LIMITATIONS TO MOLLIFYING $\zeta(s)$.

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ABSTRACT. We establish limitations to how well one can mollify $\zeta(s)$ on the critical line with mollifiers of arbitrary length. Our result gives a non-trivial lower bound for the contribution of the off-diagonal terms to mollified moments of ζ. On the Riemann Hypothesis, we establish a connection between the mollified moment and Montgomery's Pair Correlation Function.

1. INTRODUCTION

The zero-distribution of an meromorphic function and the distribution of its size are closely related problems as can be seen from Jensen's inequality in complex analysis. For this reason, when studying the zeros of the Riemann ζ-function it is advantegeous to reduce the size of $\zeta(s)$ and to count instead the zeros of $\zeta(s)M(s)$ with $M(s)$ a *mollifier*: an entire function $M(s)$ pretending to behave as $1/\zeta(s)$ [6]. A natural choice for $M(s)$ is

$$M(s) = \sum_{n\geq 1} \frac{\mu(n)W(n)}{n^s}$$

with W a smooth function ensuring the absolute convergence of the sum.

Away from the neighborhood of a zero of $\zeta(s)$, mollifiers are good pointwise approximations to $1/\zeta(s)$ (see [8], Lemma 1) Since there are at most a few zeroes in the strip $\sigma > \frac{1}{2}+\varepsilon$, a mollifier is on average an excellent pointwise approximation to $1/\zeta(s)$ to the right of the critical line. On the critical line a mollifier is no longer a good pointwise approximation to $1/\zeta(s)$ because a positive proportion of the zeros lies on the critical line [11]. For this reason on the half-line we consider

$$\mathcal{I} = \mathcal{I}(M) := \frac{1}{T}\int_T^{2T} \left|1-\zeta(\tfrac{1}{2}+\mathrm{i}t)M(\tfrac{1}{2}+\mathrm{i}t)\right|^2 \mathrm{d}t.$$

The integral $\mathcal{I}$ is related to the horizontal distribution of the zeros of $\zeta(s)$, for example via the inequality $\sum_{T\leq\gamma\leq 2T} |\beta-\frac{1}{2}| \ll T\log(1+\mathcal{I}(M))$. valid for any choice of Dirichlet polynomial M. Understanding $\mathcal{I}$, and in particular how small $\mathcal{I}(M)$ can be for various choices of M, forms the principal focus of this paper.

The mollifier

$$\mathcal{L}_\theta(s) := \sum_{n\leq T^\theta} \frac{\mu(n)}{n^s}\cdot\left(1-\frac{\log n}{\log T^\theta}\right)$$

achieves $\mathcal{I}(\mathcal{L}_\theta)\sim 1/\theta$ for $\theta < \frac{4}{7}$ by a deep result of Conrey [7] (see also [2]). It is conjectured by Farmer [8] that, with this choice of mollifier, $\mathcal{I}(\mathcal{L}_\theta)\sim 1/\theta$ for all $\theta > 0$. As we later show,

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among all Dirichlet polynomials

$$M_\theta(s) = \sum_{n \le T^\theta} \frac{a(n)}{n^s} \text{ with } a(n) \ll n^\varepsilon \text{ and } a(1) = 1 \tag{1}$$

with $\theta < \frac{1}{2}$ fixed, the mollifier $\mathcal{L}_\theta(s)$ minimizes $\mathcal{I}$. We would like to understand if $\mathcal{I}(M_\theta)$ can be much smaller than $1/\theta$ when $M_\theta(s)$ is a longer mollifier, say with $\theta > 1$. We show that the answer is "no". In fact, unconditionally, there is an absolute constant $c > 0$ such that $\mathcal{I}(M_\theta) \ge c/\theta$ for all $\theta > 0$ and all M_θ as in (1).

Theorem 1. *Let $\theta > 0$ be given. There is an absolute constant $c > 0$ such that for all T large enough, and all M_θ as in (1)*

$$\mathcal{I}(M_\theta) := \frac{1}{T} \int_T^{2T} \left|1 - \zeta(\tfrac{1}{2} + \mathrm{i}t) M_\theta(\tfrac{1}{2} + \mathrm{i}t)\right|^2 \mathrm{d}t \ge \frac{c}{\theta}.$$

The constant c in Theorem 1 depends on the proportion of the zeros of $\zeta(s)$ lying on the critical line. The constant c cannot be greater than one, since $c > 1$ would contradict Farmer's conjecture in [8]. For $\theta < \frac{1}{2}$ we show that $c = 1$, using an asymptotic formula for $\mathcal{I}$, due to Balasubramanian, Conrey and Heath-Brown. Proposition B below is due to Prof. Soundararajan.

Proposition B (Soundararajan)**.** *Let M_θ be as in (1). If $\theta < \frac{1}{2}$, then, as $T \to \infty$,*

$$\mathcal{I}(M_\theta) \sim \sum_{m,n \le T^\theta} \frac{a(m)\overline{a(n)}}{[m,n]} \cdot \left(\log \frac{T(m,n)^2}{2\pi mn} + 2\log 2 + 2\gamma - 1 \right) - 1 \ge \frac{1}{\theta} + o(1)$$

Similar quadratic forms have been considered by Selberg [11] and Iwaniec-Sarnak [10]. To the best of the authors knowledge this is the first time that the proof of such a lower bound appears in the litterature.

Proposition B suggests that most likely $c = 1$ for all $\theta > 0$. Assuming the Riemann Hypothesis and the Pair Correlation conjecture we show that $c \ge 1 - \varepsilon$ for all $\theta > \theta_0(\varepsilon)$ large enough. This is interesting because one naively expects the problem to become more difficult for large θ.

Theorem 2. *Let $\theta > 0$ be given. Assume the Riemann Hypothesis and the Pair Correlation Conjecture. Let M_θ be as in (1) and assume in addition that $a(p^k) \ll 1$. Then, as $T \to \infty$,*

$$\mathcal{I}(M_\theta) := \frac{1}{T} \int_T^{2T} \left|1 - \zeta(\tfrac{1}{2} + \mathrm{i}t) M_\theta(\tfrac{1}{2} + \mathrm{i}t)\right|^2 \mathrm{d}t \ge \frac{1}{0.5 + \theta} \cdot (1 + o_\theta(1)).$$

Remark. *The condition $a(p^k) \ll 1$ can be dispensed with.*

The size of $\mathcal{I}(M_\theta)$ depends on the distribution of the zeros of $\zeta(s)$ in small interval of length $2\pi/(1+\theta)\log T$, around zeros of $\zeta(s)$. When θ is large, the Pair Correlation Conjecture allows to control the number of zeros in such thin intervals, thus giving increasingly better lower bounds for $\mathcal{I}(M_\theta)$.

On the Dirichlet polynomial side, an average of length T such as in Theorem 2 detects the first T coefficients of a Dirichlet series $F(s) = \sum a(n)n^{-s}$. This leads to a "trivial" lower bound (see [3]),

$$\int_T^{2T} |F(\tfrac{1}{2}+\mathrm{i}t)|^2 \mathrm{d}t \gg T \sum_{n\le T} \frac{|a(n)|^2}{n}.$$

Let $F(s) = 1 - \zeta(s)M(s)$ with $M(s) = \sum_{n\le T} \mu(n)n^{-s}$. Then the first T coefficients of $1-\zeta(s)M(s)$ are zero making the above lower bound vacuous. As another example let's consider the Dirichlet series $F(s) = 1-\zeta(s)M(s)$ with $M(s) = \mathcal{L}_\theta(s)$. The trivial lower bound leads to $cT/(1+\theta)^2$ while Theorem 1 gives cT/θ.

Theorems 1 and 2 beat the trivial lower bound by exploiting the relationship between $1-\zeta(s)M(s)$ and the zeros of $\zeta(s)$. This is made explicit in Proposition A below.

Proposition A. *Let $\varepsilon > 0$ and $\theta > 0$ be given. Then for T large, and for S any $\delta := 2\pi A/\log T$ well-spaced subset of zeros of $\zeta(s)$ with ordinates in $[T;2T]$, we have for all M_θ as in (1)*

$$\frac{1}{T}\int_T^{2T} |1-\zeta(\tfrac{1}{2}+\mathrm{i}t)M_\theta(\tfrac{1}{2}+\mathrm{i}t)|^2 \mathrm{d}t \ge \frac{1+O(\varepsilon)}{1+\theta+\frac{1}{A}} \cdot \frac{\mathrm{Card}(S)}{\frac{T}{2\pi}\log T} + O(T^\varepsilon).$$

The main idea in the proof of Proposition A is to connect, using Sobolev's inequality, the value of $1-\zeta(s)M(s)$ at a zero with a continuous average of $1-\zeta(s)M(s)$ around that zero. Using this idea we can also give an elementary proof of a result of Baez-Duarte, Balazard, Landreau and Saias [1]: For a mollifier $M(s)$ of length N,

$$\int_{\mathbb{R}} \left| \frac{1-\zeta(\frac{1}{2}+it)M(\frac{1}{2}+it)}{\frac{1}{2}+it} \right|^2 dt \ge \frac{C}{\log N} \tag{2}$$

Their proof depends on functional analysis: by Plancherel (2) is related to the L^2 behavior of the function $\rho(x) = \{1/x\}$. Re-proving (2) was the starting point for this paper.

On the Riemann Hypothesis we obtain an analogue of Proposition A involving Montgomery's Pair Correlation function,

$$F(\alpha, T) := \frac{2\pi}{T\log T} \sum_{T\le \gamma,\gamma' \le 2T} T^{\mathrm{i}\alpha(\gamma-\gamma')} \cdot w(\gamma-\gamma') \text{ where } w(x) = \frac{4}{4+x^2}.$$

The function $F(\alpha,T)$ describes the vertical distribution of the zeros of $\zeta(s)$. Following Montgomery it is well known that $F(\alpha,T) = \alpha + o(1)$ for $\varepsilon \le \alpha \le 1$ and $F(\alpha,T) \ge o(1)$ for all α. The Pair Correlation Conjecture is equivalent to $F(\alpha,T) = 1+o(1)$ in $1 \le \alpha \le M$ for every fixed $M > 1$. Theorem 2 follows from Theorem 3 below.

Theorem 3. *Let $\theta > 0$ be given. Assume the Riemann Hypothesis. Let M_θ be as in (1) and assume in addition that $a(p^k) \ll 1$. Then, for T large,*

$$\mathcal{I}(M_\theta) = \frac{1}{T}\int_T^{2T} |1-\zeta(\tfrac{1}{2}+\mathrm{i}t)M_\theta(\tfrac{1}{2}+\mathrm{i}t)|^2 \mathrm{d}t \ge \left(\frac{1}{2} + \int_1^{1+\theta+\varepsilon} F(\alpha,T)\mathrm{d}\alpha\right)^{-1}.$$

Remark. *As in Theorem 2 the requirement $a(p^k) \ll 1$ can be dispensed with*

In Theorem 3, choosing $M_\theta(s) = \mathcal{L}_\theta(s)$ for $\theta < \frac{4}{7}$ and applying Conrey's result [7] we have $\mathcal{I}(\mathcal{L}_\theta) \sim \frac{1}{\theta}$ for $\frac{1}{2} < \theta < \frac{4}{7}$ and thus, for $\frac{1}{2} < \theta < \frac{4}{7}$,

$$\int_1^{1+\theta} F(\alpha,T)\mathrm{d}\alpha > \theta - \tfrac{1}{2} + o(1).$$

as $T \to \infty$. In a subsequent paper, we will improve this result assuming the Generalized Riemann Hypothesis. Further we will investigate limitations to mollifying $\zeta(s)$ in the context of Levinson's method.

Theorems 1 and 2 have analogues for double-mollifiers $M(s) = \sum a(m,n) m^{-s} n^{-1+s}$. In Theorem 1, for θ bounded away from zero, say $\theta > \frac{1}{100}$, we can take $M(s) := \int \lambda^{-s} \mathrm{d}\mu(\lambda)$ with $\mu(\cdot)$ a finite measure, supported in $[1; T^\theta]$ and such that $\int_{1 \leq x \leq t} \mathrm{d}|\mu(x)| \ll t^A$ for some $A > 1$. In particular, for θ bounded away from zero, the assumption $a(n) \ll n^\varepsilon$ in Theorem 1 can be relaxed to $a(n) \ll n^A$ for some fixed $A > 0$.

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2. Key ideas

Sobolev's inequality

$$|f(x)| \leqslant \frac{1}{b-a} \int_a^b |f(u)| \,\mathrm{d}u + \int_a^b |f'(x)| \,\mathrm{d}x,$$

bounds a function f at a particular point $a \leqslant x \leqslant b$, by an average of f and f'. For a Dirichlet polynomial $A(\cdot)$ we prove a Sobolev inequality without an average over A'.

Lemma 1. *Let A be a Dirichlet polynomial supported on integers n with $M \leqslant n \leqslant N$. If f is a smooth function such that $f(x) = 1$ for $\log M \leqslant 2\pi x \leqslant \log N$, then for all real u,*

$$A(\mathrm{i}u) = \int_{-\infty}^{\infty} A(\mathrm{i}t) \hat{f}(t-u) \,\mathrm{d}t.$$

Proof Expanding $A(s) = \sum_{M \leqslant n \leqslant N} a(n) n^{-s}$ and using Fourier inversion,

$$\begin{aligned} \int_{-\infty}^{\infty} A(\mathrm{i}t) \hat{f}(t-u) \,\mathrm{d}t &= \sum_{M \leqslant n \leqslant N} a(n) \int_{-\infty}^{\infty} n^{-\mathrm{i}t} \cdot \hat{f}(t-u) \,\mathrm{d}t \\ &= \sum_{M \leqslant n \leqslant N} a(n) n^{-\mathrm{i}u} \cdot f\left(\frac{\log n}{2\pi}\right). \end{aligned}$$

By assumptions, $f(\log n/(2\pi)) = 1$ for $M \leqslant n \leqslant N$, and so the right-hand side is equal to $A(\mathrm{i}u)$. □

In the above lemma we can take $\zeta(s)$ or $1 - \zeta(s) A(s)$ instead of $A(s)$ because $\zeta(s)$ is approximated very well by a Dirichlet polynomial.

Lemma 2. *There is a smooth function $w(x)$ with $0 \leqslant w(x) \leqslant 1$, $w(0) = 1$, such that for $T \leqslant t \leqslant 2T$, $T_1 = T^{1+\varepsilon}$, and any fixed $v > 0$,*

$$\zeta(s) = \sum_{n \leqslant T_1} n^{-s} \cdot w\left(\frac{n}{T_1}\right) + O_v\left(T^{-v}\right).$$

Proof This is Proposition 1 in Bombieri-Friedlander [5]. □

If M is a long mollifier and s is away from a zero of $\zeta(s)$ (on a scale of $2\pi/\log|s|$) then $1 - \zeta(s) M(s) \approx 0$. On the other hand, if on the same scale s is close to a zero of $\zeta(s)$, then $\zeta(s)M(s) \approx 0$ and therefore $1-\zeta(s)M(s) \approx 1$. Given a smooth $\hat{f}(x)$ concentrated in $|x| \ll 2\pi/\log T$, the function

$$\sum_{\rho} \hat{f}\left(t-\gamma\right), T \leq t \leq 2T \tag{3}$$

exhibits a similar behavior to that of $1-\zeta(s)M(s)$. However, understanding the mean-square of (3) is much simpler.

Lemma 3. *Let S be a finite set and f be a smooth function. If K is a smooth function with $K \geqslant f^2$, then,*

$$\int_{-\infty}^{\infty} \left|\sum_{\gamma\in S} \hat{f}\left(t-\gamma\right)\right|^2 \mathrm{d}t \leqslant \sum_{\gamma,\gamma'} \hat{K}\left(\gamma-\gamma'\right).$$

Proof Notice that,

$$\begin{aligned} \sum_{\gamma\in S} \hat{f}\left(t-\gamma\right) &= \sum_{\gamma\in S} \int_{-\infty}^{\infty} f\left(v\right)\cdot e^{2\pi \mathrm{i} v(t-\gamma)} \mathrm{d}v \\ &= \int_{-\infty}^{\infty} e^{2\pi \mathrm{i} vt}\cdot f\left(v\right) \sum_{\gamma\in S} e^{-2\pi\mathrm{i}\gamma v} \mathrm{d}v. \end{aligned}$$

Therefore by Plancherel,

$$\begin{aligned} \int_{-\infty}^{\infty} \left|\sum_{\gamma\in S} \hat{f}\left(t-\gamma\right)\right|^2 \mathrm{d}t &= \int_{-\infty}^{\infty} \left|\sum_{\gamma\in S} e^{-2\pi\mathrm{i}\gamma v}\right|^2 \cdot \left|f\left(v\right)\right|^2 \mathrm{d}v \\ &\leqslant \int_{-\infty}^{\infty} \left|\sum_{\gamma\in S} e^{-2\pi\mathrm{i}\gamma v}\right|^2 \cdot K\left(v\right) \mathrm{d}v. \end{aligned}$$

Expanding the square, we find

$$\sum_{\gamma,\gamma'\in S} \int_{-\infty}^{\infty} e^{2\pi\mathrm{i} v(\gamma-\gamma')}\cdot K\left(v\right)\mathrm{d}v = \sum_{\gamma,\gamma'\in S} \hat{K}\left(\gamma-\gamma'\right),$$

as desired. □

For a δ-well-spaced set S it is convenient to pick a K such that $\hat{K}\left(x\right)=0$ when $|x| \geqslant \delta$. For such a choice of K,

$$\sum_{\gamma,\gamma'\in S} \hat{K}\left(\gamma-\gamma'\right) = \hat{K}\left(0\right)\cdot \mathrm{Card}\left(S\right).$$

We construct in the lemma below a set of functions with this property. These are known as the Beurling-Selberg majorants.

Lemma 4. *Let $\delta > 0$. For any interval $I=[a,b]$, there exists an even entire function $K\left(w\right)$ such that,*

- $K\left(u\right) \geqslant \chi_I\left(u\right)$
- $\hat{K}\left(0\right) = b-a+1/\delta$
- $\hat{K}\left(x\right) = 0$ *for* $|x| > \delta$.

Proof Beurling [4] considered the function,

$$B(z)=\left(\frac{\sin\pi z}{\pi}\right)^{2}\cdot\left(\frac{1}{z^{2}}+\sum_{n=0}^{\infty}\frac{1}{(z-n)^{2}}-\sum_{n=1}^{\infty}\frac{1}{(z+n)^{2}}\right).$$

The function $B(z)$ is entire, has the property that $B(x)\geqslant\operatorname{sgn}(x)$, and

$$\int_{-\infty}^{\infty}B(x)-\operatorname{sgn}(x)\,\mathrm{d}x=1.$$

From the definition of $B(z)$ it is easy to see that $B(z)=O\left(e^{2\pi|\mathrm{Imz}|}\right)$. Therefore, by Paley-Wiener $\hat{B}(x)=0$ when $|x|\geqslant 1$. Given an interval $I=[a,b]$ we define

$$K(z)=\frac{1}{2}\cdot B(\delta(z-a))+\frac{1}{2}\cdot B(\delta(b-z)).$$

Then by a direct check using the properties of $B(z)$ we find that, $K(x)\geqslant\chi_I(x)$ for all real x, $\hat{K}(x)=0$ for $|x|\geqslant\delta$, and $\hat{K}(0)=\int_{\mathbb{R}}K(x)\,\mathrm{d}x=b-a+1/\delta$, as desired. □

3. Proof of Proposition A.

We denote by t the imaginary part of s. Let $\eta>0$. By Lemma 2 there is a smooth function $w(x)$ with $0\leqslant w(x)\leqslant 1$, $w(0)=1$, and such that for $T\leq t\leq 2T$,

$$\zeta(s)=\sum_{n\leqslant T^{1+\eta}}\frac{1}{n^{s}}\cdot w\left(\frac{n}{T^{1+\eta}}\right)+O_v\left(T^{-v}\right).$$

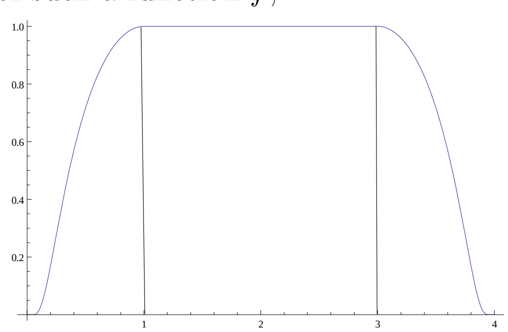
Multiplying by a Dirichlet polynomial $M(s)=\sum a(n)\,n^{-s}$ of length $N=T^{\theta}$ and with coefficients bounded by N we obtain a Dirichlet polynomial $B(s)$ of length $T^{1+\eta}\cdot N=T^{1+\eta+\theta}$ for which,

$$\zeta(s)\,M(s)=B(s)+O_v\left(T^{-v}\right). \tag{4}$$

Since $1-B(s)$ is a Dirichlet polynomial of length $T^{1+\eta}\cdot N$, by Lemma 1, for any smooth function f with $f(x)=1$ in $1\leqslant 2\pi x\leqslant\log(T^{1+\eta}\cdot N)$,

$$1-B\left(\tfrac{1}{2}+\mathrm{i}u\right)=\int_{-\infty}^{\infty}\left(1-B\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\hat{f}(t-u)\,\mathrm{d}t. \tag{5}$$

We choose a function f supported on the interval $0\leq 2\pi x\leq\log(T^{1+\eta}\cdot N)+1$, equal to one in $1\leq 2\pi x\leq\log(T^{1+\eta}\cdot N)$ and bounded between 0 and 1, with $f^{(\ell)}(x)\ll_{\ell}1$ for any given $\ell>0$. Here is an example of such a function f,



For any fixed v, $\hat{f}(x) \ll (\log T) \cdot (1 + |x| \log T)^{-v}$. Therefore for $T + T^{\eta} \leqslant u \leqslant 2T - T^{\eta}$ and $t \notin [T; 2T]$, we have $\hat{f}(t-u) \ll_v T^{-\eta v/2} \cdot (1+|x-u|\log T)^{-v/2} \ll_{\eta,v} T^{-v} \cdot (1+|x-u|\log T)^{-v}$. Since $1 - B(\frac{1}{2} + \mathrm{i}t)$ is trivially bounded by $N^2 \ll T^{2\theta}$ we get for $T + T^{\eta} \leq u \leq 2T - T^{\eta}$,

$$\int_{-\infty}^{\infty} (1 - B(\tfrac{1}{2} + \mathrm{i}t))\hat{f}(t-u)\mathrm{d}t = \int_{T}^{2T} (1 - B(\tfrac{1}{2} + \mathrm{i}t))\hat{f}(t-u)\mathrm{d}t + O_{\eta,v}(T^{-v}). \tag{6}$$

Combining (5) with (6) and (4) we obtain

$$1 - \zeta\left(\tfrac{1}{2} + \mathrm{i}u\right) M\left(\tfrac{1}{2} + \mathrm{i}u\right) = \int_{T}^{2T} \left(1 - \zeta\left(\tfrac{1}{2} + \mathrm{i}t\right) M\left(\tfrac{1}{2} + \mathrm{i}t\right)\right) \hat{f}(t-u)\,\mathrm{d}t + O_{\eta,v}\left(T^{-v}\right).$$

In the above equation take $u = \gamma$, with γ the ordinate of a zero of $\zeta(s)$ lying on the half-line and with $T + T^{\eta} \leqslant \gamma \leqslant 2T - T^{\eta}$. Summing over any set S of such zeros, we get

$$\mathrm{Card}(S) = \int_{T}^{2T} \left(1 - \zeta\left(\tfrac{1}{2} + \mathrm{i}t\right) M\left(\tfrac{1}{2} + \mathrm{i}t\right)\right) \sum_{\gamma \in S} \hat{f}(t-\gamma)\,\mathrm{d}t + O_{\eta,v}\left(T^{-v}\right).$$

By Cauchy-Schwarz

$$\mathrm{Card}(S) \leqslant \left(\int_{T}^{2T} \left|1 - \zeta\left(\tfrac{1}{2} + \mathrm{i}t\right) M\left(\tfrac{1}{2} + \mathrm{i}t\right)\right|^2 \mathrm{d}t\right)^{1/2} \cdot \left(\int_{\mathbb{R}} \left|\sum_{\gamma \in S} \hat{f}(t-\gamma)\right|^2 \mathrm{d}t\right)^{1/2} + O_{\eta,v}\left(T^{-v}\right).$$

By Lemma 3, for any K such that $K \geqslant f^2$,

$$\int_{\mathbb{R}} \left|\sum_{\gamma \in S} \hat{f}(t-\gamma)\right|^2 \mathrm{d}t \leqslant \sum_{\gamma,\gamma' \in S} \hat{K}(\gamma - \gamma').$$

Since $0 \leqslant f \leqslant 1$ and f is supported in $I = [0; (1/2\pi) \cdot \log(eT^{1+\eta} \cdot N)]$ the condition $K \geqslant f^2$ is satisfied whenever $K \geqslant \chi_I$. Using Lemma 4, we pick a function K such that, $K \geqslant \chi_I$, $\hat{K}(x) = 0$ for $|x| \geqslant \delta := 2\pi A/\log T$, and $\hat{K}(0) = |I| + 1/\delta$. Since the set S is δ well-spaced,

$$\begin{aligned}\sum_{\gamma,\gamma' \in S} \hat{K}(\gamma - \gamma') &= \hat{K}(0) \cdot \mathrm{Card}(S) = (|I| + 1/\delta) \cdot \mathrm{Card}(S) \\ &= (1 + O(\eta)) \cdot \frac{\log T}{2\pi} \cdot (1 + \theta + \tfrac{1}{A}) \cdot \mathrm{Card}(S).\end{aligned}$$

Combining the above three equations, we conclude

$$T \cdot \frac{1 + O(\eta)}{1 + \theta + \frac{1}{A}} \cdot \frac{\mathrm{Card(S)}}{\frac{T}{2\pi}\log T} \leqslant \int_{T}^{2T} \left|1 - \zeta\left(\tfrac{1}{2} + \mathrm{i}t\right) M\left(\tfrac{1}{2} + \mathrm{i}t\right)\right|^2 \mathrm{d}t + O_{\eta,v}(T^{-v}).$$

At the price of an additional error term $O(T^{\eta} \cdot \log T)$ we can add to S an arbitrary set of zeros with ordinates γ in the interval $[T; T + T^{\eta}] \cup [2T - T^{\eta}; 2T^{\eta}]$. Taking $\eta \to 0$ very slowly as $T \to \infty$ we obtain the claim.

4. Deduction of Theorem 1

Theorem 1 follows from Proposition A and the existence of a well spaced set of zeros, lying on the critical line, with ordinates in $[T; 2T]$ and cardinality $\gg N(T) \asymp T \log T$.

Lemma 5. *There is a set S of zeros of $\zeta(s)$ with $\beta = \frac{1}{2}$ and $T \leqslant \gamma \leqslant 2T$, such that*

- *The elements of S are $2\pi A / \log T$ well-spaced, for some absolute constant $A > 0$.*
- *The set S has $\gg T \log T$ elements.*

Proof Selberg's proof ([12], 10.22, p. 279) shows that there is an $h = 2\pi A / \log T$, with $A > 0$ constant, for which the set

$$E = \left\{ T \leqslant t \leqslant 2T : \gamma \in (t; t+h) \text{ for some } \rho = \tfrac{1}{2} + \mathrm{i}\gamma \right\},$$

has meas $\{E\} \geqslant c \cdot T$ with $c > 0$ constant. Hence at least $c \cdot T/h$ intervals $(T + nh; T + (n+1)h)$ contain a t such that there is a zero with $\beta = \frac{1}{2}$ and $\gamma \in (t; t+h)$. It follows that at least $c \cdot T/(2h)$ intervals $(T + (n-1)h; T + (n+1)h)$ contain the ordinate of a zero lying on the half-line. Taking every third such intervals produces a sequence of $c \cdot T/6h$ intervals of length $2h$, and spaced by at least h, each containing the ordinate of a zero on the half-line. Thus we obtain a h well-spaced set S of at least $\geqslant c \cdot T/6h$ zeros of $\zeta(s)$ lying on the half-line, with ordinates in $T \leqslant \gamma \leqslant 2T$. $\square$

Proof of Theorem 1 By Proposition A, given $\varepsilon > 0$, for any $2\pi A / \log T$-well spaced set of zeros S of $\zeta(s)$ lying on the critical line and with ordinates in $[T; 2T]$,

$$\frac{1}{T} \int_T^{2T} \left| 1 - \zeta\left(\tfrac{1}{2} + \mathrm{i}t\right) M_\theta\left(\tfrac{1}{2} + \mathrm{i}t\right) \right|^2 \mathrm{d}t \geqslant \frac{\mathrm{Card}(S)}{\frac{T}{2\pi} \log T} \cdot \frac{1 + O(\varepsilon)}{1 + \theta + 1/A} \tag{7}$$

We pick S as in Lemma 5. Then, for $\theta > \frac{1}{2}$ the above lower bound is,

$$\geq c_1 \frac{(1 + O(\varepsilon))}{1 + \theta} \geq c_2 \frac{1 + O(\varepsilon)}{\theta}$$

with $c_1, c_2 > 0$ absolute constants. Since $\varepsilon > 0$ is arbitrary, it follows that the liminf of the left-hand side of (7) is at least c/θ, as desired. On the other hand when $\theta < \frac{1}{2}$, Theorem 1 follows from Proposition B. $\square$

5. Preliminaries for Theorem 2 and 3

The proof of Theorem 2 follows the lines of proof of Proposition A. There are two main differences. The first is that for $n \ll T^{1-\varepsilon}$ we exploit cancellations in the sum,

$$\sum_{T \leqslant \gamma \leqslant 2T} n^{-\mathrm{i}\gamma}.$$

This is possible because we assume the Riemann Hypothesis.

Lemma 6. *Assume the Riemann Hypothesis. Uniformly in integer $n \geqslant 2$,*

$$\sum_{T \leqslant \gamma \leqslant 2T} n^{-1/2 - \mathrm{i}\gamma} = -\frac{T}{2\pi} \cdot \frac{\Lambda(n)}{n} + O\left((\log T)^2 \cdot n \right).$$

Proof See Gonek's paper [9]. $\square$

Lemma 7. *Let $A(s) = \sum a(n) \cdot n^{-s}$ be a Dirichlet polynomial of length N. Let f be a smooth test function. Then, for real u,*

$$\int_{-\infty}^{\infty} A(\mathrm{i}u)\, \hat{f}(t-\gamma)\, \mathrm{d}t = \sum_{n \leqslant M} \frac{a(n)}{n^{\mathrm{i}u}} \cdot f\left(\frac{\log n}{2\pi}\right).$$

Proof Expanding $A(s) = \sum a(n) \cdot n^{-s}$ and using Fourier inversion,

$$\begin{aligned} \int_{-\infty}^{\infty} A(\mathrm{i}t)\, \hat{f}(t-u)\, \mathrm{d}t &= \sum_{n \leqslant N} a(n) \int_{-\infty}^{\infty} n^{-\mathrm{i}t} \cdot \hat{f}(t-u)\, \mathrm{d}t \\ &= \sum_{n \leqslant N} a(n)\, n^{-\mathrm{i}u} \cdot f\left(\frac{\log n}{2\pi}\right). \end{aligned}$$

as claimed. □

The second difference with the proof of Proposition A, is that on the Riemann Hypothesis we can estimate asymptotically sums of the form

$$\sum_{T \leqslant \gamma, \gamma' \leqslant 2T} \hat{K}(\gamma - \gamma').$$

In application $\hat{K}(x)$ will be concentrated in $|x| \ll 1/\log T$, so that by the uncertainty principle, $K(x)$ will be spread out on intervals of length $\asymp \log T$ (or longer). If the Pair Correlation conjecture is not assumed then the lemma below is true with Montgomery's Pair Correlation $F(\alpha, T)$ instead of its limit $F(\alpha)$.

Lemma 8. *Assume the Riemann Hypothesis. Let $h \geqslant 0$ denote a smooth, non-zero, and compactly supported function. Let $K(x) = h(2\pi x/\log T)$. Then, as $T \to \infty$,*

$$\sum_{T + T^{\varepsilon} \leqslant \gamma, \gamma' \leqslant 2T - T^{\varepsilon}} \hat{K}(\gamma - \gamma') = T \cdot \left(\frac{\log T}{2\pi}\right)^2 \int_{-\infty}^{\infty} h(\alpha) \cdot F(\alpha, T)\, \mathrm{d}x + O(T^{1-\varepsilon})$$

with $F(\alpha, T)$ Montgomery's Pair Correlation function.

Proof Since $K(x) = K(\log T/2\pi \cdot x)$ the Fourier transform of K is given by,

$$\hat{K}(x) = \frac{\log T}{2\pi} \cdot \hat{h}\left(\frac{\log T}{2\pi} \cdot x\right).$$

By definition

$$\sum_{T \leqslant \gamma, \gamma' \leqslant 2T} \hat{h}\left(\frac{\log T}{2\pi} \cdot (\gamma - \gamma')\right) w(\gamma - \gamma') = \frac{T \cdot \log T}{2\pi} \int_{-\infty}^{\infty} h(\alpha)\, F(\alpha, T)\, \mathrm{d}\alpha.$$

with the weight $w(x) = 4/(4 + x^2)$. Multiplying by $\log T/2\pi$, we obtain,

$$\sum_{T \leqslant \gamma, \gamma' \leqslant 2T} \hat{K}(\gamma - \gamma')\, w(\gamma - \gamma') \sim T \cdot \left(\frac{\log T}{2\pi}\right)^2 \int_{-\infty}^{\infty} h(\alpha)\, F(\alpha)\, \mathrm{d}\alpha. \tag{8}$$

One removes the weight $w(\gamma - \gamma')$ by a standard argument which we omit. Since h is smooth, and compactly supported we have $\hat{K}(x) \ll_v (\log T) \cdot (1 + \log T \cdot |x|)^{-v}$ for any fixed v. Thus, for any γ,

$$\sum_{T\leqslant\gamma\leqslant 2T}\hat{K}\left(\gamma-\gamma'\right)\ll\left(\log T\right)^2$$

Since there are at most $\ll T^{\varepsilon}\cdot\log T$ ordinates of zeros in $[T;T+T^{\varepsilon}]\cup[2T-T^{\varepsilon};2T]$, we can restrict the summation in (8) to $T+T^{\varepsilon}\leqslant\gamma,\gamma'\leqslant 2T-T^{\varepsilon}$ at the price of a negligible error term $\ll T^{\varepsilon}\cdot(\log T)^3$. □

6. Proof of Theorem 2 and 3

We denote by t the imaginary part of s. Let M be a Dirichlet polynomial of length $N=T^{\theta}$. Fix a small $\frac{1}{10}>\eta>0$. Proceeding as in the proof of Proposition A, there is a Dirichlet polynomial $B(s)$ of length $T^{1+\eta}N$ such that for $T\leq t\leq 2T$ and for any fixed $v>0$,

$$\zeta\left(s\right)M\left(s\right)=B\left(s\right)+O_v\left(T^{-v}\right). \tag{9}$$

Since $a(1)=1$, $a(p^k)\ll 1$ and $a(n)\ll n^{\varepsilon}$, the coefficients $b(n)$ of $B(s)$ satisfy,

$$b\left(1\right)=1+O(T^{-1-\eta}),\ b(p^k)\ll 1,\ \text{and}\ b\left(n\right)\ll n^{\varepsilon}.$$

Let $h(x)=h_0(2\pi x/\log T)$ with $h_0\leq 1$ a smooth function supported on $[\eta;1+\theta+2\eta]$ and equal to one on $[2\eta;1+\theta+\eta]$. These requirements on h force that $\hat{h}(x)\ll_{\ell}\log T\cdot(1+\log T|x|)^{-\ell}$ for every fixed $\ell>0$.

Lemma. *We have*

$$\int_{-\infty}^{\infty}\left(1-B\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\sum_{\gamma\in S}\hat{h}\left(t-\gamma\right)\mathrm{d}t=\left(1+O\left(\eta\right)\right)N\left(T\right). \tag{10}$$

Proof Write $h=f-g$ with $f(x)=f_0(2\pi x/\log T), g(x)=g_0(2\pi x/\log T)$ two smooth compactly supported functions such that $f_0(x)=1$ on $[0;1+\theta+\eta]$, $g_0(x)=1$ on $[0;\eta]$ and $g_0(x)$ is supported on $[-A;2\eta]$ for some $A>0$. By Lemma 7 applied to $1-B(s)$,

$$\int_{-\infty}^{\infty}\left(1-B\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\hat{g}\left(t-u\right)\mathrm{d}t=1-b\left(1\right)+\sum_{2\leqslant n\leqslant T^{2\eta}}\frac{b\left(n\right)}{n^{1/2+\mathrm{i}u}}\cdot g\big(\frac{\log n}{2\pi}\big).$$

Set $u=\gamma$, and sum over the set S of all zeros with ordinates $T+T^{\eta}\leqslant\gamma\leqslant 2T-T^{\eta}$. Using Gonek's Lemma 6 and $1-b(1)\ll T^{-1-\eta}$, $b(p^k)\ll 1$, $g\ll 1$, we get

$$\begin{aligned}\int_{-\infty}^{\infty}\left(1-B\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\sum_{\gamma\in S}\hat{g}\left(t-\gamma\right)\mathrm{d}t&=-\frac{T}{2\pi}\sum_{n\leqslant T^{2\eta}}\frac{b\left(n\right)\Lambda\left(n\right)}{n}\cdot g\big(\frac{\log n}{2\pi}\big)+O\left(T^{3\eta}\right)\\&\ll T\sum_{n\leq T^{2\eta}}\frac{\Lambda(n)}{n}\ll\eta T\log T\ll\eta N(T).\end{aligned} \tag{11}$$

Since $1-B(s)$ is of length $T^{1+\eta}N$, and $f(x)=1$ on $1\leq 2\pi x\leq\log(T^{1+\eta}N)$; we get by Lemma 1,

$$\int_{-\infty}^{\infty}\left(1-B\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\hat{f}\left(t-u\right)\mathrm{d}t=1-B(\tfrac{1}{2}+\mathrm{i}u)+O_v\left(T^{-v}\right).$$

Set $u=\gamma$ and note that by equation (9), $B(\frac{1}{2}+\mathrm{i}\gamma)=O_v(T^{-v})$. Summing over all $T+T^\eta \leq \gamma \leq 2T-T^\eta$ we obtain

$$\int_{-\infty}^{\infty}\left(1-B\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\sum_{\gamma\in S}\hat{f}\left(t-\gamma\right)\mathrm{d}t = N(T)+O(T^\eta). \tag{12}$$

Subtracting (11) from (12), and recalling that $\hat{h}=\hat{f}-\hat{g}$ (because $h=f-g$), we obtain the claim. $\square$

Since $\hat{h}(x) \ll_v \log T\cdot(1+\log T|x|)^{-v}$, we proceed exactly as in the proof of Proposition A; we truncate the integral in (10) at T and $2T$, and using (9) replace $1-B\left(s\right)$ by $1-\zeta\left(s\right)M\left(s\right)$. Thus we obtain from (10) that

$$\int_{T}^{2T}\left(1-\zeta\left(\tfrac{1}{2}+\mathrm{i}t\right)M\left(\tfrac{1}{2}+\mathrm{i}t\right)\right)\sum_{\gamma\in S}\hat{h}\left(t-\gamma\right)\mathrm{d}t = \left(1+O\left(\eta\right)\right)N\left(T\right).$$

Applying Cauchy-Schwarz leads to

$$(1+O(\eta))N(T)\leqslant\left(\int_{T}^{2T}|1-\zeta\left(\tfrac{1}{2}+\mathrm{i}t\right)M\left(\tfrac{1}{2}+\mathrm{i}t\right)|^2\mathrm{d}t\right)^{1/2}\cdot\left(\int_{\mathbb{R}}\left|\sum_{\gamma\in S}\hat{h}\left(t-\gamma\right)\right|^2\mathrm{d}t\right)^{1/2}.$$

By Lemma 3, for any smooth K with $K\geqslant h^2$,

$$\int_{\mathbb{R}}\left|\sum_{\gamma\in S}\hat{h}\left(t-\gamma\right)\right|^2\mathrm{d}t \;\leqslant\; \sum_{\gamma,\gamma'\in S}\hat{K}\left(\gamma-\gamma'\right).$$

Take $K=h^2$, and recall that $h(x)=h_0(2\pi x/\log T)\leq 1$ with h_0 supported on $[\eta;1+\theta+2\eta]$. Thus $\hat{K}(x)=\log T/2\pi\cdot\hat{h_0^2}(x\log T/2\pi)$. Applying Lemma 8 and bounding h_0 by 1 on its interval of support we obtain

$$\begin{aligned}\sum_{\gamma,\gamma'\in S}\hat{K}\left(\gamma-\gamma'\right)&=\frac{\log T}{2\pi}\sum_{\gamma,\gamma'\in S}\hat{h_0^2}\left(\frac{\log T}{2\pi}(\gamma-\gamma')\right)\\&\sim\frac{T\left(\log T\right)^2}{(2\pi)^2}\int_{-\infty}^{\infty}h_0^2\left(\alpha\right)F\left(\alpha,T\right)\mathrm{d}\alpha\\&\leqslant(1+o(1))\frac{T\left(\log T\right)^2}{(2\pi)^2}\int_{\eta}^{1+\theta+2\eta}F\left(\alpha,T\right)\mathrm{d}\alpha.\end{aligned}$$

Finally recall that $F(\alpha,T)=|\alpha|+o(1)$ uniformly for $\eta<|\alpha|<1-\eta$. Combining the above three inequalities and letting $\eta\to 0$ very slowly as $T\to\infty$ establishes Theorem 3.

To prove Theorem 2 note that on the Pair Correlation conjecture (PCC) $F(\alpha,T)=1+o(1)$ for $1\leq|\alpha|\leq M$, and any fixed $M>1$. Therefore on PCC,

$$\int_{\eta}^{1+\theta+2\eta}F(\alpha,T)d\alpha=0.5+\theta+O(\eta)$$

Combining the above four equations and letting $\eta\to 0$ we obtain Theorem 2. Alternatively, Theorem 2 is an immediate consequence of Theorem 3 as explained in the introduction.

7. Proof of Proposition B.

Our goal is to determine the minimum of the quadratic form,

$$\log(cT)\sum_{d,e\le N}\frac{a(d)\overline{a(e)}}{[d,e]}-\sum_{d,e\le N}\frac{a(d)\overline{a(e)}}{[d,e]}\cdot\log\frac{[d,e]}{(d,e)} \tag{13}$$

with $c>0$ constant (for example $c=4e^{2\gamma-1}/2\pi$). Writing $(d,e)=\sum_{\ell|d,\ell|e}\varphi(\ell)$ diagonalizes the first quadratic form,

$$\sum_{d,e\le N}\frac{a(d)\overline{a(e)}}{[d,e]}=\sum_{\ell}\frac{\varphi(\ell)}{\ell^2}\cdot|y(\ell)|^2 \tag{14}$$

where

$$y(\ell):=\sum_{d\le N}\frac{a(d\ell)}{d}$$

By Moebius inversion

$$1=\sum_{\ell\le N}\frac{y(\ell)\mu(\ell)}{\ell}$$

Thus, by Cauchy-Schwarz,

$$1\le\left(\sum_{n\le N}\frac{\mu(n)^2}{\varphi(n)}\right)^{1/2}\cdot\left(\sum_{n\le N}\frac{\varphi(\ell)}{\ell^2}\cdot|y(\ell)|^2\right)^{1/2}$$

It follows that the minimum of the quadratic form (14) is $1/G$, where

$$G:=\sum_{n\le N}\frac{\mu(n)^2}{\varphi(n)}$$

The minimum is attained when $y(\ell)=z(\ell)$ with

$$z(\ell)=\frac{\mu(\ell)}{G}\cdot\frac{\ell}{\varphi(\ell)}$$

The above discussion is subsumed in the lemma below.

Lemma 9. *With notation as above, we have,*

$$\sum_{d,e\le N}\frac{a(d)\overline{a(e)}}{[d,e]}=\frac{1}{G}+\sum_{\ell\le N}\frac{\varphi(\ell)}{\ell^2}\cdot|y(\ell)-z(\ell)|^2$$

Proof Expanding the square and using (14),

$$\begin{aligned}\sum_{\ell\le N}\frac{\varphi(\ell)}{\ell^2}\cdot|y(\ell)-z(\ell)|^2&=\sum_{\ell\le N}\frac{\varphi(\ell)}{\ell^2}|y(\ell)|^2-2\Re\sum_{\ell\le N}\frac{\varphi(\ell)}{\ell^2}y(\ell)z(\ell)+\sum_{\ell\le N}\frac{\varphi(\ell)}{\ell^2}z(\ell)^2\\&=\sum_{d,e\le N}\frac{a(d)\overline{a(e)}}{[d,e]}-\frac{2}{G}\Re\sum_{\ell\le N}\frac{\mu(\ell)}{\ell}y(\ell)+\frac{1}{G^2}\sum_{\ell\le N}\frac{\mu(\ell)^2}{\varphi(\ell)}\\&=\sum_{d,e\le N}\frac{a(d)\overline{a(e)}}{[d,e]}-\frac{2}{G}+\frac{1}{G}\end{aligned}$$

and the claim follows. $\square$

It remains to understand the second quadratic form appearing in equation (14). This is more difficult and is accomplished in the lemma below.

Lemma 10. *Let $N = T^{\theta}$. Then, for T large,*

$$-\sum_{d,e\leq N} \frac{a(d)\overline{a(e)}}{[d,e]} \cdot \log\left(\frac{[d,e]}{(d,e)}\right) \geq 1 - (\theta+\varepsilon)\log T \sum_{\ell\leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2$$

We will prove Lemma 10 later on. Assuming the result of Lemma 10 Proposition B follows immediately.
Proof of Proposition B Take $N = T^{\theta}$ with $\theta < 1$. Let $c > 0$ be a constant. By Lemma 9, and using $G \sim \log N$, we obtain

$$\log(cT) \sum_{d,e\leq N} \frac{a(d)\overline{a(e)}}{[d,e]} = \frac{1}{\theta} + \log(cT) \sum_{\ell\leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 + o(1) \tag{15}$$

as $T \to \infty$. By Lemma 10,

$$-\sum_{d,e\leq N} \frac{a(d)\overline{a(e)}}{[d,e]} \cdot \log\left(\frac{[d,e]}{(d,e)}\right) \geq 1 - (\theta+\varepsilon)\log T \sum_{\ell\leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 \tag{16}$$

Adding the equations (15) and (16), we obtain

$$\begin{aligned}\sum_{d,e\leq N} \frac{a(d)\overline{a(e)}}{[d,e]} \cdot \log\left(\frac{cT(d,e)}{[d,e]}\right) \\ \geq 1 + \frac{1}{\theta} + (1-\theta-\varepsilon)\log T \sum_{\ell\leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 + o(1)\end{aligned}$$

The sum of squares is non-negative, and $1 - \theta - \varepsilon > 0$; we thus obtain the desired lower bound $1 + 1/\theta + o(1)$. $\square$

7.1. **Proof of Lemma 10.** In order to prove Lemma 10 we start by expressing the quadratic form (13) in terms of the sequence $y(\ell)$.

Lemma 11. *We have,*

$$\begin{aligned}\sum_{d,e\leq N} \frac{a(d)\overline{a(e)}}{[d,e]} \cdot \log\frac{[d,e]}{(d,e)} = 2\sum_{p^{\alpha}\ell\leq N} \frac{\log p}{p^{\alpha}} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot \Re\big(y(\ell)\overline{y(p^{\alpha}\ell)}\big) + \\ + O\left((\log\log N)^2 \sum_{\ell\leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 + \frac{(\log\log N)^2}{\log N}\right)\end{aligned} \tag{17}$$

Proof Since $[d\ell, e\ell]/(d\ell, e\ell) = [d,e]/(d,e)$ using the formula $(d,e) = \sum_{\ell|d,\ell|e} \varphi(\ell)$ we find

$$\sum_{e,d\leq N} \frac{a(d)\overline{a(e)}}{[d,e]} \cdot \log\frac{[d,e]}{(d,e)} = \sum_{\ell\leq N} \frac{\varphi(\ell)}{\ell^2} \sum_{d,e\leq N/\ell} \frac{a(d\ell)\overline{a(e\ell)}}{de} \cdot \log\frac{[d,e]}{(d,e)}. \tag{18}$$

A prime p divides $[d,e]/(d,e)$ if and only if $|v_p(e)-v_p(d)| \geq 1$ where $v_p(n)$ denotes the p-adic valuation of n. Therefore,

$$\log \frac{[d,e]}{(d,e)} = \sum_{\substack{p^\alpha \| e, p^\beta \| d \\ |\alpha-\beta|\geq 1}} \log p$$

It follows that (18) can be expressed as

$$\sum_{|\alpha-\beta|\geq 1} \sum_{p\ell \leq N} \frac{\log p}{p^{\alpha+\beta}} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot \Big(y(p^\alpha \ell) - \frac{1}{p} y(p^{\alpha+1}\ell) \Big) \cdot \overline{\Big(y(p^\beta \ell) - \frac{1}{p} y(p^{\beta+1}\ell) \Big)} \tag{19}$$

We bound the contribution of the terms with $\alpha, \beta \geq 1$: multiplying out and repeatedly using the inequality $2ab \leq a^2 + b^2$ we find that,

$$\sum_{\substack{|\alpha-\beta|\geq 1 \\ \alpha,\beta\geq 1}} \frac{1}{p^{\alpha+\beta}} \cdot \Big(y(p^\alpha \ell) - \frac{1}{p} y(p^{\alpha+1}\ell) \Big) \cdot \overline{\Big(y(p^\beta \ell) - \frac{1}{p} y(p^{\beta+1}\ell) \Big)} \ll \sum_{\alpha \geq 1} \frac{|y(p^\alpha \ell)|^2}{p^\alpha} \cdot \frac{1}{p}$$

Inserting this back into (19), bounds the contributions of the terms with $\alpha, \beta \geq 1$ by

$$\begin{aligned} \ll \sum_{\substack{p^\alpha \ell \leq N \\ \alpha \geq 1}} \frac{\log p}{p^{\alpha+1}} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot |y(p^\alpha \ell)|^2 &= \sum_{m \leq N} \Bigg(\sum_{\substack{p^\alpha \ell = m \\ \alpha \geq 1}} \frac{\log p}{p^{\alpha+1}} \cdot \frac{\varphi(\ell)}{\ell^2} \Bigg) \cdot |y(m)|^2 \\ &\ll (\log\log N)^2 \cdot \sum_{\ell \leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell)|^2 \\ &\ll \frac{(\log\log N)^2}{\log N} + (\log\log N)^2 \sum_{\ell \leq N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 \end{aligned} \tag{20}$$

On the other hand the remaining terms with $\alpha = 0, \beta \geq 1$ and $\beta = 0, \alpha \geq 1$ in (19) telescope to

$$2 \sum_{p\ell \leq N} \frac{\log p}{p} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot \Re\Big(y(\ell)\overline{y(p\ell)} - \frac{1}{p} \cdot |y(p\ell)|^2 \Big)$$

To the above sum we add the contribution of the terms with $p^\alpha \ell \leq N$ and $\alpha > 1$. This contribution is estimated by in the same way as in (20) and therefore negligible. This leads us to a final estimate of

$$2 \sum_{p^\alpha \ell \leq N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot \Re\Big(y(\ell)\overline{y(p^\alpha \ell)} \Big)$$

plus the same error as in (20). $\square$

Write

$$\begin{aligned} y(\ell)\overline{y(p^\alpha \ell)} = (y(\ell) - z(\ell)) \cdot \overline{(y(p^\alpha \ell) - z(p^\alpha \ell))} + \\ + z(\ell) \cdot \overline{(y(p^\alpha \ell) - z(p^\alpha \ell))} + \overline{z(p^\alpha \ell)} \cdot (y(\ell) - z(\ell)) + \overline{z(p^\alpha \ell)} z(\ell). \end{aligned} \tag{21}$$

It follows from the above identity and Lemma 11 that

$$\sum_{d,e \leq N} \frac{a(d)\overline{a(e)}}{[d,e]} \cdot \log \frac{[d,e]}{(d,e)} = S_1 - S_2 + S_3$$

where

$$S_1 := 2 \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot (y(\ell) - z(\ell))\overline{(y(p^\alpha \ell) - z(p^\alpha \ell))}$$

$$S_2 := 2 \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot \left(z(\ell)\overline{(y(p^\alpha \ell) - z(p^\alpha \ell))} + \overline{z(p^\alpha \ell)}(y(\ell) - z(\ell))\right)$$

$$S_3 := 2 \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot z(\ell)\overline{z(p^\alpha \ell)}$$

Lemma 10 follows upon computing S_1, S_2 and S_3 and combining the resulting estimate. We perform the necessary computations in the three lemma below.

Lemma 12. *We have,*

$$|S_1| \le (\log N + O(\log\log N)) \cdot \sum_{\ell} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2$$

Proof Applying to $2(y(\ell) - z(\ell))\overline{(y(p^\alpha \ell) - z(p^\alpha \ell))}$ the inequality $2|ab| \le |a|^2 + |b|^2$ we find

$$\begin{aligned} |S_1| &\le \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 + \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot |y(p^\alpha \ell) - z(p^\alpha \ell)|^2 \\ &\le \sum_{\ell \le N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2 \cdot \log(N/\ell) + \sum_{m \le N} \left(\sum_{p^\alpha \ell = m} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \right) \cdot |y(m) - z(m)|^2 \end{aligned}$$

For $m = p^\alpha \ell$ we have $\varphi(\ell)/\ell = \varphi(m)/m \cdot (1 + O(1/p))$. Therefore,

$$\sum_{p^\alpha \ell = m} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} = \frac{\varphi(m)}{m^2} \cdot \log m + O\left(\frac{\varphi(m)}{m^2} \cdot \log\log m \right)$$

Therefore the sums with $\log m$ cancel out and we obtain the bound

$$(\log N + O(\log\log N)) \cdot \sum_{\ell \le N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2.$$

as desired. $\square$

Lemma 13. *We have*

$$|S_2| \ll \frac{\log\log N}{\sqrt{\log N}} \cdot \sum_{\ell \le N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell) - z(\ell)|^2$$

Proof On the one hand,

$$\begin{aligned} 2 \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot z(\ell)\overline{(y(p^\alpha \ell) - z(p^\alpha \ell))} &= \frac{2}{G} \sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \frac{\mu(\ell)}{\ell} \cdot \overline{(y(p^\alpha \ell) - z(p^\alpha \ell))} \\ &= \frac{2}{G} \sum_{m \le N} \left(\sum_{p^\alpha \ell = m} \frac{\log p}{p^\alpha} \cdot \frac{\mu(\ell)}{\ell} \right) \cdot \overline{(y(m) - z(m))} \\ &= -\frac{2}{G} \sum_{m \le N} \frac{\mu(m)\log m}{m} \cdot \overline{(y(m) - z(m))} \qquad (22) \end{aligned}$$

On the other hand,

$$2\sum_{p^\alpha \ell \le N} \frac{\log p}{p^\alpha} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot \overline{z(p^\alpha \ell)}(y(\ell)-z(\ell)) = -\frac{2}{G} \sum_{\substack{p^\alpha \ell \le N \\ (p,\ell)=1}} \frac{\log p}{p^\alpha} \cdot \frac{\mu(\ell)}{\ell} \cdot (y(\ell)-z(\ell))$$
$$= -\frac{2}{G}\sum_{\ell \le N} \frac{\mu(\ell)}{\ell} \cdot (y(\ell)-z(\ell)) \sum_{\substack{p^\alpha \le N/\ell \\ (p,\ell)=1}} \frac{\log p}{p^\alpha}$$

Since $\sum_{\substack{p^\alpha \le N/\ell \\ (p,\ell)=1}} \frac{\log p}{p^\alpha} = \log N/\ell + O(\log\log N)$ and

$$\frac{1}{G}\sum_{\ell \le N} \frac{\mu(\ell)}{\ell} \cdot (y(\ell)-z(\ell)) = 0$$

the sum simplifies to

$$\frac{2}{G}\sum_{\ell \le N} \frac{\mu(\ell)\log \ell}{\ell} \cdot (y(\ell)-z(\ell)) + O\left(\frac{\log\log N}{\sqrt{\log N}} \cdot \sum_{\ell \le N} \frac{\varphi(\ell)}{\ell^2} \cdot |y(\ell)-z(\ell)|^2\right) \tag{23}$$

Adding (22) and (23) the main terms cancel and we obtain the bound for $|S_2|$. $\square$

Lemma 14. *We have,*

$$S_3 = -1 + O\left(\frac{\log\log N}{\log N}\right).$$

Proof Since, for $\ell \le N$,

$$\sum_{\substack{p \le N/\ell \\ (p,\ell)=1}} \frac{\log p}{p} = \log(N/\ell) + O(\log\log N)$$

We have

$$2\sum_{p\ell \le N} \frac{\log p}{p} \cdot \frac{\varphi(\ell)}{\ell^2} \cdot z(\ell)z(p\ell) = -\frac{2}{G^2} \sum_{\substack{p\ell \le N \\ (p,\ell)=1}} \frac{\log p}{p} \cdot \frac{\mu(\ell)^2}{\varphi(\ell)}$$
$$= -\frac{2}{G^2}\sum_{\ell \le N} \frac{\mu(\ell)^2}{\varphi(\ell)} \cdot \big(\log(N/\ell) + O(\log\log N)\big)$$
$$= -1 + O\left(\frac{\log\log N}{\log N}\right)$$

as desired. $\square$

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