SIMPLE ZEROS OF PRIMITIVE DIRICHLET L-FUNCTIONS AND THE ASYMPTOTIC LARGE SIEVE

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Abstract. Assuming the Generalized Riemann Hypothesis (GRH), we show using the asymptotic large sieve that 91% of the zeros of primitive Dirichlet L-functions are simple. This improves on earlier work of Özlük which gives a proportion of at most 86%. We further compute an q-analogue of the Pair Correlation Function $F(\alpha)$ averaged over all primitive Dirichlet L-functions in the range $|\alpha| < 2$. Previously such a result was available only when the average included all the characters χ.

1. Introduction

Montgomery [4] was the first to consider the Pair Correlation of the zeros of the Riemann zeta-function. Montgomery's results suggested that the distribution of the zeros of the Riemann zeta-function follows the same laws as the distribution of the eigenvalues of a random unitary matrix. This connection was further expanded on, and is responsible for much of the subsequent activity in the theory of L-functions (see for example [2], [3], [6]).

One can similarly investigate the distribution of the low-lying zeros in a family of L-function. Özlük [5] considered a q-analogue of Montgomery's results. His motivation was to understand the low-lying zeros of $L(s,\chi)$ on average over χ modulo q and $Q \leq q \leq 2Q$. Since the family is larger, one can obtain better results than in the case of the Riemann zeta-function.

One defect in Özlük's work was that it concerns an average over all characters χ rather than just the primitive characters χ. As a result, in applications this often leads to inferior results.

Recently, Conrey, Soundararajan, and Iwaniec developed an *asymptotic large sieve* [1]. They devised a method to obtain asymptotic estimates for rather general averages over primitive characters. In this paper we revisit Özlük's work in the light of these recent developments, obtaining results for primitive characters rather than all characters. As a consequence we obtain that, in a suitable sense, 91% of the zeros of primitive Dirichlet L-functions are simple, on the assumption of the Generalized Riemann Hypothesis (GRH).

Let Φ be a smooth function which is real and compactly supported in (a,b) with $0 < a < b$, and define its Mellin transform

$$\widehat{\Phi}(s) = \int_0^\infty \Phi(x) x^{s-1}\, dx.$$

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Let

$$N_\Phi(Q) := \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma_\chi} |\widehat{\Phi}(i\gamma_\chi)|^2$$

with W a smooth function, compactly supported in $(1,2)$, the second sum being over primitive characters χ, and the last sum being over all non-trivial zeros $1/2+i\gamma_\chi$ of Dirichlet L-function $L(s,\chi)$. As we will see later (in Lemma 1)

$$N_\Phi(Q) \sim \frac{A}{2\pi} Q \log Q \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 \, dx$$

where

$$A = \widehat{W}(1) \prod_p \left(1 - \frac{1}{p^2} - \frac{1}{p^3}\right). \tag{1.1}$$

Our work yields the following theorem.

Theorem 1. *Assume GRH. The proportion of simple zeros of all primitive Dirichlet L-functions is greater than or equal to $\frac{11}{12}$ in the sense of the inequality*

$$\frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \geq \frac{11}{12} + o(1)$$

with the sum being over primitive characters and with Φ chosen so that $\widehat{\Phi}(ix) = (\sin x/x)^2$.

We note that the function Φ satisfying $\widehat{\Phi}(ix) = (\sin x/x)^2$ is not smooth, but we can still apply Theorem 2 to Φ since the condition $\widehat{\Phi}(ix) \ll |x|^{-2}$ is good enough in our proof and can replace the smoothness.

Özlük obtains a similar lower bound but for *all* Dirichlet L-functions rather than just the primitive L-functions. This yields an over-count and as a result Özlük's method is only capable of delivering a proportion of $0.8688\ldots$ simple zeros. This should be compared with our proportion $11/12 = 0.917\ldots$. We will explain the number $0.8688\ldots$ in Section 6.

Following Özlük we consider the q analogue of the Pair Correlation Function, which is defined as

$$F_\Phi(Q^\alpha; W) = \frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \left| \sum_{\gamma_\chi} \widehat{\Phi}\left(i\gamma_\chi\right) Q^{i\gamma_\chi \alpha} \right|^2.$$

Our main result is the following.

Theorem 2. *Assume GRH. Let $\epsilon > 0$ and A be defined as in (1.1). Then*

$$\begin{aligned} F_\Phi(Q^\alpha; W) =& (1+o(1)) \left(f(\alpha) + \Phi(Q^{-|\alpha|})^2 \log Q \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\widehat{\Phi}(ix)\right|^2 dx \right)^{-1} \right) \\ &+ O(\Phi(Q^{-|\alpha|}) \sqrt{f(\alpha) \log Q}) \end{aligned}$$

holds uniformly for $|\alpha| \leq 2 - \epsilon$ as $Q \to \infty$, where

$$f(\alpha) := \begin{cases} |\alpha| & \text{for } |\alpha| \leq 1 \\ 1 & \text{for } |\alpha| > 1. \end{cases} \tag{1.2}$$

Since primitive Dirichlet L-functions form a unitary family, we conjecture that for $\alpha \geq 2$ the same asymptotic formula continues to hold. We obtain Theorem 2 by applying the asymptotic large sieve. The proof of Theorem 2 starts with the explicit formula, for $X \geq 1$

$$\sum_{\gamma} \widehat{\Phi}(i\gamma) X^{i\gamma} = E(\chi)\widehat{\Phi}(\tfrac{1}{2})X^{1/2} - \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \Phi\left(\frac{n}{X}\right) + \Phi\left(\frac{1}{X}\right) \log\frac{q}{\pi} + O\big(\min(X^{1/2}, X^{-1/2}\log q \log(1+X))\big), \quad (1.3)$$

where $E(\chi) = 0$ or 1 according as $\chi \neq \chi_0$ or $\chi = \chi_0$ and where $\frac{1}{2} + i\gamma$ ranges over non-trivial zeros of $L(s,\chi)$. The term $\Phi(1/X)\log q/\pi$ contributes only when X is small. Thus, Theorem 2 is essentially equivalent to the following Proposition.

Proposition 1. *Assume GRH. Let $\epsilon > 0$ and $X = Q^{\alpha}$. Then*

$$\sum_{q} \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \left| \sum_{n \leq X} \frac{\Lambda(n)\chi(n)\Phi\,(n/X)}{n^{1/2}} \right|^2 \sim f(\alpha) N_{\Phi}(Q)$$

uniformly for $|\alpha| \leq 2 - \epsilon$ as $Q \to \infty$, where $f(\alpha)$ is defined in (1.2).

The deduction of Theorem 1 from Theorem 2 can be found in Özlük's paper in section 6, but we reproduce it in Section 5 for completeness. The remainder of this paper is devoted to the proof of Proposition 1.

The bulk of the proof of Proposition 1 is devoted to the estimation of the contribution of the off-diagonal terms. When $\alpha > 1$ we extract an additional main term from the terms with $|m - n| \asymp Q$. Indeed it is explained in [1]: "Besides the primary terms of the diagonal, a secondary source for contribution to the main term is not so obvious as the diagonal one; it rests in narrow strips parallel to the diagonal. A substantial contribution may come out of the terms $a_m b_n F(m,n)$ with $|m - n| \asymp Q$, but not strips of much smaller width".

2. Lemmas

As announced in the introduction we start out by evaluating asymptotically $N_{\Phi}(Q)$.

Lemma 1. *Assume GRH. We have,*

$$N_{\Phi}(Q) \sim \frac{A}{2\pi} Q \log Q \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx$$

as $Q \to \infty$, with

$$A = \widehat{W}(1) \prod_{p} \left(1 - \frac{1}{p^2} - \frac{1}{p^3}\right).$$

Proof. Let $N(\chi, T)$ denote the number of zeros of $L(s,\chi)$ in the rectangle $0 < \sigma < 1$ and $-T \leq t \leq T$. It is a standard fact (see [7]) that if the conductor of χ is q, then

$$N(\chi, T) = \frac{T}{\pi} \log\frac{qT}{2\pi e} + O\left(\frac{\log(qT)}{\log\log(qT+3)}\right).$$

uniformly in $qT > 1$. Integrating by parts we find

$$\sum_{\gamma_\chi} |\widehat{\Phi}(i\gamma_\chi)|^2 = \int_0^{\infty} |\widehat{\Phi}(it)|^2 dN(\chi, t) = \frac{1}{\pi}\log q \int_0^{\infty} |\widehat{\Phi}(ix)|^2 dx + O\Big(\frac{\log q}{\log\log q}\Big)$$

$$= \frac{\log q}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx + O\bigg(\frac{\log q}{\log \log q} \bigg)$$

and

$$N_\Phi(Q) := \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\gamma_\chi} |\widehat{\Phi}(i\gamma_\chi)|^2$$
$$= \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx \cdot \sum_q \frac{W(q/Q)}{\varphi(q)} \cdot \frac{\log q}{2\pi} \cdot \varphi^*(q) + O\left(\frac{Q \log Q}{\log \log Q} \right)$$

where $\varphi^*(q) = \sum_{cd=q} \varphi(d)\mu(c)$ is the number of primitive characters modulo q. Since W is compactly supported in $(1,2)$ we have $\log q = \log Q + O(1)$ in the summation. Therefore,

$$N_\Phi(Q) \sim \frac{\log Q}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx \sum_q W(q/Q) \frac{\varphi^*(q)}{\varphi(q)}. \tag{2.1}$$

Since φ^* and φ are multiplicative, we have

$$\sum_q \frac{\varphi^*(q)}{\varphi(q) q^s} = \prod_p \left(1 + \frac{\varphi^*(p)}{\varphi(p) p^s} + \frac{\varphi^*(p^2)}{\varphi(p^2) p^{2s}} + \cdots \right)$$
$$= \zeta(s) \prod_p \left(1 - \frac{1}{(p-1)p^s} + \frac{1}{(p-1)p^{2s}} - \frac{1}{p^{2s+1}} \right) = \zeta(s) g(s), \tag{2.2}$$

where $g(s)$ is absolutely convergent for $\mathrm{Re}(s) > 0$ and bounded on $\mathrm{Re}(s) \geq \varepsilon$ for any $\varepsilon > 0$. Using the Mellin inversion formula,

$$W(x) = \frac{1}{2\pi i} \int_{(c)} \widehat{W}(s) x^{-s} ds, \quad c > 1$$

and (2.2), we obtain that

$$\sum_q W(q/Q) \frac{\varphi^*(q)}{\varphi(q)} = \frac{1}{2\pi i} \int_{(c)} \widehat{W}(s) \zeta(s) g(s) Q^s ds = \widehat{W}(1) g(1) Q + O(Q^\varepsilon) \tag{2.3}$$

by shifting the contour to $\mathrm{Re}(s) = \varepsilon$. Combining (2.1) and (2.3) we conclude that

$$N_\Phi(Q) \sim \frac{Q \log Q}{2\pi} \widehat{W}(1) g(1) \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx = \frac{AQ \log Q}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx.$$

□

The next four lemmas correspond to estimates of various types of prime sums. The proofs are standard, but we present them here for a completeness.

Lemma 2. *Assume GRH for $L(s, \chi^2)$. Then we have*

$$\sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \Phi\left(\frac{n}{X} \right) = \sum_p \frac{\Lambda(p)\chi(p)}{\sqrt{p}} \Phi\left(\frac{p}{X} \right) + O(1). \tag{2.4}$$

Proof. By splitting the sum into three cases $n = p$, $n = p^2$ and $n = p^k$ with $k > 2$, we get

$$\sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \Phi\left(\frac{n}{X} \right) = \sum_p \frac{\Lambda(p)\chi(p)}{\sqrt{p}} \Phi\left(\frac{p}{X} \right) + \sum_p \frac{\Lambda(p^2)\chi(p^2)}{p} \Phi\left(\frac{p^2}{X} \right) + O(1).$$

Since Φ has a compact support in (a,b) for some $0<a<b$, the last sum is

$$\begin{aligned}\left|\sum_p \frac{\Lambda(p^2)\chi(p^2)}{p}\Phi\left(\frac{p^2}{X}\right)\right| &\ll \sum_{\sqrt{aX}<p<\sqrt{bX}} \frac{\log p}{p} \\ &= \log\sqrt{bX}-\log\sqrt{aX}+O(1) \\ &= O(1).\end{aligned}$$

Hence, we prove the lemma. $\square$

Lemma 3. *As $X\to\infty$,*

$$\sum_p \frac{\log^2 p}{p}\Phi^2\left(\frac{p}{X}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(it)|^2dt\log X+O(1).$$

Proof. Note that

$$\sum_p \frac{\log^2 p}{p}\Phi^2\left(\frac{p}{X}\right)=\sum_n \frac{\Lambda(n)\log n}{n}\Phi^2\left(\frac{n}{X}\right)+O(1).$$

By the Mellin inversion we have

$$\sum_n \frac{\Lambda(n)\log n}{n}\Phi^2\left(\frac{n}{X}\right)=\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\widehat{\Phi}(s_1)\widehat{\Phi}(s_2)\sum_n\frac{\Lambda(n)\log n}{n^{1+s_1+s_2}}X^{s_1+s_2}ds_2ds_1$$

for $c_1,c_2>0$. Since $\sum_n \Lambda(n)(\log n)n^{-s}=(\zeta'/\zeta)'(s)$, the above integral equals

$$\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(c_2)}\widehat{\Phi}(s_1)\widehat{\Phi}(s_2)(\zeta'/\zeta)'(1+s_1+s_2)X^{s_1+s_2}ds_2ds_1. \tag{2.5}$$

By shifting the contour integral to $\mathrm{Re}(s_2)=-c_1-\varepsilon$, we pick up a double pole at $s_2=-s_1$. Hence we have (2.5) equals

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(c_1)}\widehat{\Phi}(s_1)\widehat{\Phi}(-s_1)\log X\,ds_1+O(1)\\ &\quad+\frac{1}{(2\pi i)^2}\int_{(c_1)}\int_{(-c_1-\varepsilon)}\widehat{\Phi}(s_1)\widehat{\Phi}(s_2)(\zeta'/\zeta)'(1+s_1+s_2)X^{s_1+s_2}ds_2ds_1.\end{aligned}$$

For the first integral we shift the contour to $\mathrm{Re}(s_1)=0$ without passing any poles and it becomes

$$\frac{\log X}{2\pi}\int_{-\infty}^{\infty}\widehat{\Phi}(it)\widehat{\Phi}(-it)dt=\frac{\log X}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(it)|^2dt.$$

The double integral is easily bounded by $O(X^{-\varepsilon})$. $\square$

Lemma 4. *Assume GRH for $L(s,\Psi)$. If Ψ is a principal character, then*

$$\sum_p \frac{\Psi(p)\Phi\left(\frac{p}{X}\right)\log p}{\sqrt{p}}=\widehat{\Phi}\left(\frac{1}{2}\right)\sqrt{X}+O(Q^{\varepsilon}).$$

If Ψ is not a principal character, then

$$\sum_p \frac{\Psi(p)\Phi\left(\frac{p}{X}\right)\log p}{\sqrt{p}}\ll_{\varepsilon} Q^{\varepsilon}$$

for any $\varepsilon > 0$.

Proof. By the Mellin inversion of Φ, we have

$$\sum_p \frac{\Psi(p)\Phi\left(\frac{p}{X}\right)\log p}{\sqrt{p}} = \frac{1}{2\pi i}\int_{(c)} \widehat{\Phi}(s)X^s \sum_p \frac{\Psi(p)\log p}{p^{1/2+s}}ds.$$

The sum over p has an analytic continuation via

$$\sum_p \frac{\Psi(p)\log p}{p^{1/2+s}} = -\frac{L'}{L}(1/2+s,\Psi) + G(s),$$

where $G(s)$ is analytic in $\mathrm{Re}(s) > 0$ and is uniformly bounded for $\mathrm{Re}(s) \geq \epsilon > 0$. By moving the contour to $\mathrm{Re}(s) = \varepsilon$, we can prove the Lemma. □

Lemma 5. *For $|Re(z)| \leq \varepsilon$ and $Re(s) < 0$, we have*

$$\sum_p \frac{\log p \cdot \Phi(p/X)}{p^{1/2+z}} B_{-s}(p)R_{-s}(p) = \widehat{\Phi}(\frac{1}{2}-z)X^{1/2-z} + O(X^{2\varepsilon}\log(2+|z|)),$$

where

$$B_s(m) = \prod_{p|m}\left(1 - \frac{1}{p^{s+1}}\right),$$

$$R_s(m) = \prod_{p|m}\left(1 + \frac{1}{(p-1)p^{s+1}}\right)^{-1}.$$

Proof. By Mellin inversion of Φ, we have

$$\sum_p \frac{\log p \cdot \Phi(p/X)}{p^{1/2+z}} B_{-s}(p)R_{-s}(p) = \frac{1}{2\pi i}\int_{(c)} \widehat{\Phi}(w)X^w \sum_p \frac{\log p B_{-s}(p)R_{-s}(p)}{p^{1/2+z+w}}dw \tag{2.6}$$

for $c > 1/2 + \varepsilon$. Define a function $H(w,s)$ by

$$\begin{aligned} H(w,s) &:= \frac{\zeta'}{\zeta}(w) + \sum_p \frac{\log p B_{-s}(p)R_{-s}(p)}{p^w} \\ &= \frac{\zeta'}{\zeta}(w) + \sum_p \frac{\log p}{p^w}(1 - \frac{1}{p^{1-s}})(1 + \frac{1}{(p-1)p^{1-s}})^{-1}. \end{aligned}$$

If $\mathrm{Re}(s) < 0$, then $H(w,s)$ is an analytic function of w in $\mathrm{Re}(w) > 1/2$ and bounded on $\mathrm{Re}(w) \geq 1/2 + \epsilon' > 1/2$. Applying this identity to (2.6) and shifting the contour to 2ε, we have

$$\begin{aligned} \sum_p \frac{\log p \cdot \Phi(p/X)}{p^{1/2+z}} B_{-s}(p)R_{-s}(p) =& \frac{1}{2\pi i}\int_{(c)} \widehat{\Phi}(w)X^w(-\frac{\zeta'}{\zeta}(\frac{1}{2}+z+w) + H(\frac{1}{2}+z+w,s))dw \\ =& \widehat{\Phi}(\frac{1}{2}-z)X^{1/2-z} + O(X^{2\varepsilon}\log(2+|z|)). \end{aligned}$$

□

The next lemma can be proved by changing the sum to its Euler product. The proof is quite standard and we omit it.

Lemma 6. *Suppose that* $(a,m)=1$. *Then*

$$\sum_{(d,m)=1} \frac{1}{\varphi(ad)d^s} = \frac{1}{\varphi(a)}\zeta(1+s)K(s)B_s(m)R_s(a)R_s(m),$$

where B_s *and* R_s *are defined in Lemma 5 and*

$$K(s) = \prod_p \left(1 + \frac{1}{(p-1)p^{s+1}}\right).$$

3. Proof of Proposition 1

Proposition 1 is equivalent to

$$S = \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \left| \sum_p \frac{\log p \ \chi(p)}{\sqrt{p}} \Phi\left(\frac{p}{X}\right)\right|^2 \sim f(\alpha)N_\Phi(Q) \tag{3.1}$$

by Lemma 2. For notational convenience we let

$$a_p = \frac{\log p}{\sqrt{p}}\Phi\left(\frac{p}{X}\right) \tag{3.2}$$

and define

$$\Delta(p,r) = \sum_{\substack{q \\ (q,pr)=1}} \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \chi(p)\overline{\chi}(r) \tag{3.3}$$

for primes p and r, then we have

$$S = \sum_{p,r} a_p a_r \Delta(p,r) = \sum_p a_p^2 \Delta(p,p) + \sum_{\substack{p,r \\ p\neq r}} a_p a_r \Delta(p,r) = S_D + S_N,$$

where S_D is the sum of diagonal terms and S_N is the sum of non-diagonal terms.

3.1. **The diagonal term** S_D**.** By (3.3) and the Mellin inversion, we obtain that

$$\Delta(p,p) = \sum_{\substack{q \\ (q,p)=1}} \frac{\varphi^*(q)}{\varphi(q)} W\left(\frac{q}{Q}\right) = \frac{1}{2\pi i}\int_{(c)} \widehat{W}(s) \left(\sum_{\substack{q \\ (q,p)=1}} \frac{\varphi^*(q)}{\varphi(q)q^s}\right) Q^s \, ds$$

for $c > 1$. By applying (2.2) and then shifting the contour to the line $\mathrm{Re}(s) = \varepsilon > 0$, we have

$$\begin{aligned}\Delta(p,p) &= \frac{1}{2\pi i}\int_{(c)} \widehat{W}(s)\zeta(s)g(s)\left(1-\frac{1}{p^s}\right)\left(1 - \frac{1}{(p-1)p^s} + \frac{1}{(p-1)p^{2s}} - \frac{1}{p^{2s+1}}\right)^{-1} Q^s \, ds \\ &= \widehat{W}(1)g(1)\left(1-\frac{1}{p}\right)\left(1 - \frac{1}{(p-1)p} + \frac{1}{(p-1)p^2} - \frac{1}{p^3}\right)^{-1} Q + O(Q^\varepsilon).\end{aligned}$$

By this equation and Lemma 3, we then obtain

$$\begin{aligned} S_D =& \widehat{W}(1)g(1)Q\sum_p \frac{\log^2 p}{p}\Phi^2\left(\frac{p}{X}\right) + O(Q) \\ =& AQ\log X \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{\Phi}(it)|^2 dt + O(Q),\end{aligned}$$

where A is defined as in (1.1).

3.2. **The non-diagonal term S_N.** We first observe that

$$\begin{aligned}\Delta(p,r) &= \sum_{\substack{q\\(q,pr)=1}} \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \chi(p)\overline{\chi}(r)\\ &= \sum_{\substack{q\\(q,pr)=1}} \frac{W(q/Q)}{\varphi(q)} \sum_{\substack{d|q\\d|p-r}} \varphi(d)\mu\left(\frac{q}{d}\right)\\ &= \sum_{\substack{d\\d|p-r}} \varphi(d) \sum_{\substack{c\\(cd,pr)=1}} \frac{W(cd/Q)\mu(c)}{\varphi(cd)}\end{aligned} \tag{3.4}$$

for primes p and r. We want to replace the condition $d|p-r$ by the character sum using

$$\sum_{\Psi \,(\mathrm{mod}\ d)} \Psi(p)\overline{\Psi}(r) = \begin{cases}\varphi(d) & \text{for } d|p-r, (pr,d)=1\\ 0 & \text{otherwise .}\end{cases}$$

However, it is not effective in our application when d is large. Hence we introduce a new parameter C and we split the above sum according to $c \le C$ or $c > C$ in order to handle the condition $d|p-r$ differently when d is large. Thus we define

$$\begin{aligned}U(p,r) &= \sum_{\substack{d\\d|p-r}} \varphi(d) \sum_{\substack{c>C\\(cd,pr)=1}} \frac{W(cd/Q)\mu(c)}{\varphi(cd)},\\ L(p,r) &= \sum_{\substack{d\\d|p-r}} \varphi(d) \sum_{\substack{c\le C\\(cd,pr)=1}} \frac{W(cd/Q)\mu(c)}{\varphi(cd)}\end{aligned} \tag{3.5}$$

so that

$$\Delta(p,r) = U(p,r) + L(p,r).$$

Then by calculating the sums

$$\begin{aligned}S_U &:= \sum_{p\ne r} a_p a_r U(p,r),\\ S_L &:= \sum_{p\ne r} a_p a_r L(p,r),\end{aligned}$$

we can evaluate the sum

$$S_N = S_U + S_L. \tag{3.6}$$

Since W is supported in $(1,2)$, we have $cd \asymp Q$. If $c > C$ then $d \ll Q/C$ and replacing the condition $d|p-r$ by a character sum modulo d in $U(p,r)$ is efficient and leads to good estimates for S_U. We perform this computation in section 3.3.

On the other hand, in the case $c \le C$, we have large $d \gg Q/C$ and the above method using modulo d character sums does not work. So we write $de = |p-r|$, we replace the condition $d|p-r$ by $e|p-r$ and we eliminate d from our sum by expressing d as $|p-r|/e$. Now we have $e \ll XC/Q$ which is small enough, so that the modulo e character sums replacing $e|p-r$

works well. This allows us to resume our argument in the case of the sum S_L. We consider the sum S_L in section 3.4. For a technical reason the above idea will be modified slightly.

3.3. **Evaluating** S_U**.** We first consider the sum $U(p,r)$ defined in (3.5). Replacing the condition $d|p-r$ by a character sum, we have

$$U(p,r)=\sum_{c>C}\mu(c)\sum_{\substack{d\\(cd,pr)=1}}\frac{W(cd/Q)}{\varphi(cd)}\sum_{\Psi\ (\mathrm{mod}\ d)}\Psi(p)\overline{\Psi}(r).$$

We denote the sum corresponding to $\Psi=\Psi_0$ in $U(p,r)$ by

$$U_0(p,r)=\sum_{c>C}\mu(c)\sum_{\substack{d\\(cd,pr)=1}}\frac{W(cd/Q)}{\varphi(cd)}$$

and the others by

$$U_E(p,r)=\sum_{c>C}\mu(c)\sum_{\substack{d\\(cd,pr)=1}}\frac{W(cd/Q)}{\varphi(cd)}\sum_{\substack{\Psi\ (\mathrm{mod}\ d)\\\Psi\neq\Psi_0}}\Psi(p)\overline{\Psi}(r).$$

Then $U(p,r)=U_0(p,r)+U_E(p,r)$ and $S_U=S_{U_0}+S_{U_E}$, where $S_{U_0}:=\sum_{p\neq r}a_pa_rU_0(p,r)$ and $S_{U_E}:=\sum_{p\neq r}a_pa_rU_E(p,r)$.

We consider the sum S_{U_0}. Since $\sum_{c|k}\mu(c)=1$ for $k=1$ and 0 for $k>1$, we have

$$\begin{aligned}U_0(p,r)&=\sum_{c>C}\mu(c)\sum_{\substack{d\\(cd,pr)=1}}\frac{W(cd/Q)}{\varphi(cd)}=\sum_{\substack{k\\(k,pr)=1}}\frac{W(k/Q)}{\varphi(k)}\sum_{\substack{c|k\\c>C}}\mu(c)\\&=W(1/Q)-\sum_{\substack{k\\(k,pr)=1}}\frac{W(k/Q)}{\varphi(k)}\sum_{\substack{c|k\\c\le C}}\mu(c)=-\sum_{c\le C}\mu(c)\sum_{\substack{d\\(cd,pr)=1}}\frac{W(cd/Q)}{\varphi(cd)}.\end{aligned}$$

By Mellin inversion, we have that

$$U_0(p,r)=-\sum_{\substack{c\le C\\(c,pr)=1}}\mu(c)\frac{1}{2\pi i}\int_{(2)}\widehat{W}(s)\frac{Q^s}{c^s}\sum_{\substack{d\\(d,pr)=1}}\frac{1}{\varphi(cd)d^s}ds.$$

By Lemma 6, we obtain that

$$U_0(p,r)=-\sum_{\substack{c\le C\\(c,pr)=1}}\frac{\mu(c)}{\varphi(c)}\frac{1}{2\pi i}\int_{(2)}\widehat{W}(s)\frac{Q^s}{c^s}\zeta(1+s)K(s)B_s(pr)R_s(c)R_s(pr)\,ds.$$

We move the contour integral to $\mathrm{Re}(s)=-1+\varepsilon$ and pick a simple pole at $s=0$. Then

$$U_0(p,r)=-\widehat{W}(0)K(0)B_0(pr)R_0(pr)\sum_{\substack{c\le C\\(c,pr)=1}}\frac{\mu(c)R_0(c)}{\varphi(c)}+O\left(\frac{C}{Q}Q^\varepsilon\right)$$

and so

$$S_{U_0}=-\widehat{W}(0)K(0)\sum_{\substack{p,r\\p\neq r}}a_pa_rB_0(pr)R_0(pr)\sum_{\substack{c\le C\\(c,pr)=1}}\frac{\mu(c)R_0(c)}{\varphi(c)}+O\left(|\sum_p a_p|^2\frac{C}{Q}Q^\varepsilon\right).\tag{3.7}$$

Now we evaluate the main term of S_{U_0}. The condition $(c, pr) = 1$ can be disregarded with an additional error term C^ε. Now the sum over p and r is

$$\sum_{\substack{p,r \\ p \neq r}} a_p a_r B_0(p) B_0(r) R_0(p) R_0(r) = \sum_{p,r} a_p a_r B_0(p) B_0(r) R_0(p) R_0(r) - \sum_p a_p^2 B_0(p)^2 R_0(p)^2$$

$$= \left(\sum_p a_p B_0(p) R_0(p) \right)^2 + O(X^\varepsilon) \tag{3.8}$$

by adding and subtracting diagonal terms. Similarly to Lemma 6, we can obtain

$$\sum_p a_p B_0(p) R_0(p) = \sqrt{X} \widehat{\Phi} \left(\frac{1}{2} \right) + O(X^\varepsilon). \tag{3.9}$$

Therefore, since the sum in the error term of (3.7) is $\sum_p a_p \ll \sqrt{X}$, we have

$$S_{U_0} = -\widehat{\Phi}(1/2)^2 \widehat{W}(0) K(0) X \sum_{c \leq C} \frac{\mu(c) R_0(c)}{\varphi(c)} + O\left(X^{\frac{1}{2}+\varepsilon} + \frac{XC}{Q} Q^\varepsilon \right) \tag{3.10}$$

by (3.7)–(3.9).

The next lemma shows the contribution of U_E is small, so that we can conclude

$$S_U = S_{U_0} + S_{U_E} = -\widehat{\Phi}(1/2)^2 \widehat{W}(0) K(0) X \sum_{c \leq C} \frac{\mu(c) R_0(c)}{\varphi(c)} + O\left(X^{\frac{1}{2}+\varepsilon} + \frac{XC}{Q} Q^\varepsilon + \frac{Q^{1+\varepsilon}}{C} \right) \tag{3.11}$$

by (3.10) and Lemma 7.

Lemma 7. *Assume GRH. We have*

$$S_{U_E} = \sum_{\substack{p,r \\ p \neq r}} a_p a_r U_E(p, r) \ll \frac{Q^{1+\varepsilon}}{C}. \tag{3.12}$$

Proof. We write

$$S_{U_E} = \sum_{p,r} \frac{\log p \log r}{\sqrt{pr}} \Phi\left(\frac{p}{X} \right) \Phi\left(\frac{r}{X} \right) U_E(p, r) - \sum_p \frac{\log^2 p}{p} \Phi^2\left(\frac{p}{X} \right) U_E(p, p). \tag{3.13}$$

By Lemma 4, the first sum in (3.13) is

$$\sum_{c > C} \mu(c) \sum_d \frac{W(cd/Q)}{\varphi(cd)} \sum_{\substack{\Psi \ (\mathrm{mod}\ d) \\ \Psi \neq \Psi_0}} \left| \sum_{p \nmid cd} \frac{\Psi(p) \Phi\left(\frac{p}{X}\right) \log p}{\sqrt{p}} \right|^2$$

$$\ll \sum_{c > C} \frac{1}{\varphi(c)} \sum_{d \leq \frac{2Q}{c}} \frac{1}{\varphi(d)} \varphi(d) Q^\varepsilon \ll \frac{Q^{1+\varepsilon}}{C}.$$

The second sum in (3.13) is also bounded by

$$\log^2 X \sum_{c > C} \frac{1}{\varphi(c)} \sum_{d \leq \frac{2Q}{c}} \frac{1}{\varphi(d)} \varphi(d) \ll \frac{Q^{1+\varepsilon}}{C}$$

in a similar way. These prove the lemma. □

3.4. **Evaluating S_L.** Recall that

$$L(p,r)=\sum_{c\le C}\mu(c)\sum_{\substack{d|p-r\\(cd,pr)=1}}\frac{W(cd/Q)}{\varphi(cd)}\varphi(d)$$

for primes p and r. For distinct prime p,r, the condition $d|p-r$ implies $(d,pr)=1$. So we can erase the condition $(d,pr)=1$, getting

$$L(p,r)=\sum_{\substack{c\le C\\(c,pr)=1}}\mu(c)\sum_{d|p-r}\frac{W(cd/Q)}{\varphi(cd)}\varphi(d).$$

Using the identity

$$\frac{\varphi(d)}{\varphi(cd)}=\frac{1}{\varphi(c)}\prod_{p|(d,c)}\left(1-\frac{1}{p}\right)=\frac{1}{\varphi(c)}\sum_{a|c,a|d}\frac{\mu(a)}{a},$$

we have

$$L(p,r)=\sum_{\substack{c\le C\\(c,pr)=1}}\frac{\mu(c)}{\varphi(c)}\sum_{d|p-r}W\left(\frac{cd}{Q}\right)\sum_{a|c,a|d}\frac{\mu(a)}{a}=\sum_{\substack{c\le C,a|c\\(c,pr)=1}}\frac{\mu(a)\mu(c)}{a\varphi(c)}\sum_{ad|p-r}W\left(\frac{acd}{Q}\right).$$

Letting $ade=|p-r|$, we change the sum over d to the sum over e as follows

$$L(p,r)=\sum_{\substack{c\le C,a|c\\(c,pr)=1}}\frac{\mu(a)\mu(c)}{a\varphi(c)}\sum_{ae|p-r}W\left(\frac{c|p-r|}{Qe}\right).$$

Now we can replace the condition $ae|p-r$ by a character sum modulo ae, getting

$$L(p,r)=\sum_{\substack{c\le C,a|c\\(c,pr)=1}}\frac{\mu(a)\mu(c)}{a\varphi(c)}\sum_{e}W\left(\frac{c|p-r|}{Qe}\right)\frac{1}{\varphi(ae)}\sum_{\Psi\,(\mathrm{mod}\ ae)}\Psi(p)\overline{\Psi}(r).$$

Similarly to the sum $U(p,r)$, we split the sum $L(p,r)$ into two parts $L_0(p,r)$ and $L_E(p,r)$, where $L_0(p,r)$ is the sum coming from the principal character $\Psi=\Psi_0$ and $L_E(p,r)$ is the sum coming from the remaining non-principal characters.

We compute the contribution from $L_0(p,r)$. Define

$$S_{L_0}:=\sum_{\substack{p,r\\p\ne r}}a_pa_rL_0(p,r),$$

where $a_p=\frac{\log p}{\sqrt{p}}\Phi(p/X)$. By the Mellin inversion, we get

$$S_{L_0}=\sum_{\substack{p,r\\p\ne r}}a_pa_r\sum_{\substack{a,c,e\\a|c,c\le C\\(ce,pr)=1}}\frac{\mu(a)}{a\varphi(ae)}\frac{\mu(c)}{\varphi(c)}W\left(\frac{c|p-r|}{eQ}\right)$$

$$= \frac{1}{2\pi i} \int_{(-\varepsilon)} \widehat{W}(s) \sum_{\substack{p,r \\ p\neq r}} a_p a_r \sum_{\substack{a,c,e \\ a|c, c\le C \\ (ce,pr)=1}} \frac{\mu(a)}{a\varphi(ae)} \frac{\mu(c)}{\varphi(c)} \left(\frac{c|p-r|}{eQ} \right)^{-s} ds.$$

In order to separate the sums of p and r, we need the following Mellin transform

$$|p-r|^{-s} = \frac{1}{2\pi i} \int_{(\delta)} \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} \big(p^{z-s} r^{-z} + r^{z-s} p^{-z}\big) dz$$

for $p \neq r$, $\delta > 0$ and $\text{Re}(s) < 0$. Note that the (absolute) convergence of the z integral is ensured by the fact that the Gamma factors decay like $|z|^{-1+\text{Re}(s)}$. Using the above identity we have

$$S_{L_0} = \frac{2}{(2\pi i)^2} \int_{(-\varepsilon)} \int_{(\delta)} \widehat{W}(s) Q^s \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} \cdot \sum_{\substack{p,r \\ p\neq r}} \frac{a_p}{p^{s-z}} \frac{a_r}{r^z} \sum_{\substack{c\le C \\ (c,pr)=1}} \frac{\mu(c)}{c^s \varphi(c)} \sum_{a|c} \frac{\mu(a)}{a} \sum_{(e,pr)=1} \frac{e^s}{\varphi(ae)} \, dz \, ds$$

for $\varepsilon > 0$. Note that the sums over a, c, p and r have only finitely many terms, so that there are no convergence issues on them. The sum over e is

$$\sum_{(e,pr)=1} \frac{e^s}{\varphi(ae)} = \frac{1}{\varphi(a)} \zeta(1-s) K(-s) B_{-s}(pr) R_{-s}(a) R_{-s}(pr)$$

by Lemma 6, where the functions K, B_s and R_s are defined in Lemmas 5 and 6. The sum over a is

$$\sum_{a|c} \frac{\mu(a) R_{-s}(a)}{a\varphi(a)} = \prod_{\ell | c} \left(1 - \frac{R_{-s}(\ell)}{\ell(\ell-1)} \right).$$

Hence, we deduce

$$S_{L_0} = \frac{2}{(2\pi i)^2} \int_{(-2\varepsilon)} \int_{(\varepsilon)} \widehat{W}(s) Q^s \zeta(1-s) K(-s) \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} \cdot \sum_{\substack{p,r \\ p\neq r}} \frac{a_p}{p^{s-z}} \frac{a_r}{r^z} B_{-s}(pr) R_{-s}(pr) \sum_{\substack{c\le C \\ (c,pr)=1}} \frac{\mu(c)}{c^s \varphi(c)} \prod_{\ell|c} \left(1 - \frac{R_{-s}(\ell)}{\ell(\ell-1)} \right) dz \, ds.$$

We can remove the condition $(pr, c) = 1$ with an additional error $O(C^{2\varepsilon} Q^{-2\varepsilon} \sqrt{X})$. The double sum over primes p and r is

$$\begin{aligned}
&\sum_{p\neq r} \frac{a_p}{p^{s-z}} \frac{a_r}{r^z} B_{-s}(p) B_{-s}(r) R_{-s}(p) R_{-s}(r) \\
&= \sum_p \frac{a_p}{p^{s-z}} B_{-s}(p) R_{-s}(p) \cdot \sum_r \frac{a_r}{r^z} B_{-s}(r) R_{-s}(r) - \sum_p \frac{a_p^2}{p^s} B_{-s}(p)^2 R_{-s}(p)^2 \\
&= \left(\widehat{\Phi}(\frac{1}{2} - s + z) X^{1/2-s+z} + O(X^{4\varepsilon} \log(2+|s-z|)) \right) \\
&\quad \times \left(\widehat{\Phi}(\frac{1}{2} - z) X^{1/2-z} + O(X^{\varepsilon} \log(2+|z|)) \right) + O(X^{\varepsilon})
\end{aligned}$$

by Lemma 5. The big-oh terms only contribute $O(Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon})$ to S_{L_0}, so that

$$\begin{aligned}S_{L_0} =&\frac{2}{(2\pi i)^2}\int_{(-2\varepsilon)}\int_{(\varepsilon)}\widehat{W}(s)Q^s\zeta(1-s)K(-s)\frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)}\widehat{\Phi}(\frac{1}{2}-s+z)\widehat{\Phi}(\frac{1}{2}-z)X^{1-s}\\ &\cdot\sum_{c\le C}\frac{\mu(c)}{c^s\varphi(c)}\prod_{\ell|c}\left(1-\frac{R_{-s}(\ell)}{\ell(\ell-1)}\right)\,dz\,ds+O(Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon}).\end{aligned}$$

To evaluate the integral, we split into two cases.

Case 1: $X = Q^\alpha$, where $1<\alpha<2$. In this case, we shift the contour of s to $\mathrm{Re}(s) = 1+\varepsilon$ and get

$$\begin{aligned}S_{L_0} &= -(\text{Residue at } s=0) - (\text{Residue at } s=1)\\ &\quad+\frac{2}{(2\pi i)^2}\int_{(1+\varepsilon)}\int_{(\varepsilon)}\widehat{W}(s)Q^s\zeta(1-s)K(-s)\frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)}\widehat{\Phi}(\frac{1}{2}-s+z)\widehat{\Phi}(\frac{1}{2}-z)X^{1-s}\\ &\qquad\cdot\sum_{c\le C}\frac{\mu(c)}{c^s\varphi(c)}\prod_{\ell|c}\left(1-\frac{R_{-s}(\ell)}{\ell(\ell-1)}\right)\,dz\,ds+O(Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon})\\ &= -(\text{Residue at } s=0) - (\text{Residue at } s=1)+O(Q^{1+\varepsilon}X^{-\varepsilon}+Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon}).\end{aligned}$$

Three functions in the integrand have poles at $s=0$ or $s=1$. $\zeta(1-s)$ has a simple pole at 0, $\Gamma(1-s)$ has a simple pole at $s=1$ and $K(-s)$ has a simple pole at $s=1$, since

$$K(-s)=\prod_\ell\left(1+\frac{1}{(\ell-1)\ell^{1-s}}\right)=\zeta(2-s)\prod_\ell\left(1+\frac{1}{(\ell-1)\ell^{2-s}}-\frac{1}{(\ell-1)\ell^{3-2s}}\right).$$

Hence, the residue at the simple pole $s=0$ is

$$\begin{aligned}&-\frac{1}{\pi i}\int_{(\varepsilon)}\widehat{W}(0)K(0)\frac{\Gamma(z)}{\Gamma(1+z)}\widehat{\Phi}(\frac{1}{2}+z)\widehat{\Phi}(\frac{1}{2}-z)X\sum_{c\le C}\frac{\mu(c)}{\varphi(c)}\prod_{\ell|c}\left(1-\frac{R_0(\ell)}{\ell(\ell-1)}\right)dz\\ &=-\frac{1}{\pi i}\int_{(\varepsilon)}\frac{1}{z}\widehat{\Phi}(\frac{1}{2}+z)\widehat{\Phi}(\frac{1}{2}-z)dz\cdot\widehat{W}(0)K(0)X\sum_{c\le C}\frac{\mu(c)R_0(c)}{\varphi(c)}\\ &=-\widehat{\Phi}(1/2)^2\widehat{W}(0)K(0)X\sum_{c\le C}\frac{\mu(c)R_0(c)}{\varphi(c)}.\end{aligned}$$

The residue at the double pole $s=1$ is

$$\begin{aligned}&\frac{1}{\pi i}\int_{(\varepsilon)}\widehat{\Phi}(-\frac{1}{2}+z)\widehat{\Phi}(\frac{1}{2}-z)dz\cdot\zeta(0)\widehat{W}(1)Q(\log\frac{Q}{X}+O(1))\sum_{c\le C}\frac{\mu(c)}{c\varphi(c)}\prod_{\ell|c}\left(1-\frac{R_{-1}(\ell)}{\ell(\ell-1)}\right)\\ &=-\frac{1}{2\pi i}\int_{(1/2)}\widehat{\Phi}(-\frac{1}{2}+z)\widehat{\Phi}(\frac{1}{2}-z)dz\cdot B\widehat{W}(1)Q\log\frac{Q}{X}+O(Q)\\ &=-\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(it)|^2dt\cdot B\widehat{W}(1)Q\log\frac{Q}{X}+O(Q),\end{aligned}$$

where

$$B=\sum_c\frac{\mu(c)}{c\varphi(c)}\prod_{\ell|c}\left(1-\frac{R_{-1}(\ell)}{\ell(\ell-1)}\right)=\prod_p\left(1-\frac{1}{p^2}-\frac{1}{p^3}\right).$$

By (1.1) and the above, the residue at $s = 1$ is

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(it)|^2dt \cdot AQ(\log\frac{Q}{X} + O(1)).$$

Combining all together, we get

$$S_{L_0} = \widehat{\Phi}(1/2)^2\widehat{W}(0)K(0)X\sum_{c\le C}\frac{\mu(c)R_0(c)}{\varphi(c)} + \frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(it)|^2dt \cdot AQ(\log\frac{Q}{X} + O(1)) + O(Q + Q^{-2\epsilon}C^{2\epsilon}X^{1/2+4\epsilon}) \quad (3.14)$$

for $1 < \alpha < 2$ with $X = Q^\alpha$. Note that the first term in S_{L_0} is cancelled with the main term of S_{U_0} in Equation (3.11).

Case 2: $X = Q^\alpha$, where $0 \le \alpha \le 1$. In this case, we shift the contour of s to $\mathrm{Re}(s) = \varepsilon$ and get

$$\begin{aligned}S_{L_0} &= -(\text{Residue at } s = 0)\\ &\quad + \frac{2}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)}\widehat{W}(s)Q^s\zeta(1-s)K(-s)\frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)}\widehat{\Phi}(\frac{1}{2}-s+z)\widehat{\Phi}(\frac{1}{2}-z)X^{1-s}\\ &\quad \cdot\sum_{c\le C}\frac{\mu(c)}{c^s\varphi(c)}\prod_{\ell|c}\left(1-\frac{R_{-s}(\ell)}{\ell(\ell-1)}\right)\,dz\,ds + O(Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon})\\ &= -(\text{Residue at } s = 0) + O(Q^\varepsilon X^{1-\varepsilon} + Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon}).\end{aligned}$$

Since $0 \le \alpha \le 1$, we obtain that

$$Q^\varepsilon X^{1-\varepsilon} = Q^{\alpha+(1-\alpha)\varepsilon} \ll Q.$$

By the same argument as in Case 1, we get that

$$S_{L_0} = \widehat{\Phi}(1/2)^2\widehat{W}(0)K(0)X\sum_{c\le C}\frac{\mu(c)R_0(c)}{\varphi(c)} + O(Q + Q^{-2\varepsilon}C^{2\varepsilon}X^{1/2+4\varepsilon}),$$

and the first term is cancelled with the main term of S_{U_0}.

The contribution from $L_E(p,r)$ is small by the following Lemma.

Lemma 8. *We have*

$$S_{L_E} := \sum_{\substack{p,r\\ p\ne r}}\frac{\log p\log r}{\sqrt{pr}}\Phi\left(\frac{p}{X}\right)\Phi\left(\frac{r}{X}\right)L_E(p,r) \ll \frac{X^{1+\varepsilon}C^{1+\varepsilon}}{Q}. \quad (3.15)$$

Proof. Let a_p be defined as in (3.2). We have that

$$S_{L_E} = \sum_{\substack{a,c,e\\ ac\le C}}\frac{\mu(a)\mu(ac)}{a\varphi(ac)\varphi(ae)}\sum_{\substack{\Psi \pmod{ae}\\ \Psi\ne\Psi_0}}\sum_{\substack{p,r\\ p\ne r,(c,pr)=1}}a_pa_r\Psi(p)\overline{\Psi}(r)W\left(\frac{ac|p-r|}{Qe}\right).$$

Since W is supported in $(1,2)$, $\frac{ac|p-r|}{Qe} \geq 1$ and $e \leq \frac{ac|p-r|}{Q} \leq \frac{acX}{Q}$. Proceeding similarly to S_{L_0} we obtain

$$S_{L_E} = \frac{1}{2\pi i}\int_{(-\varepsilon)} \widehat{W}(s)Q^s \sum_{\substack{a,c,e\\ ac\leq C, e\leq acX/Q}} \frac{\mu(a)\mu(ac)e^s}{a^{1+s}c^s\varphi(ac)\varphi(ae)} \sum_{\substack{\Psi \ (\mathrm{mod}\ ae)\\ \Psi\neq\Psi_0}} \sum_{\substack{p,r\\ p\neq r\\ (c,pr)=1}} a_p a_r \Psi(p)\overline{\Psi}(r)|p-r|^{-s}ds$$

$$= \frac{2}{(2\pi i)^2}\int_{(-\varepsilon)}\int_{(\varepsilon)} \widehat{W}(s)Q^s \frac{\Gamma(1-s)\Gamma(z)}{\Gamma(1-s+z)} \sum_{\substack{a,c,e\\ ac\leq C, e\leq acX/Q}} \frac{\mu(a)\mu(ac)e^s}{a^{1+s}c^s\varphi(ac)\varphi(ae)} \times$$

$$\sum_{\substack{\Psi \ (\mathrm{mod}\ ae)\\ \Psi\neq\Psi_0}} \sum_{\substack{p,r\\ p\neq r\\ (c,pr)=1}} a_p a_r \Psi(p)\overline{\Psi}(r)p^{z-s}r^{-z}dzds.$$

The double sum over p and r is

$$\sum_{\substack{p,r\\ p\neq r\\ (c,pr)=1}} a_p a_r \Psi(p)\overline{\Psi}(r)p^{z-s}r^{-z}$$

$$= \sum_{\substack{p\\ (c,p)=1}} \frac{\log p\Psi(p)}{p^{1/2+s-z}}\Phi\left(\frac{p}{X}\right) \sum_{\substack{r\\ (c,r)=1}} \frac{\log r\overline{\Psi}(r)}{r^{1/2+z}}\Phi\left(\frac{r}{X}\right) - \sum_{\substack{p\\ (aec,p)=1}} \frac{(\log p)^2}{p^{1+s}}\Phi\left(\frac{p}{X}\right)^2$$

and bounded by X^ε assuming GRH. The lemma easily follows from this bound. □

3.5. **Conclusion of the proof of Proposition 1.** In the beginning of Section 3, we have shown that the sum S splits into

$$S = S_D + S_N$$

with S_D the diagonal terms and S_N the off-diagonal terms. In Section 3.1 we have shown that the diagonal terms S_D contribute

$$S_D \sim \frac{A}{2\pi}Q\log X\int_{-\infty}^{\infty}|\widehat{\Phi}(ix)|^2dx$$

In Sections 3.2–3.4 we have shown that $S_N = S_U + S_L$ is at most $O(Q)$ if $X = Q^\alpha$ with $0 \leq \alpha \leq 1$ and that if $X = Q^\alpha$ with $1 < \alpha < 2$ then S_N is

$$S_N \sim \frac{A}{2\pi}Q\log(Q/X)\int_{-\infty}^{\infty}|\widehat{\Phi}(ix)|^2dx,$$

by (3.11), (3.14) and (3.15), and choosing $C = Q^\varepsilon$. Combining the above estimates we conclude that

$$S \sim f(\alpha)\tfrac{A}{2\pi}Q\log Q\int_{-\infty}^{\infty}|\widehat{\Phi}(ix)|^2dx = f(\alpha)N_\Phi(Q)$$

for $0 \leq \alpha < 2$, where $f(\alpha)$ is defined in 1.2. This gives the desired estimate.

4. Proof of Theorem 2

Recall that

$$F_\Phi(Q^\alpha;W) := \frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\; q)} \Big| \sum_{\gamma_\chi} \widehat{\Phi}(i\gamma_\chi) X^{i\gamma_\chi} \Big|^2.$$

Since W is supported in $(1,2)$, there is no primitive character in the sum over χ. Then by the Cauchy-Schwarz inequality, we have

$$F_\Phi(Q^\alpha;W) = M_1 + M_2 + O(\sqrt{M_1M_2}),$$

where

$$M_1 := \frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\; q)} \Big| \sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \Phi\left(\frac{n}{X}\right) \Big|^2$$

and

$$M_2 := \frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\; q)} \Big| \Phi\left(\frac{1}{X}\right) \log\frac{q}{\pi} \Big|^2.$$

By Proposition 1, $M_1 \sim f(\alpha)$ for $|\alpha| \le 2-\epsilon$, where $f(\alpha)$ is defined in (1.2). Also by a partial summation and (2.3) we have

$$\begin{aligned} M_2 &\sim \frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \varphi^*(q) \Phi(X^{-1})^2 \log^2 Q \\ &\sim \Phi(Q^{-|\alpha|})^2 \log Q \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \Big|\widehat{\Phi}(ix)\Big|^2 dx \right)^{-1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_\Phi(Q^\alpha;W) = {}& (1+o(1)) \left(f(\alpha) + \Phi(Q^{-|\alpha|})^2 \log Q \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \Big|\widehat{\Phi}(ix)\Big|^2 dx \right)^{-1} \right) \\ &+ O(\Phi(Q^{-|\alpha|})\sqrt{f(\alpha)\log Q}). \end{aligned}$$

5. Proof of Theorem 1

We reproduce here the argument from Özlük's paper [5]. First we need a lemma.

Lemma 9. *Assume GRH. If $1 < \alpha < 2$ is fixed, and the function Φ satisfies $\Phi(x) = \Phi(x^{-1})$, then*

$$\frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\; q)} \sum_{\gamma_\chi, \gamma'_\chi} \left(\frac{\sin(\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q)}{\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q} \right)^2 \widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi) \sim \left(1 + \frac{1}{3\alpha^2}\right).$$

Proof. We follow the argument given in [4]. Let

$$r(u) = \left(\frac{\sin \pi\alpha u}{\pi\alpha u}\right)^2$$

and we use the identity

$$\frac{1}{N_\Phi(Q)}\sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma_\chi,\gamma'_\chi} r\left(\frac{(\gamma_\chi-\gamma'_\chi)\log Q}{2\pi}\right)\widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi) = \int_{-\infty}^{\infty} F_\Phi(Q^\beta;W)\tilde{r}(\beta)d\beta \tag{5.1}$$

where $\tilde{r}(\beta)$ is the *Fourier* transform of r defined as

$$\tilde{r}(\beta) = \int_{-\infty}^{\infty} r(t)e^{-2\pi i\beta t}dt.$$

In this case

$$\tilde{r}(\beta) = \begin{cases} (\alpha-|\beta|)/\alpha^2 & \text{if } |\beta|<\alpha \\ 0 & \text{otherwise.} \end{cases}$$

We plug in $F_\Phi(Q^\beta;W)$ from Theorem 2 to the right-hand side of (5.1), obtaining that the right-hand side of (5.1) is

$$(1+o(1))\int_{-\alpha}^{\alpha}\left(f(\beta)+\Phi(Q^{-|\beta|})^2\log Q\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{\Phi}(ix)\right|^2dx\right)^{-1}\right)\tilde{r}(\beta)d\beta$$

with $f(\beta)$ is defined in (1.2). For $1<\alpha<2$, we have

$$\begin{aligned}\int_{-\alpha}^{\alpha} f(\beta)\tilde{r}(\beta)d\beta &= \frac{2}{\alpha^2}\int_0^1 \beta\cdot(\alpha-\beta)d\beta + \frac{2}{\alpha^2}\int_1^\alpha(\alpha-\beta)d\beta \\ &= 1+\frac{1}{3\alpha^2}-\frac{1}{\alpha}\end{aligned}$$

and

$$\begin{aligned}&\log Q\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{\Phi}(ix)\right|^2dx\right)^{-1}\cdot\int_{-\alpha}^{\alpha}\Phi(Q^{-|\beta|})^2\tilde{r}(\beta)d\beta \\ &\sim \log Q\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{\Phi}(ix)\right|^2dx\right)^{-1}\cdot\frac{2}{\alpha^2}\int_0^1\Phi(Q^{-\beta})^2(\alpha-\beta)d\beta \\ &\sim \frac{2}{\alpha}\int_0^{\log Q}\Phi(e^{-u})^2du\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{\Phi}(ix)\right|^2dx\right)^{-1} \\ &\sim \frac{2}{\alpha}\int_0^{\infty}\Phi(e^{-u})^2du\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{\Phi}(ix)\right|^2dx\right)^{-1} \\ &= \frac{1}{\alpha}.\end{aligned}$$

The last equality is obtained by the Plancherel's theorem for the Mellin transform in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{\Phi}(ix)\right|^2dx = \int_{-\infty}^{\infty}\Phi(e^{-u})^2du = \int_{-\infty}^{\infty}\Phi(e^{-|u|})^2du$$

and the fact that the function Φ satisfies $\Phi(x)=\Phi(x^{-1})$.

□

Proof of Theorem 1. Pick $\widehat{\Phi}(s) = ((e^s-e^{-s})/2s)^2$ so that $\widehat{\Phi}(i\gamma) = (\sin\gamma/\gamma)^2$. We need to check that this choice is possible, that is, that Φ is real and compactly supported in (a,b)

for some $a, b > 0$. Indeed, by Mellin inversion we have

$$\Phi(x) = \frac{1}{2\pi i}\int_{(c)}\left(\frac{e^s - e^{-s}}{2s}\right)^2 \cdot x^{-s} ds$$
$$= \begin{cases} \frac{1}{2} - \frac{1}{4}\log x & \text{for } 1 \le x \le e^2, \\ \frac{1}{2} + \frac{1}{4}\log x & \text{for } e^{-2} \le x \le 1, \\ 0 & \text{otherwise ,} \end{cases}$$

so it satisfies the required conditions. Note that Φ satisfies $\Phi(x) = \Phi(x^{-1})$.

Let m_ρ be the multiplicity of the zero $\rho = \frac{1}{2} + i\gamma$. We count zeros according to multiplicity. In particular,

$$\sum_{\gamma_\chi} m_\rho \widehat{\Phi}(i\gamma_\chi)^2 = \sum_{\substack{\gamma_\chi, \gamma'_\chi \\ \gamma_\chi = \gamma'_\chi}} \widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi)$$

because on both sides a given zero is counted with weight $m_\rho^2 \widehat{\Phi}(i\gamma_\chi)^2$. We have

$$\sum_{\substack{\gamma_\chi \\ \text{simple}}} \widehat{\Phi}(i\gamma_\chi)^2 \ge \sum_{\gamma_\chi} (2 - m_\rho)\widehat{\Phi}(i\gamma_\chi)^2$$
$$\ge 2\sum_{\gamma_\chi} \widehat{\Phi}(i\gamma_\chi)^2 - \sum_{\gamma_\chi, \gamma'_\chi}\left(\frac{\sin \alpha/2(\gamma_\chi - \gamma'_\chi)\log Q}{\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q}\right)^2 \widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi).$$

Hence

$$\sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} \widehat{\Phi}(i\gamma_\chi)^2$$
$$\ge 2\sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma_\chi} \widehat{\Phi}(i\gamma_\chi)^2$$
$$- \sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma_\chi, \gamma'_\chi}\left(\frac{\sin(\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q)}{\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q}\right)^2 \widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi)$$

We take $\alpha = 2 - \delta$, with $\delta > 0$, in the previous lemma, and observe that

$$\sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma_\chi, \gamma'_\chi}\left(\frac{\sin(\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q)}{\alpha/2(\gamma_\chi - \gamma'_\chi)\log Q}\right)^2 \widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi) \;\le\; \left(\frac{13}{12} + \varepsilon\right) N_\Phi(Q)$$

with some $\varepsilon \to 0$ as $\delta \to 0^+$. Combining the above two equations and using the fact that

$$\sum_q \frac{W(q/Q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\gamma_\chi} \widehat{\Phi}(i\gamma_\chi)^2 = N_\Phi(Q)$$

we prove the theorem. $\square$

6. Discussion of Özlük's result

In this section we explain why heuristically one expects that Özlük's result provides at most a proportion of 86% simple zeros. It is reasonable to suppose that as $t \to \infty$, there exists a κ such that

$$\sum_{q \le t} \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \sim \kappa \frac{t \log t}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx. \tag{6.1}$$

Özlük proves that

$$\sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \gtrsim \frac{11}{12} \frac{Q \log Q}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx.$$

We re-write the left-hand side as follows

$$\sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 = \sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{d | q} \sideset{}{^*}\sum_{\chi^* \,(\mathrm{mod}\ d)} \sum_{\substack{\gamma_{\chi^*} \\ \text{simple}}} |\widehat{\Phi}(i\gamma_{\chi^*})|^2,$$

where χ^* is the primitive character inducing χ. Note that the nontrivial zeros of $L(s,\chi)$ and $L(s,\chi^*)$ coincide. Therefore, we get

$$\begin{aligned}
\sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 &= \sum_{dq \le Q} \frac{1}{\varphi(dq)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \\
&\le \sum_{d \le Q} \frac{1}{\varphi(d)} \sum_{q \le Q/d} \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \\
&\lesssim \sum_{d \le Q} \frac{1}{\varphi(d)} \kappa \frac{Q}{d} \log \frac{Q}{d} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx \\
&\sim \kappa \sum_{d=1}^{\infty} \frac{1}{d\varphi(d)} \frac{Q \log Q}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx.
\end{aligned}$$

It thus follows that

$$\kappa \ge \frac{11}{12} \left(\sum_{d=1}^{\infty} \frac{1}{d\varphi(d)} \right)^{-1},$$

or equivalently

$$\frac{1}{N'_\Phi(Q)} \sum_{q \le Q} \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \ge \frac{11}{12} \left(\sum_{d=1}^{\infty} \frac{1}{d\varphi(d)} \right)^{-1} A_0^{-1},$$

where

$$\begin{aligned}
N'_\Phi(Q) &:= \sum_{q \le Q} \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \sum_{\gamma_\chi} |\widehat{\Phi}(i\gamma_\chi)|^2 \\
&\sim \frac{1}{2\pi} \log Q \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx \sum_{q \le Q} \frac{\varphi^*(q)}{\varphi(q)}
\end{aligned}$$

$$\sim A_0 \frac{Q \log Q}{2\pi} \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2 dx$$

and

$$A_0 = \prod_p \left(1 - \frac{1}{p^2} - \frac{1}{p^3}\right).$$

Therefore from Özlük's work we obtain a proportion of

$$\frac{11}{12} \left(\sum_{d=1}^{\infty} \frac{1}{d\varphi(d)} \right)^{-1} A_0^{-1} \approx 0.86883781 \ldots.$$

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