SUPPLEMENT TO LECTURE 8 (1/23/09)

Write $\mathbb{Z}_{\geq 0}$ for the nonnegative integers, and $\mathbb{N}$ for the positive integers. For a collection of spaces $\{X_j \mid j \in \mathbb{N}\}$, we write $\sqcup_{j\in\mathbb{N}} X_j$ for the disjoint union. For simplicity, we'll consider collections of spaces indexed by $\mathbb{N}$, but the arguments extend to collections of spaces of arbitrary cardinality.

A *graded ring* is a ring R with underlying abelian group structure

$$R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A_i.$$

(We ignore the possibility of nontrivial A_i for $i < 0$ since such rings don't come up in topology.) For each i there is a natural inclusion of A_i into R, and so we may regard A_i as a subset of R. If $a \in A_i \setminus \{0\} \subset R$, we say that a is a *pure* element of R, and that the *dimension* of a (or $\dim(a)$) is i. That R is *graded* means that, for any two pure elements a and b either $ab = 0$, or

$$\dim(ab) = \dim(a) + \dim(b).$$

An arbitrary element of a graded ring is a finite sum of pure elements.

A *graded homomorphism* of graded rings is a ring homomorphism $h\colon R \to S$ so that if x is pure and $h(x) \neq 0$, then $h(x)$ is pure and $\dim(h(x)) = \dim(x)$.

If there are only finitely many factors, the direct product in the category of graded rings coincides with the direct product in the category of rings. On the other hand, if the number of factors is infinite these direct products must differ. Indeed, suppose that, for each $i \in \mathbb{N}$, we have $R_i = H^*(\mathbb{R}P^2; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$, where $\dim(\alpha) = 1$. Let P be the product

$$\prod_{i \in \mathbb{N}} R_i = \{(r_1, r_2, \ldots) | r_i \in R_i \text{ for all } i \in \mathbb{N}\}$$

in the category of rings. There is no natural way to make P into a graded ring, since any element of a graded ring is a finite sum of pure elements, but an element of P like $(\alpha, \alpha^2, \ldots, \alpha^i, \ldots)$ cannot be written as a finite sum of pure elements. So either arbitrary products don't exist in the category of graded rings, or they are different somehow from those in the category of graded rings.

For each $j \in \mathbb{N}$, let $R_j = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A_{j,i}$ be a graded ring. Let P be the direct product of $\{R_j \mid j \in \mathbb{N}\}$ in the category of rings. We define a ring P' with underlying abelian group

$$P' = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \prod_{j \in \mathbb{N}} A_{j,i}.$$

Multiplication of pure elements is defined componentwise: if $\dim(a_j) = k$ and $\dim(b_j) = l$ for all $j \in \mathbb{Z}$, then the product

$$(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots)$$

is pure of dimension $k+l$. Multiplication of arbitrary elements is then given by the distributive law. There is an inclusion of rings $i\colon P' \to P$ given by:

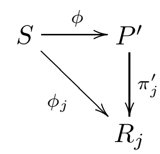
$$\sum_{i=1}^{n}(a_{i,1}, a_{i,2}, \ldots) \mapsto (\sum_{i=1}^{n} a_{i,1}, \sum_{i=1}^{n} a_{i,2}, \ldots).$$

Composing this inclusion with the projection maps $\pi_j\colon P \to R_j$ gives projection maps $\pi'_j\colon P' \to R_j$.

Lemma 0.1. *The ring P' just defined is the direct product of $\{R_j \mid j \in \mathbb{N}\}$ in the category of graded rings.*

Proof. We must verify the universal property. Let $S = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} B_i$ be a graded ring, and suppose that for each $j \in \mathbb{N}$, we have a *graded* homomorphism $\phi_j\colon S \to R_j$. We must find a graded homomorphism $\phi\colon S \to P'$, so that the triangles

(1)



commute, and show ϕ is unique.

If s is pure of dimension k, then so is $\phi_j(s)$ for each $j \in \mathbb{N}$, and we define $\phi(s)$ to be the (pure) element $(\phi_1(s), \phi_2(s), \ldots)$. Extending by linearity, this determines a graded ring homomorphism which makes the triangles in the definition of the universal property for products commute.

To see uniqueness, let $\phi'\colon S \to P'$ is another map making the triangles (1) commute. If ϕ and ϕ' differ, they must differ on some pure element s of S. If $\phi(s) \neq \phi'(s)$, then $\pi'_j\phi'(s)$ must differ from $\pi_j\phi(s)$ for some j, which contradicts the assumption that both maps make the triangles (1) commute. $\square$

Proposition 0.2. *Let $\{X_j\}_{j\in\mathbb{N}}$ be a collection of CW complexes, and let R be a commutative ring with* 1. *There is an isomorphism of graded rings between $H^*(\sqcup_{j\in\mathbb{N}} X_j; R)$ and $\prod_{j\in\mathbb{N}} H^*(X_j; R)$.*

Proof. By the Lemma, we just need to find an isomorphism of graded rings between $H^*(\sqcup_{\alpha\in I} X_j; R)$ and

$$P = \bigoplus_{i\geq 0} \prod_{j=1}^{\infty} H^i(X_j; R).$$

For each $j \in \mathbb{N}$, let $\iota_j\colon X_j \to \sqcup_{\alpha\in I} X_j$ be the inclusion. Then define

$$q\colon H^*(\sqcup_{\alpha\in I} X_j; R) \to P$$

by defining it on pure elements (i.e., cohomology classes), and then extending linearly: If $\alpha \in H^k(\sqcup_{\alpha\in I} X_j; R)$, let

$$q(\alpha) = (\iota_1^*(\alpha), \iota_2^*(\alpha), \ldots).$$

(i.e., the jth coordinate of $q(\alpha)$ is the cocycle obtained by restricting α to X_j.) It can then be checked that q is an isomorphism of graded rings. $\square$

In class, I sketched an argument that, if all the X_j are connected CW complexes, and for each j, x_j is some 0–cell of X_j, then the *reduced* cohomology of $\bigvee_{j\in\mathbb{N}} X_j$ is the same as the cohomology of the pair $(\sqcup_{j\in\mathbb{N}} X_j, \{x_j \mid j \in \mathbb{N}\})$. Essentially the same argument as in the Proposition above will show that the reduced cohomology ring $\tilde{H}^*(\bigvee_{j\in\mathbb{N}} X_j; R)$ is isomorphic to the product (in the category of graded rings) of the rings $\tilde{H}^*(X_j; R)$.