DOUBLE HILBERT TRANSFORM ALONG REAL-ANALYTIC SURFACES IN $\mathbb{R}^{d+2}$

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Abstract. We give a necessary and sufficient condition for the double Hilbert transform on $\mathbb{R}^{d+2}$ to be bounded on L^p, $1 < p < \infty$. This generalizes a result of Carbery, Wainger and Wright [1] for $d = 1$.

1. Introduction

The purpose of this paper is to establish a necessary and sufficient condition for L^r-boundedness ($1 < r < \infty$) of the double Hilbert transform along real-analytic surfaces in $\mathbb{R}^{d+2}$. The double Hilbert transform is given by

$$H[P]f(x,y,z) = \int_{\substack{|s|\leq c\\|t|\leq c}} f(x-s, y-t, z-P(s,t))\,\frac{ds\,dt}{st}, \tag{1.1}$$

where $x, y \in \mathbb{R}$, $z = (z_1, \cdots, z_d) \in \mathbb{R}^d$, and $P(s,t) = (P_1, \cdots, P_d)(s,t)$ is a vector-valued real-analytic function in s and t, defined on an open set containing $\{(s,t) : |s|, |t| \leq c\}$ for some small positive number c. Without loss of generality we assume $P(0,0) = 0$, and $\nabla P(0,0) = 0$. When the underlying collection of functions P is unambiguous, we sometimes drop the argument and denote the double Hilbert transform simply by H.

When P_i-s are monomials, $P_i(s,t) = s^{m_i}t^{n_i}$, $1 \leq i \leq d$, Ricci and Stein [6] proved that H is bounded on L^r for $1 < r < \infty$ if $(m_i, n_i) \neq (m_j, n_j)$ for $i \neq j$ and $m_i n_i$ is even for every i. The case when $d = 1$ and P_1 is a polynomial has been considered by Carbery, Wainger and Wright (in fact, their proof goes through with minor modifications even when P_1 is real-analytic). In [1], they showed that H is bounded on L^r for $1 < r < \infty$ if and only if for each corner point (m,n) of the Newton diagram of P_1, mn is even. Let us recall that for a real-analytic function P_1 given by $P_1(s,t) = \sum_{p,q\geq 0} a_{p,q}s^p t^q$, the *Newton polygon* of P_1, denoted by $\mathcal{N}(P_1)$, is defined as the convex hull of the set

$$\bigcup_{\substack{p,q\geq 0\\ a_{p,q}\neq 0}} \{(x,y) : x \geq p,\ y \geq q\}.$$

The boundary of the Newton polygon is called the *Newton diagram*, and is denoted by $\partial\mathcal{N}(P_1)$. If $P_1 \equiv 0$, both $\mathcal{N}(P_1)$ and $\partial\mathcal{N}(P_1)$ are taken to be the empty set. A point $(m,n) \in \partial\mathcal{N}(P_1)$ is said to be a *corner point* if there exist two line segments of the Newton diagram with distinct slopes that contain (m,n).

In this paper we extend the result in [1] to $d \geq 2$. We begin with a few definitions, needed to state our main result.

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Definition 1.1. *A scalar-valued real-analytic function Q is said to be even if $Q \equiv 0$ or for every corner point (m, n) in the Newton diagram of Q, mn is even.*

Definition 1.2. *A vector-valued real-analytic function $P = (P_1, \cdots, P_d)$ is even if P_i is even for $1 \le i \le d$.*

Definition 1.3. *A vector-valued real-analytic P is said to be even in $GL(d)$ if for every invertible $d \times d$ matrix A, PA^t is even.*

We observe that $P(s,t) = (s^2t^2, s^2t^2 + s^3t^3)$ is even but not even in $GL(2)$, since

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s^2t^2 \\ s^2t^2 + s^3t^3 \end{bmatrix} = \begin{bmatrix} s^2t^2 \\ s^3t^3 \end{bmatrix}.$$

On the other hand, if P is even, and the Newton diagrams of P_i and P_j are disjoint for $i \neq j$, then P is even in $GL(d)$.

The main result in this paper is the following :

Theorem 1.4. *Let P be a vector-valued real-analytic function defined in a neighborhood of the origin in $\mathbb{R}^2$. The double Hilbert transform $H[P]$ is bounded on L^r for $1 < r < \infty$ if and only if P is even in $GL(d)$.*

1.1. **Remarks.** (1) In fact, our proof will show that there exists a finite family of matrices $\mathcal{A} \subseteq GL(d)$, explicitly computable in terms of P, such that H is bounded on L^r, $1 < r < \infty$ if and only if PA^t is even for every $A \in \mathcal{A}$.

(2) The assumption of real-analyticity is crucial here. The result fails for P in C^∞, even if P is of finite type. It has been pointed out by Carbery, Wainger and Wright [2] [1] that for $P(s,t) = s^2t + \gamma(s)$, the double Hilbert transform $H[P]$ is unbounded on L^2 if $s\gamma'(s) - \gamma(s)$ fails to be doubling.

(3) A maximal operator that arises naturally in this context is the following :

$$M[P]f(x,y,z) = \sup_{r_1, r_2 > 0} \frac{1}{r_1 r_2} \int_{\substack{|s| \le r_1 \\ |t| \le r_2}} |f(x - s, y - t, z - P(s,t))| \, ds \, dt, \tag{1.2}$$

and it is natural to ask whether $M[P]$ is bounded on L^r, $1 < r < \infty$. This has been proved in Ricci-Stein [6] when P has monomial entries (not necessarily even). More generally, when P is any vector-valued polynomial (not necessarily even in $GL(d)$), Cho, Hong, Kim and Yang [3] have shown $M[P]$ to be bounded on L^r. Their proof goes through with minor modifications for real-analytic P as well. A stronger result for $d = 1$ is due to Patel [5], who has proved that for a scalar-valued real polynomial P, $M[P]$ is weak-type L^1. The point to be emphasized about Theorem 1.4 is therefore the condition that P is even in $GL(d)$, which can be viewed as the correct "cancellation" condition on the kernel under which the singular integral operator $H[P]$ is L^r-bounded.

(4) It is important to note that $H[P]$ is not invariant under reparametrizations; therefore Theorem 1.4 need not be true for other representations of the real-analytic surface. In particular, if the surface is parametrized as $\mathcal{P}(s,t)$, where $\mathcal{P}$ is a $(d+2)$-dimensional vector with real-analytic entries, then our proof shows that even-ness of $\mathcal{P}$ is sufficient for boundedness of the corresponding double Hilbert transform. However, it is not difficult to see that this condition is no longer necessary. Understanding the right cancellation condition in this general case is more complicated, and we hope to return to this issue in a future paper.

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1.2. **Motivation of proof.** The proof of Theorem 1.4 depends largely on the arguments in [1]. In particular, we first decompose H into operators supported in the regions $|s| \sim 2^{-p}$, $|t| \sim 2^{-q}$ (throughout this paper, $A \sim B$ implies that there exists a constant $C > 0$ depending only on the P_j-s such that $C^{-1} \leq A/B \leq C$). Then following [1], we use the Newton diagrams of P_i to divide the (p, q)-plane into a finite number of cones, each with vertex at the origin. These cones have the property that on each of them, P can be replaced "essentially" by a unique d-tuple of monomials. The double Hilbert transform corresponding to monomial surfaces can then be treated by Ricci and Stein. Of course, since P_i may not exactly be a monomial, a significant part of the proof is devoted to showing that the error operator is bounded on L^r.

However, there are certain aspects of the higher dimensional problem that are absent for $d = 1$. For instance, one is naturally led to consider a finite family of linear transformations in $GL(d)$, one for each cone mentioned above. These transformations ensure that the monomials "representing" P have distinct sets of exponents, so that Ricci and Stein can be invoked. Another facet of this is that a naive generalization of [1], namely the condition that P is even is not sufficient for L^r boundedness of H. We clarify this by a few examples.

Suppose first that $P(s,t) = (s^2t^2, s^2t^2)$, which is even in $GL(2)$. Then the multiplier m_P of $H[P]$ is given by

$$m_P(\xi) = m_{\tilde{P}}(\xi_1, \xi_2, \xi_3 + \xi_4),$$

where $\tilde{P}(s,t) = s^2t^2$. We know that $H[\tilde{P}]$ is bounded on L^r, $1 < r\infty$ by [1]. It is not difficult to conclude from this that $H[P]$ is also bounded on L^r. To see this, we set $u = (x, y, z_1, z_2) = (u_1, u_2, u_3, u_4)$ and write

$$\begin{aligned} H[P]f(u) &= \mathcal{F}^{-1}\left(m_P \widehat{f}\right)(u) \\ &= \int e^{i\xi\cdot(u-y)} m_{\tilde{P}}(\xi_1, \xi_2, \xi_3 + \xi_4)\, d\xi\; f(y)\, dy \\ &= \int e^{i\xi'\cdot(u'-y')} m_{\tilde{P}}(\xi') e^{i\xi_4(u_4 - y_4 - u_3 + y_3)}\, f(y)\, dy\, d\xi'\, d\xi_4 \\ &= \int e^{i\xi'\cdot(u'-y')} m_{\tilde{P}}(\xi') f(y', u_4 - u_3 + y_3)\, dy'\, d\xi' \\ &= \int K_{\tilde{P}}(u' - y') f(y', u_4 - u_3 + y_3)\, dy'. \end{aligned}$$

Here at the penultimate step, we have made use of the oscillatory integral representation of the Dirac delta distribution, and $K_{\tilde{P}}$ at the last step denotes the kernel of the transform $H[\tilde{P}]$. The notation u' and y' stand for the three-dimensional vectors consisting of the first three components of u and y respectively. Now,

$$\begin{aligned} ||H[P]f||_r^r &= \int \left| \int K_{\tilde{P}}(u' - y') f(y', u_4 - u_3 + y_3)\, dy' \right|^r du'du_4 \\ &\leq C \int \left[\int |f(u)|^r\, du' \right] du_4 = ||f||_r^r. \end{aligned}$$

Suppose on the other hand that $P(s,t) = (s^2t^2, s^2t^2 + s^3t^3)$. We have shown earlier in this section that P is not even in $GL(2)$. The multiplier m_P of $H[P]$ is given by

$$m_P(\xi) = m_{\tilde{P}}(\xi_1, \xi_2, \xi_3 + \xi_4, \xi_4),$$

where $\tilde{P}(s,t) = (s^2t^2, s^3t^3)$. We prove in section 3 that $m_{\tilde{P}}$ is unbounded, which means that $H[P]$ is not bounded on L^2.

1.3. **Notation.** We now develop the notation that will be needed in the proof of the theorem.

• For $z \in \mathbb{R}^d$, $|z| = (|z_1|, \cdots, |z_d|)$, and $||z|| = \sqrt{|z_1|^2 + \cdots |z_d|^2}$. For any scalar $c > 0$, $c^z = (c^{z_1}, \cdots, c^{z_d})$.

• C and σ denote positive constants that may vary from one line to the next.

• Given a $d \times d$ matrix B, let $f^B(x,y,z) := f(x,y,zB^t)$.

Remark : If B is invertible, we ask the reader to verify that

(1) $||f^B||_r = ||f||_r / |\det(B)|^{1/r}$,
(2) $(H[P]f)^B(x,y,z) = (H[P(B^{-1})^t])f^B(x,y,z)$, and
(3) $||H[PB^t]||_{L^r \to L^r} = ||H||_{L^r \to L^r}$.

• Given $B \in GL(d)$, and a d-dimensional vector P of real-analytic functions, the k-th entry of PB^t will be denoted by

$$\left(PB^t\right)_k (s,t) := \sum_{m,n \geq 0} a_{m,n}(B,k) s^m t^n.$$

• Given a scalar-valued real-analytic function Q, let $\{(m_j(Q), n_j(Q)) \,:\, j \geq 1\}$ denote the set of corner points of the Newton diagram of Q, indexed so that $m_j(Q) < m_{j+1}(Q)$.

• We denote by $\lambda_j(Q)$ the absolute value of the slope of the line segments joining two consecutive corner points of the Newton diagram of Q :

$$\lambda_j(Q) := \frac{n_j(Q) - n_{j+1}(Q)}{m_{j+1}(Q) - m_j(Q)}.$$

For fixed Q, the $\lambda_j(Q)$-s form a decreasing sequence, i.e., $\lambda_j(Q) > \lambda_{j+1}(Q)$. The set of all $\lambda_j(Q)$ for a fixed Q is denoted by $\Lambda(Q)$.

• We compile all slopes described in the previous item for each entry of P and under all transformations in $GL(d)$, and call the resulting set Λ :

$$\Lambda := \bigcup_{\substack{A \in GL(d) \\ 1 \leq k \leq d}} \Lambda\left((PA^t)_k\right), \tag{1.3}$$

We prove in Lemma 2.1 that Λ is a finite set. Let κ_j-s be the elements of Λ arranged in decreasing order of magnitude :

$$\kappa_0 = \infty > \kappa_1 > \cdots > \kappa_T = 0.$$

• We decompose the (p,q)-plane into M cones whose edges have slopes reciprocal to the numbers in Λ :

$$\mathcal{R}_j := \{(p,q) \,:\, \kappa_{j-1} q > p \geq \kappa_j q\}, \quad 1 \leq j \leq T.$$

• Given $\mathcal{R}_j$, $A \in GL(d)$ and an index k, $1 \leq k \leq d$, there exists a unique $i = i(j,A,k)$ such that

$$\mathcal{R}_j \subset \left\{(p,q) \,:\, \lambda_{i-1}\left(\left(PA^t\right)_k\right) q > p \geq \lambda_i\left(\left(PA^t\right)_k\right) q\right\}. \tag{1.4}$$

2. Proof of Theorem 1.4 – Sufficiency

In this section, we prove that if P is even in $GL(d)$, then H is bounded on L^r, $1 < r < \infty$. The layout of the proof is as follows. In subsection 2.1, we prove a lemma that allows us to write H as a finite sum of operators, each corresponding to a cone $\mathcal{R}_j$. Lemma 2.2 and Lemma 2.3 of subsection 2.2 contain the monomialization argument, whereby each summand is replaced by a (localized) double Hilbert transform associated to a monomial surface satisfying the hypothesis of Ricci and Stein [6]. Subsection 2.3 deals with the estimation of the error incurred by such a replacement. The endpoint cases $j = 1$ and $j = T$ give rise to additional technicalities, and these are addressed in subsection 2.4.

2.1. **Preliminaries.**

Lemma 2.1. *For a vector-valued real-analytic function P, the set Λ defined in (1.3) is a finite set.*

Proof. Since the number of line segments with distinct slopes in a single Newton diagram is always finite, it suffices to show that the collection of Newton diagrams obtained by all possible linear transformations of P is finite, i.e.,

$$\left|\left\{\partial\mathcal{N}(\overline{b}\cdot P) : \overline{b} \in \mathbb{R}^d\right\}\right| < \infty. \tag{2.1}$$

Let us define $\mathcal{M}$ to be the convex hull of

$$\bigcup_{p,q\geq 0}\left\{[p,\infty)\times[q,\infty) \,:\, \sum_{i=1}^{d}|a_{p,q}(I,i)| > 0\right\},$$

where I denotes the $d\times d$ identity matrix. Let $(\mathfrak{m}_1, \mathfrak{n}_1)$ and $(\mathfrak{m}_2, \mathfrak{n}_2)$ be the endpoints of the horizontal and vertical segments respectively of $\partial\mathcal{M}$, i.e.

$$\mathfrak{n}_1 = \min\{n \,:\, (m,n)\in\mathcal{M}\},$$
$$\mathfrak{m}_1 = \min\{m \,:\, (m,\mathfrak{n}_1)\in\partial\mathcal{M}\},$$

and $\mathfrak{m}_2$, $\mathfrak{n}_2$ are defined similarly. Since

$$\bigcup_{\substack{A\in GL(d)\\ 1\leq k\leq d}} \mathcal{N}((PA^t)_k) \subseteq \mathcal{M},$$

the q-coordinate of the horizontal edge of $\partial\mathcal{N}(PA^t)_k)$ for any k must be $\geq \mathfrak{n}_1$. The number $\mathfrak{m}_2$ has a similar property.

The statement (2.1) is proved by induction on d. Suppose $d = 2$. Without loss of generality we can take $\overline{b}$ of the form $\overline{b} = (1,\alpha)$, $\alpha\in\mathbb{R}$. If $\alpha \neq -a_{\mathfrak{m}_i,\mathfrak{n}_i}(I,1)/a_{\mathfrak{m}_i,\mathfrak{n}_i}(I,2)$ for any $i = 1,2$, then $(\mathfrak{m}_1,\mathfrak{n}_1)$ and $(\mathfrak{m}_2,\mathfrak{n}_2)$ both lie in $\partial\mathcal{N}(P_1+\alpha P_2)$, and in fact, the horizontal and vertical segments of $\partial\mathcal{N}(P_1+\alpha P_2)$ are the same as those of $\partial\mathcal{M}$. By convexity, the number of Newton diagrams with these two specified segments are at most finite in number. For the other α-s, one gets at most two new Newton diagrams.

For $d \geq 3$, one can choose $\overline{b}$ of the form $\overline{b} = (1,\alpha_2,\cdots,\alpha_d)$. Then for all α_i not satisfying

$$a_{\mathfrak{m}_j,\mathfrak{n}_j}(I,1) + \sum_{i=2}^{d}\alpha_i a_{\mathfrak{m}_j,\mathfrak{n}_j}(I,i) = 0 \tag{2.2}$$

for any $j = 1, 2$, $\partial\mathcal{N}(\bar{b} \cdot P)$ has the same horizontal and vertical line segments as $\partial\mathcal{M}$. The total number of new Newton diagrams obtained this way is at most finite. Suppose then that the α_i-s satisfy (2.2) for $j = 1$. Clearly we cannot have $a_{\mathfrak{m}_1,\mathfrak{n}_1}(I, i) = 0$ for all i, $2 \le i \le d$. Without loss of generality let $a_{\mathfrak{m}_1,\mathfrak{n}_1}(I, d) \neq 0$. Then

$$\alpha_d = B_1 + \sum_{i=2}^{d-1} B_i \alpha_i, \qquad \text{where} \qquad B_i = -\frac{a_{\mathfrak{m}_1,\mathfrak{n}_1}(I, i)}{a_{\mathfrak{m}_1,\mathfrak{n}_1}(I, d)},$$

and $\alpha_2, \cdots, \alpha_{d-1}$ are arbitrary. We therefore have a $(d - 2)$ parameter family of analytic functions

$$P_1 + \sum_{i=2}^{d} \alpha_i P_i = (P_1 + B_1 P_d) + \sum_{i=2}^{d-1} \alpha_i (P_i + B_i P_d)$$

indexed by $\alpha_2, \cdots, \alpha_{d-1}$, for which the induction hypothesis gives the desired result. $\square$

The proof of sufficiency of Theorem 1.4 is by induction on d. Suppose that P is even in $GL(d)$. The case when $d = 1$ has been proved by Carbery, Wainger and Wright [1]. We assume that H_Q is bounded on L^r if Q is a d_0-dimensional vector-valued polynomial, $d_0 < d$ and Q is even in $GL(d_0)$.

Without loss of generality, we may assume that $(PA^t)_k \not\equiv 0$ for any $A \in GL(d)$ and $1 \le k \le d$. If not, then there exists $A \in GL(d)$ and $d_0 < d$ such that $(PA^t)_k \equiv 0$ for $k > d_0$. The induction hypothesis then implies that H is bounded on L^r :

$$\begin{aligned}
||Hf||_r^r &= \frac{1}{|\det(A)|} \left|\left|(Hf)^{A^{-1}}\right|\right|_r^r \\
&= \int \left| \int f^{A^{-1}}(x - s, y - t, z - PA^t(s, t)) \frac{ds\,dt}{st} \right|^r \frac{dx\,dy\,dz}{|\det(A)|} \\
&= \int \left[\int \left| \int f^{A^{-1}} \left(x - s, y - t, z' - (PA^t(s, t))', z''\right) \frac{ds\,dt}{st} \right|^r \right. \\
&\qquad\qquad \left. \times\, dx\,dy\,dz' \right] \frac{dz''}{|\det(A)|} \\
&\le C \int \left[\int \left| f^{A^{-1}}(x, y, z', z'') \right|^r dx\,dy\,dz' \right] \frac{dz''}{|\det(A)|} \\
&\le C||f||_r^r.
\end{aligned}$$

Here at the third step, we have written $z = (z', z'')$, with $z' \in \mathbb{R}^{d_0}$ and $z'' \in \mathbb{R}^{d-d_0}$. We have also used the identities (1) and (2) from subsection 1.3.

Now let ϕ be a smooth, odd function on $\mathbb{R}$ such that $\phi(s) \ge 0$ if $s \ge 0$, $\text{supp}(\phi) \subset \{s \in \mathbb{R} : 1/2 \le |s| \le 2\}$, and $\sum_{p\in\mathbb{Z}} 2^p \phi(2^p s) = 1/s$. Following [1], we localize H to dyadic rectangles in (s, t), and denote the localized operators by $H_{p,q}$:

$$H_{p,q}[P]f(x, y, z) := 2^{p+q} \int f(x - s, y - t, z - P(s, t))\phi\Big(\frac{2^p s}{c}\Big)\,\phi\Big(\frac{2^q t}{c}\Big)\, ds\,dt, \tag{2.3}$$

where c is the small constant in (1.1) whose exact value will be prescribed in the sequel (in Lemma 2.7). Then $H = c^{-1} \sum_{(p,q)\in\mathbb{Z}^2} H_{p,q}[P]$. It suffices to consider the operator $H_0 = H_0[P] := \sum_{p,q\geq 0} H_{p,q}[P]$, which we write as a finite sum :

$$H_0 = \sum_{j=1}^{M} D_j[P], \quad \text{with} \quad D_j[P] := \sum_{\substack{(p,q)\in\mathcal{R}_j \\ p,q\geq 0}} H_{p,q}[P],$$

where $\mathcal{R}_j$ has been described in subsection 1.3. We will show that for each j, D_j is bounded on L^r, $1 < r < \infty$.

2.2. **Reduction to monomials.**

Lemma 2.2. *Given $\mathcal{R}_j$, a $d\times d$ matrix A and an index k, $1 \leq k \leq d$, let $i = i(j,A,k)$. Then the following conclusions hold :*

(a) For any (u,v) in the Newton polygon of $(PA^t)_k$ and $(p,q)\in\mathcal{R}_j$,

$$2^{-(pu+qv)} \leq 2^{-(pm_i+qn_i)}. \tag{2.4}$$

(b) For s, t such that $2^p|s|, 2^q|t| \leq c$,

$$\big|(PA^t)_k(s,t)\big| \leq C2^{-pm_i-qn_i}.$$

Proof. The proof of (a) is identical to that of Lemma 2 of [1] and we do not reproduce it here. For (b), we set $s = 2^{-p}s'$ and $t = 2^{-q}t'$. Then

$$\begin{aligned}\big|(PA^t)_k(s,t)\big| &= \left|\sum_{p,q} a_{p,q}(A,k)s^pt^q\right| \\ &\leq 2^{-pm_i-qn_i}\sum_{p,q}|a_{p,q}(A,k)|\,|s'|^p|t'|^q \\ &\leq C2^{-pm_i-qn_i},\end{aligned}$$

where at the last step we have used the fact that (s',t') lies in $[-c,c]^2$, which is contained in the domain of convergence of P. □

In view of Lemma 2.2, $a_{m,n}(I,k)s^mt^n$ in the dominant term in P_k for $|s|\sim 2^{-p}$, $|t|\sim 2^{-q}$ and $(p,q)\in\mathcal{R}_j$ if $(m,n) = (m_i,n_i)$, $i = i(j,I,k)$. Our next lemma proves that after a suitable linear transformation, the dominant terms of the entries of P can be chosen to be monomials with distinct exponent vectors.

Lemma 2.3. *Given $\mathcal{R}_j$, there exists a matrix $A\in GL(d)$ such that*

$$\big(m_{i(j,A,k)}, n_{i(j,A,k)}\big) \neq \big(m_{i(j,A,k')}, n_{i(j,A,k')}\big)$$

for $k\neq k'$, $1\leq k,k'\leq d$.

Proof. We do this inductively. Let us set $(M_1,N_1) = (m_{i(j,I,1)}, n_{i(j,I,1)})$, and define a $d\times d$ lower triangular matrix $C_1 = ((c^{(1)}_{uv}))$ as follows :

$$c^{(1)}_{uv} = \begin{cases} -a_{M_1,N_1}(I,r)/a_{M_1,N_1}(I,1) & \text{if } v=1,\ 2\leq u\leq d, \\ 1 & \text{if } u=v,\ 1\leq u\leq d, \\ 0 & \text{otherwise.}\end{cases}$$

Then the term $s^{M_1}t^{N_1}$ has a nonzero coefficient in $(PC_1^t)_l$ if and only if $l=1$.

In general at the k-th step, $2\leq k\leq d$, we set

$$B_k = C_{k-1}\cdots C_2C_1.$$

From the previous induction step, $a_{M_l,N_l}(B_k,l)s^{M_l}t^{N_l}$ is the dominant term in $(PB_k^t)_l$ for $1 \le l \le k-1$. Furthermore $(M_l,N_l) \ne (M_{l'},N_{l'})$ for $l \ne l'$, $1 \le l,l' \le k-1$. We now define

$$M_k = m_{i(j,B_k,k)},\ N_k = n_{i(j,B_k,k)}. \tag{2.5}$$

Since by construction none of the monomials $s^{M_l}t^{N_l}$ are represented in $(PB_k^t)_{l'}$ for $l \le k-1$ and $l' \ge k$, we must have $(M_k,N_k) \ne (M_l,N_l)$ for $l \le k-1$. We now define a lower triangular matrix $C_k = ((c_{uv}^{(k)}))$ as follows :

$$c_{uv}^{(k)} = \begin{cases} -a_{M_k,N_k}(B_k,r)/a_{M_k,N_k}(B_k,k) & \text{if } v=k,\ k+1 \le u \le d, \\ 1 & \text{if } u=v,\ 1 \le u \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Let us set $B_{k+1} = C_kB_k$. Then $(PB_k^t)_l \equiv (PB_{k+1}^t)_l$ for $l \le k$, so that $i(j,B_k,l) = i(j,B_{k+1},l)$ for all $l \le k$. Thus the dominant terms in the previous steps of the induction remain unchanged. Moreover, for $l \ge k$ the coefficient of $s^{M_k}t^{N_k}$ in $(PB_{k+1}^t)_l$ is nonzero if and only if $l = k$.

The process concludes in d steps and we finally choose $A = B_d = C_{d-1}\cdots C_2C_1$. □

2.3. **Error Estimation.** In the rest of the paper, for fixed j, we work with A as given by the lemma above. Let $W := W_0(A^t)^{-1}$, where $W_0(s,t)$ is the d-dimensional vector whose k-th entry is the monomial $a_{M_k,N_k}(A,k)s^{M_k}t^{N_k}$, with

$$(M_k,N_k) = (m_{i(j,A,k)}, n_{i(j,A,k)}).$$

Note that this is consistent with (2.5), and that $M_k^2 + N_k^2 \ne 0$ for each k, since $P(0,0) = 0$. The operator $H_0[W_0]$ is bounded on L^r, $1 < r < \infty$ by Ricci-Stein [6], and therefore so is $H_0[W]$, by item (3) in the remark on page 4. In this subsection, we will show that for each $1 < j < T$ and $r \in (1,\infty)$, $D_j[PA^t] - D_j[W_0]$ is bounded on L^r. (The cases $j = 1$ and $j = T$ need a little extra work, and will be dealt with separately in the next subsection). In order to achieve the L^r bound, we need further decompositions of the region $\mathcal{R}_j$ and the corresponding operator D_j, which we now describe.

Let us choose numbers $\nu_j > 0$, $1 < j < T$, satisfying $\kappa_j < \nu_j < \kappa_{j-1}$, and decompose each cone $\mathcal{R}_j$ into a lower and an upper part :

$$\mathcal{R}_j^+ = \{(p,q) \in \mathcal{R}_j \,:\, p \ge \nu_j q\} \tag{2.6}$$

$$\mathcal{R}_j^- = \{(p,q) \in \mathcal{R}_j \,:\, p < \nu_j q\}. \tag{2.7}$$

The error operator therefore splits as $D_j[PA^t] - D_j[W_0] = E_j^+ + E_j^-$, where

$$E_j^\pm = \sum_{(p,q)\in\mathcal{R}_j^\pm} \left[H_{p,q}[PA^t] - H_{p,q}[W_0]\right].$$

We prove that E_j^- is bounded on L^r, the treatment of E_j^+ being similar. Let us write $\kappa_j = p_j/q_j$ where p_j, q_j are non-negative integers that are relatively prime and $q_j > 0$. The region $\mathcal{R}_j^-$ can then be realized as the disjoint union of lines $\mathcal{R}_j^L$ that are parallel to the upper edge of the cone $\mathcal{R}_j$:

$$\mathcal{R}_j^L = \{(p,q) \in \mathcal{R}_j^- \,:\, pq_j - p_jq = Lq_j\}, \quad L \ge 0,\ Lq_j \in \mathbb{Z}.$$

Thus $E_j^- = \sum_{L\geq 0} E_j^L$, where

$$E_j^L = \sum_{(p,q)\in\mathcal{R}_j^L} \left[H_{p,q}[PA^t] - H_{p,q}[W_0]\right].$$

To show that E_j^- is bounded on L^r, it suffices to establish the following estimate :

$$||E_j^L||_{L^r\to L^r} \leq C2^{-\sigma L} \quad \text{for some } \sigma > 0. \tag{2.8}$$

In order to prove (2.8), we further split the operator E_j^L, following the anisotropic Littlewood-Paley decomposition in [1]. We choose the dilation that preserves the support of the kernel of $\sum_{(p,q)\in\mathcal{R}_j^L} H_{p,q}[W_0]$. More specifically, let δ_ℓ be the anisotropic dilation

$$\delta_\ell(\xi,\eta,\gamma) = (2^{-\sigma_1\ell}\xi, 2^{-\sigma_2\ell}\eta, 2^{-\overline{\tau}\ell}\gamma),$$

where $\sigma_1 = \kappa_j$, $\sigma_2 = 1$, $\overline{\tau} = (\tau_1,\cdots,\tau_d) = \kappa_j\overline{M} + \overline{N}$, and $2^{-\overline{\tau}\ell}\gamma$ denotes the Hadamard vector product, whose k-th entry is $2^{-\tau_k\ell}\gamma_k$. The invariance of the support of the kernel under these dilations follows from the relations $q\kappa_j + L = p$ and $q\overline{\tau} + L\overline{M} = \overline{M}p + \overline{N}q$. Further, let B_L be the dilation

$$B_L(\xi,\eta,\gamma) = (2^{-L}\xi, \eta, 2^{-\overline{M}L}\gamma),$$

so that

$$\delta_q B_L(\xi,\eta,\gamma) = (2^{-p}\xi, 2^{-q}\eta, 2^{-\overline{M}p-\overline{N}q}\gamma). \tag{2.9}$$

Let us choose $\psi \in C_0^\infty(\mathbb{R}^{d+2})$ which is supported in an annulus centered at the origin and satisfies $\sum_{\ell\in\mathbb{Z}} \psi(\delta_\ell(\cdot)) \equiv 1$. We set $\psi_\ell(\cdot) = \psi(\delta_\ell B_L(\cdot))$ and use these to decompose the identity operator into Littlewood-Paley pieces U_ℓ :

$$\widehat{U_\ell f}(\cdot) = \psi_\ell(\cdot)\widehat{f}(\cdot). \tag{2.10}$$

This yields the following decomposition : $E_j^L f = \sum_{\ell\in\mathbb{Z}} \mathcal{E}_\ell f$, where

$$\mathcal{E}_\ell = \mathcal{E}_\ell(j,L) = V_\ell[PA^t] - V_\ell[W_0], \quad \text{with} \quad V_\ell[F] := \sum_{(p,q)\in\mathcal{R}_j^L} H_{p,q}[F]\, U_{q+\ell}. \tag{2.11}$$

The main result of this subsection is the next proposition, from which (2.8) follows using standard interpolation and duality arguments.

Proposition 2.4. *For $1 < j < T$ and $\mathcal{E}_\ell = \mathcal{E}_\ell(j,L)$ as above, the following estimates hold.*

$$||\mathcal{E}_\ell||_{L^2\to L^2} \leq C2^{-\sigma L}, \tag{2.12}$$

$$||\mathcal{E}_\ell||_{L^2\to L^2} \leq C2^{-\sigma|\ell|}, \tag{2.13}$$

$$||\mathcal{E}_\ell||_{L^r\to L^r} \leq C(|\ell| + L), \quad 1 < r \leq 2. \tag{2.14}$$

The proof of Proposition 2.4, specifically that of (2.12), requires a more precise version of the estimate in (2.4) and a careful analysis of the multipliers corresponding to $H_{p,q}$. We carry this out in the following sequence of lemmas. Let $\mu_{p,q}$ denote the multiplier for the operator $H_{p,q}[PA^t] - H_{p,q}[W_0]$. Then

$$\mu_{p,q}(\xi,\eta,\gamma) = \int \exp\left(i\xi 2^{-p}s + i\eta 2^{-q}t\right)\phi\Big(\frac{s}{c}\Big)\phi\Big(\frac{t}{c}\Big) \times \left[\exp(i\gamma\cdot PA^t(2^{-p}s,2^{-q}t)) - \exp(i\gamma\cdot W_0(2^{-p}s,2^{-q}t))\right] ds\,dt, \tag{2.15}$$

where $\xi,\eta\in\mathbb{R}$, $\gamma\in\mathbb{R}^d$. We deduce estimates for $\mu_{p,q}$ analogous to Lemma 3 in [1].

Lemma 2.5. *Given j, $1 < j < T$ and k, $1 \le k \le d$, let $i = i(j,A,k)$ and $(p,q)\in\mathcal{R}_j^L$. Then there exists $\sigma > 0$ (depending only on P and the choice of the ν_j-s) such that for (u,v) in the Newton polygon of $(PA^t)_k$, $(u,v)\neq(m_i,n_i)$,*

$$\frac{2^{-pu-qv}}{2^{-pm_i-qn_i}} \le 2^{-\sigma L}.$$

Proof. The index k is fixed in this lemma, so for notational simplicity let us denote $\lambda_i = \lambda_i((PA^t)_k)$. From (1.4) it follows that $\lambda_i \le \kappa_j < \kappa_{j-1} \le \lambda_{i-1}$. We treat three subcases : $u > m_i$, $u < m_i$ and $u = m_i$. First suppose that $u > m_i$. Since the polygon is convex and contains (u,v), therefore $(n_i - v)/(u-m_i) \le \lambda_i$. This gives

$$\begin{aligned}\frac{2^{-pu-qv}}{2^{-pm_i-qn_i}} &= 2^{-p(u-m_i)}2^{q(n_i-v)}\\ &\le 2^{-p(u-m_i)}2^{q\lambda_i(u-m_i)}\\ &\le 2^{(-p+q\kappa_j)(u-m_i)}\\ &\le 2^{-L/q_j} = 2^{-\sigma L}.\end{aligned} \tag{2.16}$$

Next suppose $u < m_i$. By convexity of the Newton polygon, we have $(v - n_i)/(m_i - u) \ge \lambda_{i-1}$. We need to find $\sigma > 0$ such that

$$q(v-n_i) - p(m_i - u) \ge \sigma L.$$

Now $\sigma L = \sigma(pq_j - p_jq) \le \sigma pq_j \le \sigma q\nu_j q_j$. On the other hand,

$$\begin{aligned} q(v-n_i) - p(m_i-u) &= (m_i - u)\left[q\frac{v-n_i}{m_i-u} - p\right]\\ &\ge (m_i-u)(q\lambda_{i-1} - p)\\ &\ge q\left(\lambda_{i-1} - \frac{p}{q}\right)\\ &\ge q(\lambda_{i-1} - \nu_j).\end{aligned} \tag{2.17}$$

Choosing $0 < \sigma \le (\lambda_{i-1} - \nu_j)/(\nu_j q_j)$ completes the proof in this case.

If $u = m_i$, then $v > n_i$. We want to show that $q(v-n_i) > \sigma(pq_j - p_jq)$. The left hand side is $\ge q$. On the other hand, the right side of the inequality $\le \sigma pq_j < \sigma q\nu_j q_j$. Choosing $0 < \sigma \le 1/(q_j\nu_j)$ therefore suffices. ◻

Corollary 2.6. *Given A as in Lemma 2.3, let $\overline{M}$, $\overline{N}$ be the vectors in $\mathbb{Z}^d$ whose k-th entries are $M_k = m_{i(j,A,k)}$ and $N_k = n_{i(j,A,k)}$ respectively. Then there exist constants C and $\sigma > 0$ such that for $(p,q)\in\mathcal{R}_j^L$,*

$$|\mu_{p,q}(\xi,\eta,\gamma)| \le C2^{-\sigma L}|\gamma|\cdot 2^{-p\overline{M}-q\overline{N}}. \tag{2.18}$$

Proof. We estimate the integral in (2.15) by the mean value theorem. Using Lemma 2.5, the desired bound follows from the estimate

$$\begin{aligned}\left|\gamma \cdot \left[PA^t - W_0\right](2^{-p}s, 2^{-q}t)\right| &= \Big|\sum_{k=1}^{d} \gamma_k 2^{-pM_k - qN_k} \\ &\quad \times \Big[\sum_{(m,n)\neq(M_k,N_k)} a_{m,n}(A,k)\frac{2^{-pm-qn}}{2^{-pM_k-qN_k}} s^m t^n\Big]\Big| \\ &\leq C2^{-\sigma L}\sum_{k=1}^{d}|\gamma_k|2^{-pM_k-qN_k}\sum_{m,n}|a_{m,n}(A,k)|c^{m+n} \\ &\leq C2^{-\sigma L}|\gamma|\cdot 2^{-p\overline{M}-q\overline{N}}.\end{aligned}$$

□

Lemma 2.7. *Fix j and let $\overline{M}$ and $\overline{N}$ be as in Corollary 2.6. Then there exist constants $0 < c, \sigma_0 < 1$ and $C > 1$ (depending only on P) such that for all $(p, q) \in \mathcal{R}_j^L$ and $(\xi, \eta, \gamma) \in \mathbb{R}^{d+2}$,*

$$|\mu_{p,q}(\xi,\eta,\gamma)| \leq C||\phi||_{C^1}^2\left[|\xi|2^{-p} + |\eta|2^{-q} + |\gamma|\cdot 2^{-p\overline{M}-q\overline{N}}\right]^{-\sigma_0}. \tag{2.19}$$

The number c occurs in the definition of $\mu_{p,q}$ (see (2.15)).

Proof. We first specify c and two auxiliary constants C_0 (large) and c_1 (small). The constant C in (2.19) will depend on C_0 and c_1. Given a multi-index $\alpha = (\alpha_1, \alpha_2)$ with nonnegative integer entries, let Γ_α be the integer-valued polynomial in (m, n) given by

$$\Gamma_\alpha(m,n) = \partial^\alpha(s^m t^n)\big|_{s=t=1}.$$

Of course $\Gamma_\alpha \not\equiv 0$ if and only if $m \geq \alpha_1$ and $n \geq \alpha_2$. Let us fix two constants K and ϱ as follows,

$$K = \sum_{k=1}^{d}(M_k + N_k), \quad \text{and} \quad \varrho = \min(\varrho_1, \varrho_2, \varrho_3), \text{ where}$$

$$\varrho_1 = \frac{1}{10(d+2)}, \quad \varrho_2 = \min_{1\leq k\leq d}|a_{M_k,N_k}(A,k)| > 0, \text{ and}$$

$$\varrho_3 = \min\left\{\left|\frac{a_{M_kN_k}(A,k)}{a_{M_{k'}N_{k'}}(A,k)}\right| : k' > k,\ a_{M_{k'}N_{k'}}(A,k) \neq 0\right\}. \tag{2.20}$$

If the set in (2.20) is empty, then ϱ_3 is taken to be $+\infty$. We now choose c_1, c and C_0 satisfying $c_1 = (\varrho/(2d))^d$,

$$\sup_{\substack{|\alpha|\leq K+1\\ 1\leq k\leq d}}\sum_{m+n>|\alpha|}|a_{m,n}(A,k)|\Gamma_\alpha(m,n)c^{m+n-|\alpha|} < c_1\varrho, \tag{2.21}$$

$$\sup_{\substack{|\alpha|\leq K+1\\ 1\leq k\leq d}}\sum_{m,n}|a_{m,n}(A,k)|\Gamma_\alpha(m,n)c^{m+n-|\alpha|} < c_1(C_0-1), \text{ and} \tag{2.22}$$

To prove (2.19), we make use of a multi-variate version of Van der Corput's lemma, which we quote below (see Proposition 5 on page 342 of [8] for a proof): *Let ϕ be a smooth function supported on $[-c, c]$ for some small constant c, and let R*

be a smooth real-valued function on $\mathbb{R}^2$ satisfying $|\partial^\beta R| \geq 1$ on $[-c,c]^2$ for some multi-index β, $|\beta| > 0$. Then

$$\iint e^{i\lambda R(s,t)}\phi(s)\phi(t) \leq c_\beta(R)\,||\phi||^2_{C^1}\,\lambda^{-1/|\beta|}. \tag{2.23}$$

The constant $c_\beta(R)$ is independent of λ and ϕ, and remains bounded as long as the $C^{|\beta|+1}$ norm of R remains bounded.

We apply this result twice, by choosing

$$\lambda = 2^{-p}|\xi| + 2^{-q}|\eta| + |\widetilde{\gamma}| \cdot 2^{-p\overline{M}-q\overline{N}},$$
$$\widetilde{\gamma}_k = \sum_{k' \leq k} \gamma_{k'} a_{M_k,N_k}(A,k'),\ 1 \leq k \leq d, \text{ and}$$
$$R(s,t) = \lambda^{-1}(\xi 2^{-p}s + \eta 2^{-q}t + \gamma \cdot R_0(2^{-p}s, 2^{-q}t)),$$

where $R_0 = PA^t$ and W_0 respectively. The estimate (2.19) will follow from (2.23) (with C specified in terms of C_0, c_1 and ϱ) if we show that

(i) there exists a multi-index β, $|\beta| \leq K$ such that

$$\left|\partial^\beta R(s,t)\right| \geq 9\varrho \quad \text{for } |s|,|t| \leq c,$$

(ii) the $C^{|\beta|+1}$ norm of R on $[-c,c]^2$ is uniformly bounded by C_0, and

(iii) $\lambda \geq c_1(2^{-p}|\xi| + 2^{-q}|\eta| + |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}})$.

Let us assume (iii) for the moment. It then suffices to prove (i) and (ii) above for $R_0 = PA^t$, the proof for W_0 being similar, and in fact easier. For fixed p,q,ξ,η,γ, we choose an index k_0 such that

$$|\widetilde{\gamma}_{k_0}|2^{-pM_{k_0}-qN_{k_0}} \geq \frac{\lambda}{d+2}, \quad \text{and set} \quad \beta = (M_{k_0}, N_{k_0}). \tag{2.24}$$

(The proof when $2^{-p}|\xi|$ and $2^{-q}|\eta|$ is larger than a fraction of λ is simpler and is left to the reader). We will show that (2.19) holds with $\sigma_0 = 1/(M_{k_0}+N_{k_0})$.

The proof of (ii) is clear. For $|s|,|t| \leq c$ and any α with $|\alpha| \leq |\beta|+1$,

$$\begin{aligned}
|\partial^\alpha R(s,t)| &= \lambda^{-1}\left|\xi 2^{-p}\partial^\alpha(s) + \eta 2^{-q}\partial^\alpha(t) + \sum_{k=1}^d \gamma_k \sum_{m,n} a_{m,n}(A,k)2^{-pm-qn}\partial^\alpha(s^m t^n)\right| \\
&\leq \lambda^{-1}\left[2^{-p}|\xi| + 2^{-q}|\eta| + \sum_{k=1}^d |\gamma_k| \sum_{m,n} |a_{m,n}(A,k)|\, 2^{-pm-qn}\partial^\alpha(s^m t^n)\Big|_{s=t=c}\right] \\
&\leq 1 + \lambda^{-1}\sum_{k=1}^d |\gamma_k| 2^{-pM_k-qN_k} c_1(C_0-1) \\
&\leq 1 + \lambda^{-1}|\gamma| \cdot 2^{-p\overline{M}-q\overline{N}} c_1(C_0-1) \\
&\leq 1 + c_1^{-1}c_1(C_0-1) = C_0,
\end{aligned}$$

where the third step follows from (2.22) and (2.4), and the last step from (iii).

To prove (i), we recall from Lemma 2.3 that $(M_{k_0}, N_{k_0}) \notin \mathcal{N}((PA^t)_k)$ for $k > k_0$. Therefore we can write $\partial^\beta R$ as the sum of a dominant term $\mathfrak{A}$ and an error term $\mathfrak{B}$, and estimate

$$\left|\partial^\beta R(s,t)\right| \geq |\mathfrak{A}| - |\mathfrak{B}|$$

where

$$\mathfrak{A} := \lambda^{-1} 2^{-pM_{k_0}-qN_{k_0}} \Gamma_\beta(\beta) \sum_{k\le k_0} \gamma_k a_{M_{k_0},N_{k_0}}(A,k), \text{ and}$$

$$\mathfrak{B} := \lambda^{-1} \sum_k \gamma_k \partial^\beta \Big[\sum_{(m,n)\neq\beta} a_{m,n}(A,k) 2^{-pm-qn} s^m t^n \Big].$$

Then from (2.24) we find $|\mathfrak{A}| = \lambda^{-1} 2^{-pM_{k_0}-qN_{k_0}} |\widetilde{\gamma}_{k_0}| \ge 10\varrho_1 \ge 10\varrho$. On the other hand,

$$\partial^\beta \sum_{(m,n)\neq\beta} a_{m,n}(A,k) s^m t^n = \sum_{m+n>|\beta|} a_{m,n}(A,k) \partial^\beta(s^m t^n),$$

so it follows from (2.21), (2.4) and (iii) that

$$\begin{aligned}|\mathfrak{B}| &\le \lambda^{-1} \sum_k |\gamma_k| \sum_{m+n>|\beta|} |a_{m,n}(A,k)| 2^{-pm-qn} \partial^\beta(s^m t^n)\Big|_{s=t=c} \\ &\le \lambda^{-1} \sum_k |\gamma_k| 2^{-pM_k-qN_k} c_1\varrho \le c_1 \varrho \lambda^{-1} |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}} \le \varrho.\end{aligned}$$

Thus $|\partial^\beta R(s,t)| \ge 9\varrho$, which completes the proof of (i).

It therefore remains to prove (iii). Without loss of generality, let $\gamma \neq 0$. We use a stopping-time argument to show that there exist indices ι_0 and $k(\iota_0)$, with $1 \le \iota_0 \le d$ and $1 \le k(\iota_0) \le d$ such that

$$2^{-(p,q)\cdot\alpha(\iota_0)} |\widetilde{\gamma}_{k(\iota_0)}| \ge \Big(\frac{\varrho}{2d}\Big)^{\iota_0} |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}} \ge c_1 |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}}, \tag{2.25}$$

where $\alpha(\iota) = (M_{k(\iota)}, N_{k(\iota)})$. Let $k(1)$ be an index for which the maximum of $\{2^{-pM_k-qN_k} |\gamma_k a_{M_k N_k}(A,k)| : 1 \le k \le d\}$ is attained. Then

$$\begin{aligned}2^{-(p,q)\cdot\alpha(1)} |\gamma_{k(1)} a_{\alpha(1)}(A,k(1))| &\ge \frac{1}{d} \sum_{k=1}^d 2^{-pM_k-qN_k} |\gamma_k a_{M_k N_k}(A,k)| \\ &\ge \frac{\varrho_2}{d} |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}} \ge \frac{\varrho}{d} |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}}.\end{aligned} \tag{2.26}$$

If $|\gamma_{k(1)} a_{\alpha(1)}(A,k(1))| \ge 2|\sum_{k<k(1)} \gamma_k a_{\alpha(1)}(A,k)|$, then it follows from (2.26) that

$$\begin{aligned}2^{-(p,q)\cdot\alpha(1)} |\widetilde{\gamma}_{k(1)}| &= 2^{-(p,q)\cdot\alpha(1)} \Big| \sum_{k\le k(1)} \gamma_k a_{\alpha(1)}(A,k) \Big| \\ &\ge \frac{1}{2} 2^{-(p,q)\cdot\alpha(1)} \big| \gamma_{k(1)} a_{\alpha(1)}(A,k(1)) \big| \\ &\ge \frac{\varrho}{2d} |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}} \ge c_1 |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}},\end{aligned}$$

and (2.25) is proved for $\iota_0 = 1$. If $|\gamma_{k(1)} a_{\alpha(1)}(A,k(1))| < 2|\sum_{k<k(1)} \gamma_k a_{\alpha(1)}(A,k)|$, we proceed to step 2.

In general, at the end of step ι we have

$$2^{-(p,q)\cdot\alpha(\iota)} |\gamma_{k(\iota)} a_{\alpha(\iota)}(A,k(\iota))| \ge \frac{\varrho^\iota}{d(2d)^{\iota-1}} |\gamma| \cdot 2^{-p\overline{M}-q\overline{N}}, \tag{2.27}$$

and are forced to continue to step $(\iota+1)$ if

$$\Big| \gamma_{k(\iota)} a_{\alpha(\iota)}(A,k(\iota)) \Big| < 2 \Big| \sum_{k<k(\iota)} \gamma_k a_{\alpha(\iota)}(A,k) \Big|. \tag{2.28}$$

In this case, we pick an index $k(\iota+1) < k(\iota)$ that maximizes $\{|\gamma_k a_{\alpha(\iota)}(A,k)| : 1 \leq k < k(\iota)\}$. It follows from (2.28) that

$$|\gamma_{k(\iota+1)} a_{\alpha(\iota)}(A, k(\iota+1))| \geq \frac{1}{2d} \Big| \gamma_{k(\iota)} a_{\alpha(\iota)}(A, k(\iota)) \Big|, \tag{2.29}$$

which in particular implies that $\gamma_{k(\iota+1)} \neq 0$ and $a_{\alpha(\iota)}(A, k(\iota+1)) \neq 0$. Using (2.20), (2.4), (2.29) and (2.27) we obtain

$$\begin{aligned}
&2^{-(p,q)\cdot\alpha(\iota+1)} \Big| \gamma_{k(\iota+1)} a_{\alpha(\iota+1)}(A, k(\iota+1)) \Big| \\
&\qquad = 2^{-(p,q)\cdot\alpha(\iota+1)} \Big| \gamma_{k(\iota+1)} a_{\alpha(\iota)}(A, k(\iota+1)) \Big| \left| \frac{a_{\alpha(\iota+1)}(A, k(\iota+1))}{a_{\alpha(\iota)}(A, k(\iota+1))} \right| \\
&\qquad \geq \frac{\varrho_3}{2d} 2^{-(p,q)\cdot\alpha(\iota)} \Big| \gamma_{k(\iota)} a_{\alpha(\iota)}(A, k(\iota)) \Big| \\
&\qquad \geq \frac{\varrho^{\iota+1}}{d(2d)^{\iota}} |\gamma| \cdot 2^{-p\overline{M} - q\overline{N}}.
\end{aligned}$$

We stop at step $(\iota+1)$ if

$$\Big| \gamma_{k(\iota+1)} a_{\alpha(\iota+1)}(A, k(\iota+1)) \Big| \geq 2 \Big| \sum_{k < k(\iota+1)} \gamma_k a_{\alpha(\iota+1)}(A, k) \Big|, \tag{2.30}$$

in which case

$$\begin{aligned}
2^{-(p,q)\cdot\alpha(\iota+1)} |\widetilde{\gamma}_{k(\iota+1)}| &= 2^{-(p,q)\cdot\alpha(\iota+1)} \Big| \sum_{k \leq k(\iota+1)} \gamma_k a_{\alpha(\iota+1)}(A, k) \Big| \\
&\geq \frac{1}{2} 2^{-(p,q)\cdot\alpha(\iota+1)} \Big| \gamma_{k(\iota+1)} a_{\alpha(\iota+1)}(A, k(\iota+1)) \Big| \\
&\geq \left(\frac{\varrho}{2d} \right)^{\iota+1} |\gamma| \cdot 2^{-p\overline{M} - q\overline{N}} \geq c_1 |\gamma| \cdot 2^{-p\overline{M} - q\overline{N}},
\end{aligned}$$

and (2.25) holds with $\iota_0 = \iota + 1$. The process stops in a finite number of steps since $k(\iota)$ is a strictly decreasing sequence bounded below by 1, and (2.30) always holds for $k(\iota+1) = 1$. □

Proof of Proposition 2.4. The estimate (2.12) is obtained by combining (2.19) and (2.18). For (2.13), we observe that

$$||\delta_{\ell+q} B_L(\xi, \eta, \gamma)|| \sim 1 \text{ on } \mathrm{supp}(\psi_{\ell+q}),$$

so it follows from (2.9) that there exist $\sigma, \sigma' > 0$ such that

$$2^{\sigma' \ell} \leq 2^{-p}|\xi| + 2^{-q}|\eta| + |\gamma| \cdot 2^{-p\overline{M} - q\overline{N}} \leq 2^{\sigma\ell} \text{ for } j \neq T.$$

Combining (2.19) with the above, we have (2.13). It remains to prove (2.14). We do this by estimating $V_\ell[F]$ with $F = PA^t$ and W_0. Since $V_\ell[F]$ is a convolution operator, (2.14) follows from the Hörmander condition

$$\int_{\rho(x,y,z) > C\rho(u,v,w)} |K_\ell(x+u, y+v, z+w) - K_\ell(x,y,z)| \, dx\, dy\, dz \leq C(|\ell| + L),$$

where K_ℓ is the kernel of $V_\ell[F]$ and $\rho(x,y,z) = |x|^{1/\sigma_1} + |y|^{1/\sigma_2} + \sum_{k=1}^{d} |z_k|^{1/\tau_k}$. The arguments for proving the above inequality are the same as those in [1], p507-509, so we omit the details here. We would like to point out that on page 508 of [1], estimate (13) holds if $2^q \rho(u,v,w) \geq 1$. In other words, $\beta = 1$ in the notation of [1] $\beta = 1$. We need this to ensure that summing the estimates in (11), (12) and (13) yields $C(|\ell| + L)$.

Remark : We would like to point out that (2.14) can alternatively be proved using techniques from a paper of Duoandikoetexea and Rubio de Francia [4] on maximal and singular integral operators. We will show an application of these techniques in our analysis of the endpoint case $j = T$ in subsection 2.4, and we ask the reader to verify that the same approach works in this case as well. $\square$

2.4. **The Endpoints $j = 1$ and $j = T$.** In this subsection, we describe the case $j = T$. The case $j = 1$ is similar and is left to the reader. We choose as in subsection 2.3 a number ν_T, $0 < \nu_T < \kappa_{T-1}$, and decompose $\mathcal{R}_T$ into a lower and an upper part, following (2.6) and (2.7). We leave the reader to verify that E_T^+ can be treated exactly as before. However, since $\kappa_T = 0$, the left edge of the cone $\mathcal{R}_T^-$ is vertical, and $\mathcal{R}_T^-$ can be realized as a disjoint union of vertical lines $\mathcal{R}_T^L = \{(L, q); L < \nu_T q\}$. The analysis of subsection 2.3 therefore fails for E_T^-, since the bound on $\mu_{p,q}$ obtained in (2.18) is no longer summable in q with an uniform bound if an entry of $\overline{N}$ is zero.

We circumvent this as follows. Let $d_1 := \#\{k : N_k = 0\} \le d$. Without loss of generality, by renaming the entries of γ if necessary, we may assume that $N_k = 0$ if and only if $1 \le k \le d_1$. Let W_0 be as in subsection 2.3, and let $\widetilde{W_0}$ be the d-dimensional vector whose k-th entry is given by

$$\left(\widetilde{W_0}\right)_k (s,t) = \begin{cases} (PA^t)_k\,(s,0) & \text{if } 1 \le k \le d_1, \\ a_{M_k N_k}(A,k) s^{M_k} t^{N_k} & \text{if } d_1 < k \le d. \end{cases}$$

Since $D_T[W_0]$ is bounded on L^r, $1 < r < \infty$ by Ricci-Stein [6], the L^r-boundedness of $D_T[PA^t]$ follows from the next two propositions.

Proposition 2.8. *$D_T[PA^t] - D_T[\widetilde{W_0}]$ is bounded on L^r, $1 < r < \infty$.*

Proposition 2.9. *$D_T[\widetilde{W_0}] - D_T[W_0]$ is bounded on L^r, $1 < r < \infty$.*

The proof of Proposition 2.8 is a slight variation of our earlier arguments with suitable modifications of the dilations δ_ℓ and B_L, and we address this first. Since there is no dilation in the direction of the first variable ξ, we introduce the anisotropic dilation in the last $(d+1)$ variables only. Let

$$\tilde{\delta}_\ell(\eta, \gamma) = (2^{-\ell}\eta, 2^{-\overline{N}\ell}\gamma), \quad \tilde{B}_L(\eta, \gamma) = (\eta, 2^{-\overline{M}L}\gamma),$$

and let $\tilde{\psi}$ denote a C_0^∞ function defined on an annulus in $\mathbb{R}^{d+1}$ so that $\sum_\ell \tilde{\psi}(\tilde{\delta}_\ell(\cdot)) \equiv 1$. We now define $\tilde{\psi}_\ell$, $\tilde{U}_\ell$, and $\tilde{V}_\ell$ in the same way as their analogues in (2.10) and (2.11) without the superscript $\tilde{}$. Then

$$E_T^L = \sum_\ell \mathcal{E}_\ell = \sum_\ell \mathcal{E}_\ell(T, L),$$

where $\mathcal{E}_\ell(T, L)$ is defined as in (2.11), but with V_ℓ replaced by $\tilde{V}_\ell$. Proposition 2.8 then follows from the next proposition, using standard interpolation and duality arguments.

Proposition 2.10. *The estimates (2.12), (2.13), and (2.14) in Proposition 2.4 continue to hold with $j = T$.*

The proof of the above proposition, in particular that of (2.12), requires a refinement of Lemma 2.5, which we furnish below. Let $\widetilde{\mu}_{p,q}$ denote the multiplier for the operator $H_{p,q}[PA^t] - H_{p,q}[\widetilde{W}_0]$. Let us also henceforth write any d-dimensional vector z as $z = (z', z'')$, where $z' = (z_1, \cdots, z_{d_1})$ and $z'' = (z_{d_1+1}, \cdots, z_d)$.

Lemma 2.11. *Given k, $1 \le k \le d$, let $i = i(T, A, k)$ and $(p, q) \in \mathcal{R}_T^L$. Then for $k \le d_1$, there exists $\sigma > 0$ (depending only on P and the choice of the ν_j-s) such that for (u, v) in the Newton polygon of $(PA^t)_k$, $v \neq 0$,*

$$\frac{2^{-pu-qv}}{2^{-pm_i}} \le 2^{-\sigma(q+L)}. \tag{2.31}$$

For $k > d_1$, the conclusion of Lemma 2.5 holds, namely for (u, v) in the Newton polygon of $(PA^t)_k$, $(u, v) \neq (m_i, n_i)$,

$$\frac{2^{-pu-qv}}{2^{-pm_i-qn_i}} \le 2^{-\sigma L}. \tag{2.32}$$

Proof. We skip the proof of (2.32), as it is identical to that of Lemma 2.5. For (2.31), let us observe that since $v \ge 1$, therefore $u \le m_i$. If $u = m_i$, then

$$\frac{2^{-pu-qv}}{2^{-pm_i}} = 2^{-qv} \le 2^{-q} \le 2^{-\frac{q}{2}} 2^{-\frac{L}{\nu_T}}.$$

If $u < m_i$, then following the computation in (2.17), we obtain

$$q(v - n_i) - p(m_i - u) \ge q(\lambda_{i-1} - \nu_T).$$

This gives

$$\frac{2^{-pu-qv}}{2^{-pm_i}} \le 2^{-(\lambda_{i-1}-\nu_T)q} \le 2^{-\sigma(q+L)},$$

for some σ with $0 < \sigma < \frac{1}{2}(\lambda_{i-1} - \nu_T) \min(1, 1/\nu_T)$. □

Corollary 2.12. *Let $j = T$, and let A, $\overline{M}$, $\overline{N}$ be as in Corollary 2.6. Then there exist constants C and $\sigma > 0$ such that for $(p, q) \in \mathcal{R}_T^L$,*

$$|\widetilde{\mu}_{p,q}(\xi, \eta, \gamma)| \le C \left[2^{-\sigma(q+L)} |\gamma'| \cdot 2^{-p\overline{M}'} + 2^{-\sigma L} |\gamma''| \cdot 2^{-\left(p\overline{M}'' + q\overline{N}''\right)} \right]. \tag{2.33}$$

Proof. The proof follows from the mean value theorem and the definition of $\widetilde{W}_0$, as in the proof of Corollary 2.6. □

We also use the following lemma due to Duoandikoetexea and Rubio de Francia [4]. We provide the proof for the sake of completeness.

Lemma 2.13 ([4], p544)**.** *Let $\{\mathfrak{a}_q : q \in \mathbb{Z}\}$ denote any sequence of finite Borel measures on $\mathbb{R}^{d+2}$ such that $\sup_q ||\mathfrak{a}_q|| < \infty$, and for some $r_1 > 1$,*

$$||\mathfrak{a}^*(f)||_{r_1} \le C_0 ||f||_{r_1}, \tag{2.34}$$

where $\mathfrak{a}^*$ is the maximal operator : $\mathfrak{a}^*(f) = \sup_q ||\mathfrak{a}_q| * f|$. Then for r_0 satisfying $(2r_1)^{-1} = |2^{-1} - r_0^{-1}|$, and arbitrary functions $\{g_q\}$, the following vector-valued inequality holds :

$$\Big|\Big| \Big(\sum_q |\mathfrak{a}_q * g_q|^2 \Big)^{\frac{1}{2}} \Big|\Big|_{r_0} \le C_0 \sup_q ||\mathfrak{a}_q|| \, \Big|\Big| \Big(\sum_q |g_q|^2 \Big)^{\frac{1}{2}} \Big|\Big|_{r_0}.$$

Proof. Without loss of generality let $\sup_q ||\mathfrak{a}_q|| = 1$. It suffices to consider the case $r_0 > 2$, so that $(r_0/2)' = r_1$. Then there exists $h \in L^{r_1}$ of unit norm such that

$$||(\sum_q |\mathfrak{a}_q * g_q|^2)^{\frac{1}{2}}||_{r_0}^2 = ||\sum_q |\mathfrak{a}_q * g_q|^2||_{\frac{r_0}{2}} = \int \sum_q |\mathfrak{a}_q * g_q|^2 h.$$

Setting $\tilde{h}(x) = h(-x)$, applying Holder's inequality and using the hypothesis (2.34), we obtain

$$\begin{aligned}
||(\sum_q |\mathfrak{a}_q * g_q|^2)^{\frac{1}{2}}||_{r_0}^2 &\le \sum_q \int (|\mathfrak{a}_q| * |g_q|^2)(x) h(x)\,dx \\
&\le \sum_q \int (|g_q|(y))^2 \Big(|\mathfrak{a}_q| * \tilde{h}\Big)(-y)\,dy \\
&\le ||(\sum_q |g_q|^2)^{\frac{1}{2}}||_{r_0}^2 ||\mathfrak{a}^*(h)||_{r_1} \\
&\le C_0 ||(\sum_q |g_q|^2)^{\frac{1}{2}}||_{r_0}^2.
\end{aligned}$$

□

Proof of Proposition 2.10. We ask the reader to verify that (2.19) continues to hold with $\mu_{p,q}$ replaced by $\widetilde{\mu}_{p,q}$, with the same proof. The estimate (2.12) therefore follows by combining (2.33) with (2.19), and (2.13) from (2.19) and the definition of $\tilde{\psi}_\ell$. We also prove for $r > 1$ and $F = PA^t$ and $\widetilde{W}_0$ the following inequality which is a stronger version of (2.14):

$$||\sum_{q > p/\nu_T} H_{p,q}[F]\tilde{U}_{q+\ell}||_{L^r \to L^r} \le C_r,$$

where C_r may depend on F but is independent of p and ℓ. For this, let $\overline{\psi}$ be a smooth function of compact support which is $\equiv 1$ on the support of $\tilde{\psi}$, and for which

$$C^{-1} \le \sum_\ell \overline{\psi}(\tilde{\delta}_\ell(\eta,\gamma)) \le C, \quad (\eta,\gamma) \in \mathbb{R}^{d+1} \backslash \{0\}.$$

Then the convolution operator $\overline{U}_\ell$ given by $\widehat{\overline{U}_\ell f}(\cdot) = \overline{\psi}(\tilde{\delta}_\ell(\cdot))\widehat{f}(\cdot)$ satisfies $\overline{U}_\ell \tilde{U}_\ell = \tilde{U}_\ell$. Note also that the maximal operator $\sup_{p,q} |H_{p,q}[F]|$ is bounded pointwise by a multiple of $M[F]$ defined as in (1.2), and hence is bounded on L^{r_1} for all $r_1 > 1$. Therefore using standard facts from Littlewood-Paley theory and Lemma 2.13 with $\widehat{\mathfrak{a}}_q = \mu_{p,q}$ we obtain for all $r_0 > 1$,

$$\begin{aligned}
||\sum_{q > p/\nu_T} H_{p,q}[F]\tilde{U}_{q+\ell} f||_{r_0} &= ||\sum_{q > p/\nu_T} \overline{U}_{q+\ell} H_{p,q}[F]\tilde{U}_{q+\ell} f||_{r_0} \\
&\le C||(\sum_q \big|H_{p,q}[F]\tilde{U}_{q+\ell} f\big|^2)^{\frac{1}{2}}||_{r_0} \\
&\le C||(\sum_q |\tilde{U}_{q+\ell} f|^2)^{\frac{1}{2}}||_{r_0} \le C||f||_{r_0}.
\end{aligned}$$

□

In the remainder of this subsection, we prove Proposition 2.9. As in our earlier arguments, we make a decomposition of $D_T[\widetilde{W}_0] - D_T[W_0]$ using anisotropic Littlewood-Paley dilations that preserve the support of the kernel. Notice that now there is no scaling in the direction of the variables ξ and γ', so the appropriate dilations to use are

$$\delta_\ell^{(0)}(\eta, \gamma'') = (2^{-\ell}\eta, 2^{-\overline{N}''\ell}\gamma'') \quad \text{and} \quad B_L^{(0)}(\eta, \gamma'') = (\eta, 2^{-\overline{M}''L}\gamma'').$$

Choosing $\psi^{(0)} \in C_0^\infty$ defined on an annulus in $\mathbb{R}^{(d-d_1)+1}$ with $\sum_\ell \psi^{(0)}(\delta_\ell(\cdot)) \equiv 1$ (away from the origin), we define $\psi_\ell^{(0)}$, $U_\ell^{(0)}$ and $V_\ell^{(0)}$ as their analogues in (2.10) and (2.11) without the superscript ${}^{(0)}$. Then

$$D_T[\widetilde{W}_0] - D_T[W_0] = \sum_L \sum_\ell \mathcal{E}_\ell^{(0)}, \quad \text{where}$$

$$\mathcal{E}_\ell^{(0)} = \mathcal{E}_\ell^{(0)}(T, L) = V_\ell^{(0)}[\widetilde{W}_0] - V_\ell^{(0)}[W_0], \text{ with } V_\ell^{(0)}[F] = \sum_{L \le \nu_T q} H_{p,q}[F] U_{q+\ell}^{(0)}.$$

In order to prove Proposition 2.9, it therefore suffices to prove the following.

Proposition 2.14. *There exists $\sigma > 0$ such that for $\mathcal{E}_\ell^{(0)}$ as above the following estimates hold :*

$$||\mathcal{E}_\ell^{(0)}||_{L^2 \to L^2} \le C 2^{-\sigma(|\ell| + L)} \tag{2.35}$$

$$||\mathcal{E}_\ell^{(0)}||_{L^r \to L^r} \le C(|\ell| + L), \quad 1 < r \le 2. \tag{2.36}$$

Proof of Proposition 2.14. The proof of (2.36) is similar to its analog for $\mathcal{E}_\ell(T, L)$, so we concentrate only on (2.35). Let $\mu_{p,q}^{(0)}$ denote the multiplier for the operator $H_{p,q}[\widetilde{W}_0] - H_{p,q}[W_0]$. By Corollary 2.6 and Lemma 2.7, we have the apriori estimates

$$\begin{aligned} &\left|\mu_{p,q}^{(0)}(\xi, \eta, \gamma)\right| \le \\ &\qquad C \min\left[2^{-\sigma L}|\gamma'| \cdot 2^{-p\overline{M}'}, (1 + |\xi|2^{-p} + |\eta|2^{-q} + |\gamma| \cdot 2^{-p\overline{M} - q\overline{N}})^{-\sigma}\right], \end{aligned} \tag{2.37}$$

but once again this by itself is not sufficient to guarantee that the sum in q of $\mu_{p,q}^{(0)}$ is uniformly bounded by $2^{-\sigma(|\ell|+L)}$. Also, since the entries $(\widetilde{W}_0)_k$ and $(W_0)_k$, $1 \le k \le d_1$ are independent of t, the argument leading to Corollary 2.12 does not apply any longer. However, using certain cancellation conditions, one can obtain a stronger estimate of $\mu_{p,q}^{(0)}$ that ensures summability. We now describe the new estimate in greater detail.

Let us recall two important facts :

(i) Since ϕ is an odd function, and $N_k = 0$ if and only if $1 \le k \le d_1$, therefore $\mu_{p,q}^{(0)}(\xi, 0, \gamma', 0) \equiv 0$.

(ii) The proof of Lemma 2.7 does not use the fact that ϕ is odd, therefore the functions $2^q \partial\mu_{p,q}^{(0)}/\partial\eta$ and $2^{pM_k + qN_k}\partial\mu_{p,q}^{(0)}/\partial\gamma_k$, $1 \le k \le d$ (which are of the same form as $\mu_{p,q}^{(0)}$, except possibly for the oddness of the cutoff function) continue to satisfy the estimate (2.19).

Using (i), we compute $\mu^{(0)}_{p,q}$ as follows :

$$\begin{aligned}\mu^{(0)}_{p,q}(\xi,\eta,\gamma) &= \mu^{(0)}_{p,q}(\xi,\eta,\gamma) - \mu^{(0)}_{p,q}(\xi,0,\gamma',0)\\ &= \int_0^1 \frac{d}{du}\mu^{(0)}_{p,q}(\xi,u\eta,\gamma',u\gamma'')du\\ &= \int_0^1 \left[\eta\frac{\partial\mu^{(0)}_{p,q}}{\partial\eta} + \sum_{k=d_1+1}^{d}\gamma_k\frac{\partial\mu^{(0)}_{p,q}}{\partial\gamma_k}\right](\xi,u\eta,\gamma',u\gamma'')\,du,\end{aligned}$$

therefore by (ii),

$$\begin{aligned}\left|\mu^{(0)}_{p,q}\right| &\le C(2^{-q}|\eta| + 2^{-p\overline{M}''-q\overline{N}''}\cdot|\gamma''|)\times\\ &\qquad \int_0^1 \left(1+|\xi|2^{-p}+|u\eta|2^{-q}+2^{-p\overline{M}-q\overline{N}}\cdot|(\gamma',u\gamma'')|\right)^{-\sigma}du \qquad (2.38)\\ &\le C(2^{-q}|\eta| + 2^{-p\overline{M}''-q\overline{N}''}\cdot|\gamma''|)\left(1+|\xi|2^{-p}+2^{-p\overline{M}'}\cdot|\gamma'|\right)^{-\sigma}.\end{aligned}$$

Combining (2.37) and (2.38), we will show that

$$\sum_{q\ge\nu_T L}|\mu^{(0)}_{p,q}(\xi,\eta,\gamma)| \le C2^{-\sigma(|\ell|+L)} \text{ for } (\eta,\gamma'')\in\operatorname{supp}(\psi^{(0)}_{q+\ell}).$$

Without loss of generality let us assume that $2^{-p\overline{M}''-(q+\ell)\overline{N}''}\cdot|\gamma''| \le 2^{-(q+\ell)}|\eta| \sim 1$. If $\ell\le 0$, then we obtain from (2.37) and (2.38) that

$$\begin{aligned}\left|\mu^{(0)}_{p,q}(\xi,\eta,\gamma)\right| &\le C\min\left[2^{-\sigma L}|\gamma'|\cdot 2^{-p\overline{M}'}, 2^{-q}|\eta|, \left(|\gamma'|\cdot 2^{-p\overline{M}'}\right)^{-\sigma}\right]\\ &\le C\left(2^{-\sigma L}|\gamma'|\cdot 2^{-p\overline{M}'}\right)^{\frac{\sigma}{2(\sigma+1)}}\left(2^{-q}|\eta|\right)^{\frac12}\left(\left(|\gamma'|\cdot 2^{-p\overline{M}'}\right)^{-\sigma}\right)^{\frac{1}{2(\sigma+1)}}\\ &\le C2^{-\frac{\sigma^2}{\sigma+1}L}\left(2^{-q}|\eta|\right)^{\frac12}.\end{aligned}$$

Summing over the range of q-s where $2^{-q}|\eta|\lesssim 2^{\ell}$, we obtain $\sum_q|\mu^{(0)}_{p,q}|\lesssim 2^{-\sigma' L}2^{\ell/2} = 2^{-\sigma'(|\ell|+L)}$. If $\ell\ge 0$, then we only use (2.37) to deduce

$$\begin{aligned}|\mu^{(0)}_{p,q}| &\le C\min\left[2^{-\sigma L}|\gamma'|\cdot 2^{-p\overline{M}'}, \left(2^{-q}|\eta|\right)^{-\frac{\sigma}{2}}\left(|\gamma'|\cdot 2^{-p\overline{M}'}\right)^{-\frac{\sigma}{2}}\right]\\ &\le C\left(2^{-\sigma L}|\gamma'|\cdot 2^{-p\overline{M}'}\right)^{\frac{\sigma}{2+\sigma}}\left(\left(2^{-q}|\eta|\right)^{-\frac{\sigma}{2}}\left(|\gamma'|\cdot 2^{-p\overline{M}'}\right)^{-\frac{\sigma}{2}}\right)^{\frac{2}{2+\sigma}}\\ &\le C2^{-\frac{\sigma^2}{\sigma+2}L}\left(2^{-q}|\eta|\right)^{-\frac{\sigma}{2+\sigma}}.\end{aligned}$$

Summing over q-s with $2^{-q}|\eta|\gtrsim 2^{\ell}$, we again obtain $\sum_q|\mu^{(0)}_{p,q}|\lesssim 2^{-\sigma'(|\ell|+L)}$. □

3. Proof of Theorem 1.4 – Necessity

In this section we adapt the argument in [1] to prove that if P is not even in $GL(d)$, then H is not bounded on L^2. Without loss of generality (by replacing P by PA^t if necessary), we may assume that there exists a corner point (m^*,n^*) of $\partial\mathcal{N}(P_1)$ such that m^*n^* is odd and $a_{m^*n^*}(I,k)=\delta_{1k}$ (the Kronecker delta). The idea of the proof is as follows. The double Hilbert transform $H[P]$ is a convolution operator with respect to a tempered distribution $\Xi(P,c)$, whose action on a test function φ is given by

$$\langle \Xi(P,c), \varphi\rangle = \lim_{\substack{s_0\to 0\\ t_0\to 0}} \iint_{\substack{s_0<|s|\leq c\\ t_0<|t|\leq c}} \varphi(s,t,P(s,t))\frac{ds\,dt}{st}.$$

Using $\Xi(P,c)$ and suitable scaling, we construct a family of distributions $\{\Xi_\epsilon(P,c)\}$ satisfying two properties :

(i) The convolution operators $H_\epsilon[P]$ given by $H_\epsilon[P]f = f * \Xi_\epsilon(P,c)$ share the same L^2 operator norm as $H[P]$.

(ii) As $\epsilon \to 0$, $\Xi_\epsilon(P,c)$ converges in distribution to $\Xi(P^*,\infty)$, where

$$P^*(s,t) = (s^{m^*}t^{n^*}, 0, \cdots, 0). \tag{3.1}$$

If we assume that $H[P]$ is bounded on L^2, (i) and (ii) would imply that the operator $f \mapsto f * \Xi(P^*,\infty)$ is also bounded on L^2, which is known to be false from [6].

3.1. **Scalings of Ξ and the main result.** We now describe the dilations of $\Xi(P,c)$ in greater detail. Since (m^*,n^*) is a corner point of $\partial\mathcal{N}(P_1)$, there exist positive numbers A and B such that

$$Am^* + Bn^* < Am + Bn \text{ for every } (m,n)\in\mathcal{N}(P_1), \quad (m,n)\neq(m^*,n^*). \tag{3.2}$$

Given $0<\epsilon<1$, we define two dilation operators $\Delta_\epsilon : \mathbb{R}^2\to\mathbb{R}^2$ and $\Theta_\epsilon : \mathbb{R}^d\to\mathbb{R}^d$ as follows :

$$\Delta_\epsilon(s,t) = (\epsilon^{-A}s, \epsilon^{-B}t), \quad \Theta_\epsilon(u_1,\cdots,u_d) = (\epsilon^{-Am^*-Bn^*}u_1, u_2,\cdots,u_d).$$

This generates a one-parameter family of distributions $\Xi_\epsilon(P,c)$, and a corresponding family of convolution operators $H_\epsilon[P]$:

$$\langle \Xi_\epsilon(P,c),\varphi\rangle := \langle \Xi(P,c),\varphi_\epsilon\rangle, \qquad \text{where} \qquad \varphi_\epsilon(s,t,u) = \varphi(\Delta_\epsilon(s,t),\Theta_\epsilon(u)),$$

and $H_\epsilon[P]f = \Xi_{P,\epsilon} * f$. We leave to the reader to verify that $||H_\epsilon[P]||_{L^2\to L^2} = ||H[P]||_{L^2\to L^2}$. The main result of this section is therefore the following proposition, which is proved in subsection 3.3 below.

Proposition 3.1. *As $\epsilon\to 0$, $\Xi_\epsilon(P,c)\to\Xi(P^*,\infty)$ in the sense of distribution, where P^* is as in (3.1).*

3.2. **Some technical lemmas.** The computation of $\Xi_\epsilon(P,c)-\Xi(P^*,\infty)$ involves some technical estimates that are contained in the two lemmas below. Given a Schwartz function ψ on $\mathbb{R}^{d+2}$, and a scalar-valued real-analytic function $F(s,t) = \sum_{m,n} b_{m,n}s^m t^n$ defined on $\mathcal{D} = \{(s,t) : |s|,|t|\leq c\}$, we consider an integral $\mathcal{J}_\epsilon(F,\psi)$ that turns out to be a main ingredient in the proof of Proposition 3.1:

$$\mathcal{J}_\epsilon(F,\psi) := \int_0^1\int_{\mathcal{D}_\epsilon} F(\epsilon^A s,\epsilon^B t)\,\psi(s,t,u(s,t,r))\,\frac{ds\,dt}{st}\,dr, \tag{3.3}$$

where $u = u(s,t,r) := (r\Theta_\epsilon P\Delta_\epsilon^{-1} + (1-r)P^*)(s,t)$, and $\mathcal{D}_\epsilon = \Delta_\epsilon^{-1}\mathcal{D} = \{(s,t) : |\epsilon^A s|, |\epsilon^B t|\leq c\}$. The main estimates for (3.3) are given in Lemma 3.3.

Lemma 3.2. *Let A,B,m^*,n^* be as in (3.2). Then there exists a polynomial $\tilde{P}_1$ (independent of ϵ) such that for $(s,t)\in\mathcal{D}_\epsilon$,*

$$\left|\epsilon^{B-Am^*-Bn^*}\frac{\partial P_1}{\partial t}(\epsilon^A s,\epsilon^B t)\right| \leq \widetilde{P}_1(|s|,|t|).$$

Similar statements hold for $\partial P_1/\partial s$ and $\partial^2 P_1/(\partial s\partial t)$.

Proof. Let (p_0,q_0) and (p_1,q_1) be the corner points of the vertical and horizontal segments in $\partial\mathcal{N}(P_1)$ respectively. The number of non-negative integer points in $\mathcal{N}(P_1)$ not in $\cup_{i=0}^{1}\{p\ge p_i, q\ge q_i\}$ is at most finite. Let (p_j,q_j), $2\le j\le T_1$ be the elements of $\{(p,q): a_{p,q}(I,1)\ne 0\}\setminus\cup_{i=0}^{1}\{p\ge p_i, q\ge q_i\}$. Then P_1 can be written as a finite sum of real-analytic functions, $P_1=\sum_{j=0}^{T_1}G_j$, where

$$G_0(s,t)=\sum_{\substack{m\ge p_0\\ m\ge q_0}} a_{m,n}(I,1)s^m t^n, \tag{3.4}$$

$$G_1(s,t)=\sum_{\substack{m\ge p_1\\ q_1\le n\le q_0}} a_{m,n}(I,1)s^m t^n, \tag{3.5}$$

$$G_j(s,t)=a_{p_j,q_j}(I,1)s^{p_j}t^{q_j},\quad 2\le j\le T_1. \tag{3.6}$$

We observe that for $0\le j\le T_1$, $G_j(s,t)/(s^{p_j}t^{q_j})$ is real-analytic on $[-c,c]^2$, and so is $\widetilde{G}_j=s^{-p_j}t^{-q_j+1}\partial G_j/\partial t$. Therefore for $(s,t)\in\mathcal{D}_\epsilon$, we have

$$\begin{aligned}
\epsilon^B\epsilon^{-Am^*-Bn^*}\left|\frac{\partial P_1}{\partial t}(\epsilon^A s,\epsilon^B t)\right| &= \epsilon^{B-Am^*-Bn^*}\left|\sum_{j=0}^{T_1}\frac{\partial G_j}{\partial t}(\epsilon^A s,\epsilon^B t)\right| \\
&\le \epsilon^{B-Am^*-Bn^*}\sum_{j=0}^{T_1}\left|(\epsilon^A s)^{p_j}(\epsilon^B t)^{q_j-1}\widetilde{G_j}(\epsilon^A s,\epsilon^B t)\right| \\
&\le C\sum_{j=0}^{T_1}|s|^{p_j}|t|^{q_j-1},
\end{aligned}$$

where the last step follows from (3.2). □

Lemma 3.3. *For $\mathcal{J}_\epsilon(F,\psi)$ given by (3.3), the following conclusions hold.*

(i) If $F(0,0)=0$, then $\mathcal{J}_\epsilon(F,\psi)\to 0$ as $\epsilon\to 0$.

(ii) If $F(0,0)\ne 0$, there exists a constant $C>0$ (depending on F and ψ but independent of ϵ) such that $|\mathcal{J}_\epsilon(F,\psi)|\le C(\ln(1/\epsilon))^2$.

Proof. (i) We split the function F as $F=F_1+F_2+F_3$, where

$$F_1(s,t)=\sum_{mn\ne 0} b_{m,n}s^m t^n,\quad F_2(s)=\sum_{m\ge 1} b_{m,0}s^m,\quad F_3(t)=\sum_{n\ge 1} b_{0,n}t^n.$$

Then $\widetilde{F}_1(s,t)=F_1(s,t)/(st)$, $\widetilde{F}_2(s)=F_2(s)/s$ and $\widetilde{F}_3(t)=F_3(t)/t$ are real-analytic on $\mathcal{D}$, and $\mathcal{J}_\epsilon(F,\psi)=\sum_{j=1}^{3}\mathcal{J}_\epsilon(F_j,\psi)$. Now,

$$\begin{aligned}
|\mathcal{J}_\epsilon(F_1,\psi)| &\le \epsilon^{A+B}\int_{\mathcal{D}_\epsilon\times[0,1]}\left|\widetilde{F}_1(\epsilon^A s,\epsilon^B t)\right|\,\psi(s,t,u)\,ds\,dt\,dr \\
&\le C\epsilon^{A+B}\to 0 \text{ as } \epsilon\to 0.
\end{aligned}$$

The integrals $\mathcal{J}_\epsilon(F_2,\psi)$ and $\mathcal{J}_\epsilon(F_3,\psi)$ are treated identically, and we work with the former. Using the cancellation of the principal value integral, we have

$$\begin{aligned}
\mathcal{J}_\epsilon(F_2,\psi) &= \int_{\mathcal{D}} F_2(\epsilon^A s)\psi(s,t,u)\,\frac{ds\,dt}{st}\,dr \\
&= \epsilon^A\lim_{t_0\to 0}\int_{r=0}^{1}\int_{\mathcal{D}_\epsilon\cap\{|t|>t_0\}}\big[\psi(s,t,u(s,t,r))-\psi(s,0,u(s,0,r))\big]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad \times \widetilde{F}_2(\epsilon^A s)\,\frac{dt}{t}\,ds\,dr \\
&= \epsilon^A \int_{\mathcal{D}_\epsilon\times[0,1]} \widetilde{F}_2(\epsilon^A s)\left[\int_0^1 \frac{d}{dr'}\psi(s,r't,u(s,r't,r))\,dr'\right]\frac{dt}{t}\,ds\,dr \\
&= \epsilon^A \int_{\mathcal{D}_\epsilon\times[0,1]^2} \widetilde{F}_2(\epsilon^A s)\Big[\frac{\partial\psi}{\partial t}(v) + \nabla_u\psi(v)\cdot\frac{\partial u}{\partial t}(s,r't,r)\Big]\,ds\,dt\,dr\,dr' \\
&= \mathbf{J_1} + \mathbf{J_2},
\end{aligned}
$$

where

$$
\begin{aligned}
v &= v(s,t,r,r') = (s, r't, u(s,r't,r)), \\
\mathbf{J_1} &= \epsilon^A \int_{\mathcal{D}_\epsilon\times[0,1]^2} \widetilde{F}_2(\epsilon^A s)\frac{\partial\psi}{\partial t}(v)\,ds\,dt\,dr\,dr', \\
\mathbf{J_2} &= \epsilon^A \int_{\mathcal{D}_\epsilon\times[0,1]^2} \widetilde{F}_2(\epsilon^A s)\nabla_u\psi(v)\cdot\frac{\partial u}{\partial t}(s,r't,r)\,ds\,dt\,dr\,dr'.
\end{aligned}
$$

From the analyticity of $\widetilde{F}_2$ on $\mathcal{D}$ and decay of ψ it follows that

$$
\begin{aligned}
|\mathbf{J_1}| &\le C\epsilon^A \int_{\mathcal{D}_\epsilon\times[0,1]}\left[\int_{r'=0}^{\epsilon^B} + \int_{r'=\epsilon^B}^{1}\right]\frac{\partial\psi}{\partial t}(v)\,dr'\,ds\,dt\,dr \\
&\le C\epsilon^A\left[\epsilon^B\int_{\mathcal{D}_\epsilon}\frac{ds\,dt}{(1+|s|)^2} + \int_{\epsilon^B}^1\int_{\mathcal{D}_\epsilon}\frac{ds\,dt\;dr'}{(1+|s|)^2(1+|r't|)^2}\right] \\
&\le C\epsilon^A(1+\ln(1/\epsilon)) \to 0 \text{ as } \epsilon\to 0.
\end{aligned}
$$

We now concentrate on $\mathbf{J_2}$, which upon simplification yields

$$
\begin{aligned}
|\mathbf{J_2}| &\le C\epsilon^A\int_{\mathcal{D}_\epsilon\times[0,1]^2}\left|\nabla_u\psi(v)\cdot\left[\epsilon^B r\Theta_\epsilon\frac{\partial P}{\partial t}\Delta_\epsilon^{-1} + (1-r)\frac{\partial P^*}{\partial t}\right](s,r't)\right|\,ds\,dt\,dr\,dr' \\
&\le C(\mathbf{J_{21}}+\mathbf{J_{22}}), \qquad \text{where} \\
\mathbf{J_{21}} &:= \epsilon^A\int_{\mathcal{D}_\epsilon\times[0,1]^2}\sum_{i=2}^d\left|\epsilon^B\frac{\partial\psi}{\partial u_i}(v)\frac{\partial P_i}{\partial t}(\epsilon^A s,\epsilon^B r't)\right|\,ds\,dt\,dr\,dr', \text{ and} \\
\mathbf{J_{22}} &:= \epsilon^A\int_{\mathcal{D}_\epsilon\times[0,1]^2}\left|\frac{\partial\psi}{\partial u_1}(v)\right| \\
&\quad\times\left|\left(\epsilon^{B-Am^*-Bn^*}r\frac{\partial P_1}{\partial t}(\epsilon^A s,\epsilon^B r't) + (1-r)n^* s^{m^*}(r't)^{n^*-1}\right)\right|\,ds\,dt\,dr\,dr'.
\end{aligned}
$$

Each summand in $\mathbf{J_{21}}$ is estimated exactly as in $\mathbf{J_1}$, using the boundedness of $\partial P_i/\partial t$ on $\mathcal{D}$ and the fast decay of $\partial\psi/\partial u_i$ in s and t. This gives the bound $\mathbf{J_{21}} \le C\epsilon^{A+B}\ln(1/\epsilon)$, which $\to 0$ as $\epsilon\to 0$. For $\mathbf{J_{22}}$ we make use of Lemma 3.2, by which the integrand can be bounded above by another Schwartz function in v (independent of ϵ) and the integral is then estimated in the same way as $\mathbf{J_1}$. Thus $\mathbf{J_{22}} \le \epsilon^A\ln(1/\epsilon)\to 0$.

(ii) For the second part of the lemma, we apply the result of (i) to $F(s,t)-F(0,0)$. It therefore suffices to estimate

$$
\int_{\mathcal{D}_\epsilon\times[0,1]}\psi(s,t,u(s,t,r))\frac{ds\,dt}{st}\,dr = \int_{\mathcal{D}_\epsilon\times[0,1]}\left[\psi(s,t,u) - \psi(s,0,u(s,0,r))\right]\frac{ds\,dt}{st}\,dr
$$

$$= \int_{\mathcal{D}_\epsilon \times [0,1]^2} s^{-1} \left[\frac{\partial \psi}{\partial t}(v) + \nabla_u \psi(v) \cdot \frac{\partial u}{\partial t}(s, r't, r) \right]$$

$$= \mathbf{J_3} + \sum_{i=1}^{d} \mathbf{J_{3+i}}, \qquad \text{where}$$

$$\mathbf{J_3} = \int_{\mathcal{D}_\epsilon \times [0,1]^2} \frac{\partial \psi}{\partial t}(v) \, \frac{ds}{s} \, dt \, dr \, dr',$$

$$\mathbf{J_4} = \int_{\mathcal{D}_\epsilon \times [0,1]^2} s^{-1} \frac{\partial \psi}{\partial u_1}(v) \left[\epsilon^B \epsilon^{-Am^* - Bn^*} r \frac{\partial P_1}{\partial t}(\epsilon^A s, \epsilon^B r' t) + (1-r) n^* s^{m^*} (r't)^{n^*-1} \right],$$

$$\mathbf{J_{3+i}} = \int_{\mathcal{D}_\epsilon \times [0,1]^2} \epsilon^B r \psi_{u_i}(v) \frac{\partial P_i}{\partial t}(\epsilon^A s, \epsilon^B r' t) \frac{ds}{s} \, dt \, dr \, dr', \quad i \geq 2.$$

All the integrals above are estimated the same way, using the cancellation of the principal value integral. For example,

$$|\mathbf{J_3}| \leq \left| \int_{\mathcal{D}_\epsilon \times [0,1]^2} \left[\psi_t(s, r't, u(s, r't, r)) - \psi_t(0, r't, u(0, r't, r)) \right] \frac{ds}{s} \, dt \, dr \, dr' \right|$$

$$\leq \left| \int_{\mathcal{D}_\epsilon \times [0,1]^3} \left[\psi_{st}(w) + \psi_{ut}(w) \cdot u_s(r''s, r't, r) \right] ds \, dt \, dr \, dr' \, dr'' \right|,$$

where $w = (r''s, r't, u(r''s, r't, r))$. We now follow the same line of proof needed for estimating $\mathbf{J_1}$ and $\mathbf{J_2}$ to conclude that $|\mathbf{J_3}| \leq C(\ln(1/\epsilon))^2$. The details are left to the interested reader. □

3.3. **Proof of Proposition 3.1.**

Proof. Let us fix a test function φ. For ϵ sufficiently small, we compute

$$\langle \Xi_\epsilon(P, c), \varphi \rangle - \langle \Xi(P^*, \infty), \varphi \rangle + \iint_{\mathbb{R}^2 \setminus \mathcal{D}_\epsilon} \varphi(s, t, P^*(s,t)) \, \frac{ds \, dt}{st}$$

$$= \iint_{\mathcal{D}_\epsilon} \left[\varphi(s, t, \Theta_\epsilon P \Delta_\epsilon^{-1}(s,t)) - \varphi(s, t, P^*(s,t)) \right] \frac{ds \, dt}{st}$$

$$= \iint_{\mathcal{D}_\epsilon} \frac{ds \, dt}{st} \int_0^1 \frac{d}{dr} \left[\varphi(s, t, (r \Theta_\epsilon P \Delta_\epsilon^{-1} + (1-r) P^*)(s,t)) \right] dr$$

$$= \iint_{\mathcal{D}_\epsilon} \frac{ds \, dt}{st} \int_0^1 \nabla_u \varphi(s, t, u) \cdot \left(\Theta_\epsilon P \Delta_\epsilon^{-1} - P^* \right)(s,t) \, dr$$

$$= \mathcal{I} + \sum_{i=2}^{d} \mathcal{J}_\epsilon \Big(P_i, \frac{\partial \varphi}{\partial u_i} \Big), \qquad \text{where}$$

$$\mathcal{I} := \iint_{\mathcal{D}_\epsilon} \left(\epsilon^{-Am^* - Bn^*} P_1(\epsilon^A s, \epsilon^B t) - s^{m^*} t^{n^*} \right) \frac{\partial \varphi}{\partial u_1}(s, t, u) \, \frac{ds \, dt}{st} \, dr.$$

It is easy to check (using the fast decay of φ and the cancellation property of the principal value integral) that

$$\lim_{\epsilon \to 0} \iint_{\mathbb{R}^2 \setminus \mathcal{D}_\epsilon} \varphi(s, t, P^*(s,t)) \, \frac{ds \, dt}{st} = 0.$$

By part (i) of Lemma 3.3, we have $\mathcal{J}_\epsilon(P_i, \partial\varphi/\partial u_i) \to 0$ as $\epsilon \to 0$ for $i \geq 2$. The only term in $\langle \Xi_\epsilon(P,c), \varphi\rangle - \langle \Xi(P^*,\infty),\varphi\rangle$ that remains to be estimated is $\mathcal{I}$. For this, we write

$$\begin{aligned}\epsilon^{-Am^*-Bn^*}P_1(\epsilon^A s, \epsilon^B t) - s^{m^*}t^{n^*} &= \epsilon^{-Am^*-Bn^*}\sum_{(m,n)\neq(m^*,n^*)} a_{m,n}(I,1)(\epsilon^A s)^m(\epsilon^B t)^n \\ &= \epsilon^{-Am^*-Bn^*}\sum_{j=0}^{T_1}\mathfrak{G}_j(\epsilon^A s, \epsilon^B t) \\ &= \epsilon^{-Am^*-Bn^*}\sum_{j=0}^{T_1}\epsilon^{Ap_j+Bq_j}s^{p_j}t^{q_j}\overline{\mathfrak{G}}_j(\epsilon^A s, \epsilon^B t),\end{aligned}$$

where $\overline{\mathfrak{G}}_j = s^{-p_j}t^{-q_j}\mathfrak{G}_j$, and the $\mathfrak{G}_j$-s are defined in the same way as the G_j-s using (3.4), (3.5) and (3.6), but with the extra condition that the sums in (m,n) exclude the point (m^*,n^*). Then, $\overline{\mathfrak{G}}_j$ is real-analytic, $\overline{\mathfrak{G}}_j(0,0)\neq 0$, and

$$\mathcal{I} = \sum_{\substack{0\leq j\leq T_1 \\ (p_j,q_j)\neq(m^*,n^*)}} \epsilon^{Ap_j+Bq_j-Am^*-Bn^*}\mathcal{J}_\epsilon(\overline{\mathfrak{G}}_j, \psi_j),$$

with $\psi_j(s,t) = s^{p_j}t^{q_j}\partial\varphi/\partial u_1(s,t,u)$. It follows from (3.2) and part (ii) of Lemma 3.3 that there exists $c_0 > 0$ such that $|\mathcal{I}| \leq \epsilon^{c_0}(\ln(1/\epsilon))^2 \to 0$. This completes the proof. $\square$

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