# AN EFFECTIVE UNIVERSALITY THEOREM FOR THE RIEMANN ZETA FUNCTION

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Abstract. Let $0 < r < 1/4$, and $f$ be a non-vanishing continuous function in $|z| \le r$, that is analytic in the interior. Voronin's universality theorem asserts that translates of the Riemann zeta function $\zeta(3/4 + z + it)$ can approximate $f$ uniformly in $|z| < r$ to any given precision $\varepsilon$, and moreover that the set of such $t \in [0, T]$ has measure at least $c(\varepsilon)T$ for some $c(\varepsilon) > 0$, once $T$ is large enough. This was refined by Bagchi who showed that the measure of such $t \in [0, T]$ is $(c(\varepsilon) + o(1))T$, for all but at most countably many $\varepsilon > 0$. Using a completely different approach, we obtain the first effective version of Voronin's Theorem, by showing that in the rate of convergence one can save a small power of the logarithm of $T$. Our method is flexible, and can be generalized to other $L$-functions in the $t$-aspect, as well as to families of $L$-functions in the conductor aspect.

## 1. Introduction

In 1914 Fekete constructed a formal power series $\sum_{n=1}^{\infty} a_n x^n$ with the following *universal* property: For any continuous function $f$ on $[-1, 1]$ (with $f(0) = 0$) and given any $\varepsilon > 0$ there exists an integer $N > 0$ such that

$$\sup_{-1 \le x \le 1} \Big| \sum_{n \le N} a_n x^n - f(x) \Big| < \varepsilon.$$

In the 1970's Voronin [14] discovered the remarkable fact that the Riemann zeta-function satisfies a similar universal property. He showed that for any $r < \frac{1}{4}$, any non-vanishing continuous function $f$ in $|z| \le r$, which is analytic in the interior, and for arbitrary $\varepsilon > 0$, there exists a $T > 0$ such that

$$\max_{|z| \le r} \Big| \zeta(\tfrac{3}{4} + iT + z) - f(z) \Big| < \varepsilon. \tag{1.1}$$

Voronin obtained a more quantitative description of this phenomena, stated below.

**Voronin's universality theorem.** *Let $0 < r < \frac{1}{4}$ be a real number. Let $f$ be a non-vanishing continuous function in $|z| \le r$, that is analytic in the interior. Then, for any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \cdot \text{meas}\Big\{ T \le t \le 2T : \max_{|z| \le r} \Big| \zeta(\tfrac{3}{4} + it + z) - f(z) \Big| < \varepsilon \Big\} > 0, \tag{1.2}$$

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*where* meas *is Lebesgue's measure on* $\mathbb{R}$.

There are several extensions of this theorem, for example to domains more general than compact discs (such as any compact set $K$ contained in the strip $1/2 < \mathrm{Re}(s) < 1$ and with connected complement), or to more general $L$-functions. For a complete history of this subject, we refer the reader to [11].

The assumption that $f(z) \neq 0$ is necessary: if $f$ were allowed to vanish then an application of Rouche's theorem would produce at least $\asymp T$ zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta > \frac{1}{2} + \varepsilon$ and $T \leq \gamma \leq 2T$, contradicting the simplest zero-density theorems.

Subsequent work of Bagchi [1] clarified Voronin's universality theorem by setting it in the context of probability theory (see [7] for a streamlined proof). Viewing $\zeta(\frac{3}{4}+it+z)$ with $t \in [T, 2T]$ as a random variable $X_T$ in the space of random analytic functions (i.e $X_T(z) = \zeta(\frac{3}{4} + iU_T + z)$ with $U_T$ uniformly distributed in $[T, 2T]$), Bagchi showed that as $T \to \infty$ this sequence of random variables converges in law (in the space of random analytic functions) to a random Euler product,

$$\zeta(s, X) := \prod_p \Big(1 - \frac{X(p)}{p^s}\Big)^{-1}$$

with $\{X(p)\}_p$ a sequence of independent random variables uniformly distributed on the unit circle (and with $p$ running over prime numbers). This product converges almost surely for $\mathrm{Re}(s) > \frac{1}{2}$ and defines almost surely a holomorphic function in the half-plane $\mathrm{Re}(s) > \sigma_0$ for any $\sigma_0 > \frac{1}{2}$ (see Section 2 below). The proof of Voronin's universality is then reduced to showing that the support of $\zeta(s + 3/4, X)$ in the space of random analytic functions contains all non-vanishing analytic $f : \{z : |z| < r\} \to \mathbb{C}\backslash\{0\}$. Moreover it follows from Bagchi's work that the limit in Voronin's universality theorem exists for all but at most countably many $\varepsilon > 0$.

In this paper, we present an alternative approach to Bagchi's result using methods from hard analysis. As a result we obtain, for the first time, a rate of convergence in Voronin's universality theorem. We also give an explicit description for the limit in terms of the random model $\zeta(s, X)$.

**Theorem 1.1.** *Let* $0 < r < \frac{1}{4}$. *Let* $f$ *be a non-vanishing continuous function on* $|z| \leq (r + 1/4)/2$ *that is holomorphic in* $|z| < (r + 1/4)/2$. *Let* $\omega$ *be a real-valued continuously differentiable function with compact support. Then, we have*

$$\frac{1}{T}\int_T^{2T} \omega\left(\max_{|z|\leq r} |\zeta(\tfrac{3}{4} + it + z) - f(z)|\right) dt = \mathbb{E}\left(\omega\left(\max_{|z|\leq r} |\zeta(\tfrac{3}{4} + z, X) - f(z)|\right)\right) + O\left((\log T)^{-\frac{(3/4-r)}{11}+o(1)}\right),$$

*where the constant in the* $O$ *depends on* $f, \omega$ *and* $r$.

If the random variable $Y_{r,f} = \max_{|z|\le r} |\zeta(\frac{3}{4}+z, X) - f(z)|$ is absolutely continuous, then it follows from the proof of Theorem 1.1 that for any fixed $\varepsilon > 0$ we have

$$\frac{1}{T}\cdot \text{meas}\Big\{T \le t \le 2T : \max_{|z|\le r}\Big|\zeta(\tfrac{3}{4}+it+z) - f(z)\Big| < \varepsilon\Big\}$$
$$= \mathbb{P}\left(\max_{|z|\le r}\big|\zeta(\tfrac{3}{4}+z, X) - f(z)\big| < \varepsilon\right) + O\left((\log T)^{-\frac{(3/4-r)}{11}+o(1)}\right).$$

Unfortunately, we have not been able to even show that $Y_{r,f}$ has no jump discontinuities. We conjecture the latter to be true, and one might even hope that $Y_{r,f}$ is absolutely continuous.

A slight modification of the proof of Theorem 1.1 allows for more general domains than the disc $|z| \le r$. Furthermore, if $\omega \ge \mathbf{1}_{(0,\varepsilon)}$ (where $\mathbf{1}_S$ is the indicator function of the set $S$), then it follows from Voronin's universality theorem that the main term in Theorem 1.1 is positive. Explicit lower bounds for the limit in (1.2) (in terms of $\varepsilon$) are contained in the papers of Good [3] and Garunkstis [2].

Our approach is flexible, and can be generalized to other $L$-functions in the $t$-aspect, as well as to "natural" families of $L$-functions in the conductor aspect. The only analytic ingredients that are needed are zero density estimates, and bounds on the coefficients of these $L$-functions (the so-called Ramanujan conjecture). In particular, the techniques of this paper can be used to obtain an effective version of a recent result of Kowalski [7], who proved an analogue of Voronin's universality theorem for families of $L$-functions attached to $GL_2$ automorphic forms. In fact, using the zero-density estimates near 1 that are known for a very large class of $L$-functions (including those in the Selberg class by Kaczorowski and Perelli [6], and for families of $L$-functions attached to $GL_n$ automorphic forms by Kowalski and Michel [8]), one can prove an analogue of Theorem 1.1 for these $L$-functions, where we replace $3/4$ by some $\sigma < 1$ (and $r < 1-\sigma$).

The main idea in the proof of Theorem 1.1 is to cover the boundary of the disc $|z| \le r$ with a union of a growing (with $T$) number of discs, while maintaining a global control of the size of $|\zeta'(s+z)|$ on $|z| \le r$. It is enough to focus on the boundary of the disc thanks to the maximum modulus principle. The behavior of $\zeta(s+z)$ with $z$ localized to a shrinking disc is essentially governed by the behavior at a single point $z = z_i$ in the disc. This allows us to reduce the problem to understanding the joint distribution of a growing number of shifts $\log\zeta(s+z_i)$ with the $z_i$ well-spaced, which can be understood by computing the moments of these shifts and using standard Fourier techniques.

It seems very difficult to obtain a rate of convergence which is better than logarithmic in Theorem 1.1. We have at present no understanding as to what the correct rate of convergence should be.

## 2. Key Ingredients and detailed results

We first begin with stating certain important properties of the random model $\zeta(s, X)$. Let $\{X(p)\}_p$ be a sequence of independent random variables uniformly distributed on the unit circle. Then we have

$$-\log\left(1-\frac{X(p)}{p^s}\right)=\frac{X(p)}{p^s}+h_X(p,s),$$

where the random series

$$\sum_p h_X(p,s), \tag{2.1}$$

converges absolutely and surely for $\mathrm{Re}(s) > 1/2$. Hence, it (almost surely) defines a holomorphic function in $s$ in this half-plane. Moreover, since $\mathbb{E}(X(p)) = 0$ and $\mathbb{E}(|X(p)|^2) = 1$, then it follows from Kolmogorov's three-series theorem that the series

$$\sum_p \frac{X(p)}{p^s} \tag{2.2}$$

is almost surely convergent for $\mathrm{Re}(s) > 1/2$. By well-known results on Dirichlet series, this shows that this series defines (almost surely) a holomorphic function on the half-plane $\mathrm{Re}(s) > \sigma_0$, for any $\sigma_0 > 1/2$. Thus, by taking the exponential of the sum of the random series in (2.1) and (2.2), it follows that $\zeta(s, X)$ converges almost surely to a holomorphic function on the half-plane $\mathrm{Re}(s) > \sigma_0$, for any $\sigma_0 > 1/2$.

We extend the $X(p)$ multiplicatively to all positive integers by setting $X(1) = 1$ and $X(n) := X(p_1)^{a_1}\cdots X(p_k)^{a_k}$, if $n = p_1^{a_1}\dots p_k^{a_k}$. Then we have

$$\mathbb{E}\left(X(n)\overline{X(m)}\right)=\begin{cases}1 & \text{if } m=n,\\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

Furthermore, for any complex number $s$ with $\mathrm{Re}(s) > 1/2$ we have almost surely that

$$\zeta(s,X)=\sum_{n=1}^{\infty}\frac{X(n)}{n^s}.$$

To compare the distribution of $\zeta(s + it)$ to that of $\zeta(s, X)$, we define a probability measure on $[T, 2T]$ in a standard way, by

$$\mathbb{P}_T(S):=\frac{1}{T}\mathrm{meas}(S),\ \text{ for any } S\subseteq[T,2T].$$

The idea behind our proof of effective universality is to first reduce the problem to the discrete problem of controlling the distribution of many shifts $\log\zeta(s_j + it)$ with all of the $s_j$ contained in a compact set inside the strip $\frac{1}{2} < \mathrm{Re}(s) < 1$. One of the main ingredients in this reduction is the following result which allows us to control the maximum of the derivative of the Riemann zeta-function. This is proven in Section 4.

**Proposition 2.1.** *Let $0 < r < 1/4$ be fixed. Then there exist positive constants $b_1$, $b_2$ and $b_3$ (that depend only on $r$) such that*

$$\mathbb{P}_T\left(\max_{|z|\leq r}|\zeta'(\tfrac{3}{4}+it+z)| > e^V\right) \ll \exp\left(-b_1 V^{\frac{1}{1-\sigma(r)}}(\log V)^{\frac{\sigma(r)}{1-\sigma(r)}}\right)$$

*where $\sigma(r) = \frac{3}{4} - r$, uniformly for $V$ in the range $b_2 < V \leq b_3(\log T)^{1-\sigma}/(\log\log T)$.*

We also prove an analogous result for the random model $\zeta(s, X)$, which holds for all sufficiently large $V$.

**Proposition 2.2.** *Let $0 < r < 1/4$ be fixed and $\sigma(r) = \frac{3}{4} - r$. Then there exist positive constants $b_1$ and $b_2$ (that depend only on $r$) such that for all $V > b_2$ we have*

$$\mathbb{P}\left(\max_{|z|\leq r}|\zeta'(\tfrac{3}{4}+z, X)| > e^V\right) \ll \exp\left(-b_1 V^{\frac{1}{1-\sigma(r)}}(\log V)^{\frac{\sigma(r)}{1-\sigma(r)}}\right).$$

Once the reduction has been accomplished, it remains to understand the joint distribution of the shifts $\{\log\zeta(s_1+it), \log\zeta(s_2+it), \dots, \log\zeta(s_J+it)\}$ with $J\to\infty$ as $T\to\infty$ at a certain rate, and $s_1, \dots, s_J$ are complex numbers with $\frac{1}{2} < \mathrm{Re}(s_j) < 1$ for all $j \leq J$. Heuristically, this should be well approximated by the the joint distribution of the random variables $\{\log\zeta(s_1, X), \log\zeta(s_2, X), \dots, \log\zeta(s_J, X)\}$. In order to establish this fact (in a certain range of $J$), we first prove, in Section 5, that the moments of the joint shifts $\log\zeta(s_j+it)$ are very close to the corresponding ones of $\log\zeta(s_j, X)$, for $j \leq J$.

**Theorem 2.1.** *Fix $1/2 < \sigma_0 < 1$. Let $s_1, s_2, \dots, s_k, r_1, \dots, r_\ell$ be complex numbers in the rectangle $\sigma_0 < \mathrm{Re}(z) < 1$ and $|\mathrm{Im}(z)| \leq T^{(\sigma_0-1/2)/4}$. Then, there exist positive constants $c_3, c_4, c_5$ and a set $\mathcal{E}(T) \subset [T, 2T]$ of measure $\ll T^{1-c_3}$, such that if $k, \ell \leq c_4\log T/\log\log T$ then*

$$\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)}\left(\prod_{j=1}^{k}\log\zeta(s_j+it)\right)\left(\prod_{j=1}^{\ell}\log\zeta(r_j-it)\right)dt$$
$$= \mathbb{E}\left(\left(\prod_{j=1}^{k}\log\zeta(s_j, X)\right)\left(\prod_{j=1}^{\ell}\log\overline{\zeta(r_j, X)}\right)\right) + O\left(T^{-c_5}\right).$$

Having obtained the moments we are in position to understand the characteristic function,

$$\Phi_T(\mathbf{u}, \mathbf{v}) := \frac{1}{T}\int_T^{2T}\exp\left(i\left(\sum_{j=1}^{J}(u_j\mathrm{Re}\log\zeta(s_j+it) + v_j\mathrm{Im}\log\zeta(s_j+it))\right)\right)dt,$$

where $\mathbf{u} = (u_1, \dots, u_J) \in \mathbb{R}^J$ and $\mathbf{v} = (v_1, \dots, v_J) \in \mathbb{R}^J$. We relate the above characteristic function to the characteristic function of the probabilistic model,

$$\Phi_{\mathrm{rand}}(\mathbf{u}, \mathbf{v}) := \mathbb{E}\left(\exp\left(i\left(\sum_{j=1}^{J}(u_j\mathrm{Re}\log\zeta(s_j, X) + v_j\mathrm{Im}\log\zeta(s_j, X))\right)\right)\right).$$

This is obtained in the following theorem, which we prove in Section 6.

**Theorem 2.2.** *Fix $1/2 < \sigma < 1$. Let $T$ be large and $J \leq (\log T)^{\sigma}$ be a positive integer. Let $s_1, s_2, \dots, s_J$ be complex numbers such that $\min(\mathrm{Re}(s_j)) = \sigma$ and $\max(|\mathrm{Im}(s_j)|) < T^{(\sigma-1/2)/4}$. Then, there exist positive constants $c_1(\sigma), c_2(\sigma)$, such that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^J$ such that $\max(|u_j|), \max(|v_j|) \leq c_1(\sigma)(\log T)^{\sigma}/J$ we have*

$$\Phi_T(\mathbf{u}, \mathbf{v}) = \Phi_{\mathrm{rand}}(\mathbf{u}, \mathbf{v}) + O\left(\exp\left(-c_2(\sigma)\frac{\log T}{\log\log T}\right)\right).$$

Using this result, we can show that the joint distribution of the shifts $\log\zeta(s_j+it)$ is very close to the corresponding joint distribution of the random variables $\log\zeta(s_j, X)$. The proof depends on Beurling-Selberg functions. To measure how close are these distributions, we introduce the discrepancy $\mathcal{D}_T(s_1, \dots, s_J)$ defined as

$$\sup_{(\mathcal{R}_1,\dots,\mathcal{R}_J)\subset\mathbb{C}^J}\left|\mathbb{P}_T\Big(\log\zeta(s_j+it) \in \mathcal{R}_j, \forall j \leq J\Big) - \mathbb{P}\Big(\log\zeta(s_j, X) \in \mathcal{R}_j, \forall j \leq J\Big)\right|$$

where the supremum is taken over all $(\mathcal{R}_1, \dots, \mathcal{R}_J) \subset \mathbb{C}^J$ and for each $j = 1, \dots, J$ the set $\mathcal{R}_j$ is a rectangle with sides parallel to the coordinate axes. Our next theorem, proven in Section 7, states a bound for the above discrepancy. This generalizes Theorem 1.1 of [10], which corresponds to the special case $J = 1$.

**Theorem 2.3.** *Let $T$ be large, $\frac{1}{2} < \sigma < 1$ and $J \leq (\log T)^{\sigma/2}$ be a positive integer. Let $s_1, s_2, \dots, s_J$ be complex numbers such that*

$$\tfrac{1}{2} < \sigma := \min_j(\mathrm{Re}(s_j)) \leq \max_j(\mathrm{Re}(s_j)) < 1 \quad \textit{and} \quad \max_j(|\,\mathrm{Im}(s_j)|) < T^{(\sigma-\frac{1}{2})/4}.$$

*Then, we have*

$$\mathcal{D}_T(s_1, \dots, s_J) \ll \frac{J^2}{(\log T)^{\sigma}}.$$

With all of the above tools in place we are ready to prove Theorem 1.1. This is accomplished in the next section.

## 3. Effective universality: Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 using the results described in Section 2. First, by the maximum modulus principle, the maximum of $|\zeta(\frac{3}{4}+it+z) - f(z)|$ in the disc $\{z : |z| \leq r\}$ must occur on its boundary $\{z : |z| = r\}$. Our idea consists of first covering the circle $|z| = r$ with $J$ discs of radius $\varepsilon$ and centres $z_j$, where $z_j \in \{z : |z| = r\}$ for all $1 \leq j \leq J$, and $J \asymp 1/\varepsilon$. We call each of the discs $\mathcal{D}_j$. Then, we observe that

(3.1)
$$\max_{j\leq J}|\zeta(\tfrac{3}{4}+it+z_j) - f(z_j)| \leq \max_{|z|\leq r}|\zeta(\tfrac{3}{4}+it+z) - f(z)| \leq \max_{j\leq J}\max_{z\in\mathcal{D}_j}|\zeta(\tfrac{3}{4}+it+z) - f(z)|.$$

Using Proposition 2.1, we shall prove that for all $j \le J$ (where $J$ is a small power of $\log T$) we have

$$\max_{z\in\mathcal{D}_j} |\zeta(\tfrac{3}{4}+it+z)-f(z)| \approx |\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|$$

for all $t\in[T,2T]$ except for a set of points $t$ of very small measure. We will then deduce that the (weighted) distribution of $\max_{|z|\le r}|\zeta(\frac{3}{4}+it+z)-f(z)|$ is very close to the corresponding distribution of $\max_{j\le J}|\zeta(\frac{3}{4}+it+z_j)-f(z_j)|$, for $t\in[T,2T]$. We will also establish an analogous result for the random model $\zeta(s,X)$ along the same lines, by using Proposition 2.2 instead of Proposition 2.1. Therefore, to complete the proof of Theorem 1.1 we need to compare the distributions of $\max_{j\le J}|\zeta(\frac{3}{4}+it+z_j)-f(z_j)|$ and $\max_{j\le J}|\zeta(\frac{3}{4}+z_j,X)-f(z_j)|$. Using Theorem 2.3 we prove

**Proposition 3.1.** *Let $T$ be large, $0<r<1/4$ and $J\le(\log T)^{(3/4-r)/7}$ be a positive integer. Let $z_1,\dots,z_J$ be complex numbers such that $|z_j|\le r$. Then we have*

$$\left|\mathbb{P}_T\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\le u\right)-\mathbb{P}\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+z_j,X)-f(z_j)|\le u\right)\right|$$
$$\ll_u \frac{(J\log\log T)^{6/5}}{(\log T)^{(3/4-r)/5}}.$$

*Proof.* Fix a positive real number $u$. Let $\mathcal{A}_J(T)$ be the set of those $t$ for which $|\arg\zeta(\frac{3}{4}+it+z_j)|\le\log\log T$ for every $j\le J$. Since $\mathrm{Re}(\frac{3}{4}+it+z_j)\ge\frac{3}{4}-r$ and $\mathrm{Im}(\frac{3}{4}+it+z_j)=t+O(1)$, then it follows from Theorem 1.1 and Remark 1 of [9] that for each $j\le J$ we have

$$\mathbb{P}_T\left(|\arg\zeta(\tfrac{3}{4}+it+z_j)|\ge\log\log T\right)\ll\exp\left(-(\log\log T)^{(\frac{1}{4}+r)^{-1}}\right)\ll\frac{1}{(\log T)^4}. \tag{3.2}$$

Therefore, we obtain

$$\mathbb{P}_T\left([T,2T]\setminus\mathcal{A}_J(T)\right)\le\sum_{j=1}^{J}\mathbb{P}_T\left(|\arg\zeta(\tfrac{3}{4}+it+z_j)|\ge\log\log T\right)\ll\frac{J}{(\log T)^4}\ll\frac{1}{(\log T)^2},$$

and this implies that

$$\mathbb{P}_T\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\le u\right)=\mathbb{P}_T\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\le u\ ,\ t\in\mathcal{A}_J(T)\right)+O\left(\frac{1}{(\log T)^2}\right). \tag{3.3}$$

For each $j\le J$ consider the region

$$\mathcal{U}_j=\{z:|e^z-f(z_j)|\le u\ ,\ |\mathrm{Im}(z)|\le\log\log T\}.$$

We cover $\mathcal{U}_j$ with $K\asymp\mathrm{area}(\mathcal{U}_j)/\varepsilon^2\asymp\log\log T/\varepsilon^2$ squares $\mathcal{R}_{j,k}$ with sides of length $\varepsilon=\varepsilon(T)$, where $\varepsilon$ is a small positive parameter to be chosen later. Let $\mathcal{K}_j$ denote the set of $k\in\{1,2,\dots,K\}$ such that the intersection of $\mathcal{R}_{j,k}$ with the boundary of $\mathcal{U}_j$ is empty and write $\mathcal{K}_j^c$ for the relative complement of $\mathcal{K}_j$ with respect to $\{1, 2, \dots, K\}$. Note that $|\mathcal{K}_j^c| \asymp \log\log T/\varepsilon$. By construction,

$$\left(\bigcup_{k\in\mathcal{K}_j} \mathcal{R}_{j,k}\right) \subset \mathcal{U}_j \subset \left(\bigcup_{k\le K} \mathcal{R}_{j,k}\right).$$

Therefore (3.3) can be expressed as

$$\mathbb{P}_T\Big(\forall j \le J, \forall k \le K : \log\zeta(\tfrac{3}{4} + it + z_j) \in \mathcal{R}_{j,k}\Big) + \mathcal{E}_1$$

where by Theorem 2.3

$$\begin{aligned}\mathcal{E}_1 &\ll \sum_{j\le J}\sum_{k\in\mathcal{K}_j^c} \mathbb{P}_T\Big(\log\zeta(\tfrac{3}{4} + it + z_j) \in \ \mathcal{R}_{j,k}\Big)\\ &\ll \sum_{j\le J}\sum_{k\in\mathcal{K}_j^c}\left(\mathbb{P}_T\Big(\log\zeta(\tfrac{3}{4} + z_j, X) \in \ \mathcal{R}_{j,k}\Big) + \frac{1}{(\log T)^{3/4-r}}\right) \qquad (3.4)\\ &\ll J\cdot\frac{\log\log T}{\varepsilon}\left(\varepsilon^2 + \frac{1}{(\log T)^{3/4-r}}\right),\end{aligned}$$

and in the last step we used the fact that $\log\zeta(s, X)$ is an absolutely continuous random variable (see for example Jessen and Wintner [5]). We conclude that

$$\begin{aligned}\mathbb{P}_T\Big(\max_{j\le J}|\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \le u\Big) &= \mathbb{P}_T\Big(\forall j \le J, \forall k \le K : \log\zeta(\tfrac{3}{4} + it + z_j) \in \mathcal{R}_{j,k}\Big)\\ &\quad + O\left(\varepsilon J\log\log T + \frac{J\log\log T}{\varepsilon(\log T)^{3/4-r}}\right). \qquad (3.5)\end{aligned}$$

Additionally, it follows from Theorem 2.3 that the main term of this last estimate equals

$$\mathbb{P}\Big(\forall j \le J, \forall k \le K : \log\zeta(\tfrac{3}{4} + z_j, X) \in \mathcal{R}_{j,k}\Big) + O\left(\frac{J^2(\log\log T)^2}{\varepsilon^4(\log T)^{3/4-r}}\right). \qquad (3.6)$$

We now repeat the exact same argument but for the random model $\zeta(s, X)$ instead of the zeta function. In particular, instead of (3.2) we shall use that

$$\mathbb{P}\left(|\arg\zeta(\tfrac{3}{4} + z_j, X)| \ge \log\log T\right) \ll \exp\left(-(\log\log T)^{(\frac{1}{4}+r)^{-1}}\right) \ll \frac{1}{(\log T)^4},$$

which follows from Theorem 1.9 of [9]. Thus, similarly to (3.5) we obtain

$$\begin{aligned}\mathbb{P}\Big(\forall j \le J, \forall k \le K : \log\zeta(\tfrac{3}{4} + z_j, X) \in \mathcal{R}_{j,k}\Big) &= \mathbb{P}\left(\max_{j\le J}|\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)| \le u\right)\\ &\quad + O\left(\varepsilon J\log\log T + \frac{J\log\log T}{\varepsilon(\log T)^{3/4-r}}\right).\end{aligned}$$

Combining the above estimate with (3.5) and (3.6) we conclude that

$$\mathbb{P}_T\Big(\max_{j\le J}|\zeta(\tfrac34+it+z_j)-f(z_j)|\le u\Big)=\mathbb{P}\left(\max_{j\le J}|\zeta(\tfrac34+z_j,X)-f(z_j)|\le u\right)+O\left(\frac{J^2(\log\log T)^2}{\varepsilon^4(\log T)^{3/4-r}}+\varepsilon J\log\log T\right).$$

Finally, choosing

$$\varepsilon=\left(\frac{J\log\log T}{(\log T)^{3/4-r}}\right)^{1/5}$$

completes the proof. □

*Proof of Theorem 1.1.* We wish to estimate

$$\frac{1}{T}\int_T^{2T}\omega\left(\max_{|z|\le r}|\zeta(\tfrac34+it+z)-f(z)|\right)dt \tag{3.7}$$

with $f$ an analytic non-vanishing function, and where $\omega$ is a continuously differentiable function with compact support.

Recall that the maximum of $|\zeta(\frac34+it+z)-f(z)|$ on the disc $\{z:|z|\le r\}$ must occur on its boundary $\{z:|z|=r\}$, by the maximum modulus principle. Let $\varepsilon\le(1/4-r)/4$ be a small positive parameter to be chosen later, and cover the circle $|z|=r$ with $J\asymp 1/\varepsilon$ discs $\mathcal{D}_j$ of radius $\varepsilon$ and centres $z_j$, where $z_j\in\{z:|z|=r\}$ for all $j\le J$.

Let $\mathcal{S}_V(T)$ denote the set of those $t\in[T,2T]$ such that

$$\max_{|z|\le(r+1/4)/2}|\zeta'(\tfrac34+it+z)|\le e^V$$

where $V\le\log\log T$ is a large parameter to be chosen later, and let $L:=\max_{|z|\le(r+1/4)/2}|f'(z)|$. Then for $t\in\mathcal{S}_V(T)$, and for all $z\in\mathcal{D}_j$ we have

(3.8)

$$\Big|\zeta(\tfrac34+it+z)-f(z)-\big(\zeta(\tfrac34+it+z_j)-f(z_j)\big)\Big|=\left|\int_{z_j}^{z}\zeta'(\tfrac34+it+s)-f'(s)ds\right|$$
$$\le|z-z_j|\cdot\left(\max_{|z|\le(r+1/4)/2}|\zeta'(\tfrac34+it+z)|+L\right)\le\varepsilon(e^V+L)\le C\varepsilon e^V,$$

for some large absolute constant $C$, depending at most on $L$. Define

$$\theta(t):=\max_{|z|\le r}|\zeta(\tfrac34+it+z)-f(z)|-\max_{j\le J}|\zeta(\tfrac34+it+z_j)-f(z_j)|.$$

Then, it follows from (3.1) and (3.8) that for all $t\in\mathcal{S}_V(T)$ we have

$$0\le\theta(t)\le C\varepsilon e^V. \tag{3.9}$$

Therefore, using this estimate together with Proposition 2.1 and the fact that $\omega$ is bounded, we deduce that (3.7) equals

$$\begin{aligned}
&\frac{1}{T}\int_{t\in\mathcal{S}_V(T)} \omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|+\theta(t)\right)dt + O\left(e^{-V^2}\right)\\
&=\frac{1}{T}\int_{t\in\mathcal{S}_V(T)} \omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\right)dt + O\left(|\mathcal{E}_2|+e^{-V^2}\right)\\
&=\frac{1}{T}\int_{T}^{2T} \omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\right)dt + O\left(|\mathcal{E}_2|+e^{-V^2}\right),
\end{aligned} \tag{3.10}$$

where

$$\mathcal{E}_2=\frac{1}{T}\int_{t\in\mathcal{S}_V(T)}\int_0^{\theta(t)}\omega'\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|+x\right)dx\cdot dt \ll \varepsilon e^{V},$$

using the fact that $\omega'$ is bounded on $\mathbb{R}$ together with (3.9).

Furthermore, observe that

$$\begin{aligned}
&\frac{1}{T}\int_{T}^{2T} \omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\right)dt\\
&=-\frac{1}{T}\int_{T}^{2T}\int_{\max_{j\le J}|\zeta(\frac{3}{4}+it+z_j)-f(z_j)|}^{\infty}\omega'(u)du\cdot dt\\
&=-\int_0^{\infty}\omega'(u)\cdot\mathbb{P}_T\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\le u\right)du.
\end{aligned} \tag{3.11}$$

Since $\omega$ has a compact support, then $\omega'(u)=0$ if $u>A$ for some positive constant $A$. Furthermore, it follows from Proposition 3.1 that for all $0\le u\le A$ we have

$$\begin{aligned}
\mathbb{P}_T\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\le u\right) &= \mathbb{P}\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+z_j,X)-f(z_j)|\le u\right)\\
&\quad + O\left(\frac{(J\log\log T)^{6/5}}{(\log T)^{(3/4-r)/5}}\right).
\end{aligned}$$

Inserting this estimate in (3.11) gives that

$$\begin{aligned}
&\frac{1}{T}\int_{T}^{2T} \omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+it+z_j)-f(z_j)|\right)dt\\
&=-\int_0^{\infty}\omega'(u)\cdot\mathbb{P}\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+z_j,X)-f(z_j)|\le u\right)du + O\left(\frac{(J\log\log T)^{6/5}}{(\log T)^{(3/4-r)/5}}\right)\\
&=\mathbb{E}\left(\omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+z_j,X)-f(z_j)|\right)\right)+O\left(\frac{(J\log\log T)^{6/5}}{(\log T)^{(3/4-r)/5}}\right).
\end{aligned} \tag{3.12}$$

To finish the proof, we shall appeal to the same argument used to establish (3.10), in order to compare the (weighted) distributions of $\max_{j\le J}|\zeta(\frac{3}{4}+z_j,X)-f(z_j)|$ and $\max_{|z|\le r}|\zeta(\frac{3}{4}+z,X)-f(z)|$. Let $\mathcal{S}_V(X)$ denote the event corresponding to

$$\max_{|z|\le(r+1/4)/2}|\zeta'(\tfrac{3}{4}+z,X)|\le e^{V},$$

and let $\mathcal{S}_V^c(X)$ be its complement. Then, it follows from Proposition 2.2 that $\mathbb{P}\big(\mathcal{S}_V^c(X)\big) \ll \exp(-V^2)$. Moreover, similarly to (3.8) one can see that for all outcomes in $\mathcal{S}_V(X)$ we have, for all $z \in \mathcal{D}_j$

$$\left|\zeta(\tfrac{3}{4}+z,X) - f(z) - \big(\zeta(\tfrac{3}{4}+z_j,X) - f(z_j)\big)\right| = \left|\int_{z_j}^{z} \zeta'(\tfrac{3}{4}+s,X) - f'(s)ds\right| \ll \varepsilon e^V.$$

Thus, since the maximum of $|\zeta(\frac{3}{4}+z,X) - f(z)|$ for $|z| \le r$ occurs (almost surely) on the boundary $|z| = r$, then following the argument leading to (3.10), we conclude that

$$\begin{aligned}
&\mathbb{E}\left(\omega\left(\max_{|z|\le r}|\zeta(\tfrac{3}{4}+z,X) - f(z)|\right)\right)\\
&= \mathbb{E}\left(\mathbf{1}_{\mathcal{S}_V(X)}\,\omega\left(\max_{|z|\le r}|\zeta(\tfrac{3}{4}+z,X) - f(z)|\right)\right) + O\left(e^{-V^2}\right)\\
&= \mathbb{E}\left(\mathbf{1}_{\mathcal{S}_V(X)}\,\omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+z_j,X) - f(z)|\right)\right) + O\left(\varepsilon e^V + e^{-V^2}\right)\\
&= \mathbb{E}\left(\omega\left(\max_{j\le J}|\zeta(\tfrac{3}{4}+z_j,X) - f(z)|\right)\right) + O\left(\varepsilon e^V + e^{-V^2}\right).
\end{aligned}$$

Finally, combining this estimate with (3.10) and (3.12), and noting that $J \asymp 1/\varepsilon$ we deduce that

$$\begin{aligned}
\frac{1}{T}\int_T^{2T} \omega\left(\max_{|z|\le r}|\zeta(\tfrac{3}{4}+it+z) - f(z)|\right)dt &= \mathbb{E}\left(\omega\left(\max_{|z|\le r}|\zeta(\tfrac{3}{4}+z,X) - f(z)|\right)\right)\\
&\quad + O\left(\varepsilon e^V + e^{-V^2} + O\left(\frac{(\log\log T)^{6/5}}{\varepsilon^{6/5}(\log T)^{(3/4-r)/5}}\right)\right).
\end{aligned}$$

Choosing $\varepsilon = (\log T)^{-(3/4-r)/11}$ and $V = 2\sqrt{\log\log T}$ completes the proof. □

## 4. Controlling the derivatives of the zeta function and the random model: Proof of Propositions 2.1 and 2.2

By Cauchy's theorem we have

$$|\zeta'(\tfrac{3}{4}+it+z)| \le \frac{1}{\delta}\max_{|s-z|=\delta}|\zeta(\tfrac{3}{4}+it+s)|,$$

and hence we get

$$\max_{|z|\le r}|\zeta'(\tfrac{3}{4}+it+z)| \le \frac{1}{\delta}\max_{|s|\le r+\delta}|\zeta(\tfrac{3}{4}+it+s)|. \tag{4.1}$$

Therefore, it follows that

$$\begin{aligned}
\mathbb{P}_T\left(\max_{|z|\le r}|\zeta'(\tfrac{3}{4}+it+z)| > e^V\right) &\le \mathbb{P}_T\left(\max_{|s|\le r+\delta}|\zeta(\tfrac{3}{4}+it+s)| > \delta e^V\right)\\
&= \mathbb{P}_T\left(\max_{|s|\le r+\delta}\log|\zeta(\tfrac{3}{4}+it+s)| > V + \log\delta\right).
\end{aligned} \tag{4.2}$$

To bound the RHS we estimate large moments of $\log\zeta(\frac34+it+s)$. This is accomplished by approximating $\log\zeta(\frac34+it+s)$ by a short Dirichlet polynomial, uniformly for all $s$ in the disc $\{|s|\le r+\delta\}$. Using zero density estimates and large sieve inequalities, we can show that such an approximation holds for all $t\in[T,2T]$, except for an exceptional set of $t$'s with very small measure. We prove

**Lemma 4.1.** *Let $0<r<1/4$ be fixed, and $\delta=(1/4-r)/4$. Let $y\le\log T$ be a real number. There exists a set $\mathcal{I}(T)\subset[T,2T]$ with meas$(\mathcal{I}(T))\ll T^{1-\delta}y(\log T)^5$, such that for all $t\in[T,2T]\setminus\mathcal{I}(T)$ and all $|s|\le r+\delta$ we have*

$$\log\zeta(\tfrac34+it+s)=\sum_{n\le y}\frac{\Lambda(n)}{n^{\frac34+it+s}\log n}+O\left(\frac{(\log y)^2\log T}{y^{(1/4-r)/2}}\right).$$

To prove this result, we need the following lemma from Granville and Soundararajan [4].

**Lemma 4.2** (Lemma 1 of [4]). *Let $y\ge 2$ and $|t|\ge y+3$ be real numbers. Let $1/2\le\sigma_0<1$ and suppose that the rectangle $\{z:\sigma_0<\mathrm{Re}(z)\le1,|\mathrm{Im}(z)-t|\le y+2\}$ is free of zeros of $\zeta(z)$. Then for any $\sigma$ with $\sigma_0+2/\log y<\sigma\le1$ we have*

$$\log\zeta(\sigma+it)=\sum_{n\le y}\frac{\Lambda(n)}{n^{\sigma+it}\log n}+O\left(\log|t|\frac{(\log y)^2}{y^{\sigma-\sigma_0}}\right).$$

*Proof of Lemma 4.1.* Let $\sigma_0=1/2+\delta$. For $j=1,2$ let $\mathcal{T}_j$ be the set of those $t\in[T,2T]$ for which the rectangle

$$\{z:\sigma_0<\mathrm{Re}(z)\le1,|\mathrm{Im}(z)-t|<y+1+j\}$$

is free of zeros of $\zeta(z)$. Then, note that $\mathcal{T}_2\subseteq\mathcal{T}_1$, and for all $t\in\mathcal{T}_2$, we have $t+\mathrm{Im}(s)\in\mathcal{T}_1$ for all $|s|\le r+\delta$. Hence, by Lemma 4.2 we have

$$\log\zeta(\tfrac34+it+s)=\sum_{n\le y}\frac{\Lambda(n)}{n^{3/4+it+s}\log n}+O\left(\frac{(\log y)^2\log T}{y^{(1/4-r)/2}}\right),$$

for all $t\in\mathcal{T}_2$ and all $|s|\le r+\delta$. Let $N(\sigma,T)$ be the number of zeros of $\zeta(s)$ in the rectangle $\sigma<\mathrm{Re}(s)\le1$ and $|\mathrm{Im}(s)|\le T$. By the classical zero density estimate $N(\sigma,T)\ll T^{3/2-\sigma}(\log T)^5$ (see for example Theorem 9.19 A ot Titchmarsh [12]) we deduce that the measure of the complement of $\mathcal{T}_2$ in $[T,2T]$ is $\ll T^{1-\delta}y(\log T)^5$. $\square$

We also require a minor variant of Lemma 3.3 of [10], whose proof we will omit.

**Lemma 4.3.** *Fix $1/2<\sigma<1$, and let $s$ be a complex number such that* $\mathrm{Re}(s)=\sigma$, *and $|\mathrm{Im}(s)|\le1$. Then, for any positive integer $k\le\log T/(3\log y)$ we have*

$$\frac1T\int_T^{2T}\left|\sum_{n\le y}\frac{\Lambda(n)}{n^{s+it}\log n}\right|^{2k}dt\ll\left(\frac{c_8k^{1-\sigma}}{(\log k)^{\sigma}}\right)^{2k}$$

*and*

$$\mathbb{E}\left(|\log\zeta(s,X)|^{2k}\right) \ll \left(\frac{c_8 k^{1-\sigma}}{(\log k)^{\sigma}}\right)^{2k}$$

*for some positive constant $c_8$ that depends at most on $\sigma$.*

*Proof of Proposition 2.1.* Let $\delta = e^{-V/2}$. Taking $y = (\log T)^{5(1/4-r)^{-1}}$ in Lemma 4.1 gives for all $t \in [T, 2T]$ except for a set with measure $\ll T^{1-(1/4-r)/5}$ that

$$\log\zeta(\tfrac{3}{4}+it+s) = \sum_{n\le y} \frac{\Lambda(n)}{n^{3/4+it+s}\log n} + O\left(\frac{1}{\log T}\right), \tag{4.3}$$

for all $|s| \le r+\delta$. Furthermore, it follows from Cauchy's integral formula that

$$\left(\sum_{n\le y} \frac{\Lambda(n)}{n^{3/4+it+s}\log n}\right)^{2k} = \frac{1}{2\pi i}\int_{|z|=r+2\delta} \left(\sum_{n\le y} \frac{\Lambda(n)}{n^{3/4+it+z}\log n}\right)^{2k} \frac{dz}{z-s}.$$

Applying Lemma 4.3 we get that

(4.4)

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\left(\max_{|s|\le r+\delta}\left|\sum_{n\le y}\frac{\Lambda(n)}{n^{3/4+it+s}\log n}\right|\right)^{2k} dt \ll& \frac{1}{\delta}\int_{|z|=r+2\delta}\frac{1}{T}\int_T^{2T}\left|\sum_{n\le y}\frac{\Lambda(n)}{n^{3/4+it+z}\log n}\right|^{2k} dt|dz| \\ \ll& e^{V/2}\left(c_8(r)\frac{k^{1-\sigma'(r)}}{(\log k)^{\sigma'(r)}}\right)^{2k}\end{aligned}$$

where $\sigma'(r) = \frac{3}{4} - r - 2\delta$, and $k \le c_9 \log T/\log\log T$, for some sufficiently small constant $c_9 > 0$. We now choose $k = \lfloor c_6(r) V^{\frac{1}{(1-\sigma(r))}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \rfloor$ (so that $k^{\sigma'(r)} \asymp k^{\sigma(r)}$) where $c_6(r)$ is a sufficiently small absolute constant. Using (4.2) and (4.3) along with Chebyshev's inequality and the above estimate we conclude that there exists $c_7(r) > 0$ such that

$$\begin{aligned}\mathbb{P}_T\left(\max_{|z|\le r}|\zeta'(\tfrac{3}{4}+it+z)| > e^V\right) \ll& \mathbb{P}_T\left(\max_{|s|\le r+\delta}\left|\sum_{n\le y}\frac{\Lambda(n)}{n^{3/4+it+s}\log n}\right| > \frac{V}{4}\right) + T^{1-(1/4-r)/5} \\ \ll& e^{V/2}\left(\frac{4}{V}\cdot c_8(r)\frac{k^{1-\sigma'(r)}}{(\log k)^{\sigma'(r)}}\right)^{2k} + T^{1-(1/4-r)/5} \\ \ll& \exp\left(-c_6 V^{\frac{1}{1-\sigma(r)}}(\log V)^{\frac{\sigma(r)}{1-\sigma(r)}}\right)\end{aligned}$$

for $V \le c_7(\log T)^{1-\sigma(r)}/\log\log T$.

□

We now prove Proposition 2.2 along the same lines. The proof is in fact easier than in the zeta function case, since we can compute the moments of $\log\zeta(s,X)$, for any $s$ with $\mathrm{Re}(s) > 1/2$.

*Proof of Proposition 2.2.* Let $\delta = e^{-V/2}$. Since $\zeta(\frac34 + s, X)$ is almost surely analytic in $|s| \le r + 2\delta$, then by Cauchy's estimate we have almost surely that

$$\max_{|z|\le r} |\zeta'(\tfrac34 + z, X)| \le \frac{1}{\delta} \max_{|s|\le r+\delta} |\zeta(\tfrac34 + s, X)|.$$

Therefore, we obtain

$$\begin{aligned}\mathbb{P}\left(\max_{|z|\le r} |\zeta'(\tfrac34 + z, X)| > e^V\right) \le &\mathbb{P}\left(\max_{|s|\le r+\delta} |\zeta(\tfrac34 + s, X)| > \delta e^V\right)\\ \le &\mathbb{P}\left(\max_{|s|\le r+\delta} |\log \zeta(\tfrac34 + s, X)| > \frac{V}{2}\right).\end{aligned} \tag{4.5}$$

Let $k$ be a positive integer. By (2.2) $\log \zeta(\frac34 + s, X)$ converges almost surely to a holomorphic function in $|s| \le r + 2\delta$. Using Cauchy's integral formula as in (4.4), we obtain almost surely that

$$\left(\max_{|s|\le r+\delta} |\log \zeta(\tfrac34 + s, X)|\right)^{2k} \ll \frac{1}{\delta} \int_{|z|=r+2\delta} \left|\log \zeta(\tfrac34 + z, X)\right|^{2k} \cdot |dz|.$$

Hence, applying Lemma 4.3 we get

$$\begin{aligned}\mathbb{P}\left(\max_{|s|\le r+\delta} |\log \zeta(\tfrac34 + s, X)| > V/2\right) &\le \left(\frac{2}{V}\right)^{2k} \cdot \mathbb{E}\left(\left(\max_{|s|\le r+\delta} |\zeta(\tfrac34 + s, X)|\right)^{2k}\right)\\ &\ll \left(\frac{2}{V}\right)^{2k} e^{V/2} \int_{|z|=r+2\delta} \mathbb{E}\left(|\log \zeta(\tfrac34 + z, X)|^{2k}\right) \cdot |dz|\\ &\ll e^{V/2} \left(\frac{2c_8(r) k^{1-\sigma'(r)}}{V(\log k)^{\sigma'(r)}}\right)^{2k},\end{aligned} \tag{4.6}$$

where $\sigma'(r) = \frac34 - r - 2\delta$. Let $\sigma(r) = \frac34 - r$ and take $k = \lfloor c_6 V^{\frac{1}{1-\sigma(r)}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \rfloor$, where $c_6$ is sufficiently small (note that $k^{\sigma'(r)} \asymp k^{\sigma(r)}$), then apply (4.6) to complete the proof. □

## 5. Moments of joint shifts of $\log \zeta(s)$: Proof of Theorem 2.1

The proof of Theorem 2.1 splits into two parts. In the first part we derive an approximation to

$$\prod_{j=1}^{k} \log \zeta(s_j + it)$$

by a short Dirichlet polynomial. In the second part we compute the resulting mean-values and obtain Theorem 2.1.

5.1. **Approximating $\prod_{j=1}^{k} \log\zeta(s_j+it)$ by short Dirichlet polynomials.** Fix $1/2 < \sigma_0 < 1$, and let $\delta := \sigma_0 - 1/2$. Let $k \le \log T$ be a positive integer and $s_1, s_2, \ldots, s_k$ be complex numbers (not necessarily distinct) in the rectangle $\{z : \sigma_0 \le \mathrm{Re}(z) < 1, \text{ and } |\mathrm{Im}(z)| \le T^{\delta/4}\}$. We let $\mathbf{s} = (s_1, \ldots, s_k)$, and define

$$F_{\mathbf{s}}(n) = \sum_{\substack{n_1, n_2, \ldots, n_k \ge 2 \\ n_1 n_2 \cdots n_k = n}} \prod_{\ell=1}^{k} \frac{\Lambda(n_\ell)}{n_\ell^{s_\ell} \log(n_\ell)}.$$

Then for all complex numbers $z$ with $\mathrm{Re}(z) > 1 - \sigma_0$ we have

$$\prod_{\ell=1}^{k} \log\zeta(s_\ell + z) = \sum_{n=1}^{\infty} \frac{F_{\mathbf{s}}(n)}{n^z}.$$

The main result of this subsection is the following proposition.

**Proposition 5.1.** *Let $T$ be large, $s_1, \ldots, s_k$ be as above, and $\mathcal{E}(T)$ be as in Lemma 5.3 below. Then, there exist positive constants $a(\sigma_0), b(\sigma_0)$ such that if $k \le a(\sigma_0)(\log T)/\log\log T$ and $t \in [T, 2T] \setminus \mathcal{E}(T)$ then*

$$\prod_{j=1}^{k} \log\zeta(s_j + it) = \sum_{n \le T^{\delta/8}} \frac{F_{\mathbf{s}}(n)}{n^{it}} + O\left(T^{-b(\sigma_0)}\right).$$

This depends on a sequence of fairly standard lemmas which we now describe.

**Lemma 5.1.** *With the same notation as above, we have*

$$|F_{\mathbf{s}}(n)| \le \frac{(2\log n)^k}{n^{\sigma_0}}.$$

*Proof.* We have

$$|F_{\mathbf{s}}(n)| \le \frac{1}{n^{\sigma_0}(\log 2)^k} \sum_{\substack{n_1, n_2, \ldots, n_k \ge 2 \\ n_1 n_2 \cdots n_k = n}} \prod_{\ell=1}^{k} \Lambda(n_\ell) \le \frac{2^k}{n^{\sigma_0}} \left(\sum_{m|n} \Lambda(m)\right)^k \le \frac{(2\log n)^k}{n^{\sigma_0}}.$$

□

**Lemma 5.2.** *Let $y \ge 2$ and $|t| \ge y + 3$ be real numbers. Suppose that the rectangle $\{z : \sigma_0 - \delta/2 < \mathrm{Re}(z) \le 1, |\mathrm{Im}(z) - t| \le y + 2\}$ is free of zeros of $\zeta(z)$. Then, for all complex numbers $s$ such that $\mathrm{Re}(s) \ge \sigma_0 - \delta/4$ and $|\mathrm{Im}(s)| \le y$ we have*

$$\log\zeta(s + it) \ll_{\sigma_0} \log|t|.$$

*Proof.* This follows from Theorem 9.6 B of Titchmarsh. □

**Lemma 5.3.** *Let $s_1, \ldots, s_k$ be as above. Then, there exists a set $\mathcal{E}(T) \subset [T, 2T]$ with measure meas$(\mathcal{E}(T)) \ll T^{1-\delta/8}$, and such that for all $t \in [T, 2T] \setminus \mathcal{E}(T)$ we have $\zeta(s_j + it + z) \ne 0$ for every $1 \le j \le k$ and every $z$ in the rectangle $\{z : -\delta/2 < \mathrm{Re}(z) \le 1, |\mathrm{Im}(z)| \le 3T^{\delta/4}\}$.*

*Proof.* For every $1 \le j \le k$, let $\mathcal{E}_j(T)$ be the set of $t \in [T, 2T]$ such that the rectangle $\{z : -\delta/2 < \mathrm{Re}(z) \le 1, |\mathrm{Im}(z)| \le 3T^{\delta/4}\}$ has a zero of $\zeta(s_j + it + z)$. Then, by the classical zero density estimate $N(\sigma, T) \ll T^{3/2-\sigma}(\log T)^5$, we deduce that

$$\mathrm{meas}(\mathcal{E}_j(T)) \ll T^{\delta/4} T^{3/2-\sigma_0+\delta/2}(\log T)^5 < T^{1-\delta/4}(\log T)^5.$$

We take $\mathcal{E}(T) = \cup_{j=1}^k \mathcal{E}_j(T)$. Then $\mathcal{E}(T)$ satisfies the assumptions of the lemma, since $\mathrm{meas}(\mathcal{E}(T)) \ll T^{1-\delta/4}(\log T)^6 \ll T^{1-\delta/8}$. □

We are now ready to prove Proposition 5.1.

*Proof of Proposition 5.1.* Let $x = \lfloor T^{\delta/8} \rfloor + 1/2$. Let $c = 1 - \sigma_0 + 1/\log T$, and $Y = T^{\delta/4}$. Then by Perron's formula, we have for $t \in [T, 2T] \setminus \mathcal{E}(T)$

$$\frac{1}{2\pi i} \int_{c-iY}^{c+iY} \left( \prod_{j=1}^k \log \zeta(s_j + it + z) \right) \frac{x^z}{z} dz = \sum_{n \le x} \frac{F_{\mathbf{s}}(n)}{n^{it}} + O\left( \frac{x^c}{Y} \sum_{n=1}^{\infty} \frac{|F_{\mathbf{s}}(n)|}{n^c |\log(x/n)|} \right).$$

To bound the error term of this last estimate, we split the sum into three parts: $n \le x/2$, $x/2 < n < 2x$ and $n \ge 2x$. The terms in the first and third parts satisfy $|\log(x/n)| \ge \log 2$, and hence their contribution is

$$\ll \frac{x^{1-\sigma_0}}{Y} \sum_{n=1}^{\infty} \frac{|F_{\mathbf{s}}(n)|}{n^c} \le \frac{x^{1-\sigma_0}}{Y} \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0 + c} \log n} \right)^k \le \frac{x^{1-\sigma_0}(2\log T)^k}{Y} \ll T^{-b(\sigma_0)},$$

or some positive constant $b(\sigma_0)$, if $a(\sigma_0)$ is sufficiently small. To handle the contribution of the terms $x/2 < n < 2x$, we put $r = x - n$, and use that $|\log(x/n)| \gg |r|/x$. Then by Lemma 5.1 we deduce that the contribution of these terms is

$$\ll \frac{x^{1-\sigma_0}(3\log x)^k}{Y} \sum_{r \le x} \frac{1}{r} \ll \frac{x^{1-\sigma_0}(3\log x)^{k+1}}{Y} \ll T^{-b(\sigma_0)}.$$

We now move the contour to the line $\mathrm{Re}(s) = -\delta/4$. By Lemma 5.3, we do not encounter any zeros of $\zeta(s_j + it + z)$ since $t \in [T, 2T] \setminus \mathcal{E}(T)$. We pick up a simple pole at $z = 0$ which leaves a residue $\prod_{j=1}^k \log \zeta(s_j + it)$. Also Lemma 5.2 implies that for any $z$ on our contour we have

$$|\log \zeta(s_j + it + z)| \le c(\sigma_0) \log T,$$

for all $j$ where $c(\sigma_0)$ is a positive constant. Therefore, we deduce that

$$\frac{1}{2\pi i} \int_{c-iY}^{c+iY} \left( \prod_{j=1}^k \log \zeta(s_j + it + z) \right) \frac{x^z}{z} dz = \prod_{j=1}^k \log \zeta(s_j + it) + E_1,$$

where

$$E_1 = \frac{1}{2\pi i} \left( \int_{c-iY}^{-\delta/4 - iY} + \int_{-\delta/4-iY}^{-\delta/4+iY} + \int_{-\delta/4+iY}^{c+iY} \right) \left( \prod_{j=1}^k \log \zeta(s_j + it + z) \right) \frac{x^z}{z} dz$$
$$\ll \frac{x^{1-\sigma_0}(c(\sigma_0)\log T)^k}{Y} + x^{-\delta/4}(c(\sigma_0)\log T)^k \log Y \ll T^{-b(\sigma_0)},$$

as desired.

□

### 5.2. An Asymptotic formula for the moment of products of shifts of $\log\zeta(s)$.

*Proof of Theorem 2.1.* Let $\mathcal{E}_1(T)$ and $\mathcal{E}_2(T)$ be the corresponding exceptional sets for $\mathbf{s}$ and $\mathbf{r}$ respectively as in Lemma 5.3, and let $\mathcal{E}(T)=\mathcal{E}_1(T)\cup\mathcal{E}_2(T)$. First, note that if $t\in[T,2T]\setminus\mathcal{E}(T)$ then by Proposition 5.1 and Lemma 5.2 we have

$$\left|\sum_{n\le x}\frac{F_{\mathbf{s}}(n)}{n^{it}}\right|\ll (c(\sigma_0)\log T)^k, \text{ and } \left|\sum_{m\le x}\frac{F_{\mathbf{r}}(m)}{m^{-it}}\right|\ll (c(\sigma_0))\log T)^\ell,$$

for some positive constant $c(\sigma_0)$. Let $x=T^{(\sigma_0-1/2)/8}$. Then, it follows from Proposition 5.1 that

(5.1)
$$\begin{aligned}&\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)}\left(\prod_{j=1}^{k}\log\zeta(s_j+it)\right)\left(\prod_{j=1}^{\ell}\log\zeta(r_j-it)\right)dt\\&=\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)}\left(\sum_{n\le x}\frac{F_{\mathbf{s}}(n)}{n^{it}}\right)\left(\sum_{m\le x}F_{\mathbf{r}}(m)m^{it}dt\right)dt+O\left(T^{-b(\sigma_0)}(\log T)^{\max(k,\ell)}\right)\\&=\frac{1}{T}\int_T^{2T}\left(\sum_{n\le x}\frac{F_{\mathbf{s}}(n)}{n^{it}}\right)\left(\sum_{m\le x}F_{\mathbf{r}}(m)m^{it}\right)dt+O\left(T^{-b(\sigma_0)/2}\right).\end{aligned}$$

Furthermore, we have

$$\frac{1}{T}\int_T^{2T}\left(\sum_{n\le x}\frac{F_{\mathbf{s}}(n)}{n^{it}}\right)\left(\sum_{m\le x}F_{\mathbf{r}}(n)m^{it}\right)dt=\sum_{m,n\le x}F_{\mathbf{s}}(n)F_{\mathbf{r}}(m)\frac{1}{T}\int_T^{2T}\left(\frac{m}{n}\right)^{it}dt. \tag{5.2}$$

The contribution of the diagonal terms $n=m$ equals $\sum_{n\le x}F_{\mathbf{s}}(n)F_{\mathbf{r}}(n)$. On the other hand, by Lemma 5.1 the contribution of the off-diagonal terms $n\neq m$ is

$$\ll\frac{1}{T}\sum_{\substack{m,n\le x\\ m\neq n}}\frac{(2\log n)^k(2\log m)^\ell}{(mn)^{\sigma_0}}\frac{1}{|\log(m/n)|}\ll\frac{x^{3-2\sigma_0}(2\log x)^{k+\ell}}{T}\ll T^{-1/2}, \tag{5.3}$$

since $|\log(m/n)|\gg 1/x$.

Furthermore, it follows from (2.3) that

(5.4)
$$\mathbb{E}\left(\prod_{j=1}^{k}\log\zeta(s_j,X)\right)\left(\prod_{j=1}^{\ell}\log\overline{\zeta(r_j,X)}\right)=\sum_{n=1}^{\infty}F_{\mathbf{s}}(n)F_{\mathbf{r}}(n)=\sum_{n\le x}F_{\mathbf{s}}(n)F_{\mathbf{r}}(n)+E_2,$$

where

$$E_2\le\sum_{n>x}\frac{(2\log n)^{k+\ell}}{n^{2\sigma_0}}.$$

Since the function $(\log t)^\beta/t^\alpha$ is decreasing for $t > \exp(\beta/\alpha)$, then with the choice $\alpha = (2\sigma_0 - 1)/2$ we obtain

$$E_2 \le \frac{(2\log x)^{k+\ell}}{x^\alpha} \sum_{n>x} \frac{1}{n^{1+\alpha}} \ll \frac{(2\log x)^{k+\ell}}{x^{2\alpha}} \ll x^{-\alpha}.$$

Combining this with (5.2), (5.3), and (5.4) completes the proof. □

## 6. The characteristic function of joint shifts of $\log\zeta(s)$

*Proof of Theorem 2.2.* Let $\mathcal{E}(T)$ be as in Theorem 2.1. Let $N = [\log T/(C(\log\log T))]$ where $C$ is a suitably large constant. Then, $\Phi_T(\mathbf{u}, \mathbf{v})$ equals

$$\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \exp\left(i\left(\sum_{j=1}^{J}(u_j \mathrm{Re}\log\zeta(s_j+it) + v_j \mathrm{Im}\log\zeta(s_j+it))\right)\right) dt + O\left(T^{-c_3}\right)$$

$$= \sum_{n=0}^{2N-1} \frac{i^n}{n!}\cdot\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \left(\sum_{j=1}^{J}(u_j \mathrm{Re}\log\zeta(s_j+it) + v_j \mathrm{Im}\log\zeta(s_j+it))\right)^n dt + E_3, \tag{6.1}$$

where

$$E_3 \ll T^{-c_3} + \frac{1}{(2N)!}\left(\frac{2c_1(\log T)^\sigma}{J}\right)^{2N} \frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \left(\sum_{j=1}^{J} |\log\zeta(s_j+it)|\right)^{2N} dt.$$

Now, by Theorem 2.1 along with Lemma 4.3, we obtain that for all $1 \le j \le J$

$$\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} |\log\zeta(s_j+it)|^{2N} dt \ll \mathbb{E}\left(|\log\zeta(s_j, X)|^{2N}\right) \le \left(\frac{c_8(\sigma)N^{1-\sigma}}{(\log N)^\sigma}\right)^{2N}, \tag{6.2}$$

for some positive constant $c_8 = c_8(\sigma)$. Furthermore, by Minkowski's inequality we have

$$\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \left(\sum_{j=1}^{J} |\log\zeta(s_j+it)|\right)^{2N} dt \le \left(c_8 J \frac{N^{1-\sigma}}{(\log N)^\sigma}\right)^{2N}.$$

Therefore, we deduce that for some positive constant $c_9 = c_9(\sigma)$, we have

$$E_3 \ll T^{-c_3} + \left(c_9\frac{(\log T)^\sigma}{(N\log N)^\sigma}\right)^{2N} \ll e^{-N}.$$

Next, we handle the main term of (6.1). Let $\tilde{u_j} = (u_j + iv_j)/2$ and $\tilde{v_j} = (u_j - iv_j)/2$. Then by Theorem 2.1 we obtain

$$\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \left(\sum_{j=1}^{J}(u_j \mathrm{Re}\log\zeta(s_j+it) + v_j \mathrm{Im}\log\zeta(s_j+it))\right)^n dt$$

$$\frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \left(\sum_{j=1}^{J}(\tilde{u_j}\log\zeta(s_j+it) + \tilde{v_j}\log\zeta(s_j-it))\right)^n dt$$

$$= \sum_{\substack{k_1,\dots,k_{2J}\ge 0\\ k_1+\cdots+k_{2J}=n}} \binom{n}{k_1,k_2,\dots,k_{2J}} \prod_{j=1}^{J}\tilde{u_j}^{k_j}\prod_{\ell=1}^{J}\tilde{v_\ell}^{k_{J+\ell}}$$

$$\times \frac{1}{T}\int_{[T,2T]\setminus\mathcal{E}(T)} \prod_{j=1}^{J}(\log\zeta(s_j+it))^{k_j}\prod_{\ell=1}^{J}(\log\zeta(s_\ell-it))^{k_{J+\ell}}dt$$

$$= \sum_{\substack{k_1,\dots,k_{2J}\ge 0\\ k_1+\cdots+k_{2J}=n}} \binom{n}{k_1,k_2,\dots,k_{2J}} \prod_{j=1}^{J}\tilde{u_j}^{k_j}\prod_{\ell=1}^{J}\tilde{v_\ell}^{k_{J+\ell}}$$

$$\times \mathbb{E}\left(\prod_{j=1}^{J}(\log\zeta(s_j,X))^{k_j}\prod_{\ell=1}^{J}(\log\zeta(s_\ell,X))^{k_{J+\ell}}\right) + O\Big(T^{-c_5}\big(2c_1(\log T)^{\sigma}\big)^n\Big),$$

$$= \mathbb{E}\left(\left(\sum_{j=1}^{J}(u_j\mathrm{Re}\log\zeta(s_j,X) + v_j\mathrm{Im}\log\zeta(s_j,X))\right)^n\right) + O\Big(T^{-c_5}\big(2c_1(\log T)^{\sigma}\big)^n\Big).$$

Inserting this estimate in (6.1), we derive

$$\begin{aligned}\Phi_T(\mathbf{u},\mathbf{v}) &= \sum_{n=0}^{2N-1}\frac{i^n}{n!}\mathbb{E}\left(\left(\sum_{j=1}^{J}(u_j\mathrm{Re}\log\zeta(s_j,X) + v_j\mathrm{Im}\log\zeta(s_j,X))\right)^n\right) + O\Big(e^{-N}\Big)\\ &= \Phi_{\mathrm{rand}}(\mathbf{u},\mathbf{v}) + E_4.\end{aligned}$$

where

$$E_4 \ll e^{-N} + \frac{1}{(2N)!}\left(\frac{2c_1(\log T)^{\sigma}}{J}\right)^{2N}\mathbb{E}\left(\left(\sum_{j=1}^{J}|\log\zeta(s_j,X)|\right)^{2N}\right) \ll e^{-N}$$

by (6.2) and Minkowski's inequality. This completes the proof. □

## 7. Discrepancy estimates for the distribution of shifts

The deduction of Theorem 2.3 from Theorem 2.2 uses Beurling-Selberg functions. For $z \in \mathbb{C}$ let

$$H(z) = \left(\frac{\sin\pi z}{\pi}\right)^2\left(\sum_{n=-\infty}^{\infty}\frac{\mathrm{sgn}(n)}{(z-n)^2} + \frac{2}{z}\right) \qquad \text{and} \qquad K(z) = \Big(\frac{\sin\pi z}{\pi z}\Big)^2.$$

Beurling proved that the function $B^+(x) = H(x) + K(x)$ majorizes $\mathrm{sgn}(x)$ and its Fourier transform has restricted support in $(-1,1)$. Similarly, the function $B^-(x) =$

$H(x) - K(x)$ minorizes $\operatorname{sgn}(x)$ and its Fourier transform has the same property (see Vaaler [13] Lemma 5).

Let $\Delta > 0$ and $a, b$ be real numbers with $a < b$. Take $\mathcal{I} = [a, b]$ and define

$$F_{\mathcal{I}}(z) = \frac{1}{2}\Big(B^-(\Delta(z-a)) + B^-(\Delta(b-z))\Big).$$

The function $F_{\mathcal{I}}$ has the following remarkable properties. First, it follows from the inequality $B^-(x) \le \operatorname{sgn}(x) \le B^+(x)$ that

$$0 \le \mathbf{1}_{\mathcal{I}}(x) - F_{\mathcal{I}}(x) \le K(\Delta(x-a)) + K(\Delta(b-x)). \tag{7.1}$$

Additionally, one has

$$\widehat{F_{\mathcal{I}}}(\xi) = \begin{cases} \widehat{\mathbf{1}}_{\mathcal{I}}(\xi) + O\Big(\frac{1}{\Delta}\Big) \text{ if } |\xi| < \Delta, \\ 0 \text{ if } |\xi| \ge \Delta. \end{cases} \tag{7.2}$$

The first estimate above follows from (7.1) and the second follows from the fact that the Fourier transform of $B^-$ is supported in $(-1, 1)$. Before proving Theorem 2.3 we first require the following lemmas.

**Lemma 7.1.** *For $x \in \mathbb{R}$ we have $|F_{\mathcal{I}}(x)| \le 1$.*

*Proof.* It suffices to prove the lemma for $\Delta = 1$. Also, note that we only need to show that $F_{\mathcal{I}}(x) \ge -1$. From the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-z)^2} = \Big(\frac{\pi}{\sin \pi z}\Big)^2$$

it follows that for $y \ge 0$

$$H(y) = 1 - K(y)G(y) \tag{7.3}$$

where

$$G(y) = 2y^2 \sum_{m=0}^{\infty} \frac{1}{(y+m)^2} - 2y - 1.$$

In Lemma 5 of [13], Vaaler shows for $y \ge 0$ that

$$0 \le G(y) \le 1. \tag{7.4}$$

Also, note that for each $m \ge 1$, and $0 < y \le 1$ one has $\frac{m}{(y+m)^3} \le \int_{m-1}^{m} \frac{t}{(y+t)^3}\, dt$ so that for $0 < y \le 1$

$$G'(y) = 4y \sum_{m \ge 1} \frac{m}{(y+m)^3} - 2 \le 4y \int_0^{\infty} \frac{t}{(y+t)^3}\, dt - 2 = 0. \tag{7.5}$$

First consider the case $a \le x \le b$. By (7.3) we get that in this range

$$F_{\mathcal{I}}(x) = \frac{1}{2}\left(2 - K(x-a)(G(x-a)+1) - K(b-x)(G(b-x)+1)\right),$$

which along with (7.4) implies $F_{\mathcal{I}}(x) \geq -1$ for $a \leq x \leq b$. Now consider the case $x < a$. Since $H$ is an odd function (7.3) and (7.4) imply

$$\begin{aligned} F_{\mathcal{I}}(x) =& \frac{1}{2}\left(K(x-a)(G(a-x)-1) - K(b-x)(G(b-x)+1)\right) \\ \geq& \frac{1}{2}\left(-K(x-a) - 2K(x-b)\right), \end{aligned}$$

which is $\geq -1$ if $K(x-b) \leq 1/2$. If $K(x-b) \geq 1/2$ we also have $K(x-a) > K(x-b)$ and $0 < b-x < 1$. By this and (7.5) we have in this range as well that

$$F_{\mathcal{I}}(x) \geq \frac{1}{2}\left(K(x-b)(G(a-x) - G(b-x) - 2)\right) \geq -1.$$

Hence, $F_{\mathcal{I}}(x) \geq -1$ for $x < a$. The remaining case when $x > b$ follows from a similar argument. □

**Lemma 7.2.** *Fix $1/2 < \sigma < 1$, and let $s$ be a complex number such that $\mathrm{Re}(s) = \sigma$ and $|\mathrm{Im}(s)| \leq T^{\frac{1}{4}\cdot(\sigma - \frac{1}{2})}$. Then there exists a positive constant $c_1(\sigma)$ such that for $|u| \leq c_1(\sigma)(\log T)^{\sigma}$ we have*

$$\Phi_T(u,0) \ll \exp\left(\frac{-u}{5\log u}\right) \qquad \textit{and} \qquad \Phi_T(0,u) \ll \exp\left(\frac{-u}{5\log u}\right).$$

*Proof.* By a straightforward modification of Lemma 6.3 of [10] one has that

$$\mathbb{E}\Bigg(\exp\Big(iu\,\mathrm{Re}\log\zeta(s,X)\Big)\Bigg) \ll \exp\left(-\frac{u}{5\log u}\right),$$

and

$$\mathbb{E}\Bigg(\exp\Big(iu\,\mathrm{Im}\log\zeta(s,X)\Big)\Bigg) \ll \exp\left(-\frac{u}{5\log u}\right).$$

Using the first bound and applying Theorem 2.2 with $J = 1$ establishes the first claim. The second claim follows similarly by using the second bound and Theorem 2.2. □

*Proof of Theorem 2.3.* First, we claim that it suffices to estimate the discrepancy over $(\mathcal{R}_1, \ldots, \mathcal{R}_J)$ such that for each $j$ we have $\mathcal{R}_j \subset [-\sqrt{\log T}, \sqrt{\log T}] \times [-\sqrt{\log T}, \sqrt{\log T}]$. To see this consider $(\widetilde{\mathcal{R}_1}, \ldots, \widetilde{\mathcal{R}_J})$ where $\widetilde{\mathcal{R}_j} = \mathcal{R}_j \cap [-\sqrt{\log T}, \sqrt{\log T}] \times [-\sqrt{\log T}, \sqrt{\log T}]$. It follows that

$$\begin{aligned} &\Bigg|\mathbb{P}_T\Bigg(\log\zeta(s_j+it) \in \mathcal{R}_j, \forall j \leq J\Bigg) - \mathbb{P}_T\Bigg(\log\zeta(s_1+it) \in \widetilde{\mathcal{R}_1}, \log\zeta(s_j+it) \in \mathcal{R}_j, 2 \leq j \leq J\Bigg)\Bigg| \\ &\ll \mathbb{P}_T\Bigg(|\log\zeta(s_1+it)| \geq \sqrt{\log T}\Bigg) \ll \exp\Big(-\sqrt{\log T}\Big), \end{aligned}$$

where the last bound follows from Theorem 1.1 and Remark 1 of of [9]. Repeating this argument gives

$$\Bigg|\mathbb{P}_T\Bigg(\log\zeta(s_j+it) \in \widetilde{\mathcal{R}_j}, \forall j \leq J) - \mathbb{P}_T\Bigg(\log\zeta(s_j+it) \in \mathcal{R}_j, \forall j \leq J\Bigg)\Bigg| \ll J\exp\Big(-\sqrt{\log T}\Big).$$

Similarly,

$$\left|\mathbb{P}\Big(\log\zeta(s_j,X)\in\widetilde{\mathcal{R}_j},\forall j\le J\Big)-\mathbb{P}\Big(\log\zeta(s_j,X)\in\mathcal{R}_j,\forall j\le J\Big)\right|\ll J\exp\Big(-\sqrt{\log T}\Big).$$

Hence, the error from restricting to $(\widetilde{\mathcal{R}_1},\dots,\widetilde{\mathcal{R}_J})$ is negligible and establishes the claim.

Let $\Delta = c_1(\sigma)(\log T)^\sigma/J$ and $\mathcal{R}_j = [a_j,b_j]\times[c_j,d_j]$ for $j=1,\dots,J$, with $|b_j - a_j|, |d_j-c_j|\le 2\sqrt{\log T}$. Also, write $\mathcal{I}_j=[a_j,b_j]$ and $\mathcal{J}_j=[c_j,d_j]$. By Fourier inversion, (7.2), and Theorem 2.2 we have that

(7.6)

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T}\prod_{j=1}^J F_{\mathcal{I}_j}\Big(\operatorname{Re}\log\zeta(s_j+it)\Big)F_{\mathcal{J}_j}\Big(\operatorname{Im}\log\zeta(s_j+it)\Big)\,dt\\
&=\int_{\mathbb{R}^{2J}}\Big(\prod_{j=1}^J\widehat{F}_{\mathcal{I}_j}(u_j)\widehat{F}_{\mathcal{J}_j}(v_j)\Big)\Phi_T(\mathbf{u},\mathbf{v})\,d\mathbf{u}\,d\mathbf{v}\\
&=\int\limits_{\substack{|u_j|,|v_j|\le\Delta\\ j=1,2,\dots,J}}\Big(\prod_{j=1}^J\widehat{F}_{\mathcal{I}_j}(u_j)\widehat{F}_{\mathcal{J}_j}(v_j)\Big)\Phi_{\text{rand}}(\mathbf{u},\mathbf{v})\,d\mathbf{u}\,d\mathbf{v}+O\left(\Big(2\Delta\sqrt{\log T}\Big)^{2J}\exp\Big(-\frac{c_2\log T}{\log\log T}\Big)\right)\\
&=\mathbb{E}\bigg(\prod_{j=1}^J F_{\mathcal{I}_j}\Big(\operatorname{Re}\log\zeta(s_j,X)\Big)F_{\mathcal{J}_j}\Big(\operatorname{Im}\log\zeta(s_j,X)\Big)\bigg)+O\left(\exp\Big(-\frac{c_2\log T}{2\log\log T}\Big)\right).
\end{aligned}$$

Next note that $\widehat{K}(\xi)=\max(0,1-|\xi|)$. Applying Fourier inversion, Theorem 2.2 with $J=1$, and Lemma 7.2 we have that

$$\frac{1}{T}\int_T^{2T}K\Big(\Delta\cdot\Big(\operatorname{Re}\log\zeta(s+it)-\alpha\Big)\Big)\,dt=\frac{1}{\Delta}\int_{-\Delta}^{\Delta}\Big(1-\frac{|\xi|}{\Delta}\Big)e^{-2\pi i\alpha\xi}\Phi_T(\xi,0)\,d\xi\ll\frac{1}{\Delta},$$

where $\alpha$ is an arbitrary real number and $s\in\mathbb{C}$ satisfies $\sigma\le \operatorname{Re}(s)<1$ and $|\operatorname{Im}(s)|< T^{\frac14(\sigma-\frac12)}$. By this and (7.1) we get that

$$\frac{1}{T}\int_T^{2T}F_{\mathcal{I}_1}\Big(\operatorname{Re}\log\zeta(s_1+it)\Big)\,dt=\frac{1}{T}\int_T^{2T}\mathbf{1}_{\mathcal{I}_1}\Big(\operatorname{Re}\log\zeta(s_1+it)\Big)dt+O(1/\Delta). \tag{7.7}$$

Lemma 7.1 implies that $|F_{\mathcal{I}_j}(x)|,|F_{\mathcal{J}_j}(x)|\le 1$ for $j=1,\dots,J$. Hence, by this and (7.7)

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T}\prod_{j=1}^J F_{\mathcal{I}_j}\Big(\operatorname{Re}\log\zeta(s_j+it)\Big)F_{\mathcal{J}_j}\Big(\operatorname{Im}\log\zeta(s_j+it)\Big)\,dt\\
&=\frac{1}{T}\int_T^{2T}\mathbf{1}_{\mathcal{I}_1}\Big(\operatorname{Re}\log\zeta(s_j+it)\Big)F_{\mathcal{J}_1}\Big(\operatorname{Im}\log\zeta(s_j+it)\Big)\\
&\qquad\times\prod_{j=2}^J F_{\mathcal{I}_j}\Big(\operatorname{Re}\log\zeta(s_j+it)\Big)F_{\mathcal{J}_j}\Big(\operatorname{Im}\log\zeta(s_j+it)\Big)\,dt+O(1/\Delta).
\end{aligned}$$

Iterating this argument and using an analog of (7.7) for $\operatorname{Im}\log\zeta(s+it)$, which is proved in the same way, gives

$$\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J} F_{\mathcal{I}_j}\Big(\operatorname{Re}\log\zeta(s_j+it)\Big)F_{\mathcal{J}_j}\Big(\operatorname{Im}\log\zeta(s_j+it)\Big)\,dt = \mathbb{P}_T\Bigg(\log\zeta(s_j+it)\in\mathcal{R}_j, \forall j\le J\Bigg)+O\left(\frac{J}{\Delta}\right). \tag{7.8}$$

Similarly, it can be shown that

$$\mathbb{E}\Bigg(\prod_{j=1}^{J} F_{\mathcal{I}_j}\Big(\operatorname{Re}\log\zeta(s_j,X)\Big)F_{\mathcal{J}_j}\Big(\operatorname{Im}\log\zeta(s_j,X)\Big)\Bigg) = \mathbb{P}\Bigg(\log\zeta(s_j,X)\in\mathcal{R}_j, \forall j\le J\Bigg) + O\left(\frac{J}{\Delta}\right). \tag{7.9}$$

Using (7.8) and (7.9) in (7.6) completes the proof.

□

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