Math 108c
Introduction to Complex Analysis
Spring 2006
Homework Set 6

Solutions to this homework set should be submitted by 5:00 p.m. on Wednesday May 17.

1. We stated Hurwitz's theorem in class and outlined a proof. Fill in the details.

2. Show that for a set $\mathcal{F} \subset \mathcal{H}(G)$ the following are equivalent conditions :

   (a) $\mathcal{F}$ is normal.

   (b) For every $\epsilon > 0$, there is a number $c > 0$ such that
   $$\{cf \,:\, f \in \mathcal{F}\} \subset B(0;\epsilon).$$

3. Let G be a region and let M be a fixed constant. Show that the family
   $$\mathcal{F}_M = \left\{f \,:\, ||f||_{L^2(G)} \leq M\right\}$$
   is normal. (Hint : Show first that for $B(a;R) \subset G$, and any analytic function f on G, $|f(a)|^2$ is bounded by the average of f over $B(a;R)$.)

4. Show that
   $$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

5. (a) Let $0 < |a| < 1$ and $|z| \leq r < 1$. Show that
   $$\left|\frac{a + |a|z}{(1-\bar{a}z)a}\right| \leq \frac{1+r}{1-r}.$$

   (b) Let $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ and $\sum(1 - |a_n|) < \infty$. Show that the infinite product
   $$B(z) = \prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \bar{a}_n z}\right)$$
   converges in $\mathcal{H}(B(0;1))$, and that $|B(z)| \leq 1$. What are the zeros of B? ($B(z)$ is called a *Blaschke product.*)

   (c) Find a Blaschke product such that every point on the unit circle is a limit point of the product's set of zeros.