$$v \circ \mathrm{q}_{m,m-1},$$

   where $v \in E^\times \subset G(\mathbb{Q}) \subset G(\mathbb{A}^{\infty,p})$ is an element such that $\mathrm{val}_u(v) = 1$, $\mathrm{val}_{u^c}(v) = 0$ and $v \equiv 1 \mod (u^c)^m$.
2. *If $m \geq a$, the element $fr^{-B} \in S$ acts on J_m as*

$$(p^B, 1, 1, 1) \circ \mathrm{q}_{m,m-a} \circ frob^{-B} \circ Fr^B,$$

   where $Fr : J_m \to J_m^{(p)}$ denotes the $\bar{\mathbb{F}}_p$-linear relative Frobenius on the Igusa variety.
3. *If $m \geq B$ (and thus $m \geq a$ since $a = \lambda_1 B \leq B$), we have*

$$Frob^B = \mathrm{q}_{m-a,m-B} \circ frob^B \circ fr^{-B}.$$

Proof. — Part (1): By considering the diagram in section 3.4, when $\rho_i^{-1} = p$, and the induced isomorphisms $j_i \circ p$ are simply the restrictions of j_i on the p^{m-1}-torsions. Moreover, $\langle \mathcal{G}[p] \rangle = \mathcal{A}[u] \subset \mathcal{A}[p]$ and thus the multiplication $v : \mathcal{A} \to \mathcal{A}$ gives rise to the necessary identifications.

Part (2): We shall prove the element $(p^{-B}, fr^{-B}, 1, 1)$ acts as $\mathrm{q}_{m,m-a} \circ frob^{-B} \circ Fr^B$. Let us consider the following commutative diagram.

$$\begin{tikzcd}
\Sigma^i \arrow[r, "F^B"] \arrow[d, equal] & \Sigma^{i(p^B)} & \Sigma^{i(p^B)}[p^m] \arrow[l, hook'] \arrow[dd, hook] \arrow[ddd, bend left, "j_{m,i}^{(p^B)}"] \\
\Sigma^i \arrow[r, two heads, "\tau_i^{\lambda_i B}"] & \Sigma^i \arrow[u, "(\nu^i)^B"] & \\
\Sigma^i[p^m] \arrow[u, hook] \arrow[r, dotted] \arrow[d, "j_{m,i}"] & \Sigma^i[p^{m-a}] \arrow[u, hook] \arrow[r, "\nu^B_{m-a,i}"] \arrow[d, "j_{m,i} \circ fr^{-B}"] & \Sigma^{i(p^B)}[p^{m-a}] \arrow[d, "j_{m-a,i}^{(p^B)}"] \\
\mathcal{G}^i[p^m] \arrow[r, dotted] \arrow[d, hook] & \mathcal{G}^i/\mathcal{K}^i_{fr^{-B}}[p^{m-a}] \arrow[r, "\simeq"] \arrow[d, hook] & \mathcal{G}^{i(p^B)}[p^{m-a}] \arrow[d, hook] \\
\mathcal{G}^i \arrow[r, two heads] \arrow[d, equal] & \mathcal{G}^i/\mathcal{K}^i_{fr^{-B}} \arrow[d, "\simeq"] & \\
\mathcal{G}^i \arrow[r, "F^B"] & \mathcal{G}^{i(p^B)} & \mathcal{G}^{i(p^B)}[p^{m_i}] \arrow[l, hook']
\end{tikzcd}$$

By definition of $fr^B = \oplus_i \tau_i^{\lambda_i B} \in S$, we have that $\nu^B \circ fr^B = F^B$ on Σ, or equivalently that

$$(\nu^i)^B \circ \tau_i^{\lambda_i B} = F^B$$

on Σ^i, for all i. In particular, it follows that $\ker(F^B) = \ker(\tau_i^{\lambda_i B})$ as subgroups of Σ^i, for all i.

Thus, maintaining the notations as in section 3.4, $\mathcal{K}^i_{fr^{-B}} = \mathcal{G}^i[F^B]$, and $\mathcal{K}_{fr^{-B}} = \mathcal{G}[F^B]$, i.e. there exists an isomorphism between $\mathcal{G}/\mathcal{K}_{fr^{-B}}$ and $\mathcal{G}^{(p^B)}$, compatible with the projection $\mathcal{G} \to \mathcal{G}/\mathcal{K}_{fr^{-B}}$ and $F^B : \mathcal{G} \to \mathcal{G}^{(p^B)}$. Moreover, the isomorphisms $j_{m,i} \circ fr^{-B}$ can be identified with the restrictions of the isomorphisms $j_{m,i}^{(p^B)} \circ \nu^B_{m,i}$ over the p^{m-a}-torsion subgroups (we denote by $\nu^B_{m,i}$ the restriction of $(\nu^i)^B$ to the p^m-torsion subgroups).

Finally, we also have that $\langle \mathcal{G}[F^B] \rangle = \mathcal{A}[F^B] \subset \mathcal{A}[p^B]$ and thus, the Frobenius morphism $F^B : \mathcal{A} \to \mathcal{A}^{(p^B)}$ gives rise to an isomorphism $\mathcal{A}/\langle \mathcal{G}[F^B] \rangle \simeq \mathcal{A}^{(p^B)}$, which is compatible with the structures induced on the two quotients by the one on $\mathcal{A}$.

Part (3): The equality follows from part (2) and the equality

$$Frob^B = (p^B, 1, 1, 1) \circ \mathrm{q}_{m,m-B} \circ Fr^B.$$

□

3.5. The cohomology of the Igusa varieties. — In this section, we shall reinterpret some of the above results in terms of the cohomology with compact supports of the Igusa varieties.

We shall observe that the cohomology groups of the Igusa varieties naturally form a direct limit, under the morphism corresponding to the projections $\mathrm{q}_{m',m}$ and q_{V^p,U^p}, and also that the action of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times S \times Frob^{\mathbb{N}}$ on the system of Igusa varieties give rise to an action on the direct limit of the cohomology groups, which extends to an action of the group $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times Frob^{\mathbb{Z}}$.

Thus, the cohomology groups of the Igusa varieties are representations of the group $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times Frob^{\mathbb{Z}}$, or equivalently of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times W_{\mathbb{Q}_p}$, where the action of the Weil group is unramified.

3.5.1. Let l be a prime number, $l \neq p$, and $r \geq 1$ an integer.

For any integer $i \geq 0$, we consider the i-th étale cohomology groups with compact supports of the Igusa varieties $J_{U^p,m}$ over $\bar{\mathbb{F}}_p$, with coefficient in $\mathbb{Z}/l^r\mathbb{Z}$,

$$H_c^i(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}),$$

for any positive integer m and any open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$.

The finite étale morphisms

$$\mathrm{q}_{m',m} : J_{U^p,m'} \to J_{U^p,m}$$

and

$$\mathrm{q}_{V^p,U^p} : J_{V^p,m} \to J_{U^p,m},$$

for all positive integers $m' \geq m$ and all open compact subgroups $V^p \subset U^p$ of $G(\mathbb{A}^{\infty,p})$, induces some morphisms between the cohomology groups

$$(\mathrm{q}_{m',m})_* : H_c^i(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}) \to H_c^i(J_{U^p,m'}, \mathbb{Z}/l^r\mathbb{Z}),$$

and

$$(\mathrm{q}_{V^p,U^p})_* : H_c^i(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}) \to H_c^i(J_{V^p,m}, \mathbb{Z}/l^r\mathbb{Z}).$$

It is easy to see that the i-th cohomology groups of the Igusa varieties, together with the above morphisms, form an inductive system and we refer to the direct limit

$$H_c^i(J, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_{m,U^p} H_c^i(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}),$$

as the i-th cohomology group of the Igusa varieties, with $\mathbb{Z}/l^r\mathbb{Z}$-coefficients.

3.5.2. Let us now consider the action of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$ on the system of Igusa varieties (see sections 3.4 and 3.3). It induces an action of $G(\mathbb{A}^{\infty,p}) \times S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$ on the direct limit of the cohomology groups, $H^i_c(J, \mathbb{Z}/l^r\mathbb{Z})$, and moreover the action of Fr is trivial.

In fact, for any $(g, \rho) \in \mathbb{Q}_p^\times \times S$, the morphism $(g, \rho) : J_{U^p,m} \to J_{U^p,m-e}$ induces a morphism

$$(g, \rho)_* : H^i_c(J_{U^p,m-e}, \mathbb{Z}/l^r\mathbb{Z}) \to H^i_c(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}),$$

for any integer $m \geq e$ (where $e = e(\rho)$) and any open compact subgroup U^p.

Morover, since $(g, \rho) \circ \mathrm{q}_{m',m} = \mathrm{q}_{m'-e,m-e} \circ (g, \rho)$ and $(g, \rho) \circ \mathrm{q}_{V^p,U^p} = \mathrm{q}_{V^p,U^p} \circ (g, \rho)$, the morphisms $(g, \rho)_*$ give rise to an endomorphism of the direct limit.

Analogously, for every $g \in G(\mathbb{A}^{\infty,p})$, the morphism $g : J_{U^p,m} \to J_{g^{-1}U^pg,m}$ induces a morphism

$$g_* : H^i_c(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}) \to H^i_c(J_{g^{-1}U^pg,m}, \mathbb{Z}/l^r\mathbb{Z}),$$

for any positive integer $m \geq 0$ and any open compact subgroup U^p, and, since $g \circ \mathrm{q}_{V^p,U^p} = \mathrm{q}_{g^{-1}V^pg,g^{-1}U^pg} \circ g$ and $g \circ \mathrm{q}_{m',m} = \mathrm{q}_{m',m} \circ g$, the morphisms g_* give rise to an automorphism of the direct limit.

Similarly, the σ-semilinear morphisms $Frob, Fr : J_m \to J_m$ gives rise to an action on the étale cohomology groups, and moreover, since $Frob$ and Fr commute with the projections $\mathrm{q}_{m',m}$ and q_{V^p,U^p}, it induces an action on the groups $H^i_c(J, \mathbb{Z}/l^r\mathbb{Z})$.

We remark that the above action of Fr on the étale cohomology groups is trivial, since $Fr : J_m \to J_m$ is the absolute Frobenius.

Remark 3.8. — The action of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times S \times Frob^{\mathbb{N}}$ on $H^i_c(J, \mathbb{Z}/l^r\mathbb{Z})$ can be extended to an action of the group $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times Frob^{\mathbb{Z}}$ $(S \subset T)$.

Indeed, since $T = \langle S, p, fr^B \rangle$, in order to prove that the action of S extends to an action of T it suffices to observe that the elements $p^{-1}, fr^{-B} \in S$ act invertibly, or equivalently that the actions of $(p^{-1}, p^{-1}), (p^{-B}, fr^{-B}) \in \mathbb{Q}_p^\times \times S$ are invertible. Since the action of (p^{-1}, p^{-1}) on the Igusa varieties is given by the morphism $v \circ \mathrm{q}_{m,m-1}$, where $v \in E^\times \subset G(\mathbb{Q})$ acts isomorphically, the induced action on the cohomology groups becomes invertible once one passes to the direct limit. On the other hand, the element (p^{-B}, fr^{-B}) acts on the Igusa varieties as $\mathrm{q}_{m,m-a} \circ frob^{-B} \circ Fr^B$, and thus the induced action on the direct limit $H^i_c(J, \mathbb{Z}/l^r\mathbb{Z})$ is invertible. (Since the action of $frob$ on the Igusa varieties is invertible, such is also the induced action on the cohomology groups. On the other hand, we already remarked that the action of Fr on the étale cohomology groups is trivial and, therefore, in particular invertible.)

The same argument proves that to the action of $Frob$ on $H^i_c(J, \mathbb{Z}/l^r\mathbb{Z})$ is invertible, since $Frob = \mathrm{q}_{m,m-1} \circ Fr$.

In the following, we shall refer to the cohomology groups with compact supports of the Igusa varieties with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$ as a representation of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times W_{\mathbb{Q}_p}$, where the action of the Weil group is unramified (i.e. it factors through the

projection $W_{\mathbb{Q}_p} \twoheadrightarrow \sigma^{\mathbb{Z}}$) and the action of σ on the above spaces is defined to be equal to the action of $Frob^{-1}$.

Remark 3.9. — Let $U^p \subset G(\mathbb{A}^{\infty,p})$ be an open compact subgroup. For any integer $q \geq 0$, the $\mathbb{Z}/l^r\mathbb{Z}$-representation of $T \times W_{\mathbb{Q}_p}$

$$H^q_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_m H^q_c(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z})$$

is admissible.

In fact, for any integer $m \geq 1$, let $\Gamma^m \subset \Gamma$ be the subgroup of automorphisms of Σ which restrict to the identity on $\Sigma[p^m]$. As m vary, they form a cofinal system of compact open subgroups of T and we have

$$H^q_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z})^{\Gamma^m} = H^q_c(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}),$$

which is finite. (The latter equality follows from the existence of a trace map on cohomology and the fact that the morphisms $\mathrm{q}_{m',m}$ are finite étale, of l-prime degree.)

Let us remark that, on the other hand, the $\mathbb{Z}/l^r\mathbb{Z}$-representations $H^q_c(J, \mathbb{Z}/l^r\mathbb{Z})$ of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times W_{\mathbb{Q}_p}$ are smooth, but not *a priori* admissible (cfr. section 2.1.13).

For all integers $q \geq 0$, we define the l-adic cohomology groups of the Igusa varieties

$$H^q_c(J, \mathbb{Q}_l) = \varinjlim_{U^p,m} \varprojlim_r H^q_c(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

It follows from the definition and remark 3.8 that they are l-adic representations of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times T \times W_{\mathbb{Q}_p}$ and, moreover, are admissible.

4. A system of covers of the Newton polygon strata

In this section we shall study the geometry of the open Newton polygon strata $\bar{X}^{(\alpha)}$. For each Newton polygon α, we shall consider the product of the Igusa varieties J_m over the central leaf $C = C_\alpha$ with the reduced fiber $\bar{\mathcal{M}}^{n,d}$ of the truncated Rapoport-Zink space $\mathcal{M}^{n,d} = \mathcal{M}^{n,d}_\alpha$ (see section 2.5.10). For any positive integers m, n, d, such that $m \geq d$, we shall construct some morphisms

$$\pi_N : J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p,$$

for all positive integers N sufficiently large. (In the special case of $n = d = N = 0$, the morphisms π_0 are simply the morphisms

$$\mathrm{q}_m : J_m \to C \times \bar{\mathbb{F}}_p \hookrightarrow \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p,$$

for all $m \geq 0$.)

We shall show that the morphisms π_N are finite and surjective on geometric points for $m, n, d \gg 0$. As m, n, d, N vary, the morphisms π_N commute with the natural projections between Igusa varieties and the inclusions between the Rapoport-Zink spaces, and also $\pi_{N+1} = (Fr^B \times 1) \circ \pi_N$, for all N (Fr denotes the Frobenius morphism of $\bar{X}^{(\alpha)}$ over $\mathbb{F}_p$ and B the positive integer we defined in section 2.3.8, which depends only on α).

4.1. The action of Frobenius on the slope filtration. — The definition of the morphisms π_N is based of the following key observation regarding the action of the powers of Frobenius on the slope filtration of a Barsotti-Tate group.

4.1.1. If G be a Barsotti-Tate group over a scheme S in characteristic p we denote by $G^{(p)}$ its twist by Frobenius and by $F : G \to G^{(p)}$ the Frobenius map.

Lemma 4.1. — *Let G be a Barsotti-Tate group over a scheme S in characteristic p. Assume that G has constant Newton polygon α with slopes $\lambda_1 > \cdots > \lambda_k$ and for each i denote by b_i the denominator of λ_i (written in minimal form). We also write $B = \mathrm{lcm}(b_1, \ldots, b_k)$ and $\delta = \min(\lambda_1 - \lambda_2, \ldots, \lambda_{k-1} - \lambda_k)$.*

Suppose also that G has a slope filtration

$$(0) \subset G_1 \subset \cdots \subset G_k = G$$

over S as in theorem 2.12, and denote by G^i the corresponding subquotients.

Then, for any integer $n > 0$, there is a canonical isomorphism:

$$G^{(p^{nB})}[p^{n\delta B}] \simeq \prod_{i=1}^{k} G^{i\,(p^{nB})}[p^{n\delta B}].$$

Proof. — We prove the lemma by induction on the lenght k of the slope filtration of G. The case $k = 1$ is trivial as $G = G_1 = G^1$.

For $k \geq 2$ we write $H = G^k$, $G' = G_{k-1}$, $\lambda = \lambda_k$ and $\lambda' = \lambda_{k-1}$. As G is completely slope divisible, the quasi-isogeny $p^{-\lambda B}F^B$ is in fact an isogeny of G, H and G'. In particular, it is an isomorphism of H, since H is isoclinic of slope λ.

We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r] \arrow[d, "p^{-\lambda B}F^B"] & G \arrow[r] \arrow[d, "p^{-\lambda B}F^B"] & H \arrow[r] \arrow[d, "p^{-\lambda B}F^B"] & 0 \\
0 \arrow[r] & G'^{(p^B)} \arrow[r] & G^{(p^B)} \arrow[r] & H^{(p^B)} \arrow[r] & 0.
\end{tikzcd}$$

where the rows are exact and the vertical maps are isogenies. Since the last vertical morphism $H \to H^{(p^B)}$ is an isomorphism, it follows by the snake lemma that $G'[p^{-\lambda B}F^B] = G[p^{-\lambda B}F^B]$.

As G' is slope divisible with respect to λ' we can factor the isogeny $p^{-\lambda B}F^B$ on G' as $p^{-\lambda B}F^B = p^{(\lambda'-\lambda)B} \circ p^{-\lambda' B}F^B$. Thus $G'[p^{-\lambda B}F^B] \supset G'[p^{(\lambda'-\lambda)B}]$ and we have

$$H[p^{(\lambda'-\lambda)B}] \simeq \frac{G[p^{(\lambda'-\lambda)B}]}{G'[p^{(\lambda'-\lambda)B}]} \hookrightarrow \frac{G}{G'[p^{(\lambda'-\lambda)B}]} \to \frac{G}{G'[p^{-\lambda B}F^B]} \simeq G^{(p^B)}$$

where the composite map is a section of the natural projection

$$G^{(p^B)}[p^{(\lambda'-\lambda)B}] \twoheadrightarrow H^{(p^B)}[p^{(\lambda'-\lambda)B}] \simeq H[p^{(\lambda'-\lambda)B}].$$

We conclude that

$$G^{(p^B)}[p^{(\lambda'-\lambda)B}] \simeq G'^{(p^B)}[p^{(\lambda'-\lambda)B}] \times H^{(p^B)}[p^{(\lambda'-\lambda)B}]$$

and therefore by inductive hypothesis (as $\lambda' - \lambda \leq \delta$)

$$G^{(p^B)}[p^{\delta B}] \simeq \prod_{i=1}^{k} G^{i\,(p^B)}[p^{\delta B}].$$

Since the above argument holds also if we replace B by nB (for any integer $n > 0$), we obtain the stated result. □

Corollary 2.14 allow us to apply the previous theorem to the Barsotti-Tate group $\mathcal{G} = \epsilon\mathcal{A}[p^\infty]$ over the central leaf C.

Corollary 4.2. — *Maintaining the notations of section 3, we denote by $\mathcal{G}$ the Barsotti-Tate group $\epsilon\mathcal{A}[p^\infty]$ over the central leaf C. Let d be a positive integer.*

Then, for any integer N such that $N \geq d/\delta B$, there is a canonical isomorphism

$$\mathcal{G}^{(p^{NB})}[p^d] \simeq \prod_{i=1}^{k} \mathcal{G}^{i\,(p^{NB})}[p^d].$$

4.2. The morphisms π_N. — We denote by $\bar{\mathcal{M}}^{n,d}$ over $\operatorname{Spec}\bar{\mathbb{F}}_p$ the reduced fibers of the truncated Rapoport-Zink spaces associated to the Barsotti-Tate group Σ (see section 2.5.10). For all set of indexes $(m, n, d) \in \mathbb{Z}^3_{\geq 0}$, such that $m \geq d$, we shall introduce a system of maps

$$\pi_N : J_m \times_{\operatorname{Spec}\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p,$$

indexed by the positive integers $N \geq d/\delta B$.

4.2.1. By the universal property of $\bar{X}^{(\alpha)}$ to define such a map is equivalent to define a quadruple $(A, \lambda, i, \bar{\mu})$ over $J_m \times_{\operatorname{Spec}\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ such that the Newton polygon of the Barsotti-Tate group $\epsilon A[p^\infty]$ is constant and equal to α.

We denote by $(\mathcal{A}, \lambda, i, \bar{\mu})$ over $J_m \times_{\operatorname{Spec}\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ the pullback of the universal quadruple over J_m/C and by $(\mathcal{H}', \beta^{\mathrm{univ}})$ the pullback of the universal pair over $\bar{\mathcal{M}}^{n,d}$. Over $J_m \times_{\operatorname{Spec}\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ we also have the data of the universal isomorphisms

$$j^{\mathrm{univ}}_{m,i} : \Sigma^i[p^m] \to \mathcal{G}^i[p^m]$$

which, by corollary 4.2, induce an isomorphism

$$j^{\mathrm{univ}}_N = \oplus (j^{\mathrm{univ}}_{m,i})^{(p^{NB})}_{|[p^d]} : \Sigma^{(p^{NB})}[p^d] \to \prod_i \mathcal{G}^{i\,(p^{NB})}[p^d] \simeq \mathcal{G}^{(p^{NB})}[p^d]$$

for any $N \geq d/\delta B$.

By the definition of the space $\mathcal{M}^{n,d}$, the kernel of the isogeny $p^n\beta^{\mathrm{univ}} : \Sigma \twoheadrightarrow \mathcal{H}'$ is contained in $\Sigma[p^d]$, and thus $\ker(p^n\beta^{\mathrm{univ}})^{(p^{NB})} \subset \Sigma^{(p^{NB})}[p^d]$.

We set

$$\mathcal{K} = \mathcal{K}_N = j^{\mathrm{univ}}_N(\ker(p^n\beta^{\mathrm{univ}})^{(p^{NB})}) \subset \mathcal{G}^{(p^{NB})}[p^d].$$

We also write $\mathcal{K}_u = (\mathcal{O}_B)_u \otimes_{\mathbb{Z}_p} \mathcal{K} \subset \mathcal{A}^{(p^{NB})}[u^d]$, and $\mathcal{K}_u^\perp \subset \mathcal{A}^{(p^{NB})}[(u^c)^d]$ for the annihilator of $\mathcal{K}_u \subset \mathcal{A}^{(p^{NB})}[u^d]$ under the λ-Weil pairing.

We set

$$\langle \mathcal{K} \rangle = \mathcal{K}_u \oplus \mathcal{K}_u^{\perp} \subset \mathcal{A}^{(p^{NB})}[p^d]$$

and define the morphism π_N to be determined by the quadruple which is the quotient of the universal quadruple $(\mathcal{A}, \lambda, i, \bar{\mu})$ via the isogeny

$$\mathcal{A}^{(p^{NB})} \twoheadrightarrow \mathcal{A}^{(p^{NB})}/\langle \mathcal{K} \rangle.$$

It is clear that the abelian variety $\mathcal{A}^{(p^{NB})}/\langle \mathcal{K} \rangle$ inherites the structure of $\mathcal{A}^{(p^{NB})}$. More precisely, the induced polarization on $\mathcal{A}^{(p^{NB})}/\langle \mathcal{K} \rangle$ is defined as the unique polarization $p^d\bar{\lambda}$ which fits the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}^{(p^{NB})} \arrow[r, "p^d\lambda^{(p^{NB})}"] \arrow[d, two heads] & \mathcal{A}^{(p^{NB})\vee} \\
\mathcal{A}^{(p^{NB})}/\langle \mathcal{K} \rangle \arrow[r, "p^d\bar{\lambda}"'] & \left(\mathcal{A}^{(p^{NB})}/\langle \mathcal{K} \rangle\right)^{\vee} \arrow[u]
\end{tikzcd},$$

and the induced level structure away from p is defined as

$$V \otimes \mathbb{A}^{\infty,p} \xrightarrow{\mu^{(p^{NB})}} V^p(\mathcal{A}^{(p^{NB})}) \xrightarrow{v^{-n}(v^c)^{-d+n}} V^p(\mathcal{A}^{(p^{NB})}) \longrightarrow V^p(\mathcal{A}^{(p^{NB})}/\langle \mathcal{K} \rangle),$$

where $v \in E^{\times}$ is an element such that $\mathrm{val}_u(v) = 1$, $\mathrm{val}_{u^c}(v) = 0$ and $v \equiv 1 \mod (u^c)^m$.

4.2.2. We remark that the above constructions and definitions hold for any finite flat p^d-torsion subgroup $\mathcal{K}$ of the Barsotti-Tate group associated to an abelian variety endowed with a polarization, an action of $\mathcal{O}_B$ and a level structure away from p.

4.2.3. We observe that in the case $n = d = 0$, the space $\bar{\mathcal{M}}^{0,0}$ is just a point (namely, the point corresponding to the pair (Σ, id) over $\bar{\mathbb{F}}_p$) and the morphism π_0 on the Igusa variety J_m is simply the structure morphism

$$\mathrm{q}_m : J_m \to C \times \bar{\mathbb{F}}_p \hookrightarrow \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p,$$

for any $m \geq 0$.

4.2.4. Let us denote by Fr the Frobenius morphism of $\bar{X}^{(\alpha)}$ over $\mathbb{F}_p$, i.e. the $\mathbb{F}_p$-linear morphism defined by

$$(A, \lambda, i, \bar{\mu}) \mapsto (A^{(p)}, \lambda^{(p)}, i^{(p)}, \bar{\mu}^{(p)}),$$

and by σ the Frobenius of $\bar{\mathbb{F}}_p$.

Proposition 4.3. — *Let $m \leq m, n \leq n', d \leq d', N \leq N'$ be some positive integers and U^p a level away from p.*

Let $(g^p, g_p) \in G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^{\times} \subset G(\mathbb{A}^{\infty})$, $(\rho, Frob^r, Fr^s) \in S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$, and write $e = e(\rho)$ and $f = f(\rho)$.

1. *If $m \geq d$ and $N \geq d/\delta B$, then on $J_{m'} \times_{\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$*
$$\pi_N \circ (\mathrm{q}_{m',m} \times 1) = \pi_N.$$
2. *If $m \geq d'$, $N \geq d'/\delta B$ and $d' - d \geq (n'-n)h$, then on $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$*
$$\pi_N \circ (1 \times i^{n,d}_{n',d'}) = \pi_N.$$
3. *If $m \geq d$ and $N \geq d/\delta B$, then on $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$*
$$\pi_{N'} = (Fr^{(N'-N)B} \times 1) \circ \pi_N.$$
4. *If $m \geq d$ and $N \geq d/\delta B$, then on $J_{U^p,m} \times_{\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$*
$$\pi_N \circ ((g^p, g_p) \times 1) = ((g^p, g_p) \times 1) \circ \pi_N.$$
5. *If $m \geq d + 2e - f$ and $N \geq (d + 2e - f)/\delta B$, then on $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$*
$$\pi_N \circ ((\rho, Frob^r, Fr^s) \times (\rho, Frob^r, Fr^s)) = (Fr^s \times \sigma^{r+s}) \circ \pi_N.$$

Proof. — Part (1): It is straight forward that the definition of π_N (i.e. the definition of the isomorphism j_N^{univ}) depends only on the restrictions the isomorphisms $j_{m,i}^{\mathrm{univ}}$ over the p^d-torsion, for all i. Thus, $\pi_N \circ (\mathrm{q}_{m',m} \times 1) = \pi_N$.

Part (2): Proving that $\pi_N \circ (1 \times i^{n,d}_{n',d'}) = \pi_N$ is equivalent to proving that the definition of the abelian variety $\mathcal{A}/\langle \mathcal{K} \rangle$ and its structures, associated to π_N, is independent on n, d.

Let us denote by $\mathcal{K}^{n,d}$ (resp. $\mathcal{K}^{n',d'}$) the subgroup of $\mathcal{G}^{(p^{NB})}$ associated to the morphism π_N on $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ (resp. on $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n',d'}$).

It suffices to consider the two cases $(n', d') = (n, d+1)$ and $(n', d') = (n+1, d+1)$.

Let us first consider the case $(n', d') = (n, d+1)$. The definition of $\mathcal{K}^{n,d} = \mathcal{K}$ does not depend on d, but the definition of $\langle \mathcal{K} \rangle$ does. In particular, we have
$$(\mathcal{K}_u^{n,d})^\perp = u^c (\mathcal{K}_u^{n,d+1})^\perp.$$
Thus, the isogeny $v^c : \mathcal{A}^{(p^{NB})} \to \mathcal{A}^{(p^{NB})}$ (where we choose an element $v \in E^\times$ such that $\mathrm{val}_u(v) = 1$, $\mathrm{val}_{u^c}(v) = 0$ and $v \equiv 1 \mod (u^c)^m$) induces an isomorphism
$$\mathcal{A}^{(p^{NB})}/\langle \mathcal{K}^{n,d+1} \rangle \simeq \mathcal{A}^{(p^{NB})}/\langle \mathcal{K}^{n,d} \rangle.$$
Moreover, the following diagrams commute:

$$\begin{tikzcd}
\mathcal{A}^{(p^{NB})} \arrow[r, "v^c"'] \arrow[d] \arrow[rrr, bend left, "p^{d+1}\lambda^{(p^{NB})}"] & \mathcal{A}^{(p^{NB})} \arrow[r, "p^d\lambda^{(p^{NB})}"] \arrow[d] & (\mathcal{A}^{(p^{NB})})^\vee \arrow[r, "(v^c)^\vee"'] & (\mathcal{A}^{(p^{NB})})^\vee \\
\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d+1} \rangle} \arrow[r, "\simeq"] \arrow[rrr, bend right, "p^{d+1}\bar{\lambda}"'] & \frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d} \rangle} \arrow[r, "p^d\bar{\lambda}"] & \left(\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d} \rangle}\right)^\vee \arrow[r, "\simeq"] \arrow[u] & \left(\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d+1} \rangle}\right)^\vee \arrow[u]
\end{tikzcd}$$

where $(v^c)^\vee = v$, and

$$\begin{tikzcd}
V \otimes \mathbb{A}^{\infty,p} \arrow[r, "\mu"] & V^p(\mathcal{A}^{(p^{NB})}) \arrow[r, "v^{-n}(v^c)^{-d+n}"] \arrow[dr, bend right, "v^{-n}(v^c)^{-d-1+n}"'] & V^p(\mathcal{A}^{(p^{NB})}) \arrow[r] & V^p(\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d}\rangle}) \\
& & V^p(\mathcal{A}^{(p^{NB})}) \arrow[r] \arrow[u, "v^c"] & V^p(\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d+1}\rangle}). \arrow[u, "\simeq"']
\end{tikzcd}$$

Equivalently, the isomorphism

$$\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d+1}\rangle} \simeq \frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d}\rangle}$$

gives rise to an equivalence between the two corresponding quadruples.

Let us now consider the case $(n', d') = (n+1, d+1)$. By definition of π_N, we have that

$$\mathcal{K}^{n,d} = p(\mathcal{K}^{n+1,d+1}),$$

and also

$$\mathcal{K}_u^{n,d} = u(\mathcal{K}_u^{n+1,d+1}) \text{ and } (\mathcal{K}_u^{n,d})^\perp = (\mathcal{K}_u^{n+1,d+1})^\perp.$$

Thus, the multiplication by v on $\mathcal{A}^{(p^{NB})}$ gives rise to an isomorphism between

$$\frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n+1,d+1}\rangle} \simeq \frac{\mathcal{A}^{(p^{NB})}}{\langle \mathcal{K}^{n,d}\rangle},$$

which indeed gives rise to an equivalence between the two corresponding quadruples (by an argument completely similar to the previous one).

Part (3): The equality $\pi_{N'} = (Fr^{(N'-N)B} \times 1)\pi_N$ follows from the fact that

$$j_{N'}^{\text{univ}} = (j_N^{\text{univ}})^{(p^{(N'-N)B})}.$$

Part (4): By the very definition of the action of $G(\mathbb{A}^{\infty,p})$ on the Igusa varieties, we have that $\pi_0 \circ (g^p \times 1) = (g^p \times 1) \circ \pi_0$ on $J_{U^p,m}$, for any level U^p, m.

It suffices to remark that, for all $g^p \in G(\mathbb{A}^{\infty,p})$, we have $g^p \circ Fr = Fr \circ g^p$ on $\bar{X}_{U^p}^{(\alpha)}$ to deduce from part (3) that $\pi_N \circ (g^p \times 1) = (g^p \times 1) \circ \pi_N$ on $J_{U^p,m} \times_{\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$, for all U^p, $m \geq d$ and $N \geq d/\delta B$.

Analogously, the equality $\pi_N \circ (g_p \times 1) = (g_p \times 1) \circ \pi_N$, for any $g_p \in \mathbb{Q}_p^\times$, follows easily from the definitions and the observations in section 3.4.3.

Part (5): Let $\rho \in S$ (i.e. $(1, \rho, 1, 1) \in S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$), then the action of $\rho \in S$ on the spaces $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ is defined by the morphism

$$\rho \times \rho : J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to J_{m-e} \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n+e,d+e-f}.$$

Let us denote by $\mathcal{K}_\rho$ the unique subgroup of $\mathcal{G}$ such that $\mathcal{K}_\rho^i \simeq j_i(\ker(\rho_i^{-1}))$, for all i (i.e. the morphism ρ on the Igusa varieties is associated to the Barsotti-Tate group $\mathcal{G}/\mathcal{K}_\rho$).

In order to conclude that $\pi_N \circ ((\rho,1,1)\times(\rho,1,1)) = \pi_N$, it is enough to observe that

$$(j\rho)_N(\ker(\beta\rho)^{(p^{NB})}) = \frac{j_N(\ker(\beta)^{(p^{NB})})}{j_N(\ker(\rho^{-1})^{(p^{NB})})} = \frac{j_N(\ker(\beta)^{(p^{NB})})}{\mathcal{K}_\rho^{(p^{NB})}} \subset (\frac{\mathcal{G}}{\mathcal{K}_\rho})^{(p^{NB})},$$

and that the induced structures on the quotient abelian variety are the same.

Let us now consider the action of the element $Frob = (1, Frob, 1) \in S \times Frob^{\mathbb{N}} \times Fr^{\mathbb{N}}$. We remark that both $\pi_N \circ (Frob \times Frob)$ and $(1\times\sigma)\circ\pi_N$ are σ-semilinear, and thus it suffices to compare the associated linear morphism (which can be done in terms of the universal property of $\bar{X}^{(\alpha)}$).

Indeed, it suffices to observe that the following diagram commutes, where $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ denote respectively the Barsotti-Tate groups associated to the morphisms $(1\times\sigma)\circ\pi_N$ and $\pi_N \circ (Frob \times Frob)$.

$$\begin{tikzcd}
\mathcal{H}'^{(p^{NB})} & & & \widetilde{\mathcal{H}} \simeq \mathcal{H} \\
\Sigma^{(p^{NB+1})} \arrow[u, "p^n\beta F^{-1}"] \arrow[urr, bend left, "p^n\beta"] & \Sigma^{(p^{NB+1})}[p^{m-a}] \arrow[l, hook'] \arrow[r, "j_N^{(p)}"] & \mathcal{G}^{(p^{NB+1})}[p^{m-a}] \arrow[r, hook] & \mathcal{G}^{(p^{NB+1})} \arrow[u] \\
\Sigma^{(p^{NB})} \arrow[u, two heads, "F"] & \Sigma^{(p^{NB})}[p^m] \arrow[l, hook'] \arrow[u, two heads, dotted] \arrow[r, "j_N"] & \mathcal{G}^{(p^{NB})}[p^m] \arrow[r, hook] \arrow[u, two heads, dotted] & \mathcal{G}^{(p^{NB})} \arrow[u, two heads] \arrow[uul, bend right]
\end{tikzcd}$$

Finally, the equality $\pi_N \circ (Fr \times Fr) = (Fr \times \sigma) \circ \pi_N$ is obvious. □

4.3. The morphism Π on $\bar{\mathbb{F}}_p$-points. — In this section, we shall focus our attention on the fibers of the morphisms π_N over the $\bar{\mathbb{F}}_p$-points of $\bar{X}^{(\alpha)}$.

Let us first establish some notations. It follows from the definitions that we have

$$\bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p) = \{(A,\lambda,i,\bar{\mu})/\bar{\mathbb{F}}_p |\, \alpha(\epsilon A[u^\infty]) = \alpha\}/\sim,$$

where the abelian varieties A are considered up to prime-to-p isogenies,

$$J(\bar{\mathbb{F}}_p) = \varprojlim_m J_m(\bar{\mathbb{F}}_p) = \{(B,\lambda,i,\bar{\mu};j)/\bar{\mathbb{F}}_p |\, j : \Sigma_{\bar{\mathbb{F}}_p} \to \epsilon B[u^\infty] \text{ isomorphism }\}/\sim,$$

where the abelian varieties B are also considered up to prime-to-p isogenies, and

$$\mathcal{M}(\bar{\mathbb{F}}_p) = \varinjlim_{n,d} \mathcal{M}^{n,d}(\bar{\mathbb{F}}_p) = \{(H',\beta) |\, \beta : \Sigma_{\bar{\mathbb{F}}_p} \to H' \text{ quasi-isogeny }\}/\sim,$$

where the Barsotti-Tate groups H' are considered up to isomorphisms.

We observe that the spaces $\bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p)$ and $\mathcal{M}(\bar{\mathbb{F}}_p)$ are naturally endowed with the discrete topology and $J(\bar{\mathbb{F}}_p)$ with the inverse limit topology.

4.3.1. Let us remark that the action of the group T of the quasi-selfisogenies of $\Sigma_{\bar{\mathbb{F}}_p}$ on $\mathcal{M}$ (resp. the action of the monoid $S \subset T$ on the Igusa varieties J_m, for all m) gives rise to a continuous action on $\mathcal{M}(\bar{\mathbb{F}}_p)$ (resp. on $J(\bar{\mathbb{F}}_p)$). These two actions are defined as

$$\rho : \mathcal{M}(\bar{\mathbb{F}}_p) \to \mathcal{M}(\bar{\mathbb{F}}_p)$$

$$(H', \beta) \mapsto (H', \beta\rho),$$

for any $\rho \in T$, and

$$\rho : J(\bar{\mathbb{F}}_p) \to J(\bar{\mathbb{F}}_p)$$

$$(B, \lambda, i, \bar{\mu}; j) \mapsto (B/\langle j \ker(\rho^{-1})\rangle, \lambda', i', \bar{\mu}'; j\rho),$$

for any $\rho \in S$, where the Igusa structure on the abelian variety $B/\langle j \ker(\rho^{-1})\rangle$ is defined by the following commutative diagram

$$\begin{tikzcd}
\Sigma_{\bar{\mathbb{F}}_p} \arrow[r, "j"'] \arrow[d, "\rho^{-1}"'] & \epsilon B[u^\infty] \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\
\Sigma_{\bar{\mathbb{F}}_p} \arrow[r, "j\rho"'] & \frac{\epsilon B[u^\infty]}{j \ker(\rho^{-1})} \arrow[r, hook] & \frac{B}{\langle j \ker(\rho^{-1})\rangle},
\end{tikzcd}$$

the polarization λ' is defined as the polarization induced by the polarization $p^e\lambda$ on B (where $e = e_1(\rho)$) and the level structure $\bar{\mu}'$ is induced by the level structure $\overline{(v^c)^{-e}\mu}$ on B (for $v \in E^\times$ is such that $\mathrm{val}_u(v) = 1$ and $\mathrm{val}_{u^c}(v) = 0$.).

It is easy to see that the action of S on $J(\bar{\mathbb{F}}_p)$ extends to a continuous action of T ($S \subset T$). In fact, the above definition extends directly to all the quasi-isogenies whose inverse is an isogeny, and moreover the action of $p^{-1} \in S$ is invertible. (Indeed, the element $p^{-1} \in S$ acts as

$$(B, \lambda, i, \bar{\mu}, j) \mapsto (B/B[u], \lambda', i', \bar{\mu}', jp) \sim (B, \lambda, i, \overline{v(v^c)^{-1}\mu}, jvp^{-1}),$$

where the above equivalence is induced by the multiplication $v : B \to B$.)

4.3.2. Let $(y, z) \in J(\bar{\mathbb{F}}_p) \times \mathcal{M}(\bar{\mathbb{F}}_p)$, and $y_m \in J_m(\bar{\mathbb{F}}_p)$ be the image of y under the projection $J(\bar{\mathbb{F}}_p) \to J_m(\bar{\mathbb{F}}_p)$). Let n, d be two positive integers such that $z \in \mathcal{M}^{n,d}(\bar{\mathbb{F}}_p)$. Then, for any $m \geq d$ and $N \geq d/\delta B$, we define the point $Fr^{-NB}\pi_N(y_m, z) \in \bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p)$. It follows from proposition 4.3 that this point does not depend on the choice of the integers m, n, d, N. Thus, we can define a map

$$\Pi : J(\bar{\mathbb{F}}_p) \times \mathcal{M}(\bar{\mathbb{F}}_p) \to \bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p),$$

$$(y, z) \mapsto Fr^{-NB}\pi_N(y_m, z) \in \bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p),$$

for any set of integers m, n, d, N such that $m \geq d$, $N \geq d/\delta B$ and $(y_m, z) \in J_m(\bar{\mathbb{F}}_p) \times \mathcal{M}^{n,d}(\bar{\mathbb{F}}_p)$.

The morphism Π can be also described as follows. Let $y = (B, \lambda, i, \bar{\mu}; j; H', \beta) \in J(\bar{\mathbb{F}}_p) \times \mathcal{M}(\bar{\mathbb{F}}_p)$, and choose two positive integers n, d such that $p^n\beta$ and $p^{d-n}\beta^{-1}$ are two isogenies. We define the abelian variety A as

$$\begin{tikzcd}
\Sigma_{\bar{\mathbb{F}}_p} \arrow[r, "j"'] \arrow[d, "p^n\beta"] & \epsilon B[u^\infty] \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\
H' & \frac{\epsilon B[u^\infty]}{j\ker(p^n\beta)} \arrow[r, hook] \arrow[d, equal] & \frac{B}{\langle j\ker(p^n\beta)\rangle} \arrow[d, equal] \\
 & H & A
\end{tikzcd}$$

where the subgroup $\langle j\ker(p^n\beta)\rangle$ of B is defined as

$$(\mathcal{O}_{B_u} \otimes_{\mathbb{Z}_p} j\ker(p^n\beta)) \oplus (\mathcal{O}_{B_u} \otimes_{\mathbb{Z}_p} j\ker(p^n\beta))^\perp \subset B[u^d] \oplus B[(u^c)^d].$$

Then, the point $\Pi(y) \in \bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p)$ is the class of the abelian variety A endowed with a polarization, a $\mathcal{O}_B$-action and a level structure away from p induced by the ones of B. More precisely, the polarization $p^d\bar{\lambda}$ of A is the unique prime-to-p polarization which fits the following commutative diagram

$$\begin{tikzcd}
B \arrow[r, "p^d\lambda"] \arrow[d, two heads] & B^\vee \\
B/\langle j\ker(p^n\beta)\rangle \arrow[r, "p^d\bar{\lambda}"] & (B/\langle j\ker(p^n\beta)\rangle)^\vee \arrow[u]
\end{tikzcd}$$

and the level structure of A is defined as

$$V \otimes \mathbb{A}^{\infty,p} \xrightarrow{\mu} V^p(B) \xrightarrow{v^{-n}(v^c)^{-d+n}} V^p(B) \longrightarrow V^p(B/\langle j\ker(p^n\beta)\rangle).$$

4.3.3. It follows from the definition that the morphism Π is continuous and invariant under the action of T on $J(\bar{\mathbb{F}}_p) \times \mathcal{M}(\bar{\mathbb{F}}_p)$, since the π_N are invariant under the action of the submonoid $S \subset T$.

Proposition 4.4. — *Let x be a point of $\bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p)$. Then, the fiber $\Pi^{-1}(x)$ is a free principal homogeneous space for the continuous action of T.*

Proof. — Let us remark that the action of T on $J(\bar{\mathbb{F}}_p) \times \mathcal{M}(\bar{\mathbb{F}}_p)$ gives rise to an action of T on the fiber $\Pi^{-1}(x)$, since $\Pi \circ (\rho \times \rho) = \Pi$, for all $\rho \in T$. Moreover, since the action of T on $J(\bar{\mathbb{F}}_p) \times \mathcal{M}(\bar{\mathbb{F}}_p)$ is continuous thus is the action of T on $\Pi^{-1}(x)$.

Let us denote by $(A, \lambda_A, i_A, \bar{\mu}_A)$ a quadruple associated to x and by $H = \epsilon A[u^\infty]$ the corresponding Barsotti-Tate group.

We articulate the proof in three steps.

1. If $(B, \lambda_B, i_B, \bar{\mu}_B; j; H', \beta')$ is a 7-tuple associated to an element $y \in \Pi^{-1}(x)$, then the Barsotti-Tate group H' is isomorphic to H, or equivalently
$$(B, \lambda_B, i_B, \bar{\mu}_B; j, H', \beta') \sim (B, \lambda_B, i_B, \bar{\mu}_B; j, H, \beta),$$
where $\beta = \delta\beta'$, for any isomorphism $\delta : H' \to H$.
2. Any 7-tuple $(B, \lambda_B, i_B, \bar{\mu}_B; j, H, \beta)$, associated to an element $y \in \Pi^{-1}(x)$, is equivalent to a 7-tuple of the form
$$(A/\langle \gamma \ker(p^n\beta)^*\rangle, \lambda, i, \bar{\mu}; \bar{\gamma}, H, \beta),$$
where n, d are any two integers such that $p^n\beta$ and $p^{d-n}\beta^{-1}$ are two isogenies, $(p^n\beta)^* : H \to \Sigma_{\bar{\mathbb{F}}_p}$ denotes the unique isogeny such that $p^n\beta \circ (p^n\beta)^* = (p^n\beta)^* \circ p^n\beta = p^d$, $\gamma \in \mathrm{Aut}\,(H)$, $\bar{\gamma} : \Sigma_{\bar{\mathbb{F}}_p} \to H/\gamma \ker(p^n\beta)^*$ is the isomorphism induced by γ, and the structures of $A/\langle \gamma \ker(p^n\beta)^*\rangle$ are induced by the ones of A.
3. To any 7-tuple of the form $(A/\langle \gamma \ker(p^n\beta)^*\rangle, \bar{\gamma}, H, \beta)$ one can associate a quasi-isogeny $\hat{\beta} : \Sigma_{\bar{\mathbb{F}}_p} \to H$ in the same equivalent class of β, and the so defined map between $\Pi^{-1}(x)$ and $\mathrm{QIso}(\Sigma_{\bar{\mathbb{F}}_p}, H)$ is indeed an homeomorphism of T-spaces. (In particular, $\Pi^{-1}(x)$ is a free principal homogeneous space for the continuous action of T since $\mathrm{QIso}(\Sigma_{\bar{\mathbb{F}}_p}, H)$ is.)

Step (1): It follows from the definition of Π that the isomorphism $j : \Sigma_{\bar{\mathbb{F}}_p} \to G$ induces an isomorphism between the quotients $H' \to H$.

Step (2): Let us choose a prime-to-p isogeny
$$\varphi : B/\langle j(\ker p^n\beta)\rangle \to A$$
which gives rise to an equivalence between the corresponding two quadruple associated to the point $x = \Pi(y)$, and consider the following commutative diagram, where γ is the unique automorphism of H which makes the diagram commute.

$$\begin{tikzcd}
\Sigma \arrow[r, "j"] \arrow[d, two heads] \arrow[dd, bend right=60, "p^n\beta"'] \arrow[ddd, bend right=80, "p^d"'] & G \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \arrow[dddd, bend left=80, "p^d"] \\
\frac{\Sigma}{\ker(p^n\beta)} \arrow[r, "\bar{j}"] \arrow[d, equal] & \frac{G}{j(\ker(p^n\beta))} \arrow[r, hook] \arrow[d, "\varphi_{|G}"] & \frac{B}{\langle j(\ker(p^n\beta))\rangle} \arrow[d, "\varphi"] \\
H \arrow[r, "\gamma"] \arrow[d, two heads] \arrow[dd, bend right=60, "(p^n\beta)^*"'] & H \arrow[r, hook] \arrow[d, two heads] & A \arrow[d, two heads] \\
\frac{H}{\ker(p^n\beta)^*} \arrow[r, "\bar{\gamma}"] \arrow[d, equal] & \frac{H}{\gamma(\ker(p^n\beta)^*)} \arrow[r, hook] & \frac{A}{\langle \gamma \ker(p^n\beta)^*\rangle} \\
\Sigma \arrow[r, "j"'] & G \arrow[r, hook] \arrow[u, "\psi_{|G}"'] & B \arrow[u, dotted, "\psi"']
\end{tikzcd}$$

It follows from the commutativity of the diagram that there exists an isogeny

$$\psi : B \to A/\langle \gamma \ker(p^n\beta)^*\rangle$$

which fits in the diagram and also that ψ has degree prime to p (since the restriction of ψ to G give rise to an isomorphism between the pertinent Barsotti-Tate groups).

The isogeny ψ gives rise to an equivalence of 7-tuples

$$(B, \lambda_B, i_B, \bar{\mu}_B; j; H, \beta) \sim (A/\langle \gamma \ker(p^n\beta)^*\rangle, \lambda', i, \bar{\mu}'; \bar{\gamma}, H, \beta),$$

where λ' is the polarization on the quotient $A \twoheadrightarrow A/\langle \gamma \ker(p^n\beta)^*\rangle$ induced by the $p^d\lambda_A$ and $\bar{\mu}'$ is the level structure induced by $p^{-d}v^n(v^c)^{d-n}\bar{\mu}_A = v^{n-d}(v^c)^{-n}\bar{\mu}_A$ (where $v \in E^\times$ is an element such that $\mathrm{val}_u(v) = 1$ and $\mathrm{val}_{u^c}(v) = 0$).

<u>Step (3)</u> We now consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[rr, bend left, "p^d"] & A/\langle \gamma \ker(p^n\beta)^*\rangle \arrow[r, two heads] & A/A[p^d] \arrow[r, equal] & A \\
H \arrow[r, two heads] \arrow[u, hook] & H/\gamma \ker(p^n\beta)^* \arrow[r, two heads] \arrow[u, hook] & H/H[p^d] \arrow[r, equal] \arrow[u, hook] & H \arrow[u, hook] \\
H \arrow[r, dotted, "(p^n\beta)^*"'] \arrow[u, dotted, "\gamma"] & \Sigma \arrow[r, dotted, "p^n\beta"'] \arrow[u, "\bar{\gamma}"] \arrow[urr, "p^n\hat{\beta}"'] & & H \arrow[u, dotted, "\gamma"']
\end{tikzcd}$$

It exists a unique quasi-isogeny $\hat{\beta} : \Sigma_{\bar{\mathbb{F}}_p} \to H$ in the same equivalent class of β, which fits in the diagram (i.e. in the diagram $\hat{\beta} = \gamma\beta$).

In order to show that the map

$$f : \Pi^{-1}(x) \to \mathrm{QIso}(\Sigma_{\bar{\mathbb{F}}_p}, H)$$

$$(A/\langle \gamma \ker(p^n\beta)^*\rangle, \lambda', i, \bar{\mu}'; \bar{\gamma}, H, \beta) \mapsto \hat{\beta}$$

is a bijection, it suffices to construct its inverse.

We define the map

$$g : \mathrm{QIso}(\Sigma, H) \to \Pi^{-1}(x)$$

$$\hat{\beta} \mapsto (A/\langle \ker(p^n\hat{\beta})^*\rangle, \lambda', i, \bar{\mu}'; \bar{1}, H, \hat{\beta}),$$

for some integers $n, d \geq 0$ such that $p^n\hat{\beta}$ and $p^{d-n}\hat{\beta}^{-1}$ are isogenies.

We check that the definition of $g(\hat{\beta})$ does not depend on the choice of the integers n, d. It suffices to consider the two cases when we replace n, d by $n, d+1$ and by $n+1, d+1$.

Let us denote by $(p^n\hat{\beta})^*$ and δ the two isogenies such that $(p^n\hat{\beta})^*(p^n\hat{\beta}) = (p^n\hat{\beta})(p^n\hat{\beta})^* = p^d$ and $\delta p^n\hat{\beta} = p^n\hat{\beta}\delta = p^{d+1}$. Then, we have $p(p^n\hat{\beta})^*(p^n\hat{\beta}) =$

$(p^n\hat{\beta})p(p^n\hat{\beta})^* = p^{d+1}$, or equivalently $\delta = p(p^n\hat{\beta})^*$. Thus, the multiplication by p on H gives rise to an isomorphism

$$H/\gamma\ker\delta \simeq H/\gamma\ker(p^n\hat{\beta})^*,$$

i.e. $p(\gamma\ker\delta) = \gamma\ker(p^n\hat{\beta})^* \subset H[p^d]$. It follows that

$$(\mathcal{O}_{B_u}\otimes_{\mathbb{Z}_p}\gamma\ker\delta)^\perp = (\mathcal{O}_{B_u}\otimes_{\mathbb{Z}_p}\gamma\ker(p^n\hat{\beta})^*)^\perp \subset A[(u^c)^d],$$

and therefore $u\langle\gamma\ker\delta\rangle = \langle\gamma\ker(p^n\hat{\beta})^*\rangle \subset A[p^d]$, i.e. the multiplication $v : A \to A$ gives rise to an isomorphism

$$A/\langle\gamma\ker\delta\rangle \simeq A/\langle\gamma\ker(p^n\hat{\beta})^*\rangle.$$

It is easy to check that, under the above identification, the induced structures on the quotients abelian varieties agree.

Let us suppose now that $p^n\hat{\beta}$ and $p^{d-n}\hat{\beta}^{-1}$ are isogenies, then also $p^{n+1}\hat{\beta}$ and $p^{(d+1)-(n-1)}\hat{\beta}^{-1}$ are isogenies. Let $(p^n\hat{\beta})^*$ be the isogeny such that $(p^n\hat{\beta})^*(p^n\hat{\beta}) = (p^n\hat{\beta})(p^n\hat{\beta})^* = p^d$, then $(p^n\hat{\beta})^*(p^{n+1}\hat{\beta}) = (p^{n+1}\hat{\beta})(p^n\hat{\beta})^* = p^{d+1}$.

Let us write $\mathcal{K}_i = (\mathcal{O}_{B_u}\otimes_{\mathbb{Z}_p}\gamma\ker(p^n\hat{\beta}))^\perp \subset A[(u^c)^i]$ for $i = d, d+1$, then

$$\mathcal{K}_d = u^c\mathcal{K}_{d+1},$$

and therefore the multiplication $v^c : A \to A$ gives rise to an isomorphism

$$A/\langle\gamma\ker(p^n\hat{\beta})^*\rangle_{d+1} \simeq A/\langle\gamma\ker(p^n\hat{\beta})^*\rangle_d.$$

Again, it is easy to check that, under the above identification, the induced structures on the quotients abelian varieties agree.

The same diagram we used to define the quasi-isogeny $\hat{\beta}$ shows that the maps f and g are inverse of each others. Moreover, it is a direct consequence of the definitions that the morphisms f, g are continuous, i.e. homeomorphisms.

Finally, in order to prove that the bijection f is compatible with the action of the group T, we consider the following diagram, for any $\rho \in T$. (Without loss of generality, we may assume that both $\hat{\beta}\rho$ and ρ^{-1} are isogenies.)

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[rrr, bend left, "p^{d'}"] & A/\langle\ker(\hat{\beta})^*\rangle \arrow[r, two heads] & A/\langle\ker(\hat{\beta}\rho)^*\rangle \arrow[r, two heads] & A \\
H \arrow[u, hook] \arrow[r, two heads] \arrow[dr, "\hat{\beta}^*"'] & H/\ker(\hat{\beta})^* \arrow[u, hook] \arrow[r, two heads] \arrow[d, equal] & H/\ker(\hat{\beta}\rho)^* \arrow[u, hook] \arrow[r, two heads] \arrow[d, equal] & H \arrow[u, hook] \\
& \Sigma \arrow[r, "\rho^{-1}"'] & \Sigma \arrow[ur, "\hat{\beta}\rho"'] &
\end{tikzcd}$$

The commutativity of the diagram implies that the quasi-isogeny associated to the image of $(A/\langle\ker(p^n\beta)^*\rangle, \lambda', i, \bar{\mu}'; \bar{1}, H, \hat{\beta})$ via ρ maps to the quasi-isogeny $\hat{\beta}\rho$. $\square$

Let us remark that it follows from the above proposition that $\Pi^{-1}(x)$ is not empty, for any $x \in \bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p)$. In fact, for any $x \in \bar{X}^{(\alpha)}(\bar{\mathbb{F}}_p)$, the associated Barsotti-Tate group over $\bar{\mathbb{F}}_p$ has Newton polygon equal to $\alpha = \alpha(\Sigma)$, and any two Barsotti-Tate groups over a perfect field of characteristic p with the same Newton polygon are isogenous.

Moreover, under the identification $\Pi^{-1}(x) \simeq \mathrm{QIso}(\Sigma_{\bar{\mathbb{F}}_p}, H)$, the natural map $\Pi^{-1}(x) \twoheadrightarrow \mathcal{M}(\bar{\mathbb{F}}_p)$ corresponds to the projection

$$\mathrm{QIso}(\Sigma_{\bar{\mathbb{F}}_p}, H) \twoheadrightarrow \mathrm{Aut}\,(H)\backslash\mathrm{QIso}(\Sigma_{\bar{\mathbb{F}}_p}, H).$$

4.3.4. Let us also remark that it follows from proposition 3.3 that, for any integer $m > 0$, the projection

$$\mathrm{q}_{\infty,m} : J(\bar{\mathbb{F}}_p) \twoheadrightarrow J_m(\bar{\mathbb{F}}_p)$$

is surjective and the action of $\Gamma = \mathrm{Aut}\,(\Sigma_{\bar{\mathbb{F}}_p}) \subset T$ on $J(\bar{\mathbb{F}}_p)$ is such that

$$J_m(\bar{\mathbb{F}}_p) \simeq J(\bar{\mathbb{F}}_p)/\Gamma_m,$$

where $\Gamma_m \subset \Gamma$ is the subgroup of the automorphisms of $\Sigma_{\bar{\mathbb{F}}_p}$ which induce the identity on the p^m-torsion subgroup.

4.3.5. Finally, we remark that all the above results remain true is in place of $\bar{\mathbb{F}}_p$ we consider any algebraically closed field k over $\bar{\mathbb{F}}_p$.

4.3.6. The following results are implied by the previous analysis of the fibers of Π.

Proposition 4.5. — *For any positive integers m, n, d, N such that $m \geq d$ and $N \geq d/\delta B$, the morphism $\pi_N : J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is quasi-finite.*

Proof. — Let x be a point of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, defined over an algebraically closed field k, and denote by H the corresponding Barsotti-Tate group. We claim that $\pi_N^{-1}(x)$ is finite.

It follows from the equality $\Pi = Fr^{-NB}\pi_N$ and the surjectivity of the projection $\mathrm{q}_{\infty,m}$ that

$$\pi_N^{-1}(x) = (\mathrm{q}_{\infty,m} \times 1)(\Pi^{-1}(Fr^{-NB}x) \cap J(k) \times \mathcal{M}^{n,d}(k)),$$

where we can identify

$$\Pi^{-1}(Fr^{-NB}x) \cap J(k) \times \mathcal{M}^{n,d}(k) \simeq \mathrm{QIso}(\Sigma, H^{(p^{-NB})})^{n,d} = \mathrm{QI}^{n,d},$$

the subset of quasi-isogenies β such that $p^n\beta$ and $p^{d-n}\beta^{-1}$ are isogenies. Moreover, under the above identification, the projection $\pi_N^{-1}(x) \to \mathcal{M}^{n,d}(k)$ corresponds to the projection

$$\mathrm{QI}^{n,d} \twoheadrightarrow \mathrm{Aut}\,(H^{(p^{-NB})})\backslash\mathrm{QI}^{n,d}.$$

We claim that the quotient $\mathrm{Aut}\,(H^{(p^{-NB})})\backslash\mathrm{QI}^{n,d}$ is finite. In fact, if we choose an element $\beta_0 \in \mathrm{QI}$, then under the corresponding isomorphism $T \simeq \mathrm{QI}$ (which is defined by $\rho \mapsto \beta_0\rho$), the subset $\mathrm{QI}^{n,d}$ corresponds to a compact subset K of T and the quotient $\mathrm{Aut}\,(H^{(p^{-NB})})\backslash\mathrm{QI}^{n,d}$ to the quotient $\beta_0\mathrm{Aut}\,(H^{(p^{-NB})})\beta_0^{-1}\backslash K$.

Since $\beta_0 \mathrm{Aut}\,(H^{(p^{-NB})})\beta_0^{-1}$ is an open subgroup of T, the quotient

$$\beta_0 \mathrm{Aut}\,(H^{(p^{-NB})})\beta_0^{-1}\backslash K$$

is indeed finite.

It remains to prove that the image under $\mathrm{q}_{\infty,m} \times 1$ of the coset $\mathrm{Aut}\,(H^{(p^{-NB})})\beta$ is finite, for any $\beta \in \mathrm{QI}^{n,d}$. Equivalently, it suffices to know that there is an open subgroup R of $\mathrm{Aut}\,(H^{(p^{-NB})})$ such that the image of $R\beta$ is constant. Indeed, the subgroup $\mathrm{Aut}\,(H^{(p^{-NB})})^{m+d}$ of the automorphisms of $H^{(p^{-NB})}$ which induce the identity over the p^{m+d}-torsion subgroup has such property. □

Proposition 4.6. — *If the positive integers m, n, d, N are sufficientely large ($m \geq d$ and $N \geq d/\delta B$), the morphism π_N is surjective on geometric points.*

Proof. — Since the scheme $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is of finite type over $\mathbb{F}_p$, it suffices to show that for any geometric point x of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ there exist some integers $m, n, d, N \geq 0$ ($m \geq d$ and $N \geq d/\delta B$) such that the set

$$\pi_N^{-1}(x) = \{(y,t) \in J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \mid \pi_N(y,t) = x\}$$

is not empty. Let us recall that

$$\pi_N^{-1}(x) = (\mathrm{q}_{\infty,m} \times 1)(\Pi^{-1}(Fr^{-NB}x) \cap J(k) \times \mathcal{M}^{n,d}(k)),$$

and that, for all $x' \in \bar{X}^{(\alpha)}(k)$, the fibers $\Pi^{-1}(x')$ are not empty. Since $\mathcal{M}(k) = \varinjlim_{n,d} \mathcal{M}^{n,d}(k)$, it follows that the set $\pi_N^{-1}(x)$ is also not empty. □

4.4. The leaves are closed. — From the fact that the morphisms π_N are quasi finite we can deduce that the leafs are closed subschemes of the Newton polygon strata. (We recall that the following result is originally due to Oort, see [**26**].)

Proposition 4.7. — *For any Barsotti-Tate group $H/\bar{\mathbb{F}}_p$ the corresponding leaf C_H is a closed smooth subscheme of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$.*

Proof. — By proposition 2.7, we already know that the leaves are smooth locally closed subschemes of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. We now show that they are also closed, by showing that all the leaves have the same dimension and that if a given leaf C_H is not closed, then there exists a Barsotti-Tate group $H'/\bar{\mathbb{F}}_p$, not isomorphic to H, such that $\bar{C}_H \supset C_{H'}$ (which implies that $C_{H'} \subset \bar{C}_H - C_H$ since $C_{H'} \cap C_H = \varnothing$). These two facts are clearly in contradiction, therefore we conclude that all the leaves of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ are closed.

Let H be any Barsotti-Tate group defined over $\bar{\mathbb{F}}_p$ with Newton polygon equal to α, and choose an isogeny $\gamma : \Sigma \twoheadrightarrow H$. We also choose a positive integer d such that $p^d\gamma^{-1}$ is an isogeny.

Let N be an integer such that $N \geq d/\delta B$ and define $H' = H^{(p^{-NB})}$ and $\beta = \gamma^{(p^{-NB})}$. Then, the pair (H', β) defines a point $t \in \bar{\mathcal{M}}^{0,d}(\bar{\mathbb{F}}_p)$.

Let $m \geq d$ and consider the morphism

$$f = \pi_N \circ (1 \times t) : J_m \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p.$$

It follows from the definition that, for every point $y \in J_m$, we have

$$(f^*\mathcal{G})_y = (\mathcal{G}^{(p^{NB})}/\mathcal{K}_\beta)_y \simeq \Sigma^{(p^{NB})}/\ker \beta^{(p^{NB})} \simeq H'^{(p^{NB})} = H.$$

Thus, the morphism f factors through the leaf $C_H \subset \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. Moreover, $f : J_m \to C_H$ is quasi-finite and surjective.

In particular, we deduce that the dimension of C_H is equal to the dimension of J_m, or equivalently to the dimension of the central leaf $C = C_\alpha$.

Let us now suppose that there is a leaf C_H which is not a closed subspace of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. Then, there exists a Barsotti-Tate group $H'/\bar{\mathbb{F}}_p$ (H' not isomorphic to H) such that $C_{H'} \cap \bar{C}_H$ is not empty (e.g. for any closed point $x = (A_x, \lambda_x, i_x, \bar{\mu}_x) \in \bar{C}_H - C_H$ the Barsotti-Tate group $H' = \mathcal{G}_x$ satisfies the above assumption). We claim that $C_{H'} \subset \bar{C}_H$.

Let x be a point of $C_{H'} \cap \bar{C}_H$. Then there exists a point $y \in C_H$ which specializes to x, or equivalently (by Serre-Tate's Theorem) there exists a local domain $R/\bar{\mathbb{F}}_p$, with residue field $k(x)$ and fraction field $k(y)$, and a morphism $p_x : \operatorname{Spec} R \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ associated the data of the quadruple $(A_x, \lambda_x, i_x, \bar{\mu}_x)$ and a deformation G/R of the Barsotti-Tate group H', such that $G_{k(y)} \simeq H$.

Then, for any other point $z = (A_z, \lambda_z, i_z, \bar{\mu}_z) \in C_{H'}$, let us choose an isomorphism $\mathcal{G}_z \simeq H'$ and define $p_z : \operatorname{Spec} R \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ to be the morphism associated the data of the quadruple $(A_z, \lambda_z, i_z, \bar{\mu}_z)$ and the Barsotti-Tate group G/R, viewed as a deformation of $\mathcal{G}_z$ via the choosen isomorphism $\mathcal{G}_z \simeq H'$. Then, the generic point η of $\operatorname{Spec} R$ give rise to a point $t \in C_H$ which specialises to z, and thus $C_{H'} \subset \bar{C}_H$. $\square$

It follows from the fact that the central leaf is closed that the morphisms π_N are proper.

Proposition 4.8. — *For any positive integers m, n, d, N such that $m \geq d$ and $N \geq d/\delta B$, the morphism $\pi_N : J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is proper.*

Proof. — By the Valuative Criterium of Properness (see [**15**], Theorem 4.7, p. 101) it suffices to show that:

– if R is a complete discrete valuation ring over $\bar{\mathbb{F}}_p$, K its fraction field and $\eta : \operatorname{Spec} K \to \operatorname{Spec} R$ the morphism corresponding to the natural inclusion of R in K, then for any pair of morphisms (F, f) such that $\pi_N \circ F = f \circ \eta$ there exists a

map $\phi : \operatorname{Spec} R \to J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ such that the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "F"] \arrow[d, "\eta"] & J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \arrow[d, "\pi_N"] \\
\operatorname{Spec} R \arrow[r, "f"'] \arrow[ur, dotted, "\phi"] & \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p
\end{tikzcd}$$

A morphism $f : \operatorname{Spec} R \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ corresponds to a quadruple $(A, \lambda, i, \bar{\mu})$ defined over R, and a morphism $F = (F_1, F_2)$ corresponds to a $(6+k)$-tuple $(B, \lambda', i', \bar{\mu}'; j_{m,1}, \dots, j_{m,k}; H', \beta)$ defined over K. The equality $f \circ \eta = \pi_N \circ F$ implies that the quadruple $(A_K, \lambda_K, i_K, \bar{\mu}_K)$ is equivalent to the quotient of the quadruple $(B, \lambda', i', \bar{\mu}')^{(p^{NB})}$ via the projection $B^{(p^{NB})} \twoheadrightarrow B^{(p^{NB})}/\langle \mathcal{K} \rangle$, where $\mathcal{K} = j_N(\ker p^n \beta)^{(p^{NB})}$. Indeed, we may substitute the quadruple associated to f so that its generic fiber is isomorphic to the quotient of the quadruple corresponding to $B^{(p^{NB})}$.

Finally, defining a morphism ϕ such that the above diagram commutes is equivalent to defining an integral model $(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}'; \hat{j}_{m,1}, \dots, \hat{j}_{m,k}; \hat{H}', \hat{\beta})$ over R of the $(6+k)$-tuple $(B, \lambda', i', \bar{\mu}'; j_{m,1}, \dots, j_{m,k}; H', \beta)$, with the propety that there exists a prime-to-p isogeny between A and $\hat{B}^{(p^{NB})}/\langle \hat{\mathcal{K}} \rangle$ (where $\hat{\mathcal{K}} = \hat{j}_N(\ker p^n \hat{\beta})^{(p^{NB})}$), compatible with the given structures on the abelian varieties. In particular, this property implies the existence of an isomorphism between the quotient $\hat{G}/\hat{\mathcal{K}}$ of $\hat{G} = \epsilon \hat{B}[u^\infty]$ and $H = \epsilon A[u^\infty]$ over R.

Let us consider the isogeny

$$\psi : B^{(p^{NB})} \twoheadrightarrow B^{(p^{NB})}/\langle \mathcal{K} \rangle \simeq A_K,$$

then $p^d \psi^{-1} : A_K \to B^{(p^{NB})}$ is also an isogeny, of degree a power of p, with kernel contained in the p^d-torsion subgroup. If we consider the subgroup $\mathcal{F} \subset H[p^d]$ which is the closure of $\ker(p^d \psi^{-1}_{|H}) \subset H[p^d]_K$ in $H[p^d]$, then the quotient $A/\langle \mathcal{F} \rangle$, endowed with the induced structures, has generic fiber equivalent to the quadruple $(B, \lambda', i', \bar{\mu}')^{(p^{NB})}$.

This fact implies the existence of a quadruple $(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}')$, defined over R, whose generic fiber is equivalent to $(B, \lambda', i', \bar{\mu}')$ and such that $(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}')^{(p^{NB})}$ is equivalent to the quadruple associated to $A/\langle \mathcal{F} \rangle$. In fact, the quadruple associated to the abelian variety $A/\langle \mathcal{F} \rangle$ over R defines a morphism $g : \operatorname{Spec} R \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ such that $g \circ \eta = (Fr^{NB} \times 1) \circ (\mathrm{q}_m \circ F_1)$. Since the morphism $Fr \times 1$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is finite, there exists a morphism $g' : \operatorname{Spec} R \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ such that $g' \circ \eta = (\mathrm{q}_m \circ F_1)$, or equivalently a quadruple $(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}')$, defined over R, with the above properties.

We remark that, since the abelian variety $\hat{B}$ is isogenous to A, the Barsotti-Tate group $\hat{G} = \epsilon \hat{B}[u^\infty]$ has constant Newton polygon equal to α and thus the quadruple $(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}')/R$ defines a morphism

$$g_1 : \operatorname{Spec} R \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$$

such that $g_1 \circ \eta = \mathrm{q}_m \circ F_1$.

Moreover, since the map $g_1 \circ \eta = \mathrm{q}_m \circ F_1$ factors through the central leaf $C \times \bar{\mathbb{F}}_p \subset \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, which is a closed subscheme of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, we deduce that the morphism g_1 also factors through the $C \times \bar{\mathbb{F}}_p$.

Finally, from the equality $g_1 \circ \eta = \mathrm{q}_m \circ F_1$, where $\mathrm{q}_m : J_m \to C \times \bar{\mathbb{F}}_p$ is finite, we deduce that the morphism g_1 can be lifted to a morphism

$$\phi_1 : \operatorname{Spec} R \to J_m$$

(i.e. such that $\mathrm{q}_m \circ \phi_1 = g_1$) with the property that $\phi_1 \circ \eta = F_1$. Equivalently, the quadruple $(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}')/R$ can be extended to a $(4+k)$-tuple

$$(\hat{B}, \hat{\lambda}', \hat{i}', \bar{\mu}'; \hat{j}_{m,1}, \ldots, \hat{j}_{m,k})/R$$

whose generic fiber is $(B, \lambda', i', \bar{\mu}'; j_{m,1}, \ldots, j_{m,k})$.

Let us now consider the isogeny

$$\Psi : \hat{B}^{(p^{NB})} \to A,$$

whose generic fiber is ψ, and define $\hat{\mathcal{K}} = \ker(\Psi_{|\hat{G}^{(p^{NB})}}) \subset \hat{G}^{(p^{NB})}[p^d]$.

Then, the isogeny

$$\Sigma_R \twoheadrightarrow \Sigma_R / \hat{j}_N^{-1}(\hat{\mathcal{K}})$$

defines a morphism $\phi_2 : \operatorname{Spec} R \to \bar{\mathcal{M}}^{n,d}$ such that $\phi_2 \circ \eta = F_2$.

Therefore, the morphism $\phi = (\phi_1, \phi_2)$ makes the above diagram commute. □

Corollary 4.9. — *For any positive integers m, n, d, N such that $m \geq d$ and $N \geq d/\delta B$, the morphism $\pi_N : J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is finite.*

Proof. — It follows from propositions 4.5 and 4.8, together with the general fact that a morphism is finite if it is proper and quasi finite. □

5. Group action on cohomology

In this section we shall show that the action of S on the systems of covers $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ induces an action on the corresponding étale cohomology with compact supports, which extends to an action of T. Moreover, we shall see that via such an action of T it is possible to recover the cohomology with compact supports of $\bar{X}^{(\alpha)}$ from the cohomology with compact supports of the spaces $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$. More precisely, we shall prove that for any abelian torsion étale sheaf $\mathcal{L}$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (with torsion orders prime to p), there exists a spectral sequence involving the group homology of T and the étale cohomology with compact supports of the covers $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$, with coefficients in the pull back of $\mathcal{L}$, which converges to the étale cohomology with compact supports of $\bar{X}^{(\alpha)}$, with coefficient in $\mathcal{L}$.

We are especially grateful to J. de Jong for his help in finding correct statements and proofs of the following results.

5.1. The cohomology of étale sheaves with the action of a group. — In the following, we shall introduce some general results regarding the cohomology with compact supports of an abelian torsion étale sheaf, endowed with the action of an abstract p-adic group which acts trivially on the scheme.

5.1.1. We first recall some notations and results from the theory of representations of a p-adic group over $\mathbb{Z}/l^r\mathbb{Z}$, for a prime number $l \neq p$ and an integer $r \geq 1$. (see [**5**] for a survey of the theory over $\mathbb{C}$).

Let G be a p-adic group (e.g. $G = T$). Thus, G is a topological group such that the unit element has a basis of open neighborhoods consisting of open compact subgroups K of G. Further more, there exists an open compact subgroup K_0 of G which is a pro-p-group, i.e. for any open subgroup $K' \subset K_0$ the index $[K_0 : K]$ is a power of p. In the following, any time we consider a open compact subgroup of G we always mean a open compact subgroup contained in K_0. (In the case of $G = T$, one can choose $K_0 = \Gamma_1$). Finally, let us choose a left invariant Haar measure μ on G, with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$, such that $\mu(K_0) = 1$, i.e. for any open compact subgroup $K \subset K_0$, we set $\mu(K) = [K_0 : K]^{-1}$.

We define the Hecke algebra of G with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$, $\mathcal{H}_r(G)$, to be the space of locally constant compactly supported functions on G with values in $\mathbb{Z}/l^r\mathbb{Z}$. Then $\mathcal{H}_r(G)$ has a natural structure of algebra without a unit on $\mathbb{Z}/l^r\mathbb{Z}$. Let $f \in \mathcal{H}_r(G)$, then there exist an open compact subgroup K of G, finitely many elements $g_i \in G$ and constants $c_i \in \mathbb{Z}/l^r\mathbb{Z}$ such that $f = \sum_i c_i \chi_{g_i K}$, where we denote by χ_C the characteristic function of C, for any open compact subset C of G.

Let V be a representation of G, with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$. We say that V is smooth if $V = \varinjlim_K V^K$, where K varies among the open compact subgroups of G and V^K denotes the submodule of the K-invariant elements of V. If V is a smooth representation of G, with coefficient in $\mathbb{Z}/l^r\mathbb{Z}$, then there is a natural action of $\mathcal{H}_r(G)$ on V. (If we write $f = \sum_i c_i \chi_{g_i K} \in \mathcal{H}_r(G)$ and $v \in V^K$, for some open compact subgroup K of G, then $f \cdot v = \mu(K) \sum_i c_i g_i v$.)

5.1.2. We say that a $\mathcal{H}_r(G)$-module V is non degerate if the natural map

$$\mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} V \to V$$

is an epimorphism. Any G-smooth representation V is non degenerate as $\mathcal{H}_r(G)$-module.

In general, for any $\mathcal{H}_r(G)$-module V, the above morphism gives rise to an isomorphism

$$\mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} V \xrightarrow{\sim} \varinjlim_K e_K V,$$

where K varies among the open compact subgroups of G and $e_K = \mu(K)^{-1}\chi_K$. In fact, for any $f \in \mathcal{H}_r(G)$ there exists an open compact subgroup K such that $f = fe_K = e_K f$, which implies that the image of $\mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} V$ in V is exactly $\varinjlim_K e_K \cdot V$. Moreover, suppose $\sum_i f_i \otimes v_i$ is an element in the kernel of the map,

and choose K an open compact subgroup such that $f_i = f_i e_K = e_K f_i$ for all i, then $\sum_i f_i \otimes v_i = e_K \otimes \sum_i f_i \cdot v_i$. Saying that the image is zero is equivalent to saying that $(\sum_i f_i \cdot v_i) = 0$, which implies that $e_K \otimes \sum_i f_i \cdot v_i = 0$.

It follows, in particular, that $\mathcal{H}_r(G)$ is a flat $\mathcal{H}_r(G)$-module (for all K, the functors $V \mapsto e_K V$ are exact and the direct limit functor is also exact).

5.1.3. For any smooth representation V of G, we denote by V_G the module of the coinvariants of V, then

$$V_G \simeq \Lambda \otimes_{\mathcal{H}_r(G)} V,$$

where $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$ is the trivial representation of G (thus the action of $f = \sum_i c_i \chi_{g_i K} \in \mathcal{H}_r(G)$ on $1 \in \Lambda$ is defined as $f \cdot 1 = \mu(K)(\sum_i c_i)$). In fact, let us consider the natural morphism $V \twoheadrightarrow V_G$, $v \mapsto [v]$. For any $f = \sum_i c_i \chi_{g_i K} \in \mathcal{H}_r(G)$ and $v \in V^K$, the equality $fv = \mu(K) \sum_i c_i g_i v$ implies that $[fv] = \mu(K)(\sum_i c_i)[v]$. We deduce that the morphism $V \twoheadrightarrow V_G$ gives rise to a morphism $\Lambda \otimes_{\mathcal{H}_r(G)} V \to V_G$, which is obviously surjective. Indeed, it is an isomorphism. Let $1 \otimes v \in \Lambda \otimes_{\mathcal{H}_r(G)} V$ be an element in the kernel of the above map, then there exist finitely many $g_i \in G$ and $v_i \in V$ such that $v = \sum_i (g_i - 1) v_i$. Let K be an open compact subgroup of G such that $v_i \in V^K$, for all i. Then $\mu(K)v = \sum_i (\chi_{g_i K} - \chi_K) v_i$, and thus $1 \otimes \mu(K)v = 0$. Since $\mu(K) \in (\mathbb{Z}/l^r\mathbb{Z})^\times$, it follows that $1 \otimes v = 0$.

5.1.4. Let W be a $\mathbb{Z}/l^r\mathbb{Z}$-module, we denote by $c - Ind_{\{1\}}^G(W) = C_c^\infty(G, W)$ the space of locally constant functions $G \to W$ with compact supports. Then,

$$c - Ind_{\{1\}}^G(W) \simeq \mathcal{H}_r(G) \otimes_{\mathbb{Z}/l^r\mathbb{Z}} W.$$

(The natural morphism of G-representations $\mathcal{H}_r(G) \otimes_{\mathbb{Z}/l^r\mathbb{Z}} W \to c - Ind_{\{1\}}^G(W)$, which sends any element $f \otimes w$ to the map $g \mapsto f(g^{-1})w$, is indeed an isomorphism.)

It follows that $c - Ind_{\{1\}}^G$ is an exact functor. (It is clearly left exact and, from the above equality, it is also right exact.)

We deduce from the above isomorphisms that, for any $\mathbb{Z}/l^r\mathbb{Z}$-module W, we have $c - Ind_{\{1\}}^G(W)_G \simeq W$. Moreover, the G-representation $c - Ind_{\{1\}}^G(W)$ is acyclic for the coinvariant functor. In fact, let us consider the two functors $W \mapsto c - Ind_{\{1\}}^G(W)$ and $V \mapsto V_G$. Since $c - Ind_{\{1\}}^G()$ is exact, in order to compute the derived functors of $()_G \circ c - Ind_{\{1\}}^G()$ as the composition of the derived functors of $()_G$ and $c - Ind_{\{1\}}^G()$, it is enough to check that, for any free $\mathbb{Z}/l^r\mathbb{Z}$-module L, $c - Ind_{\{1\}}^G(L)$ is a flat $\mathcal{H}_r(G)$-module (and indeed $c - Ind_{\{1\}}^G(L) \simeq \mathcal{H}_r(G) \otimes_{\mathbb{Z}/l^r\mathbb{Z}} L$ is flat, since $\mathcal{H}_r(G)$ is a flat $\mathcal{H}_r(G)$-module). Since $()_G \circ c - Ind_{\{1\}}^G()$ is simply the identity on the category of $\mathbb{Z}/l^r\mathbb{Z}$-modules, it follows that all the higher derived functors of $()_G$ vanish on the image of $c - Ind_{\{1\}}^G()$.

5.1.5. We denote by $G - \mathfrak{Ab}(X)$ the category of abelian l^r-torsion étale sheaves over X, together with an action of G which is trivial on X.

Definition 5.1. — We say that a sheaf $\mathcal{F} \in G - \mathfrak{Ab}(X)$ is smooth if

$$\mathcal{F} = \varinjlim_K \mathcal{F}^K,$$

where K varies among the open compact subgroups of G and $\mathcal{F}^K \in G - \mathfrak{Ab}(X)$ is the subsheaf of the K-invariants section of $\mathcal{F}$.

We denote by $G - \mathfrak{SmAb}(X)$ the full subcategory of $G - \mathfrak{Ab}(X)$ whose objects are the smooth objects of $G - \mathfrak{Ab}(X)$.

5.1.6. We write $G - \mathfrak{Ab}$ for the category of abelian l^r-torsion groups, together with an action of G, and $G - \mathfrak{SmAb}$ for the full subcategory of $G - \mathfrak{Ab}$ whose objects are smooth for the action of G.

Then, the functors of étale cohomology with compact supports on X on $\mathfrak{Ab}(X)$ give rise to some functors

$$H^i_c(X, -) : G - \mathfrak{SmAb}(X) \to G - \mathfrak{SmAb}.$$

In fact, for any sheaf $\mathcal{F} \in G - \mathfrak{SmAb}(X)$, we have

$$H^i_c(X, \mathcal{F}) = H^i_c(X, \varinjlim_K \mathcal{F}^K) = \varinjlim_K H^i_c(X, \mathcal{F}^K),$$

and the action of K on $H^i_c(X, \mathcal{F}^K)$ is trivial, for all open compact subgroups K.

5.1.7. Let us denote by $\mathcal{H}_r(G) - \mathfrak{Mod}(X)$ the category of sheaf of $\mathcal{H}_r(G)$-modules over X. Then, there is a natural inclusion

$$G - \mathfrak{SmAb}(X) \hookrightarrow \mathcal{H}_r(G) - \mathfrak{Mod}(X).$$

In fact, let $\mathcal{F}, \mathcal{G} \in G - \mathfrak{SmAb}(X)$ and $\phi : \mathcal{F} \to \mathcal{G}$. For any étale open U of X, $\mathcal{F}(U), \mathcal{G}(U)$ are two smooth representations of G and $\phi(U)$ a morphism compatible with the action of G, thus $\mathcal{F}(U), \mathcal{G}(U)$ are also two $\mathcal{H}_r(G)$-modules and the morphism $\phi(U)$ a morphism of $\mathcal{H}_r(G)$-modules.

5.1.8. We remark that, if $\mathcal{F}$ is a smooth G-sheaf, then the sheaf $C(\mathcal{F})$, which is defined as

$$C(\mathcal{F})(U) = \{f : U \to \prod_{x \in X} \mathcal{F}_x \mid f(x) \in \mathcal{F}_x \forall x\},$$

for any étale open U of X, is not a smooth sheaf, but it is naturally an object of $\mathcal{H}_r(G) - \mathfrak{Mod}(X)$. For any open compact subgroup K of G, it follows from the equality $e_K \mathcal{F}_x = (e_K \mathcal{F})_x$ that $e_K C(\mathcal{F}) = C(e_K \mathcal{F})$, and indeed $C(\mathcal{F}) \neq \varinjlim_K C(e_K \mathcal{F})$.

5.1.9. Let us consider the derived functor

$$\Lambda \otimes^{L^\cdot}_{\mathcal{H}_r(G)} () : D^-(X, \mathcal{H}_r(G) - \mathfrak{Mod}) \to D^-(X, \mathcal{H}_r(G) - \mathfrak{Mod}).$$

If $\Lambda.$ is a flat resolution of Λ, then $\Lambda \otimes^{L^\cdot}_{\mathcal{H}_r(G)} \mathcal{K}. \simeq \Lambda. \otimes_{\mathcal{H}_r(G)} \mathcal{K}.$ (see [**1**], proposition 4.1.7, p.73). We remark that it is possible to choose a flat resolution of Λ such that

all the modules Λ_i are of the form $L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)$, for some free $\mathbb{Z}/l^r\mathbb{Z}$-module L. Let us also remark that, for any $\mathcal{F} \in \mathcal{H}_r(G) - \mathfrak{Mod}$ and $x \in X$, we have

$$(\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} \mathcal{F})_x = \Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} \mathcal{F}_x.$$

Theorem 5.2. — *(see* [**7**]*, Section 4.9.1, pp. 95-96.) Let* $\mathcal{K}_{\cdot} \in D^-(X, \mathcal{H}_r(G) - \mathfrak{Mod})$*, then*

$$\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} R^{\cdot} f_!(\mathcal{K}_{\cdot}) \simeq R^{\cdot} f_!(\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} \mathcal{K}_{\cdot})$$

Proof. — In [**7**] (Section 4.9.1, pp. 95-96), this statement is proved under some conditions on the algebra which are not satisfied by $\mathcal{H}_r(G)$. Nevertheless the same argument works.

Deligne's first remark is that we can assume without loss of generality that f is proper. In fact, for any f, we have $Rf_! = R\bar{f}_* j_!$, for some open embedding j and some proper map $\bar{f}$. Since taking the tensor product commutes with the extension by zero, it suffices to prove the statement for $\bar{f}$.

Given any complex of sheaves of $\mathcal{H}_r(G)$-modules $\mathcal{K}_{\cdot}$, we can replace $\mathcal{K}_{\cdot}$ by the complex of its truncated Godement resolutions, which has the property of being acyclic for the functor Rf_*.

Let L be a free $\mathbb{Z}/l^r\mathbb{Z}$-module and consider the $\mathcal{H}_r(G)$-module $L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)$. Let us first assume L of finite type. Then

$$L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} Rf_*(K_{\cdot}) \simeq Rf_*(L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} K_{\cdot})$$

In fact, for any $\mathcal{H}_r(G)$-module V, we have $L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} V \simeq L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \varinjlim_K e_K V \simeq \varinjlim_K e_K(L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} V)$. Since L is free of finite type over $\mathbb{Z}/l^r\mathbb{Z}$ and the functor Rf_* commutes with direct limits and finite direct sums, it suffices to check that $Rf_*(e_K\mathcal{K}_{\cdot}) \simeq e_K Rf_*(\mathcal{K}_{\cdot})$. Such an equality follows from the observations of section 5.1.8 (which apply since the sheaves of the complex $\mathcal{K}_{\cdot}$ are all of the form $C(\mathcal{F})$, for some sheaf $\mathcal{F}$ of $\mathcal{H}_r(G)$-modules). In particular

$$Rf_*(L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} K_{\cdot}) \simeq f_*(L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G) \otimes_{\mathcal{H}_r(G)} K_{\cdot}).$$

By passing to the direct limit, one shows that the same holds for any free $\mathbb{Z}/l^r\mathbb{Z}$-module L.

We now consider a flat resolution $\Lambda_{\cdot}$ of the $\mathcal{H}_r(G)$-module Λ, such that all the modules Λ_i are of the form $L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)$, for some free $\mathbb{Z}/l^r\mathbb{Z}$-module L. Then

$$R^{\cdot} f_*(\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} \mathcal{K}_{\cdot}) \simeq R^{\cdot} f_*(\Lambda_{\cdot} \otimes_{\mathcal{H}_r(G)} \mathcal{K}_{\cdot}) \simeq f_*(\Lambda_{\cdot} \otimes_{\mathcal{H}_r(G)} \mathcal{K}_{\cdot}) \simeq$$

$$\simeq \Lambda_{\cdot} \otimes_{\mathcal{H}_r(G)} f_*(\mathcal{K}_{\cdot}) \simeq \Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} R^{\cdot} f_*(\mathcal{K}_{\cdot}).$$

□

5.1.10. We are interested in applying the above theorem to the following case.

Definition 5.3. — We say that an object $\mathcal{F} \in G - \mathfrak{SmAb}(X)$ has property $\mathcal{P}$ if

$$\mathcal{F}_x \simeq c - Ind_{\{1\}}^G(L_x),$$

for any geometric point x in X and some abelian l^r-torsion group L_x.

Let $\mathcal{F}$ be an object in $G - \mathfrak{SmAb}(X)$, which has property $\mathcal{P}$, and consider the complex $\mathcal{H}_\cdot(G, \mathcal{F}) := \Lambda \otimes^{L^\cdot}_{\mathcal{H}_r(G)} \mathcal{F}$. Then, it follows from the acyclicity of the stalks of $\mathcal{F}$ that

$$\mathcal{H}_i(G, \mathcal{F}) = \begin{cases} \Lambda \otimes_{\mathcal{H}_r(G)} \mathcal{F} \text{ for } i = 0, \\ 0 \text{ for } i \neq 0. \end{cases}$$

Corollary 5.4. — *Let $\mathcal{F}$ be an object in $G - \mathfrak{SmAb}(X)$ which has property $\mathcal{P}$. Then there is a spectral sequence*

$$E_2^{p,q} = H_p(G, H_c^q(X, \mathcal{F})) \Rightarrow H_c^{p+q}(X, \mathcal{F}_G).$$

Proof. — By applying theorem 5.2 to the sheaf $\mathcal{F}$, we obtain a quasi-isomorphism of complexes

$$\Lambda \otimes^{L^\cdot}_{\mathcal{H}_r(G)} R^\cdot f_!(\mathcal{F}) \simeq R^\cdot f_!(\Lambda \otimes^{L^\cdot}_{\mathcal{H}_r(G)} \mathcal{F}) \simeq R^\cdot f_!(\Lambda \otimes_{\mathcal{H}_r(G)} \mathcal{F}).$$

On one hand, the homology of the complex $R^\cdot f_!(\Lambda \otimes_{\mathcal{H}_r(G)} \mathcal{F})$ is simply $H_c^n(X, \mathcal{F}_G)$. On the other hand, the homology of $\Lambda \otimes^{L^\cdot}_{\mathcal{H}_r(G)} R^\cdot f_!(\mathcal{F})$ is computed by the two spectral sequences associated to the double complex. In particular, the spectral sequence $E_2^{p,q} = H_p(G, H_c^q(X, \mathcal{F}))$ abuts to it. □

5.2. The étale sheaf $\mathcal{F}$. — We now return to the study of the Newton polygon stratum $\bar{X}^{(\alpha)}$, for some polygon α. Let $\mathcal{L}$ be an abelian torsion étale sheaf over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, with torsion orders prime to p. Let m, n, d ($m \geq d$) be some posivite integers. In section 4.2, for any integer $N \geq d/\delta B$, we constructed a morphism $\pi_N : J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. We now consider the restrictions $\dot{\pi}_N$ of the morphisms π_N to the open $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{U}^{n,d}$ in $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$. (The need for substituting the morphism π_N with its restriction on the open $J_m \times_{\operatorname{Spec} k} \bar{U}^{n,d}$ is purely technical. It corresponds to the fact that the description of $\bar{\mathcal{M}}$ as the union of an increasing sequence of opens, namely the $\bar{U}^{n,d}$, is the appropriate one to be considered when computing the cohomology with compact supports.)

For each m, n, d ($m \geq d$), we define the abelian étale sheaf over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$

$$\mathcal{F}_m^{nd} = (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}),$$

for some index N sufficiently large (e.g. any $N \geq d/\delta B$). We shall see that the definition of $\mathcal{F}_m^{n,d}$ does not depend on N. We shall also prove that the sheaves $\mathcal{F}_m^{n,d}$

form a direct system and thus, to the abelian torsion étale sheaf $\mathcal{L}$ over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, we may associate the abelian étale sheaf over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$:

$$\mathcal{F} = \varinjlim_{m,n,d} \mathcal{F}_m^{nd},$$

together with a natural morphism $\mathcal{F} \to \mathcal{L}$.

We shall show that the action of S on the covering spaces $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ induces an action on the sheaf $\mathcal{F}$ (trivial on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$), which extends to a smooth action of the group $T \supset S$ with the property of leaving invariant the morphism $\mathcal{F} \to \mathcal{L}$. Moreover, we shall prove that, if the sheaf $\mathcal{L}$ is endowed with an action of $W_{\mathbb{Q}_p}$, which is compatible with the action of Frobenius $1 \times \sigma$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, then the action of $Frob^{\mathbb{N}}$ on the $J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ enable us to define an action of $W_{\mathbb{Q}_p}$ on the sheaf $\mathcal{F}$, also compatible with the action of Frobenius $1 \times \sigma$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, which commutes with the morphism $\mathcal{F} \to \mathcal{L}$ and with the action of T on $\mathcal{F}$.

Finally, for any point x in $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, we shall prove that

$$\mathcal{F}_x = C_c^\infty(\Pi^{-1}(x), \mathcal{L}_x) \simeq c - Ind_{\{1\}}^G(\mathcal{L}_x).$$

In particular, for all $\mathcal{L}$, the associated sheaf $\mathcal{F} \in T - \mathfrak{SmAb}(\bar{X}^{(\alpha)})$ has property $\mathcal{P}$ (see definition 5.3).

5.2.1. We start by showing that it is possible to define a sheaf $\mathcal{F}$ as as above.

Proposition 5.5. — *Let $\mathcal{L}$ be an abelian sheaf over $\bar{X}^{(\alpha)}$, with torsion orders relatively prime to p. For any m, n, d ($m \geq d$), we define*

$$\mathcal{F}_m^{nd} = (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}),$$

for some integer $N \geq d/\delta B$.

1. *The sheaves $\mathcal{F}_m^{n,d}$ are independent on the integers N.*
2. *The sheaves $\mathcal{F}_m^{n,d}$ form a direct system under the morphisms*

$$(\mathrm{q}_{m',m} \times 1)^* : \mathcal{F}_m^{n,d} \to \mathcal{F}_{m'}^{n,d}$$

and

$$(1 \times i_{n',d'}^{n,d})_! : \mathcal{F}_m^{n,d} \to \mathcal{F}_m^{n',d'},$$

for all integers $m' \geq m, d' - d \geq (n' - n)h \geq 0$.

3. *There exists a natural morphism $\varsigma_! : \mathcal{F} \to \mathcal{L}$.*

Proof. — Part (1): Let m, n, d be some positive integers such that $m \geq d$. For any integers $N' \geq N \geq d/\delta B$, we have an equality of morphisms on $J_m \times \bar{U}^{n,d}$:

$$\pi_{N'} = (Fr^{(N'-N)B} \times 1) \circ \pi_N.$$

Thus, $(\pi_{N'})_! = (Fr^{(N'-N)B} \times 1)_!(\pi_N)_!$ and $(\pi_{N'})^* = (\pi_N)^*(Fr^{(N'-N)B} \times 1)^*$.

In particular, it follows that

$$\mathcal{F}_m^{n,d} = (Fr^{N'B} \times 1)^*(\dot{\pi}_{N'})_!(\dot{\pi}_{N'})^*(Fr^{N'B} \times 1)_!(\mathcal{L}) =$$

$$= (Fr^{N'B} \times 1)^*(Fr^{(N'-N)B} \times 1)_!(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{(N'-N)B} \times 1)^*(Fr^{N'B} \times 1)_!(\mathcal{L}) \simeq$$

$$\simeq (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L})$$

(since Fr is a purely inseparable finite morphism, there are canonical isomorphisms $1 \simeq Fr^* Fr_! \simeq Fr_! Fr^*$).

Part (2): From the equality $\dot{\pi}_N \circ \mathrm{q}_{m',m} = \dot{\pi}_N$ (for all $m' \geq m$) and the existence of a canonical morphism $q^* : \mathcal{D} \to q_! q^* \mathcal{D}$, for any étale sheaf $\mathcal{D}$ and any finite morphism q, we deduce the existence of a morphism

$$(\mathrm{q}_{m',m} \times 1)^* : \mathcal{F}_m^{n,d} \to \mathcal{F}_{m'}^{n,d}.$$

In fact, from the equality $\dot{\pi}_N \circ (\mathrm{q}_{m',m} \times 1) = \dot{\pi}_N$ we deduce that $(\dot{\pi}_N)_! \circ (\mathrm{q}_{m',m})_! = (\dot{\pi}_{m'}^{n,d})_!$ and $(\mathrm{q}_{m',m})^* \circ (\dot{\pi}_N)^* = (\dot{\pi}_{m'}^{n,d})^*$. Thus, there is a morphism

$$\begin{aligned}
\mathcal{F}_m^{nd} &= (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}) \to \\
&\to (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\mathrm{q}_{m',m})_!(\mathrm{q}_{m',m})^*(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}) = \\
&= (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}) = \mathcal{F}_{m'}^{nd}.
\end{aligned}$$

Analogously, from the equality $\dot{\pi}_m^{n',d'} \circ (1 \times i_{n',d'}^{n,d}) = \dot{\pi}_N$ (for any $N \geq d'/\delta B$) and the existence of a canonical morphism $i_! : i_! i^* \mathcal{D} \to \mathcal{D}$, for any étale sheaf $\mathcal{D}$ and any open embedding i, we deduce the existence of a morphism

$$(1 \times i_{n',d'}^{n,d})_! : \mathcal{F}_m^{n,d} \to \mathcal{F}_m^{n',d'}.$$

It is straight forward that these morphism respect the required commutativity rules and thus that the sheaves $\mathcal{F}_m^{n,d}$ form a direct limit.

Part (3): For any positive integers m, n, d, N $(m, N \geq d)$, there is a natural morphism

$$\begin{aligned}
(\dot{\pi}_N)_! : \mathcal{F}_m^{nd} = (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}) \to \\
\to (Fr^{NB} \times 1)^*(Fr^{NB} \times 1)_!(\mathcal{L}) \simeq \mathcal{L}.
\end{aligned}$$

It is clear that the morphism $(\dot{\pi}_N)_!$ does not dependent on the integer N. We define $\varsigma_! = \frac{1}{[J_m : J_1]}(\dot{\pi}_N)_!$. Then, it is straight forward that the morphisms $\varsigma_!$ on $\mathcal{F}_m^{n,d}$ commutes with the morphisms $(\mathrm{q}_{m',m} \times 1)^*$ and $(1 \times i_{n',d'}^{n,d})_!$, and thus give rise to a morphism $\varsigma_! : \mathcal{F} \to \mathcal{L}$. □

It follows from the above proposition that, to any abelian torsion sheaf $\mathcal{L}$ over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, with torsion orders prime to p, we can associate a sheaf

$$\mathcal{F} = \varinjlim_{m,n,d} \mathcal{F}_m^{nd},$$

together with a morphism $\mathcal{F} \to \mathcal{L}$.

Proposition 5.6. — *Let $\mathcal{L}$ be an abelian étale torsion sheaf over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (with torsion order prime to p), and consider the associated étale sheaf $\mathcal{F} \to \mathcal{L}$.*

The action of the monoid S on the systems of covering spaces $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ induces an action of S on the étale sheaf $\mathcal{F}$, which extends to a smooth action of the group T on $\mathcal{F}$.

The morphism $\mathcal{F} \to \mathcal{L}$ is invariant under the action of T.

Proof. — Let $\rho \in S$ and write $e = e(\rho)$ and $f = f(\rho)$, then, for all $m \geq d+2e-f$ and $N \geq (d+e-f)/\delta B$, the morphism $\rho \times \rho : J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to J_{m-e} \times \bar{\mathcal{M}}^{n+e,d+e-f}$ restricts to a morphism

$$\dot{\rho} : J_m \times \bar{U}^{n,d} \to J_{m-e} \times \bar{U}^{n+e,d+e-f}$$

such that $\dot{\pi}_N \circ \dot{\rho} = \dot{\pi}_N$, since $\pi_N(\rho \times \rho) = \pi_N$.

Let us also remind that on the Rapoport-Zink space $\bar{\mathcal{M}}$ the action of ρ is invertible. In particular, there are morphisms

$$\dot{\rho}^{-1} : \bar{U}^{n',d'} \to U^{n'-f,d'+e-f}$$

such that $\dot{\rho}^{-1} \circ \dot{\rho} = i$ and $\dot{\rho} \circ \dot{\rho}^{-1} = i$.

We consider the following diagram (where we right $s = e - f$).

$$\begin{tikzcd}
J_m \times \bar{U}^{n+f,d-s} \arrow[r, "\dot{\rho}\times 1"] \arrow[d, "1\times\dot{\rho}^{-1}"] \arrow[ddr, bend left=60, "1\times i"'] \arrow[ddrr, bend left=60, "1\times i"] & J_{m-e} \times \bar{U}^{n+f,d-s} \arrow[d, "1\times\dot{\rho}^{-1}"'] \\
J_m \times \bar{U}^{n,d} \arrow[r, "\dot{\rho}\times 1"] \arrow[d, "1\times\dot{\rho}"] \arrow[dr, "\dot{\rho}\times\dot{\rho}"] \arrow[ddr, bend right=60, "\dot{\pi}_N"'] & J_{m-e} \times \bar{U}^{n,d} \arrow[d, "1\times\dot{\rho}"] \\
J_m \times \bar{U}^{n+e,d+s} \arrow[r, "\dot{\rho}\times 1"] \arrow[d] & J_{m-e} \times \bar{U}^{n+e,d+s} \arrow[d, "\dot{\pi}_N"] \\
\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p \arrow[r, equal] & \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p
\end{tikzcd}$$

We define the action of ρ on the direct system of sheaves $\mathcal{F}_m^{n,d}$

$$\rho = (1 \times \dot{\rho}^{-1})_! \circ (\dot{\rho} \times 1)^* : \mathcal{F}_{m-e}^{n+f,d-(e-f)} \to \mathcal{F}_m^{n,d}$$

as follows (where we write $f = Fr^{NB} \times 1$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$)

$$\begin{aligned}
\mathcal{F}_{m-e}^{n+f,d-(e-f)} &= (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}) \to \\
&\to f^*(\dot{\pi}_N)_!(\dot{\rho} \times 1)_!(\dot{\rho} \times 1)^*(\dot{\pi}_N)^* f_!(\mathcal{L}) = \\
&= f^*(\dot{\pi}_N)_!(1 \times i)_!(\dot{\rho} \times 1)_!(\dot{\rho} \times 1)^*(1 \times i)^*(\dot{\pi}_N)^* f_!(\mathcal{L}) = \\
&= f^*(\dot{\pi}_N)_!(\dot{\rho} \times 1)_!(1 \times i)_!(1 \times i)^*(\dot{\rho} \times 1)^*(\dot{\pi}_N)^* f_!(\mathcal{L}) =
\end{aligned}$$

$$= f^*(\dot{\pi}_N)_!(\dot{\rho} \times 1)_!(1 \times \dot{\rho})_!(1 \times \dot{\rho}^{-1})_!(1 \times \dot{\rho}^{-1})^*(1 \times \dot{\rho}^{-1})^*(\dot{\rho} \times 1)^*(\dot{\pi}_N)^* f_!(\mathcal{L}) \to$$

$$\begin{aligned}
&\to f^*(\dot{\pi}_N)_!(\dot{\rho} \times 1)_!(1 \times \dot{\rho})_!(1 \times \dot{\rho})^*(\dot{\rho} \times 1)^*(\dot{\pi}_N)^* f_!(\mathcal{L}) = \\
&= f^*(\dot{\pi}_N)_!(\dot{\rho} \times \dot{\rho})_!(\dot{\rho} \times \dot{\rho})^*(\dot{\pi}_N)^* f_!(\mathcal{L}) = \\
&= f^*(\dot{\pi}_N)_!(\dot{\pi}_N)^* f_!(\mathcal{L}) = \mathcal{F}_m^{n,d}.
\end{aligned}$$

It follows from the definition that, for all $\rho \in S$, the morphisms ρ commute with the structure morphisms of the direct limit $(q\times 1)^*$ and $(1\times i)_!$, and also that $\dot{\pi}_{N\,!} \circ \rho = \dot{\pi}_{N\,!}$.

Therefore, the above construction gives rise to an action of S on $\mathcal{F}$, under which the morphism $\mathcal{F} \to \mathcal{L}$ is invariant.

Since $T = \langle S, p, fr^B \rangle$, the action of S on $\mathcal{F}$ extends to an action of T if the elements $p^{-1}, fr^{-B} \in S$ act invertibly on $\mathcal{F}$. The element $p^{-1} \in S$ acts isomorphically on the space $\bar{\mathcal{M}}$ and on J_m via the morphism $v(v^c)^{-1} \circ \mathrm{q}_{m',m}$, for some element $v \in E^\times$ such that $\mathrm{val}_u(v) = 1$, $\mathrm{val}_{u^c}(v) = 0$ and $v \equiv 1 \mod (u^c)^m$. Thus, the induced action on the sheaves $\mathcal{F}_m^{n,d}$ becomes invertible once one passes to the direct limit $\mathcal{F}$. On the other hand, the element $fr^{-B} \in S$ also acts isomorphically on the space $\bar{\mathcal{M}}$ and on J_m we have $fr^{-B} = (v^c)^{-B} \circ \mathrm{q}_{m,m-a} \circ frob^{-B} \circ Fr^B$, where Fr is the relative Frobenius morphism on the Igusa varieties over $\bar{\mathbb{F}}_p$ (a purely inseparable finite morphism). We can therefore deduce that the induced action of fr^{-B} on $\mathcal{F}$ is also invertible.

Finally, in order to prove that the action of T on $\mathcal{F}$ is smooth, it suffices to check that for any m, n, d ($m \geq d$) the action of Γ^m on the sheaf $\mathcal{F}_m^{n,d}$ is trivial, which follows from the fact that the action of Γ^m on the space $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ is trivial (see section 2.5.12 and proposition 3.3). □

5.2.2. We are interested in the case when the sheaf $\mathcal{L}$ is naturally endowed with an action of the Weil group $W_{\mathbb{Q}_p}$, which is compatible with the action of $W_{\mathbb{Q}_p}$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, e.g. $\mathcal{L}$ the pullback over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ of a sheaf over $\bar{X}^{(\alpha)}$ or some vanishing cycle sheaf.

Definition 5.7. — We say that a sheaf $\mathcal{L}$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ has an action of $W_{\mathbb{Q}_p}$, compatible with the action of $W_{\mathbb{Q}_p}$ on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ if, for all $\tau \in W_{\mathbb{Q}_p}$, there are some isomorphisms $(1 \times \bar{\tau})^* \mathcal{L} \simeq \mathcal{L}$, where $\bar{\tau}$ denotes the image of τ in $\sigma^{\mathbb{Z}} \subset Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$, such that $\tau \circ \tau' = \tau' \tau$.

Proposition 5.8. — *Maintaining the notations of propositon 5.6. Let us further assume that the étale sheaf $\mathcal{L}$ is endowed with an action of the Weil group $W_{\mathbb{Q}_p}$.*

Then, there is an induced action of $W_{\mathbb{Q}_p}$ on the étale sheaf $\mathcal{F}$, which commutes with the action of T on $\mathcal{F}$ and with the morphism $\mathcal{F} \to \mathcal{L}$.

Proof. — Let us consider the action of $Frob$ on the covering spaces $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$. From the equality $\pi_N \circ (Frob \times Frob) = (1 \times \sigma) \circ \pi_N$, we deduce that

$$(Fr^{NB} \times 1)^* (\dot{\pi}_N)_! (Frob \times Frob)_! (Frob \times Frob)^* (\dot{\pi}_N)^* (Fr^{NB} \times 1)_! (\mathcal{L}) =$$

$$= (1 \times \sigma)_! (Fr^{NB} \times 1)^* (\dot{\pi}_N)_! (\dot{\pi}_N)^* (Fr^{NB} \times 1)_! (1 \times \sigma)^* (\mathcal{L}).$$

Let us also recall that the action of $Frob$ on the Rapoport-Zink space is invertible. In particular, we have

$$Frob^{-1} : U^{n',d'} \to U^{n',d'+1},$$

where $Frob^{-1} \circ Frob = i$ and $Frob \circ Frob^{-1} = i$.

Let $\tau \in W_{\mathbb{Q}_p}$ and assume for the moment that $\bar{\tau} = \sigma^r$, for some $r \geq 0$. We define the action of τ on the system of sheaves $\mathcal{F}_m^{n,d}$ as

$$\tau \circ (1 \times Frob^{-r})_!(Frob^r \times 1)^* : (1 \times \sigma^r)^* \mathcal{F}_{m-r}^{n,d-r} \to$$
$$\to (1 \times \sigma^r)^*(Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(Frob^r \times Frob^r)_!(Frob^r \times Frob^r)^*(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L})$$
$$= (1 \times \sigma^r)^*(1 \times \sigma^r)_!(Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(1 \times \sigma^r)^*(\mathcal{L}) \simeq$$
$$\simeq (1 \times \sigma^r)^*(1 \times \sigma^r)_! \mathcal{F}_m^{n,d} \simeq \mathcal{F}_m^{n,d}.$$

Thus, the action of τ on $\mathcal{F}_m^{n,d}$ satisfies the equality

$$\varsigma_! \circ \tau = \tau \circ \varsigma_! : (1 \times \bar{\tau})^* \mathcal{F}_m^{n,d} \to \mathcal{L}.$$

The compatibility of the action of $Frob$ with the morphisms $\mathrm{q}_{m',m} \times 1$ and $1 \times i_{n',d'}^{n,d}$ and with the action of S on the spaces $J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d}$ implies that the above action of τ on the étale sheaves $\mathcal{F}_m^{n,d}$ gives rise to an action of τ on the direct limit $\mathcal{F}$, $(1 \times \bar{\tau})^* \mathcal{F} \to \mathcal{F}$, which commutes with the action of T on $\mathcal{F}$ and with the projection $\mathcal{F} \to \mathcal{L}$.

Moreover, since the action of $Frob$ on the Rapoport-Zink space $\mathcal{M}$ is invertible and on the Igusa varieties J_m is defined by the morphisms $Frob = \mathrm{q}_{m,m-1} \circ Fr$ (where Fr is a purely inseparable morphism), we deduce that the action of τ on $\mathcal{F}$ is invertible, and thus we can extend the above action to an action of $W_{\mathbb{Q}_p}$ on $\mathcal{F}$. □

5.2.3. We now focus our attention of the stalk $\mathcal{F}_x$ of the sheaf $\mathcal{F}$, at a point x of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. It follows from the fact that the morphisms π_N are finite that

$$\begin{aligned}
\mathcal{F}_x &= \varinjlim_{m,n,d} \left((Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L})\right)_x = \\
&= \varinjlim_{m,n,d} \left((\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L})\right)_{Fr^{NB}(x)} = \\
&= \varinjlim_{m,n,d} \prod_{\dot{\pi}_N(y) = Fr^{NB}(x)} \left((\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L})\right)_y = \\
&= \varinjlim_{m,n,d} \prod_{\dot{\pi}_N(y) = Fr^{NB}(x)} \left((Fr^{NB} \times 1)_!(\mathcal{L})\right)_{\dot{\pi}_N(y)} = \\
&= \varinjlim_{m,n,d} \prod_{\dot{\pi}_N(y) = Fr^{NB}(x)} \left((Fr^{NB} \times 1)_!(\mathcal{L})\right)_{Fr^{NB}(x)} = \\
&= \varinjlim_{m,n,d} \prod_{\dot{\pi}_N(y) = Fr^{NB}(x)} (\mathcal{L})_x.
\end{aligned}$$

Under the above identification, the morphisms

$$(\mathrm{q}_{m',m} \times 1)^* : (\mathcal{F}_m^{n,d})_x \to (\mathcal{F}_{m'}^{n,d})_x$$

is defined as

$$(\mathrm{q}_{m',m} \times 1)^*(s)_y = s_{(\mathrm{q}_{m',m} \times 1)(y)},$$

for all $y \in J_{m'} \times \bar{U}^{n,d}$ and the morphisms

$$(1 \times i_{n',d'}^{n,d})_! : (\mathcal{F}_m^{n,d})_x \to (\mathcal{F}_m^{n',d'})_x$$

as

$$(1 \times i^{n,d}_{n',d'})_!(s)_y = \begin{cases} s_y & \text{if } y \in J_m \times \bar{U}^{n,d}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in J_m \times \bar{U}^{n',d'}$.

The projection $\varsigma_! : (\mathcal{F}^{n,d}_m)_x \to \mathcal{L}_x$ is defined as

$$\varsigma_!(s) = \frac{1}{[J_m : J_1]} \sum_{\dot{\pi}_N(y) = Fr^{NB}(x)} s(y),$$

for $y \in J_m \times \bar{U}^{n,d}$.

Finally, the action of T on $\mathcal{F}_x$ is defined as

$$\rho(s)_y = s_{\rho(y)},$$

for all $\rho \in T$, $s \in \mathcal{F}_x$, $y \in J_m \times \bar{U}^{n,d}$ and $m, n, d \gg 0$.

Proposition 5.9. — *Let $\mathcal{L}$ be an abelian torsion étale sheaf on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, with torsion orders prime to p, and x a geometric point of $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. We denote by $\mathcal{F}$ the sheaf on $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ associated to $\mathcal{L}$, $\mathcal{F} \to \mathcal{L}$.*

Then,

$$\mathcal{F}_x = C^\infty_c(\Pi^{-1}(x), \mathcal{L}_x)$$

as representations of T (see section 4.3).

Proof. — We use the identification

$$\mathcal{F}_x = \varinjlim_{m,n,d} \prod_{\dot{\pi}_N(y) = Fr^{NB}(x)} (\mathcal{L})_x.$$

Then, there is a natural isomorphism of T-modules

$$\Theta : \mathcal{F}_x \to C^\infty_c(\Pi^{-1}(x), \mathcal{L}_x)$$

defined by

$$\Theta(s)(y) = s_{(\mathrm{q}_{\infty,m} \times 1)(y)} \quad (\exists m \gg 0),$$

for all $y \in \Pi^{-1}(x)$.

In fact, let $s \in \mathcal{F}_x$ and $y = (y_1, y_2) \in \Pi^{-1}(x) \subset J(k) \times \bar{\mathcal{M}}(k)$. Then, there exist two integers $n, d \geq 0$ such that $y_2 \in U^{n,d}(k)$. Then, for any $m \geq d$ and $N \geq d/\delta B$,

$$\dot{\pi}_N(\mathrm{q}_{\infty,m} \times 1)(y) = Fr^{NB}\Pi(y) = Fr^{NB}(x).$$

Further more, if m, n, d are sufficiently large (e.g. such that $s \in \mathcal{F}_x$ is the image of an element of $(\mathcal{F}^{n,d}_m)_x$), then $s_{(\mathrm{q}_{\infty,m} \times 1)(y)} \in \mathcal{L}_x$.

It is also clear that the value $s_{(\mathrm{q}_{\infty,m} \times 1)(y)} \in \mathcal{L}_x$ is independent on the choice of the integers m, n, d, N (since, for all $m' \geq m$, $\mathrm{q}_{\infty,m} = \mathrm{q}_{m',m} \circ \mathrm{q}_{\infty,m'}$).

In order to prove that the map $\Theta(s)$ is indeed an element of $C^\infty_c(\Pi^{-1}(x), \mathcal{L}_x)$ (for any $s \in \mathcal{F}_x$), it remains to prove that it has compact support and is invariant under the action of an open subgroup of T.

Let $s \in \mathcal{F}_x$ and denote by m, n, d ($m \geq d$) some positive integers such that s arises as the image of an element in $(\mathcal{F}_m^{n,d})_x$. Then, it follows from the definition that the support of $\Theta(s)$ is contained in $\Pi^{-1}(x) \cap J(k) \times \bar{\mathcal{M}}^{n,d}(k)$, which is compact, and thus is itself compact. Moreover, the function $\Theta(s)$ factors through the quotient $(\mathrm{q}_{\infty,m} \times 1)\Pi^{-1}(x)$ and in particular takes non zero values only on the set

$$(\mathrm{q}_{\infty,m} \times 1)(\Pi^{-1}(x) \cap J(k) \times \bar{\mathcal{M}}^{n,d}(k)) \subset (\mathrm{q}_{\infty,m} \times 1)\Pi^{-1}(x),$$

which is finite (see the proof of proposition 4.5). Therefore, for all $\rho \in \Gamma^m$,we have

$$\Theta(s)^\rho = \Theta(s) \circ \rho = \Theta(s).$$

It also follows directly from the definitions that the the map Θ is injective. To prove that Θ is surjective, it suffices to show that for any $f \in C_c^\infty(\Pi^{-1}(x), \mathcal{L}_x)$ there exist some positive integers m, n, d such that the support of f is contained in $\Pi^{-1}(x) \cap J(k) \times \bar{\mathcal{M}}^{n,d}(k)$ (which is equivalent to say that f has compact support) and f factors via the quotients $(\mathrm{q}_{\infty,m} \times 1)\Pi^{-1}(x) \subset J_m(k) \times \bar{\mathcal{M}}(k)$ (and in fact, it is enough to choose m such that Γ^m is contained in the open subgroup of Γ which fixes f).

Finally, the map Θ is a morphism of T-modules, because, for all $s \in \mathcal{F}_x$ and $y \in \Pi^{-1}(x)$, we have

$$\Theta(s)^\rho(y) = \Theta(s)\rho(y) = s_{(\mathrm{q}_{\infty,m}\times 1)\rho(y)} = s_{\rho(\mathrm{q}_{\infty,m}\times 1)(y)} = \dot{\rho}_*(s)_y.$$

□

Corollary 5.10. — *Maintaining the above notations. For any geometric point $x \in \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, there is a non canonical isomorphism of T-representations*

$$\mathcal{F}_x \simeq c - Ind_{\{1\}}^T(L_x).$$

In particular, the sheaf $\mathcal{F}/\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ has property $\mathcal{P}$ (see definition 5.3).

Proof. — By proposition 4.4, there exists a non canonical isomorphism $\Pi^{-1}(x) \simeq T$ and thus

$$\mathcal{F}_x = C_c^\infty(\Pi^{-1}(x), \mathcal{L}_x) \simeq C_c^\infty(T, \mathcal{L}_x) = c - Ind_{\{1\}}^T(L_x).$$

□

5.3. The cohomology of the Newton polygon strata. — We shall now apply the results of section 5.1 to the case when $G = T$, $X = \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ and $\mathcal{F}$ is the sheaf defined in section 5.2, attached to an abelian torsion étale sheaf $\mathcal{L}$ over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$.

Corollary 5.4 may be applied to obtain the following result.

Theorem 5.11. — *Let $\mathcal{L}$ be a torsion abelian étale sheaf over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (with torsion orders prime to p), endowed with an action of the Weil group $W_{\mathbb{Q}_p}$. Then, there is a spectral sequence*

$$E_2^{p,q} = H_p(T, \varinjlim_{m,n,d} H_c^q(J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{U}^{n,d}, (\dot{\pi}_N)^*\mathcal{L})) \Rightarrow H_c^{p+q}(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \mathcal{L}),$$

which is compatible with the action of the Weil group $W_{\mathbb{Q}_p}$.

Proof. — Let us consider the the abelian torsion étale sheaf over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$

$$\mathcal{F} = \varinjlim_{m,n,d} \mathcal{F}_m^{n,d} = \varinjlim_{m,n,d} (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L}).$$

Then, corollary 5.4 applied to the case we are considering implies the existence of a spectral sequence

$$H_p(T, H_c^q(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \mathcal{F})) \Rightarrow H_c^n(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \mathcal{F}_T).$$

Let us focus on the terms of the above spectral sequence. It follows from the definition of the sheaf $\mathcal{F}$ and from the fact that the morphisms π_N are finite that

$$\begin{aligned}
& H_c^q(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \mathcal{F}) = H_c^q(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \varinjlim_{m,n,d} \mathcal{F}_m^{n,d}) \\
= \; & \varinjlim_{m,n,d} H_c^q(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, \mathcal{F}_m^{n,d}) \\
= \; & \varinjlim_{m,n,d} H_c^q(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, (Fr^{NB} \times 1)^*(\dot{\pi}_N)_!(\dot{\pi}_N)^*(Fr^{NB} \times 1)_!(\mathcal{L})) \\
= \; & \varinjlim_{m,n,d} H_c^q(\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p, (\dot{\pi}_N)_!(\dot{\pi}_N)^*(\mathcal{L})) \\
= \; & \varinjlim_{m,n,d} H_c^q(J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{U}^{n,d}, (\dot{\pi}_N)^*(\mathcal{L})),
\end{aligned}$$

where the above identifications are compatible with the action of the group $T \times W_{\mathbb{Q}_p}$ (see propositions 5.6 and 5.8).

On the other hand, the morphism $\mathcal{F} \to \mathcal{L}$ gives rise to a morphism $\mathcal{F}_T \to \mathcal{L}$ which is also compatible with the action of $W_{\mathbb{Q}_p}$. Indeed, the morphism $\mathcal{F}_T \to \mathcal{L}$ is bijective, since such are the induced maps on stalks $(\mathcal{F}_T)_x = (\mathcal{F}_x)_T \to \mathcal{L}_x$, for all $x \in \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (this fact follows from corollary 5.10). □

5.4. Using Künneth formula. — In the following, we use Künneth formula for étale cohomology with compact supports to rewrite the result of the previous theorem in terms of the cohomology groups of the Igusa varieties and the Rapoport-Zink spaces, separately.

5.4.1. Let us first establish some general results relative the tensor product of smooth representations with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$ of a p-adic group G.

Let M, N be two smooth $\mathbb{Z}/l^r\mathbb{Z}$-representations of G. Then, the $\mathbb{Z}/l^r\mathbb{Z}$-module $M \otimes_{\mathbb{Z}/l^r\mathbb{Z}} N$ in naturally endowed with an action of G, namely $g(v \otimes w) = gv \otimes gw$. (Indeed, for all $i \geq 0$, the $\mathbb{Z}/l^r\mathbb{Z}$-modules $Tor^i_{\mathbb{Z}/l^r\mathbb{Z}}(M, N)$ also have a natural smooth action of G.)

We remark that the $\mathcal{H}_r(G)$-module associated to $M \otimes_{\mathbb{Z}/l^r\mathbb{Z}} N$ is not the module $M \otimes_{\mathcal{H}_r(G)} N$. On the other hand, there is a natural isomorphism

$$M^{op} \otimes_{\mathcal{H}_r(G)} N \simeq \Lambda \otimes_{\mathcal{H}_r(G)} (M \otimes_{\mathbb{Z}/l^r\mathbb{Z}} N),$$

where M^{op} denotes the right $\mathcal{H}_r(G)$-modules associated to the right $\mathbb{Z}/l^r\mathbb{Z}$-representation of G which underlying $\mathbb{Z}/l^r\mathbb{Z}$-module is M and the right action of G is defined

as $m \cdot g = g^{-1}m$, for all $g \in G$ and $m \in M$. Indeed, let us consider the natural epimorphism

$$M \otimes_{\mathbb{Z}/l^r\mathbb{Z}} N \to M^{op} \otimes_{\mathcal{H}_r(G)} N.$$

For any $m \in M$, $n \in N$, $g \in G$ and open compact subgroup K of G, we have $\chi_{gK}(m \otimes n) = \mu(K)gm \otimes gn \in M \otimes_{\mathbb{Z}/l^r\mathbb{Z}} N.$ and also

$$m \otimes n = gm \cdot g \otimes n = \mu(K)^{-1}\chi_{gK}gm \otimes n = \mu(K)^{-1}\chi_{gK}(gm \otimes n) =$$

$$= gm \otimes \mu(K)^{-1}\chi_{gK}n = gm \otimes gn \in M^{op} \otimes_{\mathcal{H}_r(G)} N.$$

Thus, the above morphism induces a morphism between $\Lambda \otimes_{\mathcal{H}_r(G)} (M \otimes_{\mathbb{Z}/l^r\mathbb{Z}} N)$ and $M^{op} \otimes_{\mathcal{H}_r(G)} N$. Such a morphism is clearly surjective and indeed is also injective.

Proposition 5.12. — *Let $M.,N.$ be two complexes, bounded from above, of smooth $\mathbb{Z}/l^r\mathbb{Z}$-representations of G, then*

$$\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} (M. \otimes^{L^{\cdot}}_{\mathbb{Z}/l^r\mathbb{Z}} N.) \simeq M^{op}_{\cdot} \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} N..$$

Proof. — First, we replace the complex $N.$ with its Cartan-Eilenberg resolution $P.$. Since any smooth representation of G admits a resolution by projetives of the form $L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)$, for some free $\mathbb{Z}/l^r\mathbb{Z}$-module L, we can assume without loss of generality that $P. = L. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)$.

We remark that, if L is a free $\mathbb{Z}/l^r\mathbb{Z}$-module, then $P = L \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)$ is a flat $\mathbb{Z}/l^r\mathbb{Z}$-module, and thus

$$M. \otimes^{L^{\cdot}}_{\mathbb{Z}/l^r\mathbb{Z}} N. \simeq M. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} (L. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)) \simeq (M. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} L.) \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)),$$

where the latter is a complex of acyclic objects for the functor $\Lambda \otimes_{\mathcal{H}_r(G)} ()$ (see section 5.1.4). Therefore,

$$\begin{aligned}
&\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} (M. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} L. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)) \simeq \\
&\qquad \simeq \quad \Lambda \otimes_{\mathcal{H}_r(G)} (M. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} L. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)) \simeq \\
&\qquad \simeq \quad M^{op}_{\cdot} \otimes_{\mathcal{H}_r(G)} (L. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)) \simeq \\
&\qquad \simeq \quad M^{op}_{\cdot} \otimes^{L^{\cdot}}_{\mathcal{H}_r(G)} (L. \otimes_{\mathbb{Z}/l^r\mathbb{Z}} \mathcal{H}_r(G)).
\end{aligned}$$

□

5.4.2. We apply the above proposition to the study of the cohomology of the open Newton polygon strata. To avoid ambiguities, let us reintroduce in our notation the datum of the level $U^p \subset G(\mathbb{A}^{\infty,p})$ of the Shimura variety we are studying. In the following, $\bar{X}^{(\alpha)}_{U^p}$ denotes the Newton polygon stratum associated to a Newton polygon α, and J_{m,U^p} denotes the Igusa varieties of level m over the central leaf of $\bar{X}^{(\alpha)}_{U^p}$.

For all $i \geq 0$, we write

$$H^i_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_m H^i_c(J_{U^p,m}, \mathbb{Z}/l^r\mathbb{Z})$$

for the cohomology groups with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$ of the Igusa varieties of level U^p, viewed as a module endowed with an action of $T \times (W_{\mathbb{Q}_p}/I_p)$ (see section 3.5).

We also write

$$H^i_c(\bar{\mathcal{M}}, \mathcal{G}) = \varinjlim_{n,d} H^i_c(\bar{U}^{n,d}, \mathcal{G}_{|\bar{U}^{n,d}})$$

for the cohomology groups with coefficients in an abelian torsion sheaf $\mathcal{G}$ (with torsion orders relatively prime to p) of the reduction of the Rapoport-Zink space without level structure. If the sheaf $\mathcal{G}$ is endowed with an action of the Weil group, we view the above cohomology groups as modules endowed with an action of $T \times W_{\mathbb{Q}_p}$ (see section 2.5.14), where the action of T on the cohomology groups arises from the opposite action of T on $\bar{\mathcal{M}}$.

Finally, we denote by $\bar{pr} : J_m \times \bar{U}^{n,d} \to \bar{\mathcal{M}}$ the projection to the second factor of the product.

Theorem 5.13. — *Let U^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$, $r \geq 1$ an integer, $\mathcal{L}$ (resp. $\mathcal{G}$) an étale sheaf of $\mathbb{Z}/l^r\mathbb{Z}$-modules over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (resp. $\bar{\mathcal{M}}$), endowed with an action of the Weil group.*

Suppose that, for any m, n, d ($m \geq d$), there exist an integer $N \geq d/\delta B$ and an isomorphism of étale sheaves over $J_m \times U^{n,d}$

$$\dot{\pi}^*_N \mathcal{L} \simeq \bar{pr}^* \mathcal{G}$$

invariant under the action of the Weil group, and also that, as m, n, d vary, these isomorphisms are compatible under the natural trasition maps.

Then, there is a spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-modules, compatible with the actions of Weil group,

$$\bigoplus_{t+s=q} Tor^p_{\mathcal{H}_r(T)}(H^s_c(\bar{\mathcal{M}}, \mathcal{G}), H^t_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z})) \Rightarrow H^{p+q}_c(\bar{X}^{(\alpha)}_{U^p} \times \bar{\mathbb{F}}_p, \mathcal{L}).$$

Proof. — Let us consider the abelian torsion étale sheaf $\mathcal{F}$ over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ associated to the sheaf $\mathcal{L}$ (see section 5.2). We also write $f : \bar{X}^{(\alpha)}_{U^p} \times \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$, $g_m : J_{m,U^p} \to \bar{\mathbb{F}}_p$ and $h^{n,d} : \bar{U}^{n,d} \to \bar{\mathbb{F}}_p$ for the structure morphisms. Then, as we remarked in the proof of theorem 5.11, it follows from the definition of $\mathcal{F}$ (and the equality $R^{\cdot}\dot{\pi}_{N\,!} \simeq \dot{\pi}_{N\,!}$) that

$$R^{\cdot} f_!(\mathcal{F}) \simeq \varinjlim_{m,n,d} R^{\cdot}(g_m \times h^{n,d})_!(\dot{\pi}^*_N \mathcal{L}).$$

Since $\dot{\pi}^*_N \mathcal{L} \simeq \bar{pr}^* \mathcal{G}$, one can use Künneth formula for étale cohomology with compact support to obtain

$$\begin{aligned}\varinjlim_{m,n,d} R^{\cdot}(g_m \times h^{n,d})_!(\dot{\pi}^*_N \mathcal{L}) &\simeq \varinjlim_{m,n,d} R^{\cdot} g_{m\,!}(\mathbb{Z}/l^r\mathbb{Z}) \otimes^{L^{\cdot}} R^{\cdot} h^{n,d}_!(\mathcal{G}) \\ &\simeq \varinjlim_m R^{\cdot} g_{m\,!}(\mathbb{Z}/l^r\mathbb{Z}) \otimes^{L^{\cdot}} \varinjlim_{n,d} R^{\cdot} h^{n,d}_!(\mathcal{G}).\end{aligned}$$

Thus, by proposition 5.12, we have

$$\begin{aligned}\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(T)} R^{\cdot} f_!(\mathcal{F}) &\simeq \Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(T)} (\varinjlim_{n,d} R^{\cdot} h^{n,d}_!(\mathcal{G}) \otimes^{L^{\cdot}} \varinjlim_m R^{\cdot} g_{m\,!}(\mathbb{Z}/l^r\mathbb{Z})) \\ &\simeq (\varinjlim_{n,d} R^{\cdot} h^{n,d}_!(\mathcal{G}))^{op} \otimes^{L^{\cdot}}_{\mathcal{H}_r(T)} \varinjlim_m R^{\cdot} g_{m\,!}(\mathbb{Z}/l^r\mathbb{Z}).\end{aligned}$$

Finally, by theorem 5.2 be conclude that there is a quasi-isomorphism

$$R^{\cdot} f_!(\mathcal{L}) \simeq R^{\cdot} f_!(\Lambda \otimes^{L^{\cdot}}_{\mathcal{H}_r(T)} \mathcal{F}) \simeq$$
$$\simeq (\varinjlim_{n,d} R^{\cdot} h_!^{n,d}(\mathcal{G}))^{op} \otimes^{L^{\cdot}}_{\mathcal{H}_r(T)} (\varinjlim_{m} R^{\cdot} g_{m\,!}(\mathbb{Z}/l^r\mathbb{Z})),$$

or equivalently that there exists a spectral sequence

$$\bigoplus_{t+s=q} Tor^p_{\mathcal{H}_r(T)}(H^s_c(\bar{\mathcal{M}}, \mathcal{G}), H^t_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z})) \Rightarrow H^{p+q}_c(\bar{X}^{(\alpha)}_{U^p} \times \bar{\mathbb{F}}_p, \mathcal{L}).$$

The compatiblity of the above spectral sequence with the actions of the Weil group follows from the fact that all the above quasi-isomorphisms commute with the action of the Weil group. □

The following description of the cohomology groups of the open Newton polygon strata is a special case of the theorem above.

Theorem 5.14. — *Let U^p be a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$, and $r \geq 1$ an integer.*

Then, there is a spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-modules, endowed with an unramified action of Weil group,

$$\bigoplus_{t+s=q} Tor^p_{\mathcal{H}_r(T)}(H^s_c(\bar{\mathcal{M}}, \mathbb{Z}/l^r\mathbb{Z}), H^t_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z})) \Rightarrow H^{p+q}_c(\bar{X}^{(\alpha)}_{U^p} \times \bar{\mathbb{F}}_p, \mathbb{Z}/l^r\mathbb{Z}).$$

Proof. — The corollary follows directly from theorem 5.13, applied to the sheaves $\mathcal{L} = \mathbb{Z}/l^r\mathbb{Z}$ and $\mathcal{G} = \mathbb{Z}/l^r\mathbb{Z}$. □

6. Formally lifting to characteristic zero

In this section we shall investigate the possibility of lifting the constructions of sections 3 and 4 to characteristic zero.

First, we shall lift the varieties over $\operatorname{Spec}\mathbb{F}_p$ (resp. $\operatorname{Spec}\bar{\mathbb{F}}_p$) to formal schemes over $\operatorname{Spf}\mathbb{Z}_p$ (resp. $\operatorname{Spf}\hat{\mathbb{Z}}^{nr}_p$, where $\hat{\mathbb{Z}}^{nr}_p = W(\bar{\mathbb{F}}_p)$). The truncated Rapoport-Zink spaces $\bar{\mathcal{M}}^{n,d}$ are by their very definition the reduced fibers of the formal schemes $\mathcal{M}^{n,d}$ over $\operatorname{Spf}\mathbb{Z}_p$. The open Newton polygon stratum $\bar{X}^{(\alpha)}$ and the central leaf C also have natural lifts to formal schemes over $\operatorname{Spf}\mathbb{Z}_p$, namely the formal completions along them of the Shimura variety $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_{E_u}$. We shall write $\mathfrak{X}^{(\alpha)}$ and $\mathfrak{C}$ for the lifts of $\bar{X}^{(\alpha)}$ and C respectively.

For the Igusa varieties J_m over $\bar{\mathbb{F}}_p$, a natural choice of lifts $\mathcal{J}_m$ over $\operatorname{Spf}\hat{\mathbb{Z}}^{nr}_p$ are their images under the equivalence between the category of finite étale covers of $C \times \bar{\mathbb{F}}_p$ and the category of finite étale covers of $\mathfrak{C} \times \operatorname{Spf}\hat{\mathbb{Z}}^{nr}_p$. Thus, the varieties $\mathcal{J}_m$ are by definition equipped with morphisms $q_m : \mathcal{J}_m \to \mathfrak{C} \times \operatorname{Spf}\hat{\mathbb{Z}}^{nr}_p$ which lift the morphism $\mathrm{q}_m : J_m \to C \times \operatorname{Spec}\bar{\mathbb{F}}_p$. In this section we shall investigate the possibility of extending the morphisms π_N on the formal schemes $\mathcal{J}_m \times_{\operatorname{Spf}\hat{\mathbb{Z}}^{nr}_p} \mathcal{M}^{n,d}$, for all positive integers m, n, d, N ($m \geq d$ and $N \geq d/\delta B$).

Let us remark that for the purpose of all the following constructions, it suffices to assume that $\mathcal{J}_0$ is any formally smooth formal scheme over $\operatorname{Spf}\hat{\mathbb{Z}}_p^{nr}$ which reduces modulo p to $J_0 = C \times \bar{\mathbb{F}}_p$ (not necessarily $\mathfrak{C} \times \hat{\mathbb{Z}}_p^{nr}$), and $\mathcal{J}_m \to \mathcal{J}_0$ are the finite étale covers corresponding to $J_m \to J_0$.

6.1. From $\bar{\mathbb{F}}_p$-schemes to formal $\hat{\mathbb{Z}}_p^{nr}$-schemes. — The goal of this section is to introduce some formal schemes over $\operatorname{Spf}\hat{\mathbb{Z}}_p^{nr}$ whose reduced fibers are naturally identified with the schemes over $\bar{\mathbb{F}}_p$ we studied in sections 3 and 4. We shall maintain the notations established in section 3.2.

6.1.1. By definition, $\bar{X}^{(\alpha)}$ (resp. C) is a locally closed subscheme of the reduction $\bar{X}$ of the Shimura variety $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_{E_u}$ (and under our assumptions $\mathcal{O}_{E_u} = \mathbb{Z}_p$). We define $\mathfrak{X}$ (resp. $\mathfrak{X}^{(\alpha)}$, $\mathfrak{C}$) to be the formal completion of $\mathcal{X}$ along $\bar{X}$ (resp. $\bar{X}^{(\alpha)}$, C). Then $\mathfrak{X}$ (resp. $\mathfrak{X}^{(\alpha)}$, $\mathfrak{C}$) is a formal scheme over $\operatorname{Spf}\mathbb{Z}_p$ with reduced fiber $\bar{X}$ (resp. $\bar{X}^{(\alpha)}$, C). Moreover, there are natural inclusions $\mathfrak{C} \hookrightarrow \mathfrak{X}^{(\alpha)} \hookrightarrow \mathcal{X}$ which lay above $C \hookrightarrow \bar{X}^{(\alpha)} \hookrightarrow \bar{X}$.

We observe that since $\mathcal{X}$ is a smooth variety over $\operatorname{Spec}\mathcal{O}_{E_u}$ the formal schemes $\mathfrak{X}$, $\mathfrak{X}^{(\alpha)}$ and $\mathfrak{C}$ are formally smooth over $\operatorname{Spf}\mathbb{Z}_p$.

6.1.2. By a result of Grothendieck (see [**12**], Exp. I, 8.4), there is an equivalence between the category of finite étale covers of $C \times \bar{\mathbb{F}}_p$ and the category of the finite étale covers of $\mathfrak{C} \times \hat{\mathbb{Z}}_p^{nr}$. For any $m \geq 0$, we define the formal scheme $\mathcal{J}_m$ over $\mathfrak{C} \times \hat{\mathbb{Z}}_p^{nr}$ to be the image of $J_m/C \times \bar{\mathbb{F}}_p$ under the equivalence of categories. Then $\mathcal{J}_m$ is characterised by the following properties:

1. $\mathcal{J}_m$ is finite étale and Galois over $\mathfrak{C} \times \hat{\mathbb{Z}}_p^{nr}$ with Galois group Γ_m;
2. the reduce fiber of $\mathcal{J}_m$ is J_m and $(q_m)^{\mathrm{red}} = \mathrm{q}_m$, where $q_m : \mathcal{J}_m \to \mathfrak{C} \times \hat{\mathbb{Z}}_p^{nr}$ is the structure morphism.

It also follows from the above equivalence of categories that there exist unique morphisms

$$q_{m',m} : \mathcal{J}_{m'} \to \mathcal{J}_m$$

such that $(q_{m',m})^{\mathrm{red}} = \mathrm{q}_{m',m}$ and $q_{m'} = q_m \circ q_{m',m}$, for all $m' \geq m$.

Moreover, by Artin's Approximation Theorem, the formal schemes $\mathcal{J}_m$ have the following universal property (for all m).

Remark 6.1. — For any formal $\hat{\mathbb{Z}}_p^{nr}$-scheme S and any two morphisms $f : S \to \mathfrak{C}$ and $\bar{f}_m : S^{\mathrm{red}} \to J_m$ such that $\mathrm{q}_m \circ \bar{f}_m = f^{\mathrm{red}}$ there exists a unique morphism $f_m : S \to \mathcal{J}_m$ such that $q_m \circ f_m = f$ and $(f_m)^{\mathrm{red}} = \bar{f}_m$.

6.1.3. In the next sections, we shall use extensively some results of Grothendieck in the theory of deformations of Barsotti-Tate groups. For conveniency we report them here below.

Theorem 6.2. — *(See* [**17**]*, Theorem 4.4, pp. 171-177, Corollary 4.7, pp. 178-179.) Let $i : S \hookrightarrow S'$ be a nil-immersion of schemes, where S' is affine.*

Suppose G is a trunctated Barsotti-Tate group over S of length n. Then

1. *There exists a truncated Barsotti-Tate group G' over S' of length n such that $i^*(G') = G$.*
2. *If there exists a Barsotti-Tate group H over S such that $G = H(n)$ (where $H(n)$ denotes the n-th truncate of H), then for any deformation G' over S' of G there exists a deformation H' over S' of H such that $G' = H'(n)$.*
3. *For any $r \leq n$, the natural map*
$$Def\,(G, i) \to Def\,(G(r), i),$$
which maps a deformation G'/S' of G to its r-th truncate $G'(r)$, is a surjection.
4. *Let N be an integer ≥ 1. Suppose that the nil-immersion i is defined by an ideal $\mathcal{I}$ such that $\mathcal{I}^2 = (0)$ and $p^N \in \mathcal{I}$. Then the map in part (3) is a bijection for all $n \geq r \geq N$.*
5. *Under the assumtion of parts (2) and (4), the map*
$$Def\,(H, i) \to Def\,(H(r), i)$$
which maps a deformation H'/S' of H to its r-th truncate $H'(r)$, is a bijection for all $r \geq N$.

Let us remark that parts (4) and (5) of the above theorem hold also without assuming that the scheme S' is affine.

6.2. The morphisms $\pi_N(t)$. — We now investigate the problem of lifting the morphism $\pi_N : J_m \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)}$ to a morphism over $\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}$, i.e. to a morphism $\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d} \to \mathfrak{X}^{(\alpha)} \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}$, for any positive integers m, n, d, N ($m \geq d$ and $N \geq d/\delta B$).

We shall show that, for any positive integer t such that $m \geq d + t/2$ and $N \geq (d+t/2)/\delta B$, it is possible to define a morphism $\pi_N(t)$ on the subscheme of $\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$ cut by the t-th power of the maximal ideal of definition $\mathcal{I}$ ($p \in \mathcal{I}$), such that
$$(\pi_N(t))^{\mathrm{red}} \circ (1 \times \widetilde{Fr})^{NB} = \pi_N,$$
where $\widetilde{Fr} = frob^{-1} \circ Fr$ on $\bar{\mathcal{M}}^{n,d}$, and also
$$\pi_N(t)^* \mathcal{H}[p^{[t/2]}] \simeq \mathcal{H}'[p^{[t/2]}],$$
where $\mathcal{H}$ and $\mathcal{H}'$ denote the universal Barsotti-Tate groups over $\mathfrak{X}^{(\alpha)} \times \mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}$ and $\mathcal{M}^{n,d}$, respectively. (For any positive integer t, we denote by $[t/2]$ the minimal integer greater than or equal to $t/2$.) Moreover, the morphisms $\pi_N(t)$ are compatible with the projections $q_{m',m} \times 1$ and with the inclusions $1 \times i^{n,d}_{n',d'}$.

6.2.1. Let t be a positive integer and $\mathcal{Y}$ a formal scheme over $\operatorname{Spf}\hat{\mathbb{Z}}_p^{nr}$. We denote by $\mathcal{Y}(t)$ the closed subscheme of $\mathcal{Y}$ which is defined by the t-th power of its maximal ideal of definition $\mathcal{I}$ ($\mathcal{I} \supset (p)$) and regard $\mathcal{Y}(t)$ as a scheme over $\operatorname{Spec}\hat{\mathbb{Z}}_p^{nr}/(p^t) = \operatorname{Spec}\hat{\mathbb{Z}}_p^{nr}(t)$. For any $t' \geq t$, we denote by $i_{t,t'}$ the natural inclusion $\mathcal{Y}(t) \hookrightarrow \mathcal{Y}(t')$. For any morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ between formal schemes over $\hat{\mathbb{Z}}_p^{nr}$, we denote by $f(t) : \mathcal{Y}_1(t) \to \mathcal{Y}_2(t)$ the restriction of f to $\mathcal{Y}_1(t)$, viewed as a morphism between $\hat{\mathbb{Z}}_p^{nr}(t)$-schemes.

6.2.2. For any slope $\lambda = \lambda_1, \dots, \lambda_k$, we fix a Barsotti-Tate group $\hat{\Sigma}_\lambda$ over $\mathbb{Z}_p$ such that $\hat{\Sigma}_\lambda \times_{\operatorname{Spf}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p \simeq \Sigma_\lambda$. We define

$$\hat{\Sigma}^i = \hat{\Sigma}_{\lambda_i}^{\oplus r_i} \qquad \text{and} \qquad \hat{\Sigma} = \hat{\Sigma}_\alpha = \oplus_i \hat{\Sigma}^i,$$

for all $i = 1, \dots k$. Thus, we have $\hat{\Sigma}^i \times_{\operatorname{Spf}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p \simeq \Sigma^i$ and $\hat{\Sigma} \times_{\operatorname{Spf}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p \simeq \Sigma$.

Proposition 6.3. — *Maintaining the same notations as in section 3.2.3. Let t be a positive integer and set $m_0 = [t/2]$ (i.e. $m_0 = \min\{m \in \mathbb{Z} \,|\, m \geq t/2\}$). For all $i = 1, \dots, k$, there exists a unique deformation $\hat{\mathcal{G}}^i$ of $\mathcal{G}^i$ over $\mathcal{J}_{m_0}(t)$ such that*

- *for all $m \geq m_0$, there is an isomorphism $\hat{j}_{m,i} : \hat{\Sigma}^i[p^m] \to \hat{\mathcal{G}}^i[p^m]$ over $\mathcal{J}_m(t)$ which lifts $j_{m,i}^{\mathrm{univ}}$;*
- *for any $m' \geq m \geq m_0$, the isomorphism $\hat{j}_{m',i} : \hat{\Sigma}^i[p^{m'}] \simeq \hat{\mathcal{G}}^i[p^{m'}]$ restricts on the p^m-torsions to the pullback of $\hat{j}_{m,i}$.*

Proof. — As a direct consequence of part (5) of theorem 6.2 (when $N = m$), we know that for any $m \geq t/2$ there exists a unique deformation $\hat{\mathcal{G}}_m^i$ over $\mathcal{J}_m(t)$ of $\mathcal{G}^i$ such that $\hat{\mathcal{G}}_m^i[p^m]$ is the deformation of $\mathcal{G}^i[p^m]$ defined as $(\hat{\Sigma}^i[p^m], (j_{m,i}^{\mathrm{univ}})^{-1})$.

It also follows from the uniqueness of construction that, for any $m \geq m_0$, $\hat{\mathcal{G}}_m^i$ over $\mathcal{J}_m(t)$ can be identified to the pullback of the Barsotti-Tate group $\hat{\mathcal{G}}^i = \hat{\mathcal{G}}_{m_0}^i / \mathcal{J}_{m_0}(t)$. Moreover, under these identifications, we obtain a compatible system of isomorphisms

$$\hat{j}_{m,i} : \hat{\Sigma}^i[p^m] \to \hat{\mathcal{G}}^i[p^m]$$

defined over $\mathcal{J}_m(t)$, for all $m \geq t/2$, which has the stated properties. □

6.2.3. We remark that the Barsotti-Tate group $\hat{\mathcal{G}}^i$ may be also interpreted as a deformation of the group $\mathcal{G}^{i\,(p^{rB})}$ via the isomorphism

$$(p^{-\lambda_i B} F^B)^{-r} : \mathcal{G}^{i\,(p^{rB})} \to \mathcal{G}^i.$$

We write $\hat{\mathcal{G}}^{i\,(p^{rB})} = \hat{\mathcal{G}}^i$ when viewed as a deformation of the Barsotti-Tate group $\mathcal{G}^{i\,(p^{rB})}$ (for each $i = 1, \dots k$).

Corollary 6.4. — *Maintaining the notations of proposition 6.3. Let t be a positive integer and set $m_0 = [t/2]$.*

For all positive integers r such that $r\delta B \geq t/2$, there exists a unique deformation $\hat{\mathcal{G}}^{(p^{rB})}$ of $\mathcal{G}^{(p^{rB})}$ over $\mathcal{J}_{m_0}(t)$ such that

$$\hat{\mathcal{G}}^{(p^{rB})}[p^{r\delta B}] \simeq \prod_{i=1}^{k} \hat{\mathcal{G}}^{i\,(p^{rB})}[p^{r\delta B}].$$

Proof. — In lemma 4.1 we proved the existence of a canonical isomorphism

$$\mathcal{G}^{(p^{rB})}[p^{r\delta B}] \simeq \prod_{i=1}^{k} \mathcal{G}^{i\,(p^{rB})}[p^{r\delta B}],$$

over the central leaf $C \times \bar{\mathbb{F}}_p$ (and therefore also over J_{m_0}).

Thus, the finite flat group scheme $\prod_i \hat{\mathcal{G}}^{i\,(p^{rB})}[p^{r\delta B}]$ over $\mathcal{J}_{m_0}(t)$ can be viewed as a deformation of $\mathcal{G}^{(p^{rB})}[p^{r\delta B}] \subset \mathcal{G}^{(p^{rB})}$.

It follows from part (5) of theorem 6.2 that, if $r\delta B \geq t/2$, then the above deformation of $\mathcal{G}^{(p^{rB})}[p^{r\delta B}]$ determines a unique deformation $\hat{\mathcal{G}}^{(p^{rB})}$ of the Barsotti-Tate group $\mathcal{G}^{(p^{rB})}$ over $\mathcal{J}_{m_0}(t)$. □

6.2.4. We remark that the previus corollary can be reformulated as follows.

Corollary 6.5. — *Let m, t be two positive integers and assume $m \geq t/2$. To any choice of a Barsotti-Tate group $\hat{\Sigma}$ as in 6.2.2, we can associate some liftings of powers of the Frobenius morphism on the Igusa variety J_m over $\bar{\mathbb{F}}_p$, i.e. some σ^{NB}-semilinear morphisms*

$$Frob^{NB} : \mathcal{J}_m(t) \to \mathcal{J}_{m-NB}(t),$$

for all integers $N \geq t/2\delta B$, which reduces to the morphisms $Frob^{NB}$ on J_m over $\bar{\mathbb{F}}_p$.

Proof. — Let us recall that the σ-semilinear morphism $Frob : J_m \to J_{m-1}$ is defined as the map associated to the linear morphism $Frob : J_m \to J_{m-1}^{(p)}$ which maps $(A, j_{m,i})$ to $(A^{(p)}, j_{m-1,i}^{(p)})$.

Let us denote by $\mathcal{J}_m^{(p)}$ the pullback of $\mathcal{J}_m$ under the Frobenius on $\hat{\mathbb{Z}}_p^{nr}$. Then, defining a σ^{NB}-semilinear morphism $Frob^{NB} : \mathcal{J}_m(t) \to \mathcal{J}_{m-NB}(t)$, which reduces to $Frob^{NB}$ over J_m, is equivalent to defining a linear morphism

$$\mathcal{J}_m(t) \to \mathcal{J}_{m-NB}^{(p^{NB})}(t),$$

which reduces to the morphism $Frob^{NB} : J_m \to J_{m-NB}^{(p^{NB})}$ (we remark that $\mathcal{J}_m^{(p)}$ reduces to the scheme $J_m^{(p)}$ over $\bar{\mathbb{F}}_p$).

By the universal property of the formal Igusa varieties (see remark 6.1), defining a morphism $\mathcal{J}_m \to \mathcal{J}_{m-N}^{(p^{NB})}$, which reduces to the NB-th power of the Frobenius morphism on the Igusa varieties over $\bar{\mathbb{F}}_p$, is equivalent to defining a deformation of the Barsotti-Tate group $\mathcal{G}^{(p^{NB})}/J_m$ over $\mathcal{J}_m$.

We define the morphism $Frob^{NB}$ on $\mathcal{J}_m(t)$ to be the lifting of the morphism $Frob^{NB}$ on J_m associated to the deformation $\hat{\mathcal{G}}^{(p^{NB})}$ of the Barsotti-Tate group $\mathcal{G}^{(p^{NB})}$

defined in corollary 6.4 (which depends on a choice of the Barsotti-Tate group $\hat{\Sigma}$ as in 6.2.2). $\square$

6.2.5. Let us focus our attention on the morphism $\widetilde{Fr} = frob^{-1} \circ Fr$ on $\bar{\mathcal{M}}$. We remark that, although $\widetilde{Fr}$ does not commutes with the action of T on $\bar{\mathcal{M}}$, its B-th power does (see section 2.5.13).

Proposition 6.6. — *Let t be a positive integer. For any integers m, n, d such that $m \geq d + t/2$, there exist some morphisms*

$$\pi_N(t) : (\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to (\mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr})(t),$$

for all $N \geq (d + t/2)/\delta B$, with the following properties:

- $\pi_N(1) \circ (1 \times \widetilde{Fr})^{NB} = \pi_N$,
- $\pi_N(1) \circ (\rho \times \rho) = \pi_N(1)$, *for all* $\rho \in S$,
- $\pi_N(t)^* \mathcal{H}[p^{[t/2]}] \simeq \mathcal{H}'[p^{[t/2]}]$,
- $\pi_N(t)(t-1) = \pi_N(t-1)$,
- $\pi_N(t) \circ (q_{m',m} \times 1)(t) = \pi_N(t)$, *for all* $m' \geq m$.
- $\pi_N(t) \circ (1 \times i_{n,d}^{n',d'})(t) = \pi_N(t)$, *for all* $d - d' \geq n - n' \geq 0$.

Proof. — Let us start by constructing some morphisms

$$\pi_N(1) : J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$$

such that $\pi_N(1) \circ (1 \times \widetilde{Fr})^{NB} = \pi_N$, for any set of positive integers m, n, d, N with $m \geq d + 1/2$ (i.e. $m \geq d + 1$) and $N \geq (d+1)/\delta B$.

We consider the following commutative diagram where we use the notations of section 4.2 and also write $\mathcal{K} = j_N(\nu^N \ker(p^n \beta))$ and $\nu = \oplus_i p^{-\lambda_i B} F^B$, the B-th power of the natural identification between Σ and $\Sigma^{(p)}$ over $\mathbb{F}_p$.

$$\begin{tikzcd}
\nu^N(\ker(p^n\beta)) \arrow[r] \arrow[d, hook] & \mathcal{K} \arrow[d, hook] & \\
\Sigma^{(p^{NB})}[p^d] \arrow[r, "j_N"] \arrow[d, hook] & \mathcal{G}^{(p^{NB})}[p^d] \arrow[d, hook] & \\
\Sigma^{(p^{NB})} & \mathcal{G}^{(p^{NB})} \arrow[r, hook] \arrow[dd, two heads] & \mathcal{B}^{(p^{NB})} \arrow[dd, two heads] \\
\Sigma \arrow[u, "\nu^N"] \arrow[d, two heads, "p^n\beta"] & & \\
\mathcal{H}' & \bar{\mathcal{H}} = \frac{\mathcal{G}^{(p^{NB})}}{\mathcal{K}} \arrow[r, hook] & \bar{\mathcal{A}} = \frac{\mathcal{B}^{(p^{NB})}}{\langle \mathcal{K} \rangle}
\end{tikzcd}$$

We define

$$\pi_N(1) : J_m \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} \bar{\mathcal{M}}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$$

to be the morphism associated to the abelian variety $\bar{\mathcal{A}}$ endowed with the structures induced from the ones of $\mathcal{B}$. It follows from the definition that

$$\pi_N(1) \circ (1 \times \widetilde{Fr})^{NB} = \pi_N,$$

and also that the morphisms $\pi_N(1)$ are compatible with the projections $\mathrm{q}_{m',m} \times 1$ and the inclusions $1 \times i_{n,d}^{n',d'}$.

Finally, we remark that the isomorphism

$$j_N : \Sigma^{(p^{NB})}[p^{d+1}] \to \mathcal{G}^{(p^{NB})}[p^{d+1}]$$

induces an isomorphism on the quotients $\mathcal{H}'[p] \simeq \bar{\mathcal{H}}[p] = \pi_N(1)[M]^*\mathcal{H}[p]$.

We claim that the morphisms $\pi_N(1)$ are invariant under the action of $S \subset T$. In fact, since the morphism $\widetilde{Fr}$ commutes with the action of T on $\bar{\mathcal{M}}$ and π_N is invariant under the action of S, the equality $\pi_N = \pi_N(1) \circ (1 \times \widetilde{Fr})^{NB}$ implies that for all $\rho \in S$

$$\pi_N(1) \circ (1 \times \widetilde{Fr})^{NB} = \pi_N(1) \circ (\rho \times \rho) \circ (1 \times \widetilde{Fr})^{NB}.$$

Since all the schemes we are considering are reduced, we deduce that $\pi_N(1) = \pi_N(1) \circ (\rho \times \rho)$.

We now construct the morphisms

$$\pi_N(t) : (\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to (\mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr})(t)$$

as extentions of the morphisms $\pi_N(1)$, when $m \geq d + t/2$ and $N \geq (d + t/2)/\delta B$.

By the universal property of $\mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}$, defining a morphism $\pi_N(t)$ is equivalent to defining a deformation over $(\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$ of the Barsotti-Tate group $\bar{\mathcal{H}}$, or also (by part (5) of theorem 6.2) to defining a deformation over $(\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$ of the truncated Barsotti-Tate group $\bar{\mathcal{H}}[p^{[t/2]}]$.

For all $i = 1, \dots k$, the isomorphisms $\hat{j}_{m,i} : \hat{\Sigma}^i[p^m] \to \hat{\mathcal{G}}^i[p^m]$ over $\mathcal{J}_m(t)$ give rise to an isomorphism

$$\hat{j}_N : \hat{\Sigma}[p^{d+[t/2]}] \to \hat{\mathcal{G}}^{(p^{NB})}[p^{d+[t/2]}] \simeq \prod_{i=1}^{k} \hat{\mathcal{G}}^{i\,(p^{NB})}[p^{d+[t/2]}],$$

which induces an isomorphism on the quotients

$$\hat{j}'_N : \mathcal{H}'[p^{[t/2]}] \simeq \bar{\mathcal{H}}[p^{[t/2]}] = \pi_N(1)^*\mathcal{H}[p^{[t/2]}].$$

We define $\pi_N(t)$ to be the morphisism associated to the deformation $\hat{\mathcal{H}}[p^{[t/2]}]$ of the truncated Barsotti-Tate group $\bar{\mathcal{H}}[p^{[t/2]}]$ defined as $(\mathcal{H}'[p^{[t/2]}], \hat{j}'^{-1}_N)$.

It is therefore tautological that $\mathcal{H}'[p^{[t/2]}] \simeq \hat{\mathcal{H}}[p^{[t/2]}] = \pi_N(t)^*\mathcal{H}[p^{[t/2]}]$, and moreover it is a direct consequence of the definition that the morphisms $\pi_N(t)$ commute with the projections $(q_{m',m} \times 1)(t)$ and the inclusions $(1 \times i_{n,d}^{n',d'})(t)$, and that $\pi_N(t)(t-1) = \pi_N(t-1)$. $\square$

Proposition 6.7. — *Let t, m, n, d, N be some positive integer such that $m \geq d + t/2$ and $N \geq (d + t/2)/\delta B$.*

The morphism

$$id \times \pi_N(t) : (\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to (\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr})(t)$$

is étale.

Proof. — Let $y = (y_1, y_2)$ be a geometric point of $J_m \times \bar{\mathcal{M}}^{n,d}$ and $x = \pi_N(t)(y) \in \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$.

We need to prove the morphism

$$(id \otimes \pi_N(t))^* : \mathcal{O}^{\wedge}_{\mathcal{J}_m, y_1} \hat{\otimes}_{\hat{\mathbb{Z}}_p^{nr}} \mathcal{O}^{\wedge}_{\mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}, x} / \mathfrak{a}^t \to \mathcal{O}_{\mathcal{J}_m, y_1} \hat{\otimes}_{\hat{\mathbb{Z}}_p^{nr}} \mathcal{O}^{\wedge}_{\mathcal{M}, y_2} / \mathcal{I}^t,$$

where $\mathfrak{a}$ and $\mathcal{I}$ denote the maximal ideal of definitions of the respective algebras, is an isomorphism.

Let us denote by B (resp. A) the abelian variety associated to the point y_1 (resp. x), and by G (resp. H) the corresponding Barsotti-Tate group $\epsilon B[u^\infty]$ (resp. $\epsilon A[u^\infty]$). We also write $j_{m,i} : \Sigma^i[p^m] \to G^i[p^m]$ for the isomorphisms associated to y_1 ($i = 1, \ldots k$) and $\beta : \Sigma \to H'$ for the quasi-isogeny associated to y_2.

We recall that the complete local rings $\mathcal{O}^{\wedge}_{\mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}, x}$ and $\mathcal{O}^{\wedge}_{\mathcal{M}, y_2}$ are by definition the deformation rings of the Barsotti-Tate groups H and H', respectively. We denote by $\mathcal{H}$ and $\mathcal{H}'$ the corresponding universal objects.

We now choose an isomorphism $j : \Sigma \to G^{(p^{NB})}$ which extends the isomorphism $\oplus_i j_{m,i}$ between the p^m-torsion subgroups. Then j induces an isomorphism j' between H and H', i.e.

$$\begin{tikzcd}
0 \arrow[r] & \ker(p^n\beta) \arrow[r] \arrow[d, "\oplus_i j_{m,i}"] & \Sigma \arrow[r, "p^n\beta"] \arrow[d, "j"] & H' \arrow[r] \arrow[d, "j'"] & 0 \\
0 \arrow[r] & K_{j,\beta} \arrow[r] & G^{(p^{NB})} \arrow[r, "\bar{s}"] & H \arrow[r] & 0.
\end{tikzcd}$$

By the very definition of the morphism $\pi_N(t)$, over $\mathcal{O}_{\mathcal{J}_m, y_1} \hat{\otimes}_{\hat{\mathbb{Z}}_p^{nr}} \mathcal{O}^{\wedge}_{\mathcal{M}, y_2} / \mathcal{I}^t$ there exists an isomorphism $\pi_N(t)^* \mathcal{H}[p^{[t/2]}] \simeq \mathcal{H}'[p^{[t/2]}]$, which reduces modulo $\mathcal{I}$ to $j'_{|[p^{[t/2]}]}$, and moreover (by part (5) of theorem 6.2) such an isomorphism extends to an isomorphism $\pi_N(t)^* \mathcal{H} \simeq \mathcal{H}'$, which reduces modulo $\mathcal{I}$ to j'. This fact is equivalent to saying that the morphism $(id \otimes \pi_N(t))^*$ is an isomorphism. □

6.3. The morphisms $\pi_N[t, V]$. — In this section, we shall discuss the possibility of extending the morphisms $\pi_N(t)$ on the formal schemes $\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$.

We shall prove that the morphism $\pi_N(t)$ may be extended Zariski locally to a morphism over $\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}$, i.e. for any open affine $V \subset \mathcal{J}_m \times U^{n,d}$ the morphism $\pi_N(t)_{|V(t)}$ lifts to a morphism on V.

Proposition 6.8. — *Let m, n, d, N, t be some positive integers such that $m \geq d+t/2$ and $N \geq (d+t/2)/\delta B$.*

For any affine open V of $\mathcal{J}_m \times U^{n,d} \subset \mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$, there exists a morphism

$$\pi_N[t,V] : V \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}$$

such that $\pi_N[t,V](t) = \pi_N(t)_{|V(t)}$ and also $\pi_N[t,V]^*\mathcal{H}[p^{[t/2]}] \simeq \mathcal{H}'[p^{[t/2]}]$.

Proof. — Let us recall that over $(\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$ there exists an isomorphism

$$\pi_N(t)^*\mathcal{H}[p^{[t/2]}] \simeq \mathcal{H}'[p^{[t/2]}].$$

Under such an identification, the finite flat group scheme $\mathcal{H}'[p^{[t/2]}]$ over $\mathcal{J}_m \times \mathcal{M}^{n,d}$ gives rise to a deformation of the group $\hat{\mathcal{H}}[p^{[t/2]}]/(\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$.

Moreover, it follows from part (2) of theorem 6.2 that over any open affine V of $\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$, there exists a deformation $\tilde{\mathcal{H}}/V$ of the Barsotti-Tate group $\hat{\mathcal{H}}/V(t)$ such that $\tilde{\mathcal{H}}[p^{[t/2]}] \simeq \mathcal{H}'[p^{[t/2]}]$.

We define $\pi_N[t,V]$ on V to be the lifting of the morphism $\pi_N(t)_{|V(t)}$ associated to the Barsotti-Tate group $\tilde{\mathcal{H}}/V$. □

Proposition 6.9. — *Maintaining the notations as above, we assume $t > 1$. Then, the morphism $\pi_N[t,V] : V \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}$ is formally smooth.*

Proof. — Let $y = (y_1, y_2)$ be a geometric point of V and $x = \pi_N[t,V](y) = \pi_N(t)(y) \in \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$.

In order to conclude, it suffices to prove the morphism

$$id\hat{\otimes}\pi_N[t,V]^* : \mathcal{O}^{\wedge}_{\mathcal{J}_m,y_1}\hat{\otimes}_{\hat{\mathbb{Z}}_p^{nr}}\mathcal{O}^{\wedge}_{\mathfrak{X}^{(\alpha)}\times\hat{\mathbb{Z}}_p^{nr},x} \to \mathcal{O}^{\wedge}_{V,y} = \mathcal{O}_{\mathcal{J}_m,y_1}\hat{\otimes}_{\hat{\mathbb{Z}}_p^{nr}}\mathcal{O}^{\wedge}_{\mathcal{M},y_2}$$

is an isomorphism. In fact, we may then deduce that the morphism $\pi_N[t,V]$ is smooth at the point y from the smoothness of the formal scheme $\mathcal{J}_m/\hat{\mathbb{Z}}_p^{nr}$ (see section 6.1.2).

From the equality $\pi_N[t,V](t) = \pi_N(t)_{|V(t)}$, we deduce that

$$(id\hat{\otimes}\pi_N[t,V])^*(t) = (id\otimes\pi_N(t))^*$$

and therefore, in particular, is an isomorphism (see proposition 6.7). For $t > 1$, this suffices to deduce that $(id\hat{\otimes}\pi_N[t,V])^*$ is an isomorphism, since the complete local rings $\mathcal{O}^{\wedge}_{\mathcal{J}_m,y_1}\hat{\otimes}_{\hat{\mathbb{Z}}_p^{nr}}\mathcal{O}^{\wedge}_{\mathfrak{X}^{(\alpha)}\times\hat{\mathbb{Z}}_p^{nr},x}$ and $\mathcal{O}^{\wedge}_{V,y}$ are both power series rings over an algebraically closed field, of the same dimension. (Indeed, it is a general fact that, if A, B are two power series ring over an algebraically closed field k, of the same dimension, and $\phi : A \to B$ a morphism of k-algebras, such that the morphism induced modulo the squares of the maximal ideals of definitions of A and B, $A/\mathfrak{a}^2 \to B/\mathfrak{b}^2$, is an isomorphisms, then ϕ is also an isomorphism.) □

6.4. The morphisms $\hat{y}_N$. — In this section, for any integers $n, d \geq 0$ and $N \geq 1$, we shall associate to a point $y \in J(\bar{\mathbb{F}}_p)$ a compatible system of points $y_m^\wedge \in \mathcal{J}_m(\hat{\mathbb{Z}}_p^{nr})$ and some morphisms $\hat{y}_N : U^{n,d} \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}$, which canonically lift the morphisms $\pi_N(1) \circ (y_m, id)$ and such that $\hat{y}_N(t) = \pi_N(t) \circ (y_m^\wedge, id)$ over $U^{n,d}(t)$, for all $t > 0$. We maintain the notations introduced in section 4.3.

6.4.1. Let $y \in J(\bar{\mathbb{F}}_p)$ be a point associated to a quintuple $(B, \lambda, i, \bar{\mu}; j)$, and write $G = \epsilon B[u^\infty]$. Thus $j : \Sigma_{\bar{\mathbb{F}}_p} \to G$ is an isomorphism.

To the point $y \in J(\bar{\mathbb{F}}_p)$, we associated a compatible system of morphisms

$$y_m^\wedge : \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr} \to \mathcal{J}_m,$$

for all $m \geq 0$. Each $y_m^\wedge$ is defined by the data of the point $y_m = \mathrm{q}_{\infty,m}(y) \in J_m(\bar{\mathbb{F}}_p)$ and the deformation $\hat{G} = (\hat{\Sigma}, j)/\hat{\mathbb{Z}}_p^{nr}$ of the Barsotti-Tate group $G/\bar{\mathbb{F}}_p$.

For any integers $n, d \geq 0$, we define the morphisms

$$\hat{y}_N : U^{n,d} \to \mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}$$

for all $N \geq 1$, as follows. For any $m \geq d+1$, we consider the morphisms

$$\hat{y}_N(1) = \pi_N(1) \circ (y_m, id) : \bar{U}^{n,d} \to J_m \times_{\bar{\mathbb{F}}_p} \bar{U}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p.$$

We remark that the morphisms $\hat{y}_N(1)$ do not depend on the choice of the integer $m \geq d+1$. If $(\mathcal{H}', \beta)$ denotes the universal object over $U^{n,d}$ and $\mathcal{H}$ the universal Barsotti-Tate group over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, then it follows from the definitions that the isomorphism $j : \Sigma_{\bar{\mathbb{F}}_p} \to G$ give rise to an isomorphism

$$\bar{j} : \mathcal{H}' \times \bar{U}^{n,d} \simeq H = (y_m, id)^* \pi_N(1)^* \mathcal{H}.$$

Thus, to extend the morphism $\pi_N(1) \circ (y_m, id)$ to a morphism from $U^{n,d}$ to $\mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}$, it suffices to define a deformation $\hat{H}$ over $U^{n,d}$ of the Barsotti-Tate group $H/\bar{U}^{n,d}$. We set $\hat{H} = (\mathcal{H}', \bar{j})$ and denote the corresponding morphism by $\hat{y}_N$.

The following properties of the morphisms $\hat{y}_N$ are direct consequences of the definition.

Proposition 6.10. — *Maintaining the above notations. Let n, d be two positive intergers and $y \in J(\bar{\mathbb{F}}_p)$.*

Then, for all $N \geq 1$, the morphisms $\hat{y}_N : U^{n,d} \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}$ satisfy the conditions

1. *over $U^{n,d}$ we have $\hat{y}_N^* \mathcal{H} \simeq \mathcal{H}'$;*
2. *for any integers $t \geq 1$, $m \geq d + t/2$ and $N \geq (d + t/2)/\delta B$, over $U^{n,d}(t)$ we have $\hat{y}_N(t) = \pi_N(t) \circ (y_m^\wedge, id)(t)$;*
3. *for any $\rho \in T$, over $U^{n,d}$ we have $\hat{y}_N \circ \rho = \hat{\rho} y_N$.*

Finally, let us remark that the same argument we used in the proof of proposition 6.7 shows the following fact.

Proposition 6.11. — *Let n, d be two positive intergers and $y \in J(\bar{\mathbb{F}}_p)$. Then, for all $N \geq 1$, the morphisms $\hat{y}_N : U^{n,d} \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}$ are étale.*

7. Shimura varieties with level structure at p

In this section we shall focus our attention to the study of Shimura varieties with level structure at p. Our goal is to compare the rigid analytic spaces associated to the Shimura varieties with level structure at p to the the Rapoport-Zink spaces with level structure.

In order to do it, we shall first define some integral models for the Shimura varieties and the Rapoport-Zink spaces, as formal schemes over $\mathfrak{X}$ and $\mathcal{M}$, respectively. These integral models naturally form a system, as the level varies, and they are endowed with an action of a certain submonoid $GL_h(\mathbb{Q}_p)^+ \subset GL_h(\mathbb{Q}_p)$ (such that $\langle GL_h(\mathbb{Q}_p)^+, p\mathbb{I}_h\rangle = GL_h(\mathbb{Q}_p)$ and $p^{-1}\mathbb{I}_h \in GL_h(\mathbb{Q}_p)^+$), which is compatible with the action of the group $GL_h(\mathbb{Q}_p)$ on the corresponding rigid analytic spaces.

For any integer $m, n, d, t > 0$, we shall consider the morphisms

$$\pi_N(t) : (\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to (\mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr})(t)$$

(for all N) and the projections $pr(t) : (\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to \mathcal{M}(t)$, and compare the two towers of covering spaces over $(\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$ which are obtained as the pullbacks of the Shimura varieties (via $\pi_N(t)$) and of the Rapoport-Zink spaces (via $pr(t)$), with level structure at p. In particular, we shall prove that, for all level $M \leq t/2$, the corresponding two spaces over $(\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$ are indeed isomorphic and that these isomorphisms are compatible with the action of $GL_h(\mathbb{Q}_p)^+$ on the two sides.

Moreover, for any open affine V of $\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$ and any level $M \leq t/2$, we shall consider the morphisms $\pi_N[t, V] : V \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr}$ and $pr_V : V \to \mathcal{M}$, which reduce over $\hat{\mathbb{Z}}_p^{nr}(t)$ to the restrictions of $\pi_N(t)$ and $pr(t)$ on $V(t)$, respectively. We shall prove that the two covering spaces of V^{rig}, which are obtained as the pullbacks of the Shimura variety with structure of level M at p and of the Rapoport-Zink space of the same level, are also isomorphic and that, as in the previous case, these isomorphisms are compatible with the action of $GL_h(\mathbb{Q}_p)^+$.

As a consequence of these two facts, we shall deduce that the pullbacks of the vanishing cycles of the Shimura varieties with level structure at p are isomorphic to the pullbacks of the vanishing cycles of the corresponding Rapoport-Zink spaces, and that such isomorphisms are compatible with the group actions.

7.1. Integral models for Shimura varieties with level structure at p. — In [**22**] Katz and Mazur develop Drinfeld's notion of full level structure for elliptic modules into the notion of full set of sections and A-generators for finite flat group schemes, where A is a finite abstract group.

In this section, we shall use their work to define some integral models for the Shimura varieties with level structure at p.

7.1.1. We first introduce some notations. Let $A = (\mathbb{Q}_p/\mathbb{Z}_p)^h$ be the abstract p-divisible group of height h and denote by $A[p^M]$ its p^M-torsion subgroup (then $A[p^M] \simeq \left(\mathbb{Z}/p^M\mathbb{Z}\right)^h$).

Let us consider the group of the quasi-isogenies of A, $GL_h(\mathbb{Q}_p)$, and define

$$GL_h(\mathbb{Q}_p)^+ = \{g \in GL_h(\mathbb{Q}_p) \,|\, g^{-1} \in M_h(\mathbb{Z}_p)\}.$$

Then $GL_h(\mathbb{Q}_p)^+$ is a submonoid of $GL_h(\mathbb{Q}_p)$ such that $\langle GL_h(\mathbb{Q}_p)^+, p\mathbb{I}_h\rangle = GL_h(\mathbb{Q}_p)$ and $p^{-1}\mathbb{I}_h \in GL_h(\mathbb{Q}_p)^+$.

For any $g \in GL_h(\mathbb{Q}_p)^+$, we denote by $A[g^{-1}]$ the kernel of the isogeny $g^{-1} : A \twoheadrightarrow A$, by $e = e(g)$ the minimal integer such that $A[g^{-1}] \subset A[p^e]$ and write $d(g) = \log_p(\#A[g^{-1}])$ (thus $d(g) \leq e(g)h$). The morphism g^{-1} induces an inclusion of groups

$$A[p^{M-e}] \hookrightarrow A[p^M]/A[g^{-1}].$$

If $g \in GL_h(\mathbb{Z}_p)$, then $A[g^{-1}] = (0)$, $e(g) = d(g) = 0$, and the corresponding inclusion $A[p^M] \hookrightarrow A[p^M]$ is simply the automorphism of $A[p^M]$ induced by restriction from $g^{-1} : A \to A$. In particular, if $g^{-1} \equiv \mathbb{I}_h \mod (p^M)$, this inclusion is just the identity.

7.1.2. For any positive integer M, we define $\mathcal{X}_M$ over $\mathcal{X}$ to be the scheme

$$\mathcal{X}_M = W(A[p^M], \mathcal{H}[p^M]/\mathcal{X}),$$

where $\mathcal{H}/\mathcal{X}$ is the Barsotti-Tate group $\epsilon\mathcal{A}[u^\infty]$ associated to the universal abelian variety $\mathcal{A}$ over $\mathcal{X}$. We recall that the scheme $W(A[p^M], \mathcal{H}[p^M]/\mathcal{X})$ is the universal space for the existence of a set of $A[p^M]$-generators $\{P_1, \dots P_h\}$ of $\mathcal{H}[p^M]/\mathcal{X}$ and is endowed with a natural action of the group of automorphisms of $A[p^M]$, namely $GL_h(\mathbb{Z}/p^M\mathbb{Z})$ (notations as in section 2.6.3).

For any $M \geq 0$, we regard the scheme $\mathcal{X}_M$ as endowed with the action of the group $GL_h(\mathbb{Z}_p)$, via the projection $GL_h(\mathbb{Z}_p) \twoheadrightarrow GL_h(\mathbb{Z}/p^M\mathbb{Z})$.

Proposition 7.1. — *Let M be a positive integer.*

1. *The scheme $\mathcal{X}_M$ is finite over $\mathcal{X}/\operatorname{Spec}\mathcal{O}_{E_u}$, and we have*

$$\mathcal{X}_M \times_{\operatorname{Spec}\mathcal{O}_{E_u}} \operatorname{Spec} E_u = X_M \times_{\operatorname{Spec} E} \operatorname{Spec} E_u.$$

2. *For any $M' \geq M$, there is a natural morphism $\phi_{M',M} : \mathcal{X}_{M'} \to \mathcal{X}_M$ over $\mathcal{X}$ which is induced by the map $p^{M'-M} : \mathcal{H}[p^{M'}] \twoheadrightarrow \mathcal{H}[p^M]$ (or equivalently by the inclusion $\mathcal{H}[p^M] \hookrightarrow \mathcal{H}[p^{M'}]$), and we have*

$$\phi_{M',M} \times 1_{E_u} = \mathrm{f}_{M',M} \times 1_{E_u},$$

where $\mathrm{f}_{M',M} : X_{M'} \to X_M$ is the natural projection.

3. *For any $g \in GL_h(\mathbb{Z}_p)$ and any $M' \geq M$, we have*

$$g \circ \phi_{M',M} = \phi_{M',M} \circ g.$$

Moreover, the restriction of the action of $GL_h(\mathbb{Z}_p)$ on $\mathcal{X}_M$ to the generic fiber $X_M \times_{\operatorname{Spec} E} \operatorname{Spec} E_u$ coincides with the restriction to $GL_h(\mathbb{Z}_p)$ of the action of $G(\mathbb{Q}_p)$ the Shimura varieties X_M.

Proof. — Part (1): The fact that $\mathcal{X}_M$ is finite over $\mathcal{X}$ follows from the more general fact that $\overline{W(A, Z/S)}$ is finite over S, for any finite abstract group A and finite flat group scheme Z/S (see section 2.6.3).

We observe that the E_u-scheme $\mathcal{X}_M \times_{\mathcal{O}_{E_u}} E_u$ is the universal space for the existence of a set of $\left(\mathbb{Z}/p^M\mathbb{Z}\right)^n$-generators on $\mathcal{H}[p^M]$ over $\mathcal{X} \times_{\mathcal{O}_{E_u}} E_u = X \times_E E_u$, i.e.

$$\mathcal{X}_M \times_{\mathcal{O}_{E_u}} E_u = W(A[p^M], \mathcal{H}[p^M]/X \times_E E_u).$$

Since p is invertible in E_u, the group scheme $\mathcal{H}[p^M]$ is étale over $X \times_E E_u$. Thus the datum of a set of $\left(\mathbb{Z}/p^M\mathbb{Z}\right)^n$-generators of $\mathcal{H}[p^M]$ is equivalent to the datum of an isomorphism

$$\left(\mathbb{Z}/p^M\mathbb{Z}\right)^n_X \to \mathcal{H}[p^M],$$

defined over $\mathcal{X}_M \times_{\mathcal{O}_{E_u}} E_u$ by setting $e_i = (0, \dots, 1, \dots 0) \mapsto P_i$, for $i = 1, \dots, n$ (see section 2.6.2).

We conclude that $\mathcal{X}_M \times_{\mathcal{O}_{E_u}} E_u = X_M \times_E E_u$ over $X \times_E E_u$, since they are defined by equivalent universal properties.

Part(2): If the morphism $\phi_{M',M}$ exists, then it follows directly the definitions that its generic fiber over E_u agrees the natural projection between Shimura varieties. Moreover, by the defining universal properties, proving the existence of the morphism $\phi_{M',M}$ is equivalent to showing that, if $\{P_1, \dots P_n\}$ is the universal set of $\left(\mathbb{Z}/p^{M'}\mathbb{Z}\right)^n$-generators of $\mathcal{H}[p^{M'}]$ over $\mathcal{X}_{M'}$, then $\{p^{M'-M}P_1, \dots, p^{M'-M}P_n\}$ is a set of $\left(\mathbb{Z}/p^M\mathbb{Z}\right)^n$-generators of $\mathcal{H}[p^M]$ over $\mathcal{X}_{M'}$. We postpone the proof of this fact to lemma 7.2.

Part (3): It follows directly from the definitions. □

Lemma 7.2. — *Let M be a positive integer and H a Barsotti-Tate group of height h over a scheme S.*

Suppose that $P_1, \dots, P_h \in H[p^{M+1}](S)$ form a set of $\left(\mathbb{Z}/p^{M+1}\mathbb{Z}\right)^n$-generators of $H[p^{M+1}]$. Then $\{pP_1, \dots, pP_h\}$ is a set of $\left(\mathbb{Z}/p^M\mathbb{Z}\right)^n$-generators of $H[p^M]/S$.

Proof. — For any affine S-scheme $\operatorname{Spec} R$, we write $B' = H^0(H[p^{M+1}]_R, \mathcal{O})$ and $B = H^0(H[p^M]_R, \mathcal{O})$. The morphism $p : H[p^{M+1}] \twoheadrightarrow H[p^M]$ induces a morphism of R-algebras $p^* : B \hookrightarrow B'$ such that B' is a locally free B-module of rank p^h. Thus, for any $g \in B$, we have $\det_{B'/R}(T - p^*(g)) = \left(\det_{B/R}(T - g)\right)^{p^h}$

The points $P_1, \dots, P_h \in H[p^{M+1}](S)$ form a set of $\left(\mathbb{Z}/p^{M+1}\mathbb{Z}\right)^h$-generators of $H[p^{M+1}]$. Thus, for any $f \in B'$, we have

$$\det_{B/R}(T - f) = \prod_{(a_i) \in (\mathbb{Z}/p^{M+1})^h} T - f\left(\sum_{i=1}^h a_i P_i\right).$$

In particular, for any $g \in B$, we have

$$\begin{aligned}\det_{B'/R}(T - p^*(g)) &= \prod_{(a_i)\in(\mathbb{Z}/p^{M+1})^h} T - p^*(g)\left(\sum_{i=1}^h a_i P_i\right)\\ &= \prod_{(a_i)\in(\mathbb{Z}/p^{M+1})^h} T - g\left(p\sum_{i=1}^h a_i P_i\right)\\ &= \left(\prod_{(a_i)\in(\mathbb{Z}/p^{M})^h} T - g\left(\sum_{i=1}^h a_i p P_i\right)\right)^{p^h}.\end{aligned}$$

Then $\left(\det_{B/R}(T-g)\right)^{p^h} = \left(\prod_{(a_i)\in(\mathbb{Z}/p^M)^h} T - g\left(\sum_{i=1}^h a_i p P_i\right)\right)^{p^h}$, which implies

$$\det_{B/R}(T-g) = \prod_{(a_i)\in(\mathbb{Z}/p^M)^h} T - g\left(\sum_{i=1}^h a_i p P_i\right).$$

□

7.1.3. In order to extend the above action of $GL_h(\mathbb{Z}_p)$ on the integral models of the Shimura varieties to an action of $GL_h(\mathbb{Q}_p)^+$, we need to introduce some other integral models over $\mathcal{O}_{E_u}$.

Let $g \in GL_h(\mathbb{Q}_p)^+$, we write $e = e(g)$ and $d = d(g)$. Let M be a positve integer, $M \geq e$. We consider the space $\mathcal{X}_M$ and denote by $a_M : A[p^M] \to \mathcal{H}[p^M](\mathcal{X}_M)$ the universal $A[p^M]$-generator over $\mathcal{X}_M$.

We define $\mathcal{X}_{M,g}/\mathcal{X}_M$ to be the universal space for the existence of a finite flat subgroup $\mathcal{E} \subset \mathcal{H}[p^e]$, of order d, such that

$$a_M(A[g^{-1}]) \subset \mathcal{E}(\mathcal{X}_{M,g}),$$

and the induced morphisms on the subquotients

$$A[p^{M-e}] \to (\mathcal{H}/\mathcal{E})[p^{M-e}](\mathcal{X}_{M,g})$$

is a $A[p^{M-e}]$-generator.

Proposition 7.3. — *Mantaning the above notations.*

1. *The scheme $\phi_{M,g} : \mathcal{X}_{M,g} \to \mathcal{X}_M$ exists and is proper. Moreover,*

$$\mathcal{X}_{M,g} \times_{\operatorname{Spec}\mathcal{O}_{E_u}} \operatorname{Spec} E_u = X_M \times_{\operatorname{Spec} E} \operatorname{Spec} E_u$$

   and $\phi_{M,g} \times 1_{E_u} = 1_{X_M} \times 1_{E_u}$.
2. *For any $M' \geq M$, there is a natural morphism $\phi_{M',M,g} : \mathcal{X}_{M',g} \to \mathcal{X}_{M,g}$ over $\mathcal{X}$, which is induced by the inclusion $\mathcal{H}[p^M] \hookrightarrow \mathcal{H}[p^{M'}]$, and we have*

$$\phi_{M,g} \circ \phi_{M',M,g} = \phi_{M',g} \circ \phi_{M',M}$$

   and thus also $\phi_{M',M,g} \times 1_{E_u} = \mathrm{f}_{M',M} \times 1_{E_u}$.

3. *There is a natural proper morphism*

$$g : \mathcal{X}_{M,g} \to \mathcal{X}_{M-e}$$

such that $g \circ \phi_{M',M,g} = \phi_{M',M} \circ g$, *for any* $M' \geq M$ *and* $g \times 1_{E_u} = g \times 1_{E_u}$.

4. *For any* $\gamma \in GL_h(\mathbb{Z}_p)$, *there is a natural identification*

$$\mathcal{X}_{M,g} \simeq \mathcal{X}_{M,g\gamma}$$

over $\mathcal{X}_M$ *and, under such identification, we have*

$$\phi_{M,g\gamma} = \gamma \circ \phi_{M,g}.$$

5. *For any positive integer* $r \leq M$, *the morphism*

$$\phi_{M,p^{-r}\mathbb{I}_h} : \mathcal{X}_{M,p^{-r}\mathbb{I}_h} \to \mathcal{X}_M$$

is an isomorphism and $f_{M,M-r} = p^{-r}\mathbb{I}_h \circ \phi^{-1}_{M,p^{-r}\mathbb{I}_h}$.

Proof. — Part (1): It follows from the general theory of Hilbert Spaces and from proposition 2.19 that the scheme $\mathcal{X}_{M,g}$ exists and is proper over $\mathcal{X}_M$. Moreover, the remark of section 2.6.2 implies that the generic fiber of $\mathcal{X}_{M,g}$ can be identified with $X_M \times_E E_u$.

Part (2): The statement follows from lemma 7.2, using the same argument of part (2) of proposition 7.1.

Part (3): Let $\mathcal{A}$ be the universal abelian variety over $\mathcal{X}_{M,g}$ and consider the subgroup $\langle \mathcal{E} \rangle \subset \mathcal{A}[p^e]$, associated to $\mathcal{E} \subset \mathcal{H}[p^e]$. We define the morphism

$$g : \mathcal{X}_{M,g} \to \mathcal{X}_{M-e}$$

to be associated to the quintuple $(\mathcal{A}/\langle \mathcal{E} \rangle, \lambda', i', \bar{\mu}'; a'_{M-e})$ where there structures on the abelian varieties $\mathcal{A}/\langle \mathcal{E} \rangle$ are induced by the ones on $\mathcal{A}$ via the isogeny $\mathcal{A} \to \mathcal{A}/\langle \mathcal{E} \rangle$:

- λ' is the polarization induced by $p^e\lambda$;
- i' is the B-action induce by i;
- μ' is the level structure induced by $\mu \circ v^e$, for $v \in E^\times$ such that $\mathrm{val}_u(v) = 0$ and $\mathrm{val}_{u^c}(v) = 1$;
- $a'_{M-e} : A[p^{M-e}] \to (\mathcal{H}/\mathcal{E})[p^{M-e}](\mathcal{X}_{M,g})$ denotes the $A[p^{M-e}]$-generator induced by a_M on the p^{M-e}-torsion subgroup of the Barsotti-Tate group $\mathcal{H}/\mathcal{E} = \epsilon\mathcal{A}/\langle \mathcal{E} \rangle[u^\infty]$.

It follows from the definition that the morphisms g commute with the projections among integral models of the Shimura varieties of different levels and that their restrictions to the generic fibers agree with the previously defined action of $GL_h(\mathbb{Q}_p)^+ \subset GL_h(\mathbb{Q}_p)$ on the Shimura varieties.

It remains to prove that the morphisms $g : \mathcal{X}_{M,g} \to \mathcal{X}_{M-e}$ are proper. By the Valuative Criterium of Properness (see [**15**], Theorem 4.7, p. 101) it suffices to show that:

– if R is a complete discrete valuation ring over $\bar{\mathbb{F}}_p$, K its fraction field and $\eta : \operatorname{Spec} K \to \operatorname{Spec} R$ the morphism corresponding to the natural inclusion of R in K, then for any pair of morphisms (F, f) such that $g \circ F = f \circ \eta$ there exists a map $\phi : \operatorname{Spec} R \to \mathcal{X}_{M,g}$ such that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "F"] \arrow[d, "\eta"] & \mathcal{X}_{M,g} \arrow[d, "g"] \\ \operatorname{Spec} R \arrow[r, "f"'] \arrow[ur, dotted, "\phi"] & \mathcal{X}_{M-e} \end{tikzcd}$$

Let $(A, \lambda, i, \bar{\mu}; E \subset H[p^e], a_M)$ (where $H = \epsilon A[u^\infty]$) be the sixtuple defined over K, associated to the morphism F, and $(\mathcal{B}, \lambda', i', \bar{\mu}'; b'_{M-e})$ the quintuple over R, associated to f. Then, the equality $g \circ F = f \circ \eta$ implies that there is an equaivalence of quintuple

$$(\mathcal{B}, \lambda', i', \bar{\mu}'; b'_{M-e})_K \simeq (A/\langle E \rangle, \lambda', i', \bar{\mu}; a'_{M-e}).$$

We fix an isogeny $\psi : \mathcal{B}_K \to A/\langle E \rangle$, giving rise to the above equivalence (then, ψ induces an isomorphism $\epsilon \mathcal{B}_K \simeq H/E$).

Let us now consider the projection $q : A \twoheadrightarrow A/\langle E \rangle$. Since $E \subset H[p^e]$, it follows that there exists an isogeny $q' : A/\langle E \rangle \twoheadrightarrow A$ such that $q \circ q' = p^e$.

Let us write $F = \psi^{-1}(\ker q') \subset \mathcal{B}_K[p^e]$ and $\mathcal{F} = \bar{F} \subset \mathcal{B}[p^e]$ its Zariski closure. Then, $\mathcal{F}$ is a finite flat subgroup of the abelian variety $\mathcal{B}$ and the quotient $\mathcal{B}/\mathcal{F}$, together with the appropriate induced structures, defines an integral model over R for the quadruple $(A, \lambda, i, \bar{\mu})$. Moreover, the finite flat subgroup $\mathcal{E} = \epsilon(\mathcal{B}[p^e]/\mathcal{F})[u^e]$ restricts over K to the subgroup E. Finally, it follows from proposition 2.19 that it is possible to define an $A[p^M]$-generator b_M of $\epsilon(\mathcal{B}/\mathcal{F})[p^M]$ over R, compatible with a_M.

The morphism ϕ, associated to the abelian variety $\mathcal{B}/\mathcal{F}$ together with the finite flat subgroup $\mathcal{E}$ and the $A[p^M]$-generator b_M, has the required property.

Part (4): Let $\gamma \in GL_h(\mathbb{Z}_p)$. Then, for any $g \in GL_h(\mathbb{Q}_p)^+$, $g\gamma \in GL_h(\mathbb{Q}_p)^+$ and moreover $A[g^{-1}] = A[(g\gamma)^{-1}]$ (thus also $e(g) = e(g\gamma)$ and $d(g) = d(g\gamma)$).

Now, suppose $\mathcal{E}$ is a subgroup of $\mathcal{H}[p^e]$ such that $a_M(A[g^{-1}]) = a_M(A[(g\gamma)^{-1}]) \subset \mathcal{E}$ and denote by

$$a'_{M-e} : A[p^{M-e}] \to (\mathcal{H}/\mathcal{E})[p^{M-e}](\mathcal{X}_{M,g})$$

the morphism of groups induced by a_M via g. Then, $a'_{M-e} \circ \gamma$ is the morphism induced by a_M via $g\gamma$. It follows, in particular, that a'_{M-e} is a $A[p^{M-e}]$-generator of $(\mathcal{H}/\mathcal{E})[p^{M-e}]$ if and only if $a'_{M-e} \circ \gamma$ is one. Thus, we may identify $\mathcal{X}_{M,g} \simeq \mathcal{X}_{M,g\gamma}$ and under this identification we have $\phi_{M,g\gamma} = \gamma \circ \phi_{M,g}$.

Part (5): Let r be an integer, $0 \leq r \leq M$. Then $p^{-r}\mathbb{I}_h \in GL_h(\mathbb{Q}_p)^+$ and we have $e(p^{-r}\mathbb{I}_h) = r$ and $d(p^{-r}\mathbb{I}_h) = rh$. Let $\mathcal{E}$ be the universal finite flat subgroup of $\mathcal{H}$ over $\mathcal{X}_{M,p^{-r}\mathbb{I}_h}$. By definition, $\mathcal{E} \subset \mathcal{H}[p^r]$ and has order rh, i.e. the same order of $\mathcal{H}[p^r]$. Therefore $\mathcal{E} = \mathcal{H}[p^r]$. (This equality of subgroup implies that the morphism $\phi_{M,p^{-r}\mathbb{I}_h}$

is a closed embedding.) Moreover, it follows from lemma 7.2 that the subgroup $\mathcal{H}[p^r]/\mathcal{X}_M$ has all the required universal properties. We therefore conclude that the morphism $\phi_{M,p^{-r}\mathbb{I}_h}$ is an isomorphism.

Finally, the equality $f_{M,M-r} = p^{-r}\mathbb{I}_h \circ \phi_{M,p^{-r}\mathbb{I}_h}$ is a direct consequence of the equality $\mathcal{E} = \mathcal{H}[p^{-r}]$. □

7.1.4. In the following we will refer to the data of the morphisms $g : \mathcal{X}_{M,g} \to \mathcal{X}_{M-e}$, for $g \in GL_h(\mathbb{Q}_p)^+$, as the action of $GL_h(\mathbb{Q}_p)^+$ on the integral models of the Shimura varieties.

We remark that the above action of $GL_h(\mathbb{Q}_p)^+$ preserves the Newton polygon stratification of the special fibers.

7.1.5. For all $M \geq 0$ and $g \in GL_h(\mathbb{Q}_p)^+$ ($M \geq e = e(g)$), we denote by $\varphi_{M,g} : \mathfrak{X}_{M,g} \to \mathfrak{X}_M$ (resp. $\varphi_M : \mathfrak{X}_M \to \mathfrak{X}$) the formal scheme over $\mathfrak{X}$ associated to $\mathcal{X}_{M,g} \to \mathcal{X}_M$ (resp. $\mathcal{X}_M \to \mathcal{X}$), and by $\varphi_{M',M,g} : \mathfrak{X}_{M',g} \to \mathfrak{X}_{M,g}$ (resp. $\varphi_{M',M} : \mathfrak{X}_{M'} \to \mathfrak{X}_M$ and $g : \mathfrak{X}_{M,g} \to \mathfrak{X}_{M-e}$) the morphism induced by $\phi_{M',M,g}$ (resp. $\phi_{M',M}$ and g), for any $M' \geq M$.

7.2. Integral models for Rapoport-Zink spaces with level structure. — In the following we define some formal schemes over $\mathcal{M}$, which are the analogues of the formal schemes $\mathfrak{X}_{M,g}/\mathfrak{X}$, such that the associated rigid-analytic spaces are the covering spaces $\mathcal{M}_M^{\mathrm{rig}}/\mathcal{M}^{\mathrm{rig}}$ by defined Rapoport and Zink (see section 2.5).

7.2.1. Let us recall that Zarisky locally the Rapoport-Zink space $\mathcal{M}$ is defined as the p-adic completion of the a closed subscheme U of a Grassmanian variety associated to the algebra of functions of $\Sigma[p^d]$, for some positive integer d.

For any such scheme $U/\hat{\mathbb{Z}}_p^{nr}$, let us denote by $\mathcal{H}'$ the universal Barsotti-Tate group over U. For any integer $M \geq 0$, we define

$$U_M = W(A[p^M], \mathcal{H}'[p^M]/U)$$

and denote by $\delta_M : U_M \to U$ the natural morphism and by $b_M : A[p^M] \to \mathcal{H}'[p^M]$ the universal $A[p^M]$-generator over U_M.

Moreover, for any $g \in GL_h(\mathbb{Q}_p)^+$ and $M \geq e = e(g)$, we define $U_{M,g}/U_M$ to be the universal space for the existence of a finite flat subgroup $\mathcal{E}' \subset \mathcal{H}'[p^e]$, of order $d = d(g)$, such that

$$b_M(A[g^{-1}]) \subset \mathcal{E}'(U_{M,g}),$$

and the induced morphisms on the subquotients

$$A[p^{M-e}] \to (\mathcal{H}'/\mathcal{E}')[p^{M-e}](\mathcal{X}_{M,g})$$

is a $A[p^{M-e}]$-generator. We denote by $\delta_{M,g} : U_{M,g} \to U$ the natural morphism.

As the p-adic completion of U varies among an open cover of $\mathcal{M}$, the p-adic completions of the spaces $U_{M,g}$ (resp. U_M) describe a formal scheme $\delta_{M,g} : \mathcal{M}_{M,g} \to \mathcal{M}$

(resp. $\delta_M : \mathcal{M}_M \to \mathcal{M}$), for all M, g. It follows from the construction that all the formal schemes $\mathcal{M}_{M,g}$ and $\mathcal{M}_M$ are formally locally of finite type over $\hat{\mathbb{Z}}_p^{nr}$.

It is also a direct consequence of the definitions that the above spaces naturally form a system, i.e. there are some morphisms $\delta_{M',M,g} : \mathcal{M}_{M',g} \to \mathcal{M}_{M,g}$ and $\delta_{M',M} : \mathcal{M}_{M'} \to \mathcal{M}_M$, associated to the inclusions $\mathcal{H}'[p^M] \hookrightarrow \mathcal{H}'[p^{M'}]$, which satisfy the obvious commutativity laws.

7.2.2. For any positive integers n, d, we denote by $\mathcal{M}_{M,g}^{n,d}$ and $\mathcal{M}_M^{n,d}$ the pullbacks over $\mathcal{M}^{n,d} \subset \mathcal{M}$ of the spaces $\mathcal{M}_{M,g}$ and $\mathcal{M}_M$ respectively, for all g, M.

Then, for any $n' \geq n$ and $d' - d \geq (n' - n)h$, the inclusions $i = i_{n',d'}^{n,d} : \mathcal{M}^{n,d} \hookrightarrow \mathcal{M}^{n',d'}$ naturally give rise to some morphisms

$$i_{M,g} = (i_{n',d'}^{n,d})_{M,g} : \mathcal{M}_{M,g}^{n,d} \to \mathcal{M}_{M,g}^{n',d'}$$

and

$$i_M = (i_{n',d'}^{n,d})_M : \mathcal{M}_M^{n,d} \to \mathcal{M}_M^{n',d'},$$

and the restriction of the morphisms $\delta_{M',M}$ and $\delta_{M',M,g}$ to some morphisms

$$\delta_{M',M,g}^{n,d} : \mathcal{M}_{M'}^{n,d} \twoheadrightarrow \mathcal{M}_M^{n,d}$$

and

$$\delta_{M',M,g}^{n,d} : \mathcal{M}_{M',g}^{n,d} \twoheadrightarrow \mathcal{M}_{M,g}^{n,d}.$$

Proposition 7.4. — *Mantaing the above notations.*

1. *The formal schemes $\delta_M : \mathcal{M}_M \to \mathcal{M}$ (resp. $\delta_{M,g} : \mathcal{M}_{M,g} \to \mathcal{M}_M$) are finite (resp. proper).*

   Moreover, there are natural identifications

   $$(\mathcal{M}_{M,g})^{\mathrm{rig}} = (\mathcal{M}_M)^{\mathrm{rig}} = \mathcal{M}_M^{\mathrm{rig}},$$

   which are compatible with the natural projections.
2. *For any $M \geq 0$, there is an action of $GL_h(\mathbb{Z}_p)$ on $\mathcal{M}_M$, which is compatible with the action of $GL_h(\mathbb{Z}_p) \subset GL_h(\mathbb{Q}_p)$ on the corresponding rigid analytic space, and commutes with the projections $\delta_{M',M}$, for any $m' \geq M$.*
3. *For any $M \geq e(g) = e$, there exist some proper morphisms*

   $$g : \mathcal{M}_{M,g} \to \mathcal{M}_{M-e}$$

   which are compatible, under the above identifications, with the action of $GL_h(\mathbb{Q}_p)^+ \subset GL_h(\mathbb{Q}_p)$ on the rigid analytic Rapoport-Zink spaces.
4. *For any $\gamma \in GL_h(\mathbb{Z}_p)$, there is a natural identification $\mathcal{M}_{M,g} \simeq \mathcal{M}_{M,g\gamma}$ over $\mathcal{M}_M$, and, under such identification, we have $\delta_{M,g\gamma} = \gamma \circ \delta_{M,g}$.*
5. *For any positive inter $r \leq M$, the morphism $\delta_{M,p^{-r}\mathbb{I}_h} : \mathcal{M}_{M,p^{-r}\mathbb{I}_h} \to \mathcal{M}_M$ is an isomorphism and $\delta_{M,M-r} = p^{-r}\mathbb{I}_h \circ \delta_{M,p^{-r}\mathbb{I}_h}^{-1}$.*

6. *There exist some σ-linear automorphisms*
$$Frob : \mathcal{M}_M \to \mathcal{M}_M \text{ and } Frob : \mathcal{M}_{M,g} \to \mathcal{M}_{M,g}$$
such that $\delta_M \circ Frob = Frob \circ \delta_M$, $\delta_{M,g} \circ Frob = Frob \circ \delta_{M,g}$ and $g \circ Frob = Frob \circ g$.
7. *For any $\rho \in T$, there exist some automorphisms*
$$\rho : \mathcal{M}_M \to \mathcal{M}_M \text{ and } \rho : \mathcal{M}_{M,g} \to \mathcal{M}_{M,g}$$
which define an action of T on the integral models of the Rapoport-Zink spaces compatible with the action of T on the corresponding rigid analytic spaces.

Moreover, for any $\rho \in T$, we have $\delta_M \circ \rho = \rho \circ \delta_M$, $\delta_{M,g} \circ \rho = \rho \circ \delta_{M,g}$ and $g \circ \rho = \rho \circ g$.

Proof. — Part (1): The same arguments of propositions 7.1 and 7.3 apply but, in order to deduce the above identifications among the corresponding rigid analytic spaces, one should also check that the construction of the space $\underline{\mathrm{Isom}}(X, Y)/S$, for X, Y two finite flat group schemes over S, commutes with analytification (and this is proved in [**6**],Theorem 3.5.6, p. 61).

Part (2): Let M be a positive integer and $\gamma \in GL_h(\mathbb{Z}_p)$. We first define, for all n, d, some morphisms
$$\gamma^{n,d} : \mathcal{M}_M^{n,d} \to \mathcal{M}_M^{n,d}.$$
Let $(\mathcal{H}, \beta : \Sigma \to \bar{\mathcal{H}}; a_M)$ be the universal triple over $\mathcal{M}_M^{n,d}$ (where $\bar{\mathcal{H}}$ denotes the restriction of $\mathcal{H}$ to the locus $p = 0$). Then, we define $\gamma^{n,d}$ to be the morphism associated to the triple
$$(\mathcal{H}, \beta : \Sigma \to \bar{\mathcal{H}}; a_M \circ \gamma_{|A[p^M]}).$$
It follows from the definition that the morphisms $\gamma^{n,d}$ commutes with the inclusions i_M, and thus give rise to a morphism $\gamma : \mathcal{M}_M \to \mathcal{M}_M$. Moreover, it is easy to see that the morphisms γ define an action of $GL_(\mathbb{Z}_p)$ on the $\mathcal{M}_M$ with the required properties.

Part (3): As in part (2), we first define, for all n, d, some morphisms
$$g : \mathcal{M}_{M,g}^{n,d} \to \mathcal{M}_{M-e}^{n,d+e}.$$
Let $(\mathcal{H}, \beta : \Sigma \to \bar{\mathcal{H}}; \mathcal{E} \subset \mathcal{H}[p^e], a_M)$ be the universal quadruple over $\mathcal{M}_{M,g}^{n,d}$ (where $\bar{\mathcal{H}}$ denotes the restriction of $\mathcal{H}$ to the locus $p = 0$). Then, we define g to be the morphism associated to the triple
$$(\mathcal{H}/\mathcal{E}, \beta \circ p_{\bar{\mathcal{E}}} : \Sigma \to \bar{\mathcal{H}}/\bar{\mathcal{E}}; a'_{M-e})$$
where $\bar{\mathcal{E}}$ denotes the restriction of $\mathcal{E}$ to the locus $p = 0$, $p_{\bar{\mathcal{E}}} : \bar{\mathcal{H}} \twoheadrightarrow \bar{\mathcal{H}}/\bar{\mathcal{E}}$ the natural projection and a'_{M-e} the induced $A[p^{M-e}]$-generator of $(\mathcal{H}/\mathcal{E})[p^{M-e}]$.

It follows from the definition that the morphisms g commutes with the inclusions $i_{M,g}$ and i_{M-e}, and thus give rise to a morphism $g : \mathcal{M}_{M,g} \to \mathcal{M}_{M-e}$.

The same argument we used to prove part (3) of proposition 7.3 shows here that the morphism g we have defined is proper. It is also an easy consequence of the

definition that the above morphism is compatible with the previously defined action of $GL_h(\mathbb{Q}_p)^+ \subset GL_h(\mathbb{Q}_p)$ on the rigid analytic Rapoport-Zink spaces.

Parts (4) and (5): The same arguments used to prove parts (4) and (5) of proposition 7.3 apply here.

Part (6): Let us identify $\mathcal{M}_M = \mathcal{M}_{M,\mathbb{I}_h}$ and define the σ-linear morphism on $\mathcal{M}_{M,g}$, for all M, g ($M \geq e(g)$). We use the universal property of $\mathcal{M}_{M,g}$ to define $Frob$ to be the morphism

$$(H, \beta; a_M, E) \mapsto (H, \beta \circ F^{-1}; a_M, E).$$

It is clear that the morphism $Frob$ as all the required properties.

Part (7): As in part (6), we identify $\mathcal{M}_M = \mathcal{M}_{M,\mathbb{I}_h}$ and define, for any $\rho \in T$, some automorphisms ρ of $\mathcal{M}_{M,g}$, for all M, g ($M \geq e(g)$). We set ρ to be the automorphism of $\mathcal{M}_{M,g}$ defined by

$$(H, \beta; a_M, E) \mapsto (H, \beta \circ \rho; a_M, E).$$

Again, it is a direct consequence of the definition that the above morphisms define an action of T with the required properties. □

7.2.3. We refer to the data of the morphisms $g : \mathcal{M}_{M,g} \to \mathcal{M}_{M-e}$, for $g \in GL_h(\mathbb{Q}_p)^+$, as the action of $GL_h(\mathbb{Q}_p)^+$ on the integral models of the Rapoport-Zink spaces.

7.3. Comparing the spaces $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$ and $\mathcal{M}_{M,g}$. — The goal of this section is to compare, for any Newton polygon α of dimension q and height h, the spaces $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$ and $\mathcal{M}_{M,g}$ (for any $g \in GL_h(\mathbb{Q}_p)^+$ and integer $M \geq e(g)$), in terms of the associated covers over $(\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$ (for all $t > 0$) and over any affine open V of $\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$.

In the first case, we shall consider the morphisms

$$\pi_N(t) : (\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr} \subset \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr},$$

for $N \geq (d + t/2)/\delta B$, and the projection onto the second factor

$$pr(t) : (\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t) \to \mathcal{M},$$

and we shall compare the two systems of spaces $\pi_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})$ and $pr(t)^*\mathcal{M}_{M,g}$ over $(\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$, as g, M vary.

In the second case, for any affine open V of $\mathcal{J}_m \times_{\mathrm{Spf}\,\hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$, we shall consider the morphism

$$\pi_N[t, V] : V \to \mathfrak{X}^{(\alpha)} \times \hat{\mathbb{Z}}_p^{nr} \subset \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr},$$

for $N \geq (d + t/2)/\delta B$, and the projection

$$pr_V = pr_{|V} : V \to \mathcal{M},$$

and compare the spaces $\pi_N[t, V]^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})$ and $pr_V^*\mathcal{M}_{M,g}$ over V.

We shall prove that, in both cases, when $[t/2] \geq M$, the pullbacks of $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$ and $\mathcal{M}_{M,g}$ we consider are indeed isomorphic.

7.3.1. Let α be a Newton polygon of dimension q and height h, and m, n, d be some positive integers. Let $g \in GL_h(\mathbb{Q}_p)^+$ and $M \geq e = e(g)$ and assume $m - d \geq M$. For any positive integers t, N such that $m - d \geq [t/2] \geq M$ and $N \geq (d + t/2)/\delta B$, we consider the spaces

$$\pi_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}) = \mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr} \times_{\mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}, \pi_N(t)[M]} (\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$$

and

$$pr(t)^* \mathcal{M}_{M,g} = \mathcal{M}_{M,g} \times_{\mathcal{M}, pr(t)} (\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t).$$

We also denote respectively by $f_{M,g}(t)$ and $g_{M,g}(t)$ the natural projections to $(\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$. (Let us remark that the morphisms $g_{M,g}(t)$ are the restrictions of some morphisms defined over $\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}$, namely $g_{M,g} : pr^* \mathcal{M}_{M,g} \to \mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d}$, but this is not true for the morphisms $f_{M,g}(t)$.)

Proposition 7.5. — *Maintaining the notations as above.*

There exist some isomorphisms

$$\xi_N(t)[M, g] : \pi_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}) \to pr(t)^* \mathcal{M}_{M,g},$$

such that $g_{M,g}(t) \circ \xi_N(t)[M, g] = f_{M,g}(t)$, which are compatible with the actions of $GL_h(\mathbb{Q}_p)^+$ on the two systems of spaces.

Proof. — Let us first consider the case $g = \mathbb{I}_h$. We write $\mathfrak{X}_{M,\mathbb{I}_h} = \mathfrak{X}_M$ and $\mathcal{M}_{M,\mathbb{I}_h} = \mathcal{M}_M$.

We recall that, for any S-scheme T, we have $W(A, Z/S)_T = W(A, Z_T/T)$ (see section 2.6.3). Thus, it follows from the definitions that

$$\pi_N(t)^*(\mathfrak{X}_M \times \hat{\mathbb{Z}}_p^{nr}) = W((\mathbb{Z}/p^M\mathbb{Z})^n, \hat{\mathcal{H}}_N[p^M]/(\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t))$$

and

$$pr(t)^* \mathcal{M}_M = W((\mathbb{Z}/p^M\mathbb{Z})^n, \mathcal{H}'[p^M]/(\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)),$$

where $\hat{\mathcal{H}}_N = \pi_N(t)^* \mathcal{H}$, and $\mathcal{H}$ and $\mathcal{H}'$ are the universal Barsotti-Tate group over $\mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}$ and the Rapoport-Zink space $\mathcal{M}^{n,d}$, respectively.

We recall that, for any two finite flat group schemes Z, Z' over a scheme S, we have $W(A, Z/S) \simeq W(A, Z'/S)$ if $Z \simeq Z'$ (see section 2.6.3). Therefore, in order to conclude, it suffices to show that there exists an isomorphism of finite flat group schemes over $(\mathcal{J}_m \times_{\operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}} \mathcal{M}^{n,d})(t)$

$$\hat{\mathcal{H}}_N[p^M] \to \mathcal{H}'[p^M].$$

Indeed, such an isomorphism exists by the very definition of the morphisms $\pi_N(t)$, since we assumed $[t/2] \geq M$. The corresponding isomorphism

$$\xi_N(t)[M] = \xi_N(t)[M, \mathbb{I}_h] : \pi_N(t)^*(\mathfrak{X}_M \times \hat{\mathbb{Z}}_p^{nr}) \to pr(t)^* \mathcal{M}_M$$

has the required properties.

Let $g \in GL_h(\mathbb{Q}_p)^+$. The above isomorphism $\xi_N(t)[M, \mathbb{I}_h]$, together with the isomorphism of finite flat group schemes $\hat{\mathcal{H}}_N[p^M] \to \mathcal{H}'[p^M]$, enables us to identify the schemes $\pi_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})$ and $pr(t)^*\mathcal{M}_{M,g}$, as spaces defined by the same universal property. Moreover, it is clear that these identifications are compatible with the action of $GL_h(\mathbb{Q}_p)^+$. □

7.3.2. Let V be an affine open of $\mathcal{J}_m \times U^{n,d}$. We proved (see proposition 6.8) that, for any positive integers t, N such that $m \geq d + t/2$ and $N \geq (d + t/2)/\delta B$, there exists a morphism $\pi_N[t, V]$ on V which restricts to $\pi_N(t)_{|V(t)}$ on $V(t)$. Thus, if we consider the spaces

$$\pi_N[t, V]^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}) = \mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr} \times_{\mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}, \pi_N[M,t,V]} V$$

and

$$pr_V^* \mathcal{M}_{M,g} = \mathcal{M}_{M,g} \times_{\mathcal{M}, pr_V} V,$$

then over $V(t)$ we have

$$(\pi_N[t, V]^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}))_{|V(t)} = (\pi_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}))_{|V(t)}$$

and

$$(pr_V^* \mathcal{M}_{M,g})_{|V(t)} = (pr(t)^* \mathcal{M}_{M,g})_{|V(t)}.$$

We investigate the possibility of extending the restrictions over $V(t)$ of the isomorphisms $\xi_N(t)[M, g]$ to an isomorphism over V.

Proposition 7.6. — *Maintaining the notations as above. For any affine open V of $\mathcal{J}_m \times U^{n,d}$, there exists an isomorphism of formal schemes over V*

$$\xi_N[t, V][M, g] : \pi_N[t, V]^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}) \to pr_V^* \mathcal{M}_{M,g}$$

which extends the isomorphism $\xi_N(t)[M, g]_{|V(t)}$ and is compatible with the actions of $GL_h(\mathbb{Q}_p)^+$ on the two systems of spaces.

Proof. — The statement follows directly from the definitions of the morphisms $\pi_N[t, V]$ and $\xi_N(t)[M, g]$. In fact, to extend the isomorphism $\xi_N(t)[M, g]_{|V(t)}$ to an isomorphism over V, compatible with the action of $GL_h(\mathbb{Q}_p)^+$, it suffices to extend the isomorphism over $V(t)$

$$\pi_N(t)^* \mathcal{H}[p^M] \to \mathcal{H}'[p^M]$$

to an isomorphism over V between the p^M-torsion subgroups of the Barsotti-Tate group $\pi_N[t, V]^* \mathcal{H}/V$ and $\mathcal{H}'/V$. Such an isomorphism exists by the very definition of $\pi_N[t, V]$ and the assumption $[t/2] \geq M$. □

7.3.3. Finally, let us recall that, in section 6.4, for all n, d, we introduced some morphisms

$$\hat{y}_N : U^{n,d} \to \mathfrak{X}^{(\alpha)} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr} \subset \mathfrak{X} \times \operatorname{Spf} \hat{\mathbb{Z}}_p^{nr}$$

associated to a point $y \in J_\infty(\bar{\mathbb{F}}_p)$, for all $N \geq 1$, such that over $U^{n,d}$ we have $\hat{y}_N(t) = \pi_N(t) \circ (y_m^\wedge, id)(t)$, for all $t \geq 1$ and m, N sufficiently large, and also $\hat{y}_N^* \mathcal{H} \simeq \mathcal{H}'$. Arguing as in the proofs of propositions 7.5 and 7.6, we conclude that there exist some isomorphisms over $U^{n,d}$

$$\xi_{y,N}[M,g] : \hat{y}_N^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}) \to U_{M,g}^{n,d},$$

which extends the isomorphism $(y_m, id)(t)^* \xi_N(t)[M,g]$ from $\hat{y}_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}) = (y_m, id)(t)^* \pi_N(t)^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})$ to $(y_m, id)(t)^* pr(t)^* \mathcal{M}_{M,g} = U_{M,g}^{n,d}(t)$, and which are compatible with the action of $GL_h(\mathbb{Q}_p)^+$.

7.4. The vanishing cycles sheaves on Shimura varieties. — We shall now use Berkovich's theory of vanishing cycles for rigid analytic spaces to prove that the vanishing cycles sheaves $R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})$ on a Shimura variety with level structure at p and the vanishing cycles sheaves $R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})$ on the Rapoport-Zink space of the same level are isomorphic once pulled back over the covering spaces $J_m \times \bar{U}^{n,d}$. Moreover, such isomorphisms between the vanishing cycles sheaves are compatible with the action of the Weil group and $GL_h(\mathbb{Q}_p)^+$.

7.4.1. Let l be a prime number, $l \neq p$. We fix an integer $r \geq 1$, a level $M > 0$ and $g \in GL_h(\mathbb{Q}_p)^+$ (where $M \geq e = e(g)$), and study the vanishing cycles of the constant étale sheaf $\mathbb{Z}/l^r\mathbb{Z}$ on the Shimura variety $\mathfrak{X}_{M,g}$.

7.4.2. Let m, n, d be some positive integers, $m \geq d + M$. We choose a finite cover $\mathcal{V}$ of affine opens V of $\mathcal{J}_m \times U^{n,d}$, and write $V_{M,g}$ for the pullback of $\mathcal{J}_m \times U_{M,g}^{n,d}$ over V, for all $V \in \mathcal{V}$.

Let $\mathcal{I}$ denote an ideal of definition of $\mathcal{J}_m \times U_{M,g}^{n,d}$, and choose a positive integer $t = t_{M,r,\mathcal{V}} \geq 2M$ such that, for all $W = V_1 \cap V_2$ for some $V_1, V_2 \in \mathcal{V}$ (possibly $V_1 = V_2$), the ideals $\mathcal{I}_{|W}^t$ satisfy the property in the statement of proposition 2.24 for $\mathfrak{X}' = W_{M,g}$ and $\mathfrak{X} = \mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$ respectively.

Let $m' \geq m$ such that $m' \geq d + t/2$ and choose $N \geq (d + t/2)/\delta B$. For each $V \in \mathcal{V}$, we write $V_{m'}$ (resp. $V_{m',M,g}$) for the pullback of $\mathcal{J}_{m'} \times U^{n,d}$ (resp. $\mathcal{J}_{m'} \times U_{M,g}^{n,d}$) over V. Then, for each $V \in \mathcal{V}$, $V_{m',M,g} \to V_{M,g}$ is finite étale with degree equal to $[J_{m'} : J_m]$, which is a p-power (and thus, in particular, relatively prime to l). For simplicity, we write $\pi_V = \pi_N[t, V_{m'}]$, $\xi_V = \xi_N[t, V_{m'}][M,g]$ and $\bar{\pi}_N = \pi_N(1)$.

For any $V \in \mathcal{V}$, we use the isomorphism ξ_V to identify the spaces $V_{m',M,g}$ and $\pi_V^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})$ and write

$$\varpi_V : V_{m',M,g} \to \mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$$

for the pullback of the morphism $\pi_V : V_{m'} \to \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}$, under the projection $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr} \twoheadrightarrow \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}$.

We remark that the morphism $(\pi_V)_s : \bar{V}_{m'} \to \bar{X}$ factors as $\bar{\pi}_N \circ (\mathrm{q}_{m',m} \times 1)_{|\bar{V}}$, and thus the morphism $(\varpi_V)_s$ factors via the projection $\bar{V}_{m',M,g} \to \bar{V}_{M,g}$, i.e.

$$(\varpi_V)_s = \bar{\varpi}_N \circ (\mathrm{q}_{m,m'} \times 1)_{|\bar{V}_{M,g}},$$

where $\bar{\varpi}_N : J_m \times \bar{U}_{M,g}^{n,d} \to \bar{X}_{M,g} \times \bar{\mathbb{F}}_p$ denotes the pullback of the morphism $\bar{\pi}_N$.

We deduce that the morphism ϖ_V gives rise to a morphism of objects in the derived category of étale sheaves over $\bar{V}_{M,g}$

$$\theta_V : \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})_\eta})_{|\bar{V}_{M,g}} \to R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{J}_m \times U_{M,g}^{n,d})_\eta})_{|\bar{V}_{M,g}}$$

(see proposition 2.24).

Proposition 7.7. — *The morphisms θ_V, for $V \in \mathcal{V}$, piece together in an isomorphism between the vanishing cycles sheaves over $J_m \times \bar{U}_{M,g}^{n,d}$*

$$\theta = \theta_{M,g} : \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})_\eta}) \to R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{J}_m \times U_{M,g}^{n,d})_\eta}),$$

Moreover, the isomorphisms $\theta_{M,g}$ are compatible with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$ on the Shimura varieties $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$ and on the spaces $\mathcal{J}_m \times U_{M,g}^{n,d}$, respectively.

Proof. — First, we prove that the morphisms θ_V, $V \in \mathcal{V}$, give rise to a global morphism between the above vanishing cycles sheaves over $J_m \times U_{M,g}^{n,d}$.

Indeed, for each $W = V_1 \cap V_2$, $V_1, V_2 \in \mathcal{V}$, we consider the restriction to $W_{m',M,g}$ of the two morphism ϖ_{V_i}, $i = 1, 2$. By definition, we have $(\pi_{V_1})_{|W_{m'}}(t) = (\pi_{V_2})_{|W_{m'}}(t)$ which implies that we also have $(\varpi_{V_1})_{|W_{m',M,g}}(t) = (\varpi_{V_2})_{|W_{m',M,g}}(t)$. It follows from proposition 2.24 and from our choice of the integer t that the induced morphisms $\theta_{V_i|\bar{W}_{M,g}}$ $(i = 1, 2)$ between the vanishing cycles sheaves agree.

Moreover, since for all $V \in \mathcal{V}$ the morphisms ϖ_V are formally smooth (because such are the morphism π_V), it follows from proposition 2.26 that the corresponding morphisms between the vanishing cycles of $\mathbb{Z}/l^r\mathbb{Z}$ over the special fiber $\bar{V}_{m',M,g}$ of $V_{m',M,g}$ are isomorphisms and thus such are also the morphisms θ_V.

Finally, the compatiblity of the $\theta_{M,g}$ with the morphisms associated to changes of level and the action of $GL_h(\mathbb{Q}_p)^+$ follows from the corresponding property of the isomorphisms ξ_V, for all V. □

Corollary 7.8. — *Let $\mathfrak{p} : \mathcal{J}_m \times U_{M,g}^{n,d} \to \mathcal{M}_{M,g}$ be the natural projection.*

There exists an isomorphism of objects in the derived category of étale sheaves over $J_m \times \bar{U}_{M,g}^{n,d}$

$$\zeta = \zeta_{M,g} : \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})_\eta}) \to \mathfrak{p}_s^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{M,g\,\eta}}),$$

which is compatible with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$ on the Shimura varieties and the Rapoport-Zink spaces, respectively.

Proof. — Since the formal Igusa varieties are formally smoth over $\hat{\mathbb{Z}}_p^{nr}$, it follows from proposition 2.26 that the morphism $\mathfrak{p}$ gives rise to an isomorphism between the vanishing cycles

$$\mathfrak{p}_s^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{M,g\,\eta}}) \simeq R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{J}_m\times U^{n,d}_{M,g})_\eta}).$$

It is also clear from the definitions that such isomorphisms is compatible with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$ on the Rapoport-Zink spaces $\mathcal{M}_{M,g}$ and on the $\mathcal{J}_m \times U^{n,d}_{M,g}$, respectively.

Thus, proposition 7.7 implies the existence of an isomorphism as in the statement. □

Under some further assumptions on the integer $t \geq 0$, it is possible to describe the stalks of the isomorphisms ζ in terms of the morphisms $\hat{y}_N$ (see section 6.4).

7.4.3. Let us assume that the affine opens $V \in \mathcal{V}$ are of the form $V = V^1 \times V^2$, where V^2 varies in an open cover $\mathcal{U}$ of $U^{n,d}$. Then, for any $V^2 \in \mathcal{U}$, we write $V^2_{M,g}$ for the pullback of V^2 over $U^{n,d}_{M,g} \to U^{n,d}$). We also assume that the integer t we chose is sufficiently large such that, for any open $V^2 \in \mathcal{U}$ and any two morphisms from $V^2_{M,g}$ to $\mathcal{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$, which coincide modulo the t-th power of the maximal ideal, the induced morphisms between the vanishing cycles of $\mathbb{Z}/l^r\mathbb{Z}$ agree.

Let $y \in J(\bar{\mathbb{F}}_p)$ and $N \geq d/\delta B$. We denote by

$$\tilde{y}_N : U^{n,d}_{M,g} \to \mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$$

the pullback under $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr} \to \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}$ of the morphism $\hat{y}_N : U^{n,d} \to \mathfrak{X} \times \hat{\mathbb{Z}}_p^{nr}$, composed with the isomorphism $\xi_{y,N}[M,g]^{-1} : U^{n,d}_{M,g} \simeq \hat{y}_N^*(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})$ (see section 7.3.3).

Proposition 7.9. — *Maintaining the above notations and assumptions. Let (y_m, z) be a geometric closed point of $J_m \times \bar{U}^{n,d}_{M,g}$.*

The morphisms

$$\zeta_{(y_m,z)} : \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr})_\eta})_{(y_m,z)} \to \mathfrak{p}_s^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{M,g\,\eta}})_{(y_m,z)}$$

agree with the morphisms

$$\psi_\eta(\tilde{y}_N, \mathbb{Z}/l^r\mathbb{Z})_z : \tilde{y}_{N\,s}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr})_\eta})_z \to R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{M,g\,\eta}})_z,$$

when we choose $y \in J(\bar{\mathbb{F}}_p)$ such that $\mathrm{q}_{\infty,m}(y) = y_m$ (and thus $\tilde{y}_N(z) = \varpi_N(y_m, z)$ and $\mathfrak{p}(y_m, z) = z$).

Proof. — Let us choose $V = V^1 \times V^2 \in \mathcal{V}$ such that $y_m^\wedge \in V^1$ and $(y_m, z) \in \bar{V}_{M,g}$. Then, over V^2, we have

$$\pi_N[t, V] \circ (y_m^\wedge, id) \equiv \hat{y}_{N\,|V^2}$$

modulo the t-th power of the maximal ideal of definition of $U^{n,d}$. Since $pr \circ (y_m, id) = id$ on $U^{n,d}$, we conclude. □

We remark that the above description of the stalks of the morphism ζ provides an alternative proof of the fact that the isomorphisms θ_V piece together, as V varies in $\mathcal{V}$.

7.4.4. We now focus our attention on the action of the inertia group I_p on the vanishing cycles sheaves $\bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr})_\eta}) \simeq \mathfrak{p}_s^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$. In particular, we are interested in the possibility of extending its action to an action of the Weil group $W_{\mathbb{Q}_p} \supset I_p$.

Let us remark that it is a direct consequence of the definitions that the action of the inertia group I_p on these vanishing cycles sheaves commutes with the isomorphisms $\zeta_{M,g}$ and with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$. We are interested in defining an action of $W_{\mathbb{Q}_p}$ with the same property.

Remark 7.10. — Maintaining the above notations.

1. Let us consider the natural identification

$$R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr})_\eta}) = R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta})$$

   over $\bar{X}_{M,g} \times \bar{\mathbb{F}}_p$. Then, the action of I_p on the left hand side is simply the restriction to I_p of the action of $W_{\mathbb{Q}_p}$ on the right hand side.

   Moreover, the action of the Weil group on the right hand side is compatible with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$ on the Shimura varieties.
2. The action of I_p on the vanishing cycles $R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$ over $\bar{\mathcal{M}}_{M,g}$ naturally extends to an action of $W_{\mathbb{Q}_p}$, which is compatible with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$ on the Rapoport-Zink spaces.

Indeed, the first statement is obvious. (To conclude the compatibility between the action of $W_{\mathbb{Q}_p}$ and the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$, it suffices to recall that both the projections $\mathfrak{X}_{M',g}\times\hat{\mathbb{Z}}_p^{nr} \to \mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr}$, for $M' \geq M$, and the action of $GL_h(\mathbb{Q}_p)^+$ on the Shimura varieties $\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr}$ arise from morphisms defined over $\mathbb{Z}_p \subset \hat{\mathbb{Z}}_p^{nr}$, and thus the corresponding induced morphisms on the vanishing cycles sheaves $R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta})$ over $\bar{X}_{M,g} \times \bar{\mathbb{F}}_p$ commute with the action of $W_{\mathbb{Q}_p}$.)

As for the second statement, the possibility of extending the action of I_p on the vanishing cycles sheaves to an action of $W_{\mathbb{Q}_p}$, which is compatible with the morphisms induced by changes of level and by the action of $GL_h(\mathbb{Q}_p)^+$, follows from the existence of a descent datum for the Rapoport-Zink spaces $\mathcal{M}_{M,g}/\hat{\mathbb{Z}}_p^{nr}$, which commutes with natural projections and with the action of $GL_h(\mathbb{Q}_p)^+$, namely the σ-linear automorphism $Frob$ of $\mathcal{M}_{M,g}$ (see part (6) of proposition 7.4).

7.4.5. Let us remark that the actions of $W_{\mathbb{Q}_p}$ on the above vanishining cycles sheaves give rise to an action of $W_{\mathbb{Q}_p}$ on their pullbacks over $J_m \times \bar{U}^{n,d}_{M,g}$.

In fact, let $\tau \in W_{\mathbb{Q}_p}$ and define $r = r(\tau)$ to be the integer such that the image of τ in the absolute Galois group of $\mathbb{F}_p$ is $\bar{\tau} = \sigma^{r(\tau)}$. Then, the actions of $\tau \in W_{\mathbb{Q}_p}$ on the above vanishing cycles sheaves are defined by some isomorphisms

$$(1 \times \sigma^r)^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta})$$

over $\bar{X}_{M,g} \times \bar{\mathbb{F}}_p$ and

$$(Frob^r)^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}) \simeq R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$$

over $\bar{\mathcal{M}}_{M,g}$.

Let us assume $m \geq d + 2 + t/2 + M$ and $N \geq (d + 1 + t/2 + M)/\delta B$, and consider the morphism

$$Frob \times Frob : J_m \times \bar{U}^{n,d}_{M,g} \to J_{m-1} \times \bar{U}^{n+1,d+1}_{M,g}.$$

Then, we have $\bar{\varpi}_N \circ (Frob \times Frob) = (1 \times \sigma) \circ \bar{\varpi}_N$ and $\bar{\mathfrak{p}} \circ (Frob \times Frob) = Frob \circ \bar{\mathfrak{p}}$.

Thus, for $r = r(\tau) \geq 0$, the above isomorphisms on the vanishing cycles sheaves over $\bar{X}_{M,g} \times \bar{\mathbb{F}}_p$ and $\bar{\mathcal{M}}_{M,g}$ give rise to some isomorphisms on the pullbacks over $J_m \times \bar{U}^{n,d}_{M,g}$, namely

$$\bar{\varpi}^*_N(\tau) : (Frob^r \times Frob^r)^* \bar{\varpi}^*_N R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq$$

$$\simeq \bar{\varpi}^*_N (1 \times \sigma^r)^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq \bar{\varpi}^*_N R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta})$$

and

$$\bar{\mathfrak{p}}^*(\tau) : (Frob^r \times Frob^r)^* \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}) \simeq$$

$$\simeq \bar{\mathfrak{p}}^*(Frob^r)^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}) \simeq R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}).$$

Proposition 7.11. — *Maintaining the above notations. Let $\tau \in W_{\mathbb{Q}_p}$ such that $r = r(\tau) \geq 0$. The isomorphism*

$$\zeta : \bar{\varpi}^*_N R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr})_\eta}) \to \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$$

satisfy the equality $\zeta \circ \bar{\varpi}^*_N(\tau) = \bar{\mathfrak{p}}^*(\tau) \circ (Frob^r \times Frob^r)^*(\zeta)$.

Proof. — First, let us remark that we already know that the statements holds for any $\tau \in I_p$, i.e. when $r(\tau) = 0$. Thus, it suffices to check that the statement for a single element $\tilde{\sigma} \in W_{\mathbb{Q}_p}$ such that $r(\tilde{\sigma}) = 1$, (i.e. for a lift $\tilde{\sigma}$ of the Frobenius element σ),

Let $\mathcal{D} = \mathcal{D}_{M,g} = \bar{\varpi}^*_N R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$. We need to show that the two morphisms

$$\bar{\varpi}^*_N(\tilde{\sigma}), \bar{\mathfrak{p}}^*(\tilde{\sigma}) : (Frob \times Frob)^* \mathcal{D} \to \mathcal{D}$$

agree.

By the universal property of $\mathfrak{X}_{M,g}$, the descent datum on $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$ is equivalent to the datum of an isomorphism

$$\sigma : (1 \times \sigma)^* \mathcal{A} \to \mathcal{A},$$

where $\mathcal{A}$ is the universal abelian variety over $(\mathfrak{X}_{M,g})_s \times \bar{\mathbb{F}}_p$, such that σ induces isomorphisms $(1 \times \sigma)^* \mathcal{H} \simeq \mathcal{H}$ and $(1 \times \sigma)^* \mathcal{E} \simeq \mathcal{E}$, where $\mathcal{H}$ and $\mathcal{E}$ ($\mathcal{E} \subset \mathcal{H} \subset \mathcal{A}$) are respectively the universal Barsotti-Tate group and flat subgroup over $\mathfrak{X}_{M,g} \times \hat{\mathbb{Z}}_p^{nr}$.

Analogously, the descent datum on $\mathcal{M}_{M,g}$ is equivalent to the datum of an isomorphism

$$\sigma : Frob^* \mathcal{H}' \to \mathcal{H}',$$

where $\mathcal{H}'$ is the universal Barsotti-Tate group over $\mathcal{M}_{M,g}$, which restricts to an isomorphism $Frob^* \mathcal{E}' \simeq \mathcal{E}'$ on the subgroup $\mathcal{E}' \subset \mathcal{H}'$.

Moreover, the identification between the vanishing cycle sheaves

$$\bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$$

arises from the isomorphism

$$H[p^M] = \mathfrak{p}^* \mathcal{H}'[p^M] \simeq \varpi_V^* \mathcal{H}[p^M],$$

over the affine opens $V_{m'}$ of $\mathcal{J}_{m'} \times U_{M,g}^{n,d}$, $V \in \mathcal{V}$.

Thus, the two actions of $\tilde{\sigma} \in W_{\mathbb{Q}_p}$ on the vanishing cycles can be interpreted as arising from the descent data

$$\varpi_V^*(\sigma) : \varpi_V^*(1 \times \sigma)^* \mathcal{H}[p^M] \to \varpi_V^* \mathcal{H}[p^M]$$

and

$$\mathfrak{p}^*(\sigma) : \mathfrak{p}^* Frob^* \mathcal{H}'[p^M] \to \mathfrak{p}^* \mathcal{H}'[p^M].$$

On $J_m \times \bar{U}_{M,g}^{n,d}$ we have

$$\bar{\varpi}_N^*(1 \times \sigma)^* = (Frob \times Frob)^* \bar{\varpi}_N^*,$$

$$\bar{\mathfrak{p}}^* Frob^* = (Frob \times Frob)^* \bar{\mathfrak{p}}^*,$$

and also, under the identification $\bar{H}[p^M] = \bar{\mathfrak{p}}^* \mathcal{H}'[p^M] \simeq \bar{\varpi}_N^* \mathcal{H}[p^M]$,

$$\bar{\varpi}_N^*(\sigma)_s = \bar{\mathfrak{p}}^*(\sigma)_s : (Frob \times Frob)^* \bar{H}[p^M] \to \bar{H}[p^M].$$

Therefore, the isomorphism

$$\bar{\varpi}_N^*(\tilde{\sigma})^{-1} \circ \bar{\mathfrak{p}}^*(\tilde{\sigma}) : \mathfrak{p}^* Frob^* \mathcal{H}'[p^M] \to \varpi_V^*(1 \times \sigma)^* \mathcal{H}[p^M]$$

can be viewed as an isomorphism between two deformations of the group scheme $(Frob \times Frob)^* \bar{H}[p^M]$, which reduces to the identity on the special fiber. It follows that it gives rise to an identification of the two deformations, and thus, equivalently, that the morphism $\bar{\varpi}_N^*(\tilde{\sigma})^{-1} \circ \bar{\mathfrak{p}}^*(\tilde{\sigma})$ on the vanishing cycle sheaves is simply the identity. $\square$

7.4.6. We now investigate the action of T on the vanishing cycles sheaves we studied.

From the equality $\bar{\pi}_N \circ (\rho \times \rho) = \bar{\pi}_N$, for any $\rho \in S$, we deduce that $\bar{\varpi}_N \circ (\rho \times \rho) = \bar{\varpi}_N$ and thus $\bar{\varpi}_N^* \simeq (\rho \times \rho)^* \circ \bar{\varpi}_N^*$. It follows that there is a natural action of S on the vanishing cycles sheaves $\bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr})_\eta})$, i.e. for any $\rho \in S$ there is an isomorphism

$$\rho : (\rho \times \rho)^* \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g}\times\hat{\mathbb{Z}}_p^{nr})_\eta}) \simeq \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M}\times\hat{\mathbb{Z}}_p^{nr})_\eta}),$$

such that $(\rho_1\rho_2)^* = \rho_2^*\rho_1^*$, for any $\rho_1, \rho_2 \in S$.

Clearly, the above action of S commutes with the action of $I_p \times GL_h(\mathbb{Q}_p)^+$, and indeed it commutes also with the action of $W_{\mathbb{Q}_p}$, since the action of $\rho \in S$ on the schemes $J_m \times \bar{U}_{M,g}^{n,d}$ commutes with the morphism $Frob \times Frob$.

On the other hand, from the equality $\bar{\mathfrak{p}} \circ (\rho \times \rho) = \rho \circ \bar{\mathfrak{p}}$, for any $\rho \in S$, we deduce that there is an isomorphism

$$(\rho \times \rho)^* \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}) \simeq \bar{\mathfrak{p}}^* \rho^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}).$$

Moreover, since the action of the monoid $S \subset T$ on the reduced fibers of the Rapoport-Zink spaces extends to an action of the group T on the Rapoport-Zink spaces over $\hat{\mathbb{Z}}_p^{nr}$ (see part (7) of proposition 7.4), there are also some isomorphisms

$$\rho^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}) \simeq R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}).$$

Thus, by composing the above two isomorphisms, we define an action of S on the vanishing cycles $\bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$, i.e. for any $\rho \in S$ we define an isomorphism

$$\rho : (\rho \times \rho)^* \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}) \simeq \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}),$$

such that $(\rho_1\rho_2)^* = \rho_2^*\rho_1^*$, for any $\rho_1, \rho_2 \in S$.

Proposition 7.12. — *Maintaining the above notations. Let $\rho \in S \subset T$. The isomorphisms*

$$\zeta : \bar{\varpi}_N^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \to \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})$$

satisfy the equality $\zeta \circ \rho = \rho \circ (\rho \times \rho)^*(\zeta)$.

Proof. — Clearly, it suffices to check that the above equality on the stalks of the geometric points of $J_m \times \bar{U}_{M,g}^{n,d}$. Without loss of generality we may assume that our choice of the integer t is compatible with the properties stated in section 7.4.3, and thus we can apply the description of the stalks of the isomorphism ζ, we gave in proposition 7.9.

Let $\rho \in S$ and (y_m, z) be a geometric point of $J_m \times \bar{U}^{n,d}_{M,g}$. We need to prove that the following diagram commutes, for all $q \geq 0$.

$$\begin{array}{ccc} R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}))_{\bar{\varpi}_N(\rho y_m,\rho z)} & = & R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}))_{\bar{\varpi}_N(y_m,z)} \\ \downarrow{\scriptstyle \zeta_{(\rho y_m,\rho z)}} & & \downarrow{\scriptstyle \zeta_{(y_m,z)}} \\ R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})_{\rho z} & \xrightarrow{\rho_z} & R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})_z \end{array}$$

Let $y \in J(\bar{\mathbb{F}}_p)$ be a point such that $\mathrm{q}_{\infty,m}(y) = y_m$. Then, $\rho(y) \in J(\bar{\mathbb{F}}_p)$ and $\mathrm{q}_{\infty,m}(\rho y) = \rho y_m$. By proposition 7.9, we know that $\zeta_{(y_m,z)} = \psi_\eta(\hat{y}_N, \mathbb{Z}/l^r\mathbb{Z})_z$ and $\zeta_{(\rho y_m,\rho z)} = \psi_\eta(\hat{\rho y}_N, \mathbb{Z}/l^r\mathbb{Z})_{\rho z}$. Thus, the commutativity of the above diagram follows from the equality $\hat{y}_N \circ \rho = \rho\hat{y}_{m\infty N}$ (in the diagram we denote by ρ_z the morphism $\psi_\eta(\rho, \mathbb{Z}/l^r\mathbb{Z})_z$). □

7.4.7. The results of this section on the vanishing cycles sheaves of the Shimura varieties and of the Rapoport-Zink spaces can be summarised in the following proposition.

In the following, we write $W^+_{\mathbb{Q}_p} = \{\tau \in W_{\mathbb{Q}_p} | \, r(\tau) \geq 0\}$. Thus, $W_{\mathbb{Q}_p} = \langle W^+_{\mathbb{Q}_p}, \tilde{\sigma}\rangle$, for some $\tilde{\sigma} \in W^+_{\mathbb{Q}_p}$ such that $r(\tilde{\sigma}) = 1$ (see section 7.4.5).

Theorem 7.13. — *With the above notations. There exist some quasi-isomorphisms of complexes in the derived category of abelian torsion étale sheaves over $J_m \times U^{n,d}$*

$$\bar{\varpi}^*_N R\Psi_\eta R(\varphi_M\varphi_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq \bar{\mathfrak{p}}^* R\Psi_\eta R(\delta_M\delta_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}),$$

compatible with the actions of $W^+_{\mathbb{Q}_p} \times S \times GL_h(\mathbb{Q}_p)^+$ and the changes of level M, g.

Proof. — In propositions 7.7, 7.11 and 7.12, we proved that the quasi-isomorphisms ζ's over $J_m \times \bar{U}^{n,d}_{M,g}$

$$\bar{\varpi}^*_N R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}) \simeq \bar{\mathfrak{p}}^* R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta}),$$

have the required properties. By applying the derived functor associated to the (proper) projections $J_m \times \bar{U}^{n,d}_{M,g} \to J_m \times \bar{U}^{n,d}$, we obtain the quasi-isomorphisms in the statement (after using the proper base change theorem and part (2) of proposition 2.22). □

8. The cohomology of Shimura varieties

In this last section, we shall compute the l-adic cohomology of the Shimura varieties as a (virtual) representation of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$ $(l \neq p)$, in terms of the l-adic cohomologies of the Igusa varieties and of the Rapoport-Zink spaces.

More precisely, we shall apply theorem 5.13 to the complex of A-R l-adic étale sheaves

$$\mathcal{L} = (R\Psi_\eta R(\varphi_M\varphi_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta})_{|\bar{X}^{(\alpha)}\times\bar{\mathbb{F}}_p})_r$$

over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$, for each Newton polygon α, to relate the cohomology groups with compact supports of the Newton polygon strata to the cohomology of the Igusa varieties and of the Rapoport-Zink space, in the case of level structure at p. As the Newton polygon stratum varies in the stratification of the reduction of the Shimura varieties and the level (both at p and away from p) changes, the above descriptions combine into a formula which computes the cohomology of the Shimura varieties, in terms of the cohomologies of the Igusa varieties and of the Rapoport-Zink spaces.

We choose to formulate the main result of this section (Theorem 8.11) as an equality of virtual l-adic representations of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$, even though what we really prove is a stronger version of this result which regards the corresponding $\mathbb{Z}_l$-representations and can be formulated as the existence of quasi-isomorphisms in the derived category of A-R l-adic systems, compatible with the action of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$. Indeed, we prove more, as we prove the existence of such quasi-isomorphisms for l^r-torsion coefficients, for all $r \geq 1$ (which translates in a result regarding the corresponding $\mathbb{Z}/l^r\mathbb{Z}$-representations).

On the other hand, since the cohomology groups of Shimura varieties both with $\mathbb{Z}/l^r\mathbb{Z}$-coefficients and with $\mathbb{Z}_l$-coefficients are not *a priori* admissible representations of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$, the above two results in the derived categories can not be stated as equalities in the appropriate Grothendieck groups, which is why we prefer to state the theorem for l-adic coefficients.

8.1. The Newton polygon decomposition. — We shall start by explaining how it is possible to compute the l-adic cohomology of the Shimura varieties in terms of the cohomology of the Newton polygon strata of their special fibers, with coefficients in the l-adic vanishing cycles sheaves.

8.1.1. We recall that our final goal is to study the virtual $\mathbb{Q}_l$-representation of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$

$$H^\bullet(X, \mathbb{Q}_l) = \sum_i (-1)^i H^i(X, \mathbb{Q}_l),$$

where

$$H^i(X, \mathbb{Q}_l) = \varinjlim_U H^i_{et}(X_U \times_E (\hat{E}^{nr}_u)^{ac}, \mathbb{Q}_l),$$

and U varies among the sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$. (In the following we shall consistently use the upper index $\bullet$ to denote the corresponding alternating sum of representations inside the appropriate Grothendieck group.)

Let us also observe that if we restrict our attention to the open compact subgroups of $G(\mathbb{A}^\infty)$ of the form

$$U = U^p(M) = U^p \times \mathbb{Z}_p^\times \times \ker(\mathcal{O}^\times_{B_u^{op}} \to (\mathcal{O}_{B_u^{op}}/u^M)^\times),$$

for some integer $M \geq 0$ and some sufficiently small open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$, then we compute

$$H^\bullet(X, \mathbb{Q}_l)^{\mathbb{Z}_p^\times} = \varinjlim_{U^p,M} H^\bullet_{et}(X_{U^p(M)} \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Q}_l).$$

8.1.2. In the following we restict our attention to such levels $U = U^p(M)$, and relay on the theory of vanishing cycles to express the cohomology of the Shimura varieties in terms of the the cohomology of the special fibers of their integral models.

Let $r \geq 1$. For any level $U^p(M)$, we consider the integral models of the Shimura varieties $\mathcal{X}_{U^p,M} = \mathcal{X}_{U^p,M,\mathbb{I}_h}$. Then, there exist quasi-isomorphisms

$$i_{U^p,M} : R\Gamma(X_U \times_E (\hat{E}_u^{nr})^{ac}, \mathbb{Z}/l^r\mathbb{Z}) \simeq R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M})_\eta})),$$

such that $g^* \circ i_{U^p,M} = i_{U^p,M} \circ g^*$, for all $g \in GL_h(\mathbb{Z}_p)$ and $p^{-1}\mathbb{I}_h^* \circ i_{U^p,M} = i_{U^p,M-1} \circ p^{-1}\mathbb{I}_h^*$, for $M \geq 1$.

Further more, for any $g \in GL_h(\mathbb{Q}_p)^+$ and $M \geq e = e(g)$, let us consider the integral models $\mathcal{X}_{U^p,M,g}$, together with the projections $\phi_{U^p,M,g} : \mathcal{X}_{U^p,M,g} \to \mathcal{X}_{U^p,M}$ and the morphisms $g : \mathcal{X}_{U^p,M,g} \to \mathcal{X}_{U^p,M-e}$. They give rise to quasi-isomorphisms

$$\phi^*_{U^p,M,g} : R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M})_\eta})) \simeq$$

$$\simeq R\Gamma(\bar{X}_{U^p,M,g} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M,g})_\eta})),$$

and to some morphisms

$$g^* : R\Gamma(\bar{X}_{U^p,M-e} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M})_\eta})) \to$$

$$\to R\Gamma(\bar{X}_{U^p,M,g} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M,g})_\eta})),$$

such that $(\phi^{*\,-1}_{U^p,M-e,g} \circ g^*) \circ i_{U^p,M} = i_{U^p,M-e} \circ g^*$, for all $g \in GL_h(\mathbb{Q}_p)^+$ and $M \geq e = e(g)$. We deduce that the morphisms $\phi^{*\,-1}_{U^p,M-e,g} \circ g^*$ define a quasi-action (i.e. an action via quasi-isomorphisms) of $GL_h(\mathbb{Q}_p)$ on the direct limit, as M varies, of the complexes $R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M})_\eta}))$. It is also clear that this quasi-action extends the previously defined action of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \subset GL_h(\mathbb{Q}_p)$.

On the other hand, as the integer r varies, the above complexes form an A-R l-adic system, also endowed with a quasi-action of $GL_h(\mathbb{Q}_p)$, as the level M varies. We us denote by

$$H^i(R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta})))$$

the i-th cohomology group of the A-R l-adic complex. Then, for all i, the $\mathbb{Q}_l$-vector spaces

$$\varinjlim_{U^p,M} H^i(R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

are admissible representations of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$, and there is an equality in the Grothendieck group of virtual l-adic representations of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$

$$H^\bullet(X, \mathbb{Q}_l)^{\mathbb{Z}_p^\times} = \varinjlim_{U^p,M} H^\bullet(R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

We remark that as virtual representations of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p} \subset G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$

$$\varinjlim_{U^p,M} H^\bullet(R\Gamma(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l =$$

$$= \varinjlim_{U^p,M} H^\bullet(\bar{X}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R^\bullet\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

8.1.3. We now consider the Newton polygon stratification of the reductions modulo p of the integral models $\mathcal{X}_{U^p,M,g}$ of the Shimura varieties. For any level U^p, M, g, the stratification gives rise to a sequence of exact triangles in the derived category computing the cohomology of the special fibers of the Shimura varieties in terms of the cohomology with compact supports of the corresponding Newton polygon strata (see [**11**], Theorem I.8.7(3), pp. 91–94). Since both the morphisms corresponding to changes of level and the action of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times GL_h(\mathbb{Q}_p)^+ \times W_{\mathbb{Q}_p}$ preserve the Newton polygon stratification of the special fibers, the corresponding exact triangles in the derived category are compatible under the morphisms $\phi^*_{U^p,M,g}$, g^* (for any $g \in GL_h(\mathbb{Q}_p)^+$) and the group action.

We deduce that, for all Newton polygons α, the morphisms $\phi^{*-1}_{U^p,M,g} \circ g^*$ define a quasi-action of $GL_h(\mathbb{Q}_p)$ on the direct limit (as the level M varies) of the complexes

$$R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}}),$$

and also of the corresponding complexes of A-R l-adic systems

$$R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}}).$$

Thus, we obtain the following decomposition of the cohomology of Shimura varieties.

Proposition 8.1. — *There is an equality of virtual $\mathbb{Q}_l$-representations of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$*

$$H^\bullet(X, \mathbb{Q}_l)^{\mathbb{Z}_p^\times} =$$

$$= \sum_\alpha \varinjlim_{U^p,M} H^\bullet(R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

where U^p varies among the sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$ and M among the positive integers.

We remark that as virtual representations of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p} \subset G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$

$$\varinjlim_{U^p,M} H^\bullet(R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l =$$

$$= \varinjlim_{U^p,M} H^\bullet_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R^\bullet\Psi_\eta(\mathbb{Z}_{l/(\mathfrak{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

8.1.4. We now focus our attention on the cohomology groups

$$H^i(R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}_{l/(\mathcal{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}}))$$

and use theorems 5.13 and 7.13 to relate them to the cohomology of the Igusa varieties and of the Rapoport-Zink spaces of the same level.

First, we recall some notations. For all $i \geq 0$, we write

$$H^i_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_m H^i_c(J_{\alpha,U^p,m}, \mathbb{Z}/l^r\mathbb{Z})$$

for the cohomology groups with coefficients in $\mathbb{Z}/l^r\mathbb{Z}$ of the Igusa varieties of level U^p and Newton polygon α, viewed as a module endowed with an action of $T_\alpha \times W_{\mathbb{Q}_p}$ (we recall that the action of the Weil group is unramified).

We also write $\mathcal{M}_{\alpha,M,g}$ (and $\mathcal{M}_{\alpha,M} = \mathcal{M}_{\alpha,M,\mathbb{I}_h}$) for the formal Rapoport-Zink space, of level M, g ($M \geq e(g)$) and Newton polygon α, and $\{U^{n,d}_{\alpha,M,g}\}_{n,d}$ for our usual choice of an open cover of $\mathcal{M}_{\alpha,M,g}$. Thus, for any abelian torsion étale sheaf $\mathcal{F}$ (with torsion orders prime to p), we have

$$H^i_c(\bar{\mathcal{M}}_{\alpha,M,g}, \mathcal{F}) = \varinjlim_{n,d} H^i_c(\bar{U}^{n,d}_{\alpha,M,g}, \mathcal{F}_{|\bar{U}^{n,d}_{\alpha,M,g}}).$$

We view the above cohomology groups as representations of $T_\alpha \times W_{\mathbb{Q}_p}$, where the action of T_α is the one induced by the opposite of the action of T_α we considered so far (see section 2.5.14).

Theorem 8.2. — *Let α be a Newton polygon of dimension q and height h. For any sufficiently small open compact subgroup $U^p \subset G(\mathbb{A}^{\infty,p})$, and any integers $M \geq 0$ and $r \geq 1$, there are some isomorphisms of $\mathbb{Z}/l^r\mathbb{Z}$-representations of $GL_h(\mathbb{Z}_p) \times W_{\mathbb{Q}_p}$*

$$H^i(R\Gamma_c(\bar{\mathcal{M}}_{\alpha,M}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{\alpha,M})_\eta})) \otimes^L_{\mathcal{H}_r(T)} R\Gamma_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z})) \simeq$$

$$\simeq H^i(R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{U^p,M})_\eta})_{|\bar{X}^{(\alpha)}_{U^p,M}})),$$

for all $i \geq 0$.

As the levels U^p, M vary, the above representations are endowed with an action of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$, and the two actions on the direct limit representations are compatible under the above isomorphisms.

Moreover, the above isomorphisms are compatible with the natural projections as the integer $r \geq 1$ varies.

Proof. — The equality $\bar{\pi}_N \circ (1 \times \widetilde{Fr})^{NB} = \dot{\pi}_N$, together with the fact that $\widetilde{Fr}$ is purely inseparable and finite, implies that, for any abelian étale sheaf $\mathcal{L}$ over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ (with torsion orders relatively prime to p), we have

$$\dot{\pi}_{N\,!}\dot{\pi}^*\mathcal{L} \simeq \bar{\pi}_{N\,!}(1 \times \widetilde{Fr})^{NB}_!(1 \times \widetilde{Fr})^{NB\,*}\bar{\pi}^*_N\mathcal{L} \simeq \bar{\pi}_{N\,!}\bar{\pi}^*_N\mathcal{L}.$$

Thus, in all the constructions of section 5, we can replace $\dot{\pi}_N$ with $\bar{\pi}_N$.

Let $g \in GL_h(\mathbb{Q}_p)^+$ ($M \geq e(g)$, and consider the complex of torsion abelian sheaves

$$\mathcal{L} = R\Psi_\eta(R(\varphi_M\varphi_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}))$$

over $\bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$. Then, theorems 5.13 and 7.13 (together with Berkovich's comparison theorem which allows us to identify classical vanishing cycles with rigid analytic vanishing cycles) imply the existence of quasi-isomorphisms, compatible with the action of $W_{\mathbb{Q}_p}$

$$R\Gamma_c(\bar{\mathcal{M}}_\alpha, R\Psi_\eta R(\delta_M \delta_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{M,g})_\eta})) \otimes^L_{\mathcal{H}_r(T)} R\Gamma_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}) \simeq$$

$$\simeq R\Gamma_c(\bar{X}^{(\alpha)}_{U^p(0)} \times \bar{\mathbb{F}}_p, R\Psi_\eta R(\varphi_M \varphi_{M,g})_*(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathfrak{X}_{M,g})_\eta}))_{|\bar{X}^{(\alpha)}_{U^p(0)} \times \bar{\mathbb{F}}_p}).$$

By part (2) of proposition 2.22, we can rewrite the above quasi-isomorphisms as

$$R\Gamma_c(\bar{\mathcal{M}}_{M,g}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{M\,\eta}})) \otimes^L_{\mathcal{H}_r(T)} R\Gamma_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z}) \simeq$$

$$\simeq R\Gamma_c(\bar{X}^{(\alpha)}_{U^p,M,g} \times \bar{\mathbb{F}}_p, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathfrak{X}_{M\,\eta}}))_{|\bar{X}^{(\alpha)}_{U^p} \times \bar{\mathbb{F}}_p}).$$

These quasi-isomorphisms commute with the action of $GL_h(\mathbb{Z}_p)$ on the two hand side when $g = \mathbb{I}_h$, and also with the maps induced by the projections $\delta_{M,g}$, $\phi_{U^p,M,g}$, and by the morphisms associated to the elements $g \in GL_h(\mathbb{Q}_p)^+$.

In particular, when $g = \mathbb{I}_h$, the above quasi-isomorphisms give rise to the isomorphisms of $\mathbb{Z}/l^r\mathbb{Z}$-representations of $GL_h(\mathbb{Z}_p) \times W_{\mathbb{Q}_p}$ in the statement.

Further more, we deduce that the morphisms $(\delta_{M,g} \times 1)^{*-1} \circ g^*$ define a quasi-action of $GL_h(\mathbb{Q}_p)$ on the direct limit (as M varies) of the complexes

$$R\Gamma_c(\bar{\mathcal{M}}_{M,g}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/\mathcal{M}_{M\,\eta}})) \otimes^L_{\mathcal{H}_r(T)} R\Gamma_c(J_{U^p}, \mathbb{Z}/l^r\mathbb{Z}),$$

which is compatible under the above isomorphisms with the quasi-action of the right hand side. Thus, we conclude. □

8.1.5. We remark that, for all level U^p, M, there are $GL_h(\mathbb{Z}_p) \times W_{\mathbb{Q}_p}$-equivariant spectral sequences

$$\oplus_{s+t+q=p'} Tor^p_{\mathcal{H}_r(T_\alpha)}(H^t_c(\bar{\mathcal{M}}_{\alpha,M}, R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), H^s_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}))$$

which abut the representations

$$H^n(R\Gamma_c(\bar{\mathcal{M}}_{\alpha,M}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}_{/(\mathcal{M}_{\alpha,M})_\eta})) \otimes^L_{\mathcal{H}_r(T)} R\Gamma_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z})),$$

for all $n = p + p' \geq 0$.

As U^p, M vary, the $\mathbb{Z}/l^r\mathbb{Z}$-modules

$$\varinjlim_{U^p,M} \oplus_{s+t+q=p'} Tor^p_{\mathcal{H}_r(T_\alpha)}(H^t_c(\bar{\mathcal{M}}_{\alpha,M}, R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), H^s_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}))$$

are naturally endowed with an action of $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$. Moreover, this action induces an action on the limit of convergence which simply is the restriction to $G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$ of the action of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$.

8.2. The cohomology of the Rapoport-Zink spaces. — In this section, we shall study the cohomology of the Rapoport-Zink more closely. In particular, we have two goals in mind. On one hand, we want to relate the cohomology of the rigid analytic Rapoport-Zink spaces to the cohomology of their special fibers, with coeffiecients in the vanishing cycles sheaves, as it appears in theorem 8.2. On the other hand, there is the proof of the admissibility of the representations associated to the cohomology groups of the Igusa varieties and the Rapoport-Zink spaces, which we prove in lemma 8.9. (The admissibiblity of these representations is an obviuos prerequisite for describing our final result as an equality of virtual representations.)

We are very grateful to L. Fargues for explaining to us the results of this section and correcting an early mistake.

8.2.1. Let α be a Newton polygon of dimension q and height h. We denote by $\mathcal{M}_\alpha$ the Rapoport-Zink space (without level structure) associated to the Barsotti-Tate group $\Sigma^{(\alpha)}$ and, for any positive integer M, we write $\mathcal{M}^{\mathrm{rig}}_{\alpha,M}$ for the Rapoport-Zink rigid analytic space of level M over $\mathcal{M}^{\mathrm{rig}}_\alpha$.

8.2.2. We start by considering the case of cohomology with l^r-torsion coefficients, for any $r \geq 1$. We recall that the j-th cohomology group of the Rapoport-Zink spaces associated to the Newton polygon α is defined to be the representation of $T_\alpha \times GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$H^j_c(\mathcal{M}_\alpha, \mathbb{Z}/l^r\mathbb{Z}) = \varinjlim_M H^j_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z})$$

(see section 2.5), where the action of T_α is the one induced by the left action of T_α on $\mathcal{M}^{\mathrm{rig}}_{\alpha,M}$ (for all M) (see [**28**], Remark 1.3(i), p. 425).

On the other hand, because of theorem 8.2, we are induced to consider the $\mathbb{Z}/l^r\mathbb{Z}$-representations of $T_\alpha \times \langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$

$$H^j_c(\bar{\mathcal{M}}_\alpha \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) = \varinjlim_M H^j_c(\bar{\mathcal{M}}_{\alpha,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})),$$

for all $j, q \geq 0$.

The goal of this section is to understand the relation between the cohomology groups of the rigid analytic Rapoport-Zink spaces and these modules.

More precisely, we shall establish a link between the $W_{\mathbb{Q}_p}$-representations

$$Tor^k_{\mathcal{H}_r(T_\alpha)}(H^j_c(\bar{\mathcal{M}}_{\alpha,M} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), \Pi)$$

and

$$Ext^s_{T_\alpha - smooth}(H^t_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), \Pi)$$

for any smooth $\mathbb{Z}/l^r\mathbb{Z}$-representation Π of T_α, and all levels M. (The above derived functors are computed in the category of smooth $\mathbb{Z}/l^r\mathbb{Z}$-representations of $T_\alpha \times W_{\mathbb{Q}_p}$).

8.2.3. Let us fix the level M of the formal Rapoport-Zink space $\mathcal{M}_{\alpha,M} = \mathcal{M}_{\alpha,M,\mathbb{I}_h}$, which we now simply denote by $\mathcal{M}$, and write $\{U^{n,d} = U^{n,d}_{\alpha,M}\}_{n,d}$ for the usual open cover of $\mathcal{M}$.

By proposition 4.6, there exist some positive integers m, n, d such that the morphism $\dot{\pi}_N : J_m \times_{\bar{\mathbb{F}}_p} \bar{U}^{n,d} \to \bar{X}^{(\alpha)} \times \bar{\mathbb{F}}_p$ is surjective on geometric points, for some $N \geq d/\delta B$. Equivalently, there exist some integers n, d such that the reduced fiber of the Rapoport-Zink space $\bar{\mathcal{M}}$ is covered by the opens ρU, as ρ varies in T and $U = \bar{U}^{n,d}$. Indeed, since the maximal open compact subgroup $\Gamma \subset T$ stabilizes the open U, it suffices to let ρ vary among a set of representatives of the cosets in T/Γ.

For any positive integer s, we set

$$(T/\Gamma)^{s+1}_{\neq} = \{t = (\rho_0\Gamma, \ldots, \rho_s\Gamma) \in (T/\Gamma)^{s+1} | \rho_i\Gamma \neq \rho_j\Gamma \forall i \neq j\}$$

and for any $t = (\rho_0\Gamma, \ldots, \rho_s\Gamma) \in (T/\Gamma)^{s+1}_{\neq}$, we write $tU = \rho_0 U \cap \cdots \cap \rho_s U$ and $j_{tU} : tU \hookrightarrow \bar{\mathcal{M}}$ for the natural inclusion.

Let $\mathcal{F}$ be an abelian étale torsion sheaf over $\bar{\mathcal{M}}$, with torsion orders prime to p. For any integer $s \geq 0$, we define an abelian étale torsion sheaf over $\bar{\mathcal{M}}$

$$C_s(U, \mathcal{F}) = \oplus_{t \in (T/\Gamma)^{s+1}_{\neq}} j_{tU\,!}(\mathcal{F}_{|tU}).$$

Moreover, let $t \in (T/\Gamma)^{s+1}_{\neq}$, $s \geq 1$, and write $t^i = (\rho_0\Gamma, \ldots, \widehat{\rho_i\Gamma}, \ldots, \rho_s\Gamma) \in (T/\Gamma)^{s}_{\neq}$, for all $i = 0, \ldots, s$. Then, the natural inclusions $tU \subset t^iU$ give rise to some morphisms

$$j_{tU\,!}(\mathcal{F}_{|tU}) \to j_{t^iU\,!}(\mathcal{F}_{|t^iU}),$$

and analogously, for any $\rho \in T$, the inclusion $\rho U \subset \bar{\mathcal{M}}$ gives rise to a morphism

$$j_{\rho U\,!}(\mathcal{F}_{|\rho U}) \to \mathcal{F}.$$

These maps induce some morphisms $C_s(U, \mathcal{F}) \to C_{s-1}(U, \mathcal{F})$ and $C_0(U, \mathcal{F}) \to \mathcal{F}$ such that the corresponding complex of sheaves over $\bar{\mathcal{M}}$

$$\cdots \to C_s(U, \mathcal{F}) \to \cdots C_0(U, \mathcal{F}) \to \mathcal{F} \to 0$$

is exact (we adopt the same alternating sign convention of the usual Cech complex). Equivalently, there is a quasi-isomorphism in the derived category of abelian torsion sheaves over $\bar{\mathcal{M}}$ between the complex $C.(U, \mathcal{F})$ and the sheaf $\mathcal{F}$.

8.2.4. Let us now focus our attention of the action of the group T on $\bar{\mathcal{M}}$. This action gives rise to an action of T on the cohomology groups $H^i_c(\bar{\mathcal{M}}, \mathcal{F})$ and $H^i_c(\bar{\mathcal{M}}, C_s(U, \mathcal{F}))$, for all i and s.

Moreover, for all $i, s \geq 0$, we can identify

$$H^i_c(\bar{\mathcal{M}}, C_s(U, \mathcal{F})) = \oplus_{t \in (T/\Gamma)^{s+1}_{\neq}} H^i_c(tU, \mathcal{F}),$$

where the action of T can be interpreted, on the right hand side, as arising from the isomorphisms

$$\gamma : tU \to \gamma tU,$$

which are induced by restriction from the action of γ on $\bar{\mathcal{M}}$, for all $\gamma \in T$ (if $t = (\rho_0\Gamma, \ldots, \rho_s\Gamma) \in (T/\Gamma)^{s+1}_{\neq}$, we write $\gamma t = (\gamma\rho_0\Gamma, \ldots, \gamma\rho_s\Gamma)$).

For any $s \geq 0$, let $t \in (T/\Gamma)^{s+1}_{\neq}$, $t = (\rho_0\Gamma, \ldots, \rho_p\Gamma)$. If $s \geq 1$, we write $\epsilon = \epsilon_t = [(\rho_0^{-1}\rho_1, \ldots, \rho^{-1}\rho_s)] \in \Gamma\backslash(T/\Gamma)^s_{\neq}$ and $U_\epsilon = U \cap \rho_0^{-1}\rho_1 U \cap \cdots \cap \rho_0^{-1}\rho_s U$. If $s = 0$, we write $\Gamma\backslash(T/\Gamma)^0_{\neq} = \{\bar{\epsilon}\}$ and $U_{\bar{\epsilon}} = U$.

Then, the action of $\rho_0 \in T$ gives rise to some isomorphisms

$$H^i_c(tU, \mathcal{F}) \simeq H^i_c(U_\epsilon, \mathcal{F}),$$

for all i, and thus to an identification of T-representations

$$\oplus_{t\in(T/\Gamma)^{s+1}_{\neq}} H^i_c(tU, \mathcal{F}) = \oplus_{\epsilon\in\Gamma\backslash(T/\Gamma)^s_{\neq}} c - Ind^T_{\Gamma_\epsilon}(H^i_c(U_\epsilon, \mathcal{F})),$$

where $\Gamma_\epsilon = \Gamma \cap \epsilon_1\Gamma\epsilon_1^{-1} \cap \cdots \cap \epsilon_s\Gamma\epsilon_s^{-1}$ is an open compact subgroup of T, for any $\epsilon = [(\epsilon_1, \ldots, \epsilon_s)] \in \Gamma\backslash(T/\Gamma)^s_{\neq}$ and $s \geq 0$ (for $s = 0$, we write $\Gamma_{\bar{\epsilon}} = \Gamma$).

8.2.5. Let $s \geq 1$, and for each $\epsilon = [(\epsilon_1, \ldots, \epsilon_s)] \in \Gamma\backslash(T/\Gamma)^s_{\neq}$ consider the open $U_\epsilon \subset \mathcal{M}$. We claim that the set

$$\{\epsilon \in \Gamma\backslash(T/\Gamma)^s_{\neq} | U_\epsilon \neq \varnothing\}$$

is finite, and empty for s sufficiently large. In fact, for any $\rho \in T$, the set $U \cap \rho U = \varnothing$ unless $\rho \in T^{d,d}$, i.e. unless both $p^d\rho$ and $p^d\rho^{-1}$ are isogenies. (This fact follows from the equalities $p^d\rho = (p^{d-n}\beta^{-1})(p^n\beta\rho)$ and $p^d\rho^{-1} = (p^{d-n}\rho^{-1}\beta^{-1})(p^n\beta)$, for some $\beta \in U \cap \rho U$.) Thus, if $U_\epsilon \neq \varnothing$, then $\epsilon \in \Gamma\backslash(T^{d,d}/\Gamma)^s_{\neq}$. Since $T^{d,d} \subset T$ is compact, the set $T^{d,d}/\Gamma$ is finite and moreover $(T^{d,d}/\Gamma)^s_{\neq} = \varnothing$ if $s > \#(T^{d,d}/\Gamma)$.

It follows that the complex $C.(U, \mathcal{F})$ is bounded (indeed $C_s(U, \mathcal{F}) = 0$ if $s > \#(T^{d,d}/\Gamma)$), and thus the above considerations give rise to the following proposition.

Proposition 8.3. — *Let $\mathcal{F}$ be an object in the derived category of abelian l^r-torsion sheaves over $\bar{\mathcal{M}}$.*

Then, there is a spectral sequence of $\mathbb{Z}/l^r\mathbb{Z}$-representations of $T \times W_{\mathbb{Q}_p}$

$$E_1^{s,t} = \oplus_{\epsilon\in\Gamma\backslash(T/\Gamma)^s_{\neq}} c - Ind^T_{\Gamma_\epsilon} H^t_c(U_\epsilon, \mathcal{F}) \Rightarrow H^{s+t}_c(\bar{\mathcal{M}}, \mathcal{F}).$$

In particular, if $\mathcal{F}$ is a constructible sheaf, the representations $H^k_c(\bar{\mathcal{M}}, \mathcal{F})$ are smooth of finite type for the action of T, for all $k \geq 0$.

Proof. — The above spectral sequence arises from the quasi-isomorphism in the derived category

$$R^{\cdot}\Gamma_c(\bar{\mathcal{M}}, \mathcal{F}) \simeq R^{\cdot}\Gamma_c(\bar{\mathcal{M}}, C.(U, \mathcal{F})) \simeq \oplus_{\epsilon\in\Gamma\backslash(T/\Gamma)^{\cdot}_{\neq}} c - Ind^T_{\Gamma_\epsilon} R^{\cdot}\Gamma_c(U_\epsilon, \mathcal{F}).$$

For $\mathcal{F}$ a constructible sheaf , the cohomology groups $H^t_c(U_\epsilon, \mathcal{F})$ are finite, for all $\epsilon, t \geq 0$. Moreover, the set $\Gamma\backslash(T/\Gamma)^s_{\neq}$ is finite, for any $s \geq 0$, and thus the representations $\oplus_{\epsilon\in\Gamma\backslash(T/\Gamma)^s_{\neq}} c - Ind^T_{\Gamma_\epsilon} H^t_c(U_\epsilon, \mathcal{F})$ are smooth of finite type for the action of T. It follows from the fact that the category of smooth representations of finite type of T is noetherian and closed under extensions (see [**8**]) that the representations $H^k_c(\bar{\mathcal{M}}, \mathcal{F})$ are also smooth of finite type. □

Lemma 8.4. — *Let Λ denote $\mathbb{Z}/l^r\mathbb{Z}$ (for some $r \geq 1$) or $\mathbb{Q}_l$, and denote by $\Lambda(T_\alpha)$ the Hecke algebra of T_α with coefficients in Λ.*

Let M be a finite smooth Λ-representation of $K \times GL_h(\mathbb{Z}_p) \times W_{\mathbb{Q}_p}$, for $K \subset T$ a open compact subgroup, and denote by $M^\vee$ its dual.

For any admissible Λ-representation Π of $T \times W_{\mathbb{Q}_p}$, and any integer $j \geq 0$, there exists an isomorphism

$$Ext^j_{T-smooth}\,(c-Ind_K^T M^\vee, \Pi) \simeq Tor^j_{\Lambda(T)}\,(c-Ind_K^T M, \Pi),$$

equivariant for the action of $W_{\mathbb{Q}_p}$.

Moreover, the above modules are finite and vanish for j sufficiently large.

Proof. — The vanishing of the modules $Ext^j_{T-smooth}\,(c-Ind_K^T M^\vee, \Pi)$ for j sufficiently large (e.g. j greater than the rank of T) is proved in [**10**] (lemma 4.3.12, pp. 69-70).

On the other hand, if $F_. \to \Pi$ is a projective resolution of Π, then one can use such resolution to compute both

$$Tor^j_{\mathcal{H}_r(T)}\,(c-Ind_K^T M, \Pi) = H^j(c-Ind_K^T M \otimes_{\Lambda(T)} F_.)$$

and

$$Ext^j_{T-smooth}\,(c-Ind_K^T M^\vee, \Pi) = H^j(Hom_{T-smooth}(c-Ind_K^T M^\vee, F_.)).$$

Thus, in order to conclude, it suffices to prove that for any smooth projective representation F, there exists an isomorphism

$$c-Ind_K^T M \otimes_{\Lambda(T)} F \simeq Hom_{T-smooth}(c-Ind_K^T M^\vee, F),$$

equivatiant for the action of $GL_h(\mathbb{Z}_p) \times W_{\mathbb{Q}_p}$, and that the above modules are finite.

By Frobenius reciprocity, we have a canonical isomorphism

$$Hom_{T-smooth}(c-Ind_K^T M^\vee, F) \simeq Hom_{K-smooth}(M^\vee, F_{|K}),$$

where we denote by $F_{|K}$ the restriction of the T-representation F to the compact subgroup $K \subset T$. Moreover, since the K-representation M is smooth and finite, there exists an open compact normal subgroup $K' \subset K$ such that $M = M^{K'}$ (and thus also $M^\vee = (M^\vee)^{K'}$). We deduce that

$$Hom_{K-smooth}(M^\vee, F_{|K}) \simeq Hom_{K-smooth}(M^\vee, F^{K'}),$$

which is indeed finite since the representation F is admissible.

Further more, we can rewrite

$$Hom_{T-smooth}(c-Ind_K^T M^\vee, F) \simeq Hom_\Lambda(M^\vee, F)^K \simeq (M^{op} \otimes_{\mathbb{Z}/l^r\mathbb{Z}} F)^K$$

and, analoguously,

$$c-Ind_K^T M \otimes_{\Lambda(T)} F \simeq (M^{op} \otimes_\Lambda F)_K.$$

For any Λ-representation V of K, the morphism $e_K : V \to V$ gives rise to an isomorphism $V_K \simeq V^K$. □

It is a direct consequence of the above lemma that, for any smooth $\mathbb{Z}/l^r\mathbb{Z}$-representations V of T of finite type and any admissible representations Π, the modules $Tor^j_{\mathcal{H}_r(T)}(V,\Pi)$ are finite. (In fact, any smooth representation of finite type V admits a resolution by representations of the form $\oplus_{i\in I} c-Ind^T_{K_i} M_i$, for some open compact subgroups $K_i \subset T$, some finite free $\mathbb{Z}/l^r\mathbb{Z}$-modules M_i, endowed with an action of K_i, and some finite set I.)

Moreover, it also follows from lemma 4.3.12 in [**10**] that the above modules vanish for j sufficiently large.

In particular, one can combine the above lemma and proposition 8.3 to obtain the following corollary.

Corollary 8.5. — *Let $\mathcal{F}$ be a constructible sheaves over $\bar{\mathcal{M}}$ and Π be an admissible $\mathbb{Z}/l^r\mathbb{Z}$-representations of $T\times W_{\mathbb{Q}_p}$.*

There is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $W_{\mathbb{Q}_p}$.

$$Tor^\bullet_{\mathcal{H}_r(T)}(H^\bullet_c(\bar{\mathcal{M}},\mathcal{F}),\Pi)=\sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^\bullet_{\neq}} Tor^\bullet_{\mathcal{H}_r(T)}(c-Ind^T_{\Gamma_\epsilon} H^\bullet_c(U_\epsilon,\mathcal{F}),\Pi).$$

8.2.6. Let us remark that, in particular, the above corallary applies to the case of $\mathcal{F}=R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})$, for any $q\geq 0$. (The complex of vanishing cycles sheaves of the integral model $\mathcal{M}$ is indeed bounded and constructible since, locally on $\bar{\mathcal{M}}$, they can be identified to the vanishing cycles sheaves of some schemes of finite type.)

Further more, in the case of the complex $\mathcal{F}=R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})$, we obtain the following result. (We reintroduce the level M in our notations.)

Corollary 8.6. — *Let Π be an admissible $\mathbb{Z}/l^r\mathbb{Z}$-representations of $T\times W_{\mathbb{Q}_p}$.*

There is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle\times W_{\mathbb{Q}_p}\subset GL_h(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$

$$\varinjlim_M Tor^\bullet_{\mathcal{H}_r(T)}(H^\bullet_c(\bar{\mathcal{M}}_M,R^\bullet\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})),\Pi)=$$

$$=\sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^\bullet_{\neq}}\varinjlim_M Tor^\bullet_{\mathcal{H}_r(T)}(c-Ind^T_{\Gamma_\epsilon} H^\bullet_c(U_{\epsilon,M},R^\bullet\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})),\Pi).$$

Proof. — It suffices to remark that the quasi-isomorphisms in the derived category

$$R^\cdot\Gamma_c(\bar{\mathcal{M}}_M,R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}))\simeq\oplus_{\epsilon\in\Gamma\backslash(T/\Gamma)^\cdot_{\neq}} c-Ind^T_{\Gamma_\epsilon} R^\cdot\Gamma_c(U_{\epsilon,M},R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z}))$$

are equivariant under the action of $GL_h(\mathbb{Z}_p)$ and the morphisms $p^{-1}\mathbb{I}_h^*$, as M varies. □

Since we are interested in applying the above corollary to the case of Π equal to the cohomology of the Igusa varieties, let us recall that the representations of $T\times W_{\mathbb{Q}_p}$ associated to the cohomology of the Igusa varieties of level U^p (for any sufficiently small open compact subgroup U^p of $G(\mathbb{A}^{\infty,p})$) are indeed admissible (see remark 3.9).

8.2.7. We now focus our attention on the cohomology groups

$$R\Gamma_c^j(U_\epsilon, R^q\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})),$$

for all integer $j, q \geq 0$ and any $\epsilon \in \Gamma\backslash(T/\Gamma)^s_{\neq}$, for some $s \geq 0$, as representations of $\Gamma_\epsilon \times GL_h(\mathbb{Z}_p) \times W_{\mathbb{Q}_p}$. We recall that $\Gamma_\epsilon \subset T$ is an open compact subgroup and U_ϵ an open subscheme, of finite type, of the reduction $\bar{\mathcal{M}}$ of the formal Rapoport-Zink space $\mathcal{M} = \mathcal{M}_{\alpha,M,g}$.

For all ϵ, we denote by U_ϵ^{cl} the closure of the open $U_\epsilon \subset \bar{\mathcal{M}}$. Then, the complement $U_\epsilon^{cl} - U_\epsilon$ is also a closed subscheme of $\bar{\mathcal{M}}$. In particular, both U_ϵ^{cl} and $U_\epsilon^{cl} - U_\epsilon$ proper schemes.

For any object $\mathcal{F}$ in the derived category of abelian torsion étale sheaves, the inclusion $U_\epsilon \subset U_\epsilon^{cl}$ gives rise to an exact triangle

$$\begin{tikzcd}
R\Gamma_c(U_\epsilon, \mathcal{F}) \arrow[rr] & & R\Gamma(U_\epsilon^{cl}, \mathcal{F}) \arrow[dl] \\
& R\Gamma(U_\epsilon^{cl} - U_\epsilon, \mathcal{F}) \arrow[ul, "+1"] &
\end{tikzcd}$$

and in particular, for $\mathcal{F} = R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})$, part (2) of proposition 2.22 implies that there is an exact triangle

$$\begin{tikzcd}
R\Gamma_c(U_\epsilon, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \arrow[r] & R\Gamma(sp^{-1}U_\epsilon^{cl} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}) \arrow[d] \\
& R\Gamma(sp^{-1}(U_\epsilon^{cl} - U_\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}) \arrow[ul, "+1"]
\end{tikzcd}$$

where $sp : \mathcal{M}_{\alpha,M}^{\mathrm{rig}} = (\mathcal{M}_{\alpha,M})_\eta \to \bar{\mathcal{M}} = (\mathcal{M}_{\alpha,M})_s$ denotes the reduction map (see section 2.7.1).

Let us recall that the rigid analytic spaces $sp^{-1}U_\epsilon^{cl}$ and $sp^{-1}(U_\epsilon^{cl} - U_\epsilon) = sp^{-1}U_\epsilon^{cl} - sp^{-1}U_\epsilon$ are by definition some open subspaces of $\mathcal{M}^{\mathrm{rig}} = \mathcal{M}_{\alpha,M}^{\mathrm{rig}}$, and therefore in particular they are smooth. Thus, Poincaré duality imples that

$$H^q(sp^{-1}U_\epsilon^{cl} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(D)) \simeq H_c^{2D-q}(sp^{-1}U_\epsilon^{cl} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z})^\vee,$$

and

$$H^q(sp^{-1}(U_\epsilon^{cl} - U_\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(D)) \simeq H_c^{2D-q}(sp^{-1}(U_\epsilon^{cl} - U_\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z})^\vee,$$

where $D = q(h-q)$ the dimension of the Rapoport-Zink spaces and $M^\vee$ denotes the dual of M, for any $\mathbb{Z}/l^r\mathbb{Z}$-module M.

8.2.8. The cohomology with compact supports of the rigid analytic space $\mathcal{M}^{\mathrm{rig}}$ can be described in terms of the above cohomology with compact supports of the opens $sp^{-1}U_\epsilon^{cl}$. Indeed, in [**10**] (section 4.3, pp. 67 – 72) Fargues proves the analogue of proposition 8.3 for the cohomology with compact support of the rigid analytic space $\mathcal{M}^{\mathrm{rig}}$. In his result, the role of the open $U = \bar{U}^{n,d} \subset \bar{\mathcal{M}}$ is played by any $V \subset \mathcal{M}^{\mathrm{rig}}$,

such that $\mathcal{M}^{\mathrm{rig}} = \cup_{\rho\in T}\,\rho V$, for V either an open subset or a closed analytic domain (e.g. $V = sp^{-1}U^{cl}, sp^{-1}U$).

In particular, there is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $W_{\mathbb{Q}_p}$

$$Ext^{\bullet}_{T-smooth}\,(H^{\bullet}_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi) =$$

$$= \sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} Ext^{\bullet}_{T-smooth}\,(c-Ind^T_{\Gamma_\epsilon}\,H^{\bullet}_c(V_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi),$$

for any admissible $\mathbb{Z}/l^r\mathbb{Z}$-representation Π of $T \times W_{\mathbb{Q}_p}$.

8.2.9. Let us now reintroduce the level at p, M, in our notations and consider how the previous construction behaves, as M varies, under the action of $GL_h(\mathbb{Q}_p)$.

Let V_0 be an open subset or a closed analytic domain of the Rapoport-Zink space with no level structure $\mathcal{M}^{\mathrm{rig}}_0$ such that $\mathcal{M}^{\mathrm{rig}}_0 = \cup_{\rho\in T}\,\rho V_0$. We denote by V_M the pullback of V_0 under the projection $\mathcal{M}^{\mathrm{rig}}_M \to \mathcal{M}^{\mathrm{rig}}_0$, for all $M \geq 1$. Then, $\mathcal{M}^{\mathrm{rig}}_M = \cup_{\rho\in T}\,\rho V_M$ and the action of $GL_h(\mathbb{Q}_p)$ on the system of rigid analytic Rapoport-Zink spaces preserves such decompositions.

We deduce that, for any admissible $\mathbb{Z}/l^r\mathbb{Z}$-representation Π of $T \times W_{\mathbb{Q}_p}$, there is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M Ext^{\bullet}_{T-smooth}\,(H^{\bullet}_c(\mathcal{M}^{\mathrm{rig}}_M \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi) =$$

$$= \sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} \varinjlim_M Ext^{\bullet}_{T-smooth}\,(c-Ind^T_{\Gamma_\epsilon}\,H^{\bullet}_c(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi).$$

Finally, we remark that the analoguous results also hold for the l-adic cohomology groups. More precisely, in [**10**] Fargues proves that, for all $M \geq 0$, there is a spectral sequence of $\mathbb{Q}_l$-representations of $T \times W_{\mathbb{Q}_p}$

$$E_1^{s,t} = \oplus_{\epsilon\in\Gamma\backslash(T/\Gamma)^{s}_{\neq}} c-Ind^T_{\Gamma_\epsilon} H^t_c(V_{\epsilon,M}, \mathbb{Q}_l(-D)) \Rightarrow H^{s+t}(\mathcal{M}^{\mathrm{rig}}_M, \mathbb{Q}_l(-D)),$$

and thus the equality of virtual $\mathbb{Q}_l$-representations of $GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M Ext^{\bullet}_{T-smooth}\,(H^{\bullet}_c(\mathcal{M}^{\mathrm{rig}}_M \times_K \bar{K}, \mathbb{Q}_l(-D)), \Pi) =$$

$$= \sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} \varinjlim_M Ext^{\bullet}_{T-smooth}\,(c-Ind^T_{\Gamma_\epsilon}\,H^{\bullet}_c(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Q}_l(-D)), \Pi),$$

for any admissible $\mathbb{Z}/l^r\mathbb{Z}$-representation Π of $T \times W_{\mathbb{Q}_p}$. (We recall that the l-adic cohomology groups of the Rapoport-Zink spaces are defined as the smooth $\mathbb{Q}_l$-representations of $T_\alpha \times W_{\mathbb{Q}_p}$

$$H^j_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M}, \mathbb{Q}_l) = \varinjlim_{n,d}(\varprojlim_r H^j_c((U^{n,d}_{\alpha,M})^{\mathrm{rig}} \times_K \times\bar{K}, \mathbb{Z}/l^r\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l),$$

for all $j \geq 0$ and any level $M \geq 0$.)

Theorem 8.7. — *Let Π be an admissible $\mathbb{Z}/l^r\mathbb{Z}$-representation of $T \times W_{\mathbb{Q}_p}$.*

Then, there is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p} \subset GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M Tor^{\bullet}_{\mathcal{H}_r(T)}\left(H^{\bullet}_c(\bar{\mathcal{M}}_M, R^{\bullet}\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), \Pi\right) =$$

$$= \varinjlim_M Ext^{\bullet}_{T-smooth}\left(H^{\bullet}_c(\mathcal{M}^{\mathrm{rig}}_M \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right),$$

where $D = q(h-q)$ the dimension of the Rapoport-Zink space $\mathcal{M}$.

Proof. — Combining lemma 8.4 and corollary 8.6 with the equalities corresponding to the exact triangles in section 8.2.7 and Poincaré duality, we obtain (for all $M \geq 0$)

$$Tor^{\bullet}_{\mathcal{H}_r(T)}\left(H^{\bullet}_c(\bar{\mathcal{M}}_M, R^{\bullet}\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), \Pi\right) =$$

$$= \sum_{\epsilon \in \Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} Tor^{\bullet}_{\mathcal{H}_r(T)}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}_c(U_\epsilon, R^{\bullet}\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), \Pi\right) =$$

$$= \sum_{\epsilon \in \Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} \Bigg(Tor^{\bullet}_{\mathcal{H}_r(T)}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}(sp^{-1}U^{cl}_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), \Pi\right) +$$

$$-Tor^{\bullet}_{\mathcal{H}_r(T)}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}(sp^{-1}(U^{cl}_\epsilon - U_\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), \Pi\right)\Bigg) =$$

$$= \sum_{\epsilon \in \Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} \Bigg(Tor^{\bullet}_{\mathcal{H}_r(T)}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}_c(sp^{-1}U^{cl}_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D))^\vee, \Pi\right) +$$

$$-Tor^{\bullet}_{\mathcal{H}_r(T)}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}_c(sp^{-1}(U^{cl}_\epsilon - U_\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D))^\vee, \Pi\right)\Bigg) =$$

$$= \sum_{\epsilon \in \Gamma\backslash(T/\Gamma)^{\bullet}_{\neq}} \Bigg(Ext^{\bullet}_{T-smooth}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}_c(sp^{-1}U^{cl}_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right) +$$

$$-Ext^{\bullet}_{T-smooth}\left(c-Ind^T_{\Gamma_\epsilon} H^{\bullet}_c(sp^{-1}(U^{cl}_\epsilon - U_\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right)\Bigg).$$

Let us consider the open $V = sp^{-1}U^{cl}$ (resp. the closed analytic domain $V = sp^{-1}U_\epsilon$). Then, for all $\epsilon \in \Gamma\backslash(T/\Gamma)^s_{\neq}$, $s \geq 0$, we have $V_\epsilon = sp^{-1}U^{cl}_\epsilon$ (resp. $V_\epsilon = sp^{-1}U_\epsilon$). It follows that there are equalities of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M Ext^{\bullet}_{T-smooth}\left(H^{\bullet}_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right) =$$

$$= \varinjlim_M \sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^\bullet_{\neq}} Ext^\bullet_{T-smooth}\left(c-Ind^T_{\Gamma_\epsilon} H^\bullet_c(sp^{-1}U^{cl}_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right)$$

$$= \varinjlim_M \sum_{\epsilon\in\Gamma\backslash(T/\Gamma)^\bullet_{\neq}} Ext^\bullet_{T-smooth}\left(c-Ind^T_{\Gamma_\epsilon} H^\bullet_c(sp^{-1}U_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right).$$

Further more, the open inclusions $sp^{-1}(U^{cl}_\epsilon - U_\epsilon) = sp^{-1}U^{cl}_\epsilon - sp^{-1}U_\epsilon \hookrightarrow sp^{-1}U^{cl}_\epsilon$ give rise to the equalities

$$H^\bullet_c(sp^{-1}(U^{cl}_\epsilon - U\epsilon) \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi) =$$

$$= H^\bullet_c(sp^{-1}U^{cl}_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi) - H^\bullet_c(sp^{-1}U_\epsilon \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi).$$

Substituting the terms on the right hand side of our formula, accordingly with the last three equalities, we deduce the equality in the statement. □

8.2.10. Finally, it is possible to improve theorem 8.7, by considering the integral models $\mathcal{M}_{\alpha,M,g}$, for all $g \in GL_h(\mathbb{Q}_p)^+$.

Let us consider the $\mathbb{Z}/l^r\mathbb{Z}$-modules

$$H^n(R\Gamma_c(\bar{\mathcal{M}}_M, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi)$$

for any admissible representation Π of $T \times W_{\mathbb{Q}_p}$ and $n \geq 0$. For all $n \geq 0$, the direct limits

$$\varinjlim_M H^n(R\Gamma_c(\bar{\mathcal{M}}_M, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi)$$

are naturally endowed with an action of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$, and also with the morphisms induced by the maps

$$\delta^{*-1}_{M,g} \circ g^* : R\Gamma_c(\bar{\mathcal{M}}_{M-e}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \to R\Gamma_c(\bar{\mathcal{M}}_M, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})),$$

for all $g \in GL_h(\mathbb{Q}_p)^+$ and $M \geq e = e(g)$.

By corollary 8.6, we can write

$$\varinjlim_M H^\bullet(R\Gamma_c(\bar{\mathcal{M}}_{\alpha,M}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi) =$$

$$= \varinjlim_M Tor^\bullet_{\mathcal{H}_r(T)}\left(H^\bullet_c(\bar{\mathcal{M}}_{\alpha,M}, R^\bullet\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})), \Pi\right)$$

as virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p} \subset GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$.

Then, Theorem 90 implies that there in an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$

$$\varinjlim_M H^\bullet(R\Gamma_c(\bar{\mathcal{M}}_{\alpha,M}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi) =$$

$$= \varinjlim_M Ext^\bullet_{T-smooth}\left(H^\bullet_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi\right),$$

where the virtual representation on the right hand side is the restriction to the subgroup $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$ of a representation of $GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$.

Proposition 8.8. — *Let Π be an admissible $\mathbb{Z}/l^r\mathbb{Z}$-representation of $T \times W_{\mathbb{Q}_p}$.*

The morphisms induced by the maps $\delta_{M,g}^{*-1} \circ g^*$, $g \in GL_h(\mathbb{Q}_p)^+$, on the modules

$$\varinjlim_M H^n(R\Gamma_c(\bar{\mathcal{M}}_M, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi),$$

define an action of $GL_h(\mathbb{Q}_p)$ which extends the action of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \subset GL_h(\mathbb{Q}_p)$ and commutes with the action of $W_{\mathbb{Q}_p}$.

Moreover, there is an equality of virtual $\mathbb{Z}/l^r\mathbb{Z}$-representations of $GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M H^\bullet(R\Gamma_c(\bar{\mathcal{M}}_{\alpha,M}, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi) =$$

$$= \varinjlim_M Ext^\bullet_{T-smooth}(H^\bullet_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi).$$

Proof. — First, let us remark that the equalities in theorem 8.7 arise from some quasi-isomorphisms

$$R\Gamma_c(\bar{\mathcal{M}}_M, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi \simeq$$

$$\simeq RHom_{T-smooth}(R\Gamma_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi)$$

which are compatible not only with the action of $\langle GL_h(\mathbb{Z}_p), p\mathbb{I}_h\rangle \times W_{\mathbb{Q}_p}$, but also with the morphisms associated to the elements $g \in GL_h(\mathbb{Q}_p)^+$, as M varies.

As the level M varies, the action of $GL_h(\mathbb{Q}_p)$ on the complex on the right hand side give rise to an action on the cohomology groups

$$\varinjlim_M H^\bullet(RHom_{T-smooth}(R\Gamma_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi)) =$$

$$\varinjlim_M Ext^\bullet_{T-smooth}(H^\bullet_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi).$$

It follows, in particular, that the morphisms induced on the complex on the left hand side by the maps $\delta_{M,g}^{*-1} \circ g^*$, $g \in GL_h(\mathbb{Q}_p)^+$, are quasi-isomorphisms and that, for all $n \geq 0$, there is an isomorphisms of $\mathbb{Z}/l^r\mathbb{Z}$ representations of $GL_h(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

$$\varinjlim_M H^n(R\Gamma_c(\bar{\mathcal{M}}_M, R\Psi_\eta(\mathbb{Z}/l^r\mathbb{Z})) \otimes^L_{\mathcal{H}_r(T)} \Pi) =$$

$$= \varinjlim_M H^n(RHom_{T-smooth}(R\Gamma_c(\mathcal{M}^{\mathrm{rig}} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}(-D)), \Pi)).$$

□

8.3. An equality in the Grothendieck group. — In the following, we shall apply combine the results of the previous sections in a formula describing the l-adic cohomology of the Shimura varieties, in terms of the l-adic cohomology of the Igusa varieties and the Rapoport-Zink spaces, as virtual representations.

8.3.1. First, we recall the notations for the l-adic-cohomology groups of the Igusa varieties.

For any level away from p, U^p and any Newton polygons α, of dimension q and height h, we define, for all integer $j \geq 0$,

$$H^j_c(J_{\alpha,U^p}, \mathbb{Q}_l) = \varinjlim_m (\varprojlim_r H^j_c(J_{\alpha,U^p,m} \times_{\mathbb{F}_p} \times \bar{\mathbb{F}}_p, \mathbb{Z}/l^r\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$$

as an admissible representation of $T_\alpha \times W_{\mathbb{Q}_p}$, and

$$H^j_c(J_\alpha, \mathbb{Q}_l) = \varinjlim_{U^p} H^j_c(J_{\alpha,U^p}, \mathbb{Q}_l)$$

as an admissible $\mathbb{Q}_l$-representation of $T_\alpha \times G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times W_{\mathbb{Q}_p}$.

Lemma 8.9. — *Let α be a Newton polygon , of dimension q and height h. For all positive integers d, e, f, the l-adic representations of $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$*

$$\varinjlim_{U^p,M} Ext^{\,d}_{T_\alpha - smooth}(H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Q}_l), H^j_c(J_\alpha, \mathbb{Q}_l)) =$$

$$= \varinjlim_{U^p,M} (\varprojlim_r Ext^{\,d}_{T_\alpha - smooth}(H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), H^j_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Moreover, they are admissible and vanish for d, e, f sufficiently large.

Proof. — It follows from the definitions that it is enough to show that, for all level U^p, M,

$$Ext^{\,d}_{T_\alpha - smooth}(H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Q}_l), H^j_c(J_\alpha, \mathbb{Q}_l)) =$$

$$= \varprojlim_r Ext^{\,d}_{T_\alpha - smooth}(H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), H^j_c(J_\alpha, \mathbb{Z}/l^r\mathbb{Z})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

and that they are finite dimensional vector spaces, which vanish for d, e, f sufficiently large.

Let us fix a level U^p, M. We choose an closed analytic domain $V_0 \subset \mathcal{M}^{\mathrm{rig}}_{\alpha,0}$ as in section 8.2.9, and denote by V_M its pullback under the projection $\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \to \mathcal{M}^{\mathrm{rig}}_{\alpha,0}$.

The equalities in section 8.2.9 reduce the proof of lemma to showing that for all $\epsilon \in \Gamma \backslash (T_\alpha/\Gamma)^s_{\neq}$, $s \geq 0$,

$$Ext^\bullet_{T_\alpha - smooth}\,(c - Ind^{T_\alpha}_{\Gamma_\epsilon} H^\bullet_c(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Q}_l), H^\bullet_c(J_{\alpha,U^p}, \mathbb{Q}_l)) =$$

$$= \varprojlim_r (Ext^\bullet_{T_\alpha - smooth}\,(c - Ind^{T_\alpha}_{\Gamma_\epsilon} H^\bullet_c(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), H^\bullet_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

or equivalently, up to Tate twist, that

$$Tor^\bullet_{\mathcal{H}(T_\alpha)}\,(c - Ind^{T_\alpha}_{\Gamma_\epsilon} H^\bullet(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Q}_l), H^\bullet_c(J_{\alpha,U^p}, \mathbb{Q}_l)) =$$

$$= \varprojlim_r (Tor^\bullet_{\mathcal{H}_r(T_\alpha)}\,(c - Ind^{T_\alpha}_{\Gamma_\epsilon} H^\bullet(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), H^\bullet_c(J_{\alpha,U^p}, \mathbb{Z}/l^r\mathbb{Z}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

where we write $\mathcal{H}(T_\alpha)$ for the Hecke algerba of T_α with coefficients in $\mathbb{Q}_l$. (The equivalence between the two equalities follows from Poincaré duality and lemma 8.4.)

Since the l-adic cohomology groups $H^i_c(V_{\epsilon,M}, \mathbb{Q}_l)$ are finite dimensional, for all $i \geq 0$, there exists an open compact normal subgroup $K_\epsilon \subset \Gamma_\epsilon$ which acts on them trivially. Indeed, we can choose K_ϵ such that is acts trivially on the representations $H^i_c(V_{\epsilon,M}, \mathbb{Z}/l^r\mathbb{Z})$, for all $i \geq 0$ and $r \geq 1$.

Let m_ϵ be a positive integer such that $\Gamma^{m_\epsilon} \subset K_\epsilon \subset \Gamma_\epsilon$ (we remark that m_ϵ exists since the subgroups Γ^m, $m \geq 0$, form a basis of open subgroups of T_α). Then, proving the previous equality of virtual $\mathbb{Q}_l$-representations of $W_{\mathbb{Q}_p}$ is equivalent to proving that

$$Tor^{\bullet}_{\mathcal{H}(\Gamma_\epsilon/K_\epsilon)}\left(H^\bullet(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Q}_l), H^\bullet_c(J_{\alpha,U^p,m_\epsilon}, \mathbb{Q}_l)\right) =$$

$$= \varprojlim_r (Tor^{\bullet}_{\mathcal{H}_r(\Gamma_\epsilon/K_\epsilon)}\left(H^\bullet(V_{\epsilon,M} \times_K \bar{K}, \mathbb{Z}/l^r\mathbb{Z}), H^\bullet_c(J_{\alpha,U^p,m_\epsilon}, \mathbb{Z}/l^r\mathbb{Z})\right)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

which follows from the next lemma. □

Lemma 8.10. — *Let G be an abstract finite group of order a power of p, and write $\mathcal{H}(G) = \mathbb{Q}_l[G]$ and $\mathcal{H}_r(G) = \mathbb{Z}/l^r\mathbb{Z}[G]$, for all $r \geq 1$.*

Let $(M_r)_{r\geq 1}$ and $(N_r)_{r\geq 1}$ be two A-R l-adic systems, such that $l^r M_r = 0 = l^r N_r$, which are endowed with an action of G. We write $M = \varprojlim_r M_r$ and $N = \varprojlim_r N_r$.

Then, for all $i \geq 0$, we have

$$Tor^{\,i}_{\mathcal{H}(G)}(M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, N \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = \varprojlim_r Tor^{\,i}_{\mathcal{H}_r(G)}(M_r, N_r) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Proof. — Let us remark that without loss of generality we may replace M_r (resp. N_r) by $M/l^r = M \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^r\mathbb{Z}$ (resp. $N/l^r = N \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^r\mathbb{Z}$), for all $r \geq 1$, since the two modules differ by a torsion module of bounded order. Thus, it suffices to prove that, for all $i \geq 0$,

$$Tor^{\,i}_{\mathcal{H}(G)}(M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, N \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = \varprojlim_r Tor^{\,i}_{\mathcal{H}_r(G)}(M/l^r, N/l^r) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Let us remark that the modules $Tor^{\,i}_{\mathcal{H}_r(G)}(M/l^r, N/l^r)$ clearly satisfy the M-L condition and thus the functors

$$N \mapsto Tor^{\,i}_{\mathcal{H}_r(G)}(M/l^r, N/l^r) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

for $i \geq 0$, form a δ-functor from the category of finitely generated $\mathbb{Z}_l$-modules, endowed with an action of G, to the category of finite dimensional $\mathbb{Q}_l$-vector spaces.

Moreover, it is easy to see that, as a δ-functor, it is effaceable (e.g. any finitely generated $\mathbb{Z}_l$-module, endowed with an action of G, admits an epimorphism from a module of the form $Ind^G_{\{1\}}(\mathbb{Z}_l^r) = \mathbb{Z}_l^{r|G|}$, for some integer $r \geq 0$, which is acyclic for the above δ-functor).

For any finitely generated $\mathbb{Z}_l$-module N, endowed with an action of G, let $N \mapsto Tor^{\,i}_G(M, N)$ be the derived functors of $N \mapsto (M^{op} \otimes_{\mathbb{Z}_l} N)_G$, for $i \geq 0$.

Then, we can identify

$$Tor^{\,i}_{\mathcal{H}(G)}(M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, N \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = Tor^{\,i}_G(M, N) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

and deduce the existence of a natural morphisms of δ-functors

$$Tor^{\,i}_{\mathcal{H}(G)}(M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, N \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \to \varprojlim_r Tor^{\,i}_{\mathcal{H}_r(G)}(M/l^r, N/l^r) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Then, in order to conclude it suffices to remark that the above morphism is indeed an isomorphism for $i = 0$. □

We have finally completed all the steps necessary to obtain the following description of the l-adic cohomology of the Shimura varieties.

Theorem 8.11. — *There is an equality of virtual representations of the group* $G(\mathbb{A}^\infty) \times W_{\mathbb{Q}_p}$

$$\sum_t (-1)^t H^t(X, \mathbb{Q}_l)^{\mathbb{Z}_p^\times} =$$

$$= \sum_{\alpha,d,e,f} (-1)^{d+e+f} \varinjlim_M Ext^d_{T_\alpha - smooth}(H^e_c(\mathcal{M}^{\mathrm{rig}}_{\alpha,M} \times_K \bar{K}, \mathbb{Q}_l(-D)), H^j_c(J_\alpha, \mathbb{Q}_l))$$

where d, e, f *are positive integers and* α *varies among the Netwon polygons of dimension* q *and height* h.

Proof. — This follows from proposition 8.1, theorem 8.2, proposition 8.8 and lemma 8.9. □

We conclude by remarking that theorem 8.11 is compatible with corollary 4.5.1 in [**10**].

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