Ma 151b – Winter 2009

Midterm Examination

Due Friday Feb. 13, at 5PM

Remember, this is a midterm, so collaboration is not allowed. You shouldn't discuss the exam with anyone except me or consult algebraic topology texts other than Hatcher. You may use any statement from Hatcher, up through Section 3.3, without proof. Tell me if you think there is a misprint in one of these problems.

1. Hatcher, problem 3.2.4. (See 2C.3 for the Lefschetz fixed point theorem.)

2. Let M_1 and M_2 be two connected closed oriented n-manifolds. The *oriented connect sum* $M_1 \# M_2$ is obtained by deleting the interiors of balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the boundary $n-1$-spheres ∂B_1 and ∂B_2 by an orientation-reversing homeomorphism. Thus $M_1 \# M_2$ is again a closed connected oriented n-manifold with an orientation which agrees with the orientation on both M_1 and M_2.

   (a) Describe the cohomology ring of $M_1 \# M_2$ (with coefficients in some ring R) in terms of the cohomology rings of M_1 and M_2. Specifically, show that $H^*(M_1 \# M_2) = (H^*(M_1) \oplus H^*(M_2))/I$, where I is the ideal generated by $[M_1]-[M_2]$ together with a certain 0–dimensional element.

   (b) Use cohomology to show that $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not homotopy equivalent to $S^2 \times S^2$.

3. (a) Show that a direct limit of short exact sequences of abelian groups is short exact.

   (b) Show that any (not necessarily proper) subgroup of $\mathbb{Q}$ is the direct limit of a directed system of cyclic groups.

   (c) Describe a locally finite CW-complex with $H_1 = \mathbb{Q}$.

4. Problem 3.3.9 in Hatcher. (Hint: You probably want to do problem 3.3.3 and at least part of problem 3.3.8 for this.)

5. Given a topological space X, consider the sub-complex $C_b^*(X;\mathbb{R}) \subset C^*(X;\mathbb{R})$ defined by

$$C_b^n(X;\mathbb{R}) = \{\phi\colon C_n(X) \to \mathbb{R} \mid \exists K,\ |\phi(\sigma)| \le K\ \forall \sigma\colon \Delta^n \to X\}.$$

   The (co)homology of this sub-complex, written $H_b^*(X;\mathbb{R})$, is called the *bounded cohomology* of X with coefficients in $\mathbb{R}$. (Here $\mathbb{R}$ is the real numbers, and $|\cdot|$ is absolute value, but we could define bounded cohomology with coefficients in any *normed* abelian group.) It turns out that bounded cohomology is an example of an *exotic* cohomology theory, i.e. one which does not satisfy all of the cohomology axioms.

(a) Describe $H_b^0(X;\mathbb{R})$. How does it differ from $H^0(X;\mathbb{R})$?

(b) Show that $H_b^1(X;\mathbb{R}) = 0$ for any space X. Conclude that there is no Mayer-Vietoris sequence for $H_b^*(\cdot,\mathbb{R})$, even for finite CW complexes.

(c) Let X be the graph pictured in Figure 1. Define a 1-cochain ϕ as

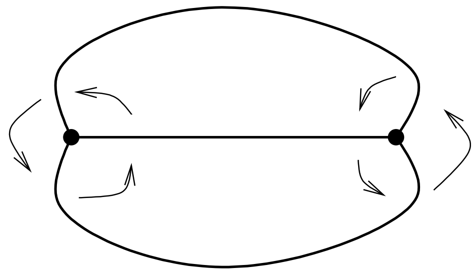

Figure 1: "Turning right" is well-defined.

follows. If $\sigma\colon [0,1] \to X$ is a 1-simplex, lift σ to the universal cover $\widetilde{X}$ of X. Because $\widetilde{X}$ is a tree, there is a unique (up to reparameterization) embedded path with the same endpoints as the lift of σ. Define $\phi(\sigma) \in \mathbb{Z}$ to be the number of times this path "turns right" minus the number of times this path "turns left". (The arrows in the picture show how to make this precise by putting a cyclic order on the edges coming into each vertex.) Show that although $\phi \notin C_b^1(X;\mathbb{R})$, the coboundary $\delta\phi \in C_b^2(X;\mathbb{R})$.

Extra Credit (worth 1/3 of a full problem): Is $[\delta\phi] = 0$ in $H_b^2(X;\mathbb{R})$, in Problem 5 above?