On the Typical Size and Cancelations Among the Coefficients of Some Modular Forms

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Abstract

We obtain a nontrivial upper bound for almost all elements of the sequences of real numbers which are multiplicative and at the prime indices are distributed according to the Sato–Tate density. Examples of such sequences come from coefficients of several L-functions of elliptic curves and modular forms. In particular, we show that $|\tau(n)| \le n^{11/2}(\log n)^{-1/2+o(1)}$ for a set of n of asymptotic density 1, where $\tau(n)$ is the Ramanujan τ function while the standard argument yields $\log 2$ instead of $-1/2$ in the power of the logarithm. Another consequence of our result is that in the number of representations of n by a binary quadratic form one has slightly more than square-root cancellations for almost all integers n.

In addition we obtain a central limit theorem for such sequences, assuming a weak hypothesis on the rate of convergence to the Sato–Tate law. For Fourier coefficients of primitive holomorphic cusp forms such a hypothesis is known conditionally assuming the automorphy of all symmetric powers of the form and seems to be within reach unconditionally using the currently established potential automorphy.

1 Introduction

1.1 Background and motivation

Let $\mathcal{A}_{\rm ST}$ be the class of infinite sequences $\{a_n\}_{n\ge1}$ of real numbers, which satisfy the following properties:

- *Multiplicativity:* for any coprime positive integers m, n we have $a_{mn} = a_m a_n$.
- *Sato–Tate distribution:* for any prime p we have $a_p \in [-2,2]$ for the angles $\vartheta_p \in [0,\pi)$ defined by $a_p = 2\cos\vartheta_p$, and any $\alpha \in [0,\pi)$ we have

$$\frac{\#\{p \le x \ : \ p \text{ prime}, \ \vartheta_p \in [0,\alpha]\}}{\pi(x)} \to \frac{2}{\pi}\int_0^\alpha \sin^2\vartheta \, d\vartheta, \quad \text{as } x \to \infty,$$

where, as usual, $\pi(x)$ denotes the prime counting function of all primes $p \le x$.

- *Growth on prime powers:* There exist a constant $\varrho > 0$ such that for any integer $a \ge 2$ and prime p we have $|a_{p^a}| \le p^{(a-1)/2-\varrho}$.

Very often such sequences come in the form $\lambda(n)/n^\gamma$ with some real $\gamma > 0$, where $\{\lambda(n)\}_{n\ge1}$ is the sequence of coefficients of a Dirichlet series of a certain L-function or of Fourier coefficients of a certain modular form.

In particular, we recall the striking results of Barnet-Lamb, Geraghty, Harris, and Taylor [2], Clozel, Harris and Taylor [4], Harris, Shepherd-Barron and Taylor [14], who established that Fourier coefficients of several types of modular forms, after an appropriate normalisation, belong to the class $\mathcal{A}_{\mathrm{ST}}$.

The two most famous examples of such sequences to which the results of [2, 4, 14] apply are:

- *Elliptic curves* and are given by $t_n(E)/n^{1/2}$, where $\{t_n(E)\}_{n\ge1}$ are the coefficients of the L-function of an elliptic curve E (see [15, Section 8.1] or [16, Section 14.4]).

- *Ramanujan τ-function* and are given by $\tau(n)/n^{11/2}$, where $\tau(n)$ is the Ramanujan τ-function (see [21]).

Here, we obtain a nontrivial upper bound on the size of $|a_n|$ that holds for almost all n and we also show that there are nontrivial cancellations in the summatory function of $\{a_n\}_{n\ge1} \in \mathcal{A}_{\mathrm{ST}}$.

We note that our results can be viewed as analogues of those of Fouvry and Michel [10, 11] concerning Kloosterman sums. However the approaches of [10, 11] and of our paper are very different. In particular, Kloosterman sums do not satisfy the multiplicativity condition in the definition of the class $\mathcal{A}_{\mathrm{ST}}$.

1.2 Notation

Throughout the paper, the implied constants in the symbols 'O', '$\ll$' and '$\gg$' may occasionally, where obvious, depend on the real parameters A and ϱ and are absolute, otherwise. We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are all equivalent to the assertion that the inequality $|U| \le c|V|$ holds for some constant $c > 0$.

We always use n to denote a positive integer and use p, q to denote primes. We write $\log x$ for the natural logarithm of $x \geq 1$. We put $\log_1 x = \max\{\log x, 1\}$ and for $k \geq 2$, we define $\log_k x$ as $\log_k x = \log_1 \log_{k-1} x$. Thus, for large x, $\log_k x$ coincides with the k fold iterated composition of the natural logarithm function evaluated in x (and it is 1 for smaller values of x).

1.3 Main Results

We say that some property holds for *almost all* n if the number of $n \in [1, x]$ for which it fails is $o(x)$ as $x \to \infty$.

Theorem 1. *For any sequence $\{a_n\}_{n\geq 1} \in \mathcal{A}_{\mathrm{ST}}$, the inequality*

$$|a_n| \leq (\log n)^{-1/2+o(1)}$$

holds for almost all positive integers n.

It is certainly possible to construct artificial sequences $a_n \in \mathcal{A}_{\mathrm{ST}}$ such that $a_n = 0$ for almost all n. However, if $a_n = t_n(E)$ and E is a non-CM elliptic curve or $a_n = \tau(n)$, then the exponent $1/2$ in Theorem 1 is optimal (see Theorem 3 below).

It is well-known the number of representations of n in terms of *a binary quadratic form* can be expressed via coefficients of some cusp forms, see [15, Chapter 11] and [23, Chapter 1], Thus, one interesting consequence of our Theorem 1 is that in the number of such representations has slightly more than square-root cancellation in the error term for almost all n (however making a precise general statements requires imposing a series of technical conditions and maybe quite cluttered).

With regards to the cancellations and sign changes, we prove the following result.

Theorem 2. *For any sequence $\{a_n\}_{n\geq 1} \in \mathcal{A}_{\mathrm{ST}}$, the estimate*

$$\sum_{n\leq x} a_n = o\left(\sum_{n\leq x} |a_n|\right)$$

holds as $x \to \infty$.

As we have mentioned before for many interesting sequences $\{a_n\}_{n\ge 1} \in \mathcal{A}_{\mathrm{ST}}$ our result is (conjecturally) optimal. Indeed, let us make the following two additional assumptions:

(A1) If $a_p \neq 0$ then $|a_{p^k}| \gg p^{-Ck}$ for some $C > 0$ and all integer $k \geq 1$.

(A2) We have, for any fixed $A > 0$,

$$\frac{\#\{p \le x \ : \ \vartheta_p \in [\alpha, \beta]\}}{\pi(x)} = \frac{2}{\pi}\int_{\alpha}^{\beta} \sin^2 \vartheta d\vartheta + O\left((\log_2 x)^{-A}\right).$$

uniformly in $\pi \ge \beta \ge \alpha \ge 0$ as $x \to \infty$.

In the case of $\tau(n)/n^{11/2}$, Assumption A1 is known (see [8, Lemma 9]), while Assumption A2 follows from the automorphy of $L(\mathrm{Sym}^k f, s)$ for $f(n) = a_n$ and every $k = 1, 2, \ldots$, (see [28]). It would be interesting to determine whether the currently known potential automorphy is enough in order to deduce Assumption A2.

For general coefficients of holomorphic cusp forms one can prove Assumption A1 for all primes with at most finitely many exceptions by combining [19, Lemma 9] and [18, Lemma 2.2]. For coefficients of elliptic curves Assumption A2 can be proven for fixed α, β, on average for most elliptic curves (see [1, 24, 25] for the currently strongest forms of this statement).

Theorem 3. *Assume that for $a_n \in \mathcal{A}_{\mathrm{ST}}$ both Assumptions A1 and A2 hold. Let $\mathcal{N}(x) = \{n \in [1, x] : a_n \neq 0\}$. Then, for fixed $v \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{\#\mathcal{N}(x)} \cdot \# \left\{ n \in \mathcal{N}(x) : \frac{\log|a_n| + \frac{1}{2}\log_2 n}{\sqrt{c \log_2 n}} \ge v \right\} = \int_v^{\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}$$

with

$$c = \frac{1}{2} + \frac{\pi^2}{12}.$$

In the recent work, Elliott and Kish [9] claim to be able to prove Theorem 3 under Assumption A2 but with a stronger rate of convergence of the form $o((\log X)^{-3})$. It is important to reduce the error term as much as possible in Assumption A2. For example, our assumption (but not that of Elliott and Kish [9]) allows us to use the current conditional results towards

Assumption A2 assuming automorphy. An unconditional effective Sato-Tate theorem would certainly have an even weaker error term, since the currently known potential automorphy is weaker than automorphy.

It is an interesting question whether one can establish unconditionally the upper bound part of Theorem 3. In the case of coefficients of half-integral weight cusp forms this has been very recently done unconditionally in [20].

2 Preparations

2.1 Preliminaries on Arithmetic Functions

For the proof of Theorem 1 and Theorem 2, we need the following results. The first one is given by [13, Theorem 01] or [17, Lemma 9.6].

Lemma 4. *Let f be a multiplicative function such that $f(n) \geq 0$ for every n. Assume that there exist positive constants A and B such that for $x > 1$ both inequalities*

$$\sum_{p \leq x} f(p) \log p \leq Ax \quad \text{and} \quad \sum_{p} \sum_{\alpha \geq 2} \frac{f(p^\alpha)}{p^\alpha} \log(p^\alpha) \leq B$$

hold. Then

$$\sum_{n \leq x} f(n) \leq (A + B + 1) \frac{x}{\log x} \sum_{n \leq x} \frac{f(n)}{n}.$$

We are now ready to prove some estimates that are of independent interest:

Lemma 5. *We have*

(i)

$$\sum_{n \leq x} \frac{|a_n|}{n} \ll (\log x)^{0.85},$$

(ii)

$$\sum_{n \leq x} |a_n|^2 \leq x (\log x)^{o(1)},$$

(ii) for $\gamma < 1$,

$$\sum_{n \le x} |a_n|^\gamma \le x(\log x)^{-\gamma/2 + c\gamma^2}$$

with some absolute constant $c > 0$,

as $x \to \infty$.

Proof. To prove (i), we note that by multiplicativity and by the bound on the values of $|a_n|$ at prime powers, we have for any fixed $\gamma \in [0, 2]$,

$$\begin{aligned}
\sum_{n \le x} \frac{|a_n|^\gamma}{n} &\le \prod_{p \le x} \left(1 + \frac{|a_p|^\gamma}{p} + \frac{|a_{p^2}|^\gamma}{p^2} + \cdots\right) \\
&= \prod_{p \le x} \left(1 + \frac{(2|\cos \vartheta_p|)^\gamma}{p} + O\left(\frac{1}{p^{1+2\varrho}}\right)\right) \\
&= \exp\left(\sum_{p \le x} \frac{(2|\cos \vartheta_p|)^\gamma}{p} + O\left(\sum_{p \ge 2} \frac{1}{p^{1+2\varrho}}\right)\right) \\
&= \exp\left(\sum_{p \le x} \frac{(2|\cos \vartheta_p|)^\gamma}{p} + O\left(1\right)\right).
\end{aligned}$$

Recalling the Sato–Tate distribution property of the sequence $\{a_n\}_{n \ge 1}$, and the Mertens formula (see [27, Chapter I.1, Theorem 9]), via partial summation we obtain

$$\sum_{p \le x} \frac{(2|\cos \vartheta_p|)^\gamma}{p} = \left(\frac{2}{\pi} \int_0^\pi (2|\cos \vartheta|)^\gamma \sin^2 \vartheta d\vartheta + o(1)\right) \log_2 x$$

as $x \to \infty$. Let

$$h(\gamma) = \frac{2}{\pi} \int_0^\pi (2|\cos \vartheta|)^\gamma \sin^2 \vartheta d\vartheta.$$

The above integral evaluates to $0.848826\ldots$ when $\gamma = 1$; hence, we obtain (i). The integral also evaluates to 1 when $\gamma = 2$. Furthermore for small $\gamma < 1$ since $h'(0) = -1/2$,

$$h(\gamma) \le 1 - \frac{\gamma}{2} + O(\gamma^2).$$

It follows that,

$$\sum_{n \le x} \frac{|a_n|^2}{n} \le (\log x)^{1+o(1)} \qquad \text{and} \qquad \sum_{n \le x} \frac{|a_n|^\gamma}{n} \le (\log x)^{1 - \frac{\gamma}{2} + c\gamma^2} \tag{1}$$

with $c > 0$ an absolute constant.

Noting that the functions $f(n) = |a_n|^2 \le 1$ and $f(n) = |a_n|^\gamma$ satisfy the conditions of Lemma 4 with some appropriate constants A and B, we derive (ii) and (iii) from the inequalities (1). $\square$

3 Proof of Theorem 1

Let $\varepsilon > 0$ be fixed. Let $\mathcal{N}_\varepsilon(x) = \#\{n \le x : |a_n| > (\log n)^{-1/2+\varepsilon}\}$. By Lemma 5 (iii),

$$\mathcal{N}_\varepsilon(x) \le \sum_{n \le x} |a_n|^\gamma \cdot (\log n)^{\gamma/2-\gamma\varepsilon} \ll x(\log x)^{-\gamma\varepsilon+c\gamma^2},$$

where $c > 1$ is an absolute constant. Choosing $\gamma = \varepsilon/(2c)$, we get

$$\mathcal{N}_\varepsilon(x) \le x(\log x)^{-\varepsilon^2/2} = o(x) \qquad (x \to \infty).$$

Hence, for any fixed $\varepsilon > 0$, we have $|a_n| < (\log n)^{-1/2+\varepsilon}$ for almost all n.

4 Proof of Theorem 2

4.1 Sums S and T and negligible sets

We let x be sufficiently large and define

$$S = \sum_{x/2<n\le x} a_n \quad \text{and} \quad T = \sum_{x/2<n\le x} |a_n|.$$

It suffices to show that $S = o(T)$ as $x \to \infty$. Note first that by considering only those n which are primes p such that $|\cos\vartheta_p| \ge 1/2$, a set of density $(4/\pi)\int_0^{\pi/3} \sin^2\vartheta d\vartheta > 0.39$, we already get that

$$T \ge \sum_{\substack{x/2<p\le x \\ |\cos\vartheta_p|\ge 1/2}} 2|\cos\vartheta_p| \gg \pi(x). \tag{2}$$

Suppose next that $\mathcal{N} \subseteq [x/2, x]$ is such that $\#\mathcal{N} = O(x/(\log x)^3)$. Then, by the Cauchy-Schwarz inequality and Lemma 5 (ii), we have,

$$|\sum_{n\in\mathcal{N}} a_n| \le \sum_{n\in\mathcal{N}} |a_n| \le (\#\mathcal{N})^{1/2}\left(\sum_{n\in\mathcal{N}} |a_n|^2\right)^{1/2} \le \frac{x}{(\log x)^{1.5+o(1)}} = o(T)$$

as $x \to \infty$. Hence, we may neglect the contribution to either S or T coming from any subsets $\mathcal{N}$ of $[x/2, x]$ of cardinality of order at most $x/(\log x)^3$.

4.2 Discarding contributions from exceptional integers

Let

$$y = \exp\left(\frac{4\log x \log_3 x}{\log_2 x}\right).$$

For a positive integer m put $P(m)$ for the largest prime factor of m with $P(1) = 1$. Let

$$\mathcal{N}_1(x) = \{x/2 < n \le x \ : \ P(n) \le y\}.$$

From the theory of smooth numbers [3], we know that in our range for y versus x,

$$\#\mathcal{N}_1(x) \ll x\exp(-(1+o(1))u\log u), \quad \text{where} \quad u = \frac{\log x}{\log y} \qquad (x \to \infty).$$

Since $u = 4\log_2 x/(\log_3 x)$, it follows that $u\log u = (4+o(1))\log_2 x$ as $x \to \infty$, and therefore

$$\#\mathcal{N}_1(x) \le \frac{x}{(\log x)^3}.$$

From now on, we discard the positive integers $n \in \mathcal{N}_1(x)$. Next let

$$\mathcal{N}_2(x) = \{n \in [x/2, x] \ : \ p^2 \mid n \text{ for some } p > y/2\}.$$

Fixing p, the number of $n \in [x/2, x]$ which are multiples of p^2 is at most $\lfloor x/(2p^2)\rfloor + 1$. Thus,

$$\#\mathcal{N}_2(x) \le \sum_{y/2\le p\le x^{1/2}} \left(\lfloor x/(2p^2)\rfloor + 1\right) \ll x\sum_{y/2\le p} \frac{1}{p^2} + \pi(\sqrt{x}) \ll \frac{x}{y} \ll \frac{x}{(\log x)^3}.$$

From now on, we also discard the positive integers $n \in \mathcal{N}_2(x)$.

Consider $n \in [x/2, x] \setminus (\mathcal{N}_1(x) \cup \mathcal{N}_2(x))$.

We write $n = P(n)m$. Then $P(n) > \max\{y, P(m)\}$ is prime and $P(n) \in [x/(2m), x/m]$. Further, any prime P in the interval $[x/(2m), x/m]$ can serve the role of $P = P(n)$ for $n = Pm$, except when it is the case that $P(m) \in [x/(2m), x/m]$. Let $\mathcal{N}_3(x)$ be the set of n of the form $n = P(n)m$, where $P(n) > P(m)$ and $P(m) \in [x/(2m), x/m]$. Then $P(m) > x/(2m) \geq P(n)/2 > y/2$, so that $P(m) \| m$ (that is, $P(m)^2 \nmid m$). Write $Q = P(m)$ and $m = Q\ell$. Then $n = PQ\ell$, $P > Q > P/2$ and $P(\ell) < Q$. Further,

$$\frac{x}{2\ell} < PQ < \frac{x}{\ell}, \quad \text{therefore} \quad \left(\frac{x}{2\ell}\right)^{1/2} < P < 2Q < 2\left(\frac{x}{\ell}\right)^{1/2}.$$

Let $\mathcal{L}(x)$ be the set of all possible values ℓ and for a fixed $\ell \in \mathcal{L}(x)$, let $\mathcal{N}_{3,\ell}(x)$ be the subset of $n = PQ\ell$ from $\mathcal{N}_3(x)$ with the corresponding ℓ. Note that P, Q and ℓ are pairwise relatively prime. Then

$$\begin{aligned}\left|\sum_{n \in \mathcal{N}_{3,\ell}(x)} a_n\right| &\leq |a_\ell| \sum_{P \in \left(\sqrt{x/(2\ell)}, 2\sqrt{x/\ell}\right)} \sum_{Q \in \left(\sqrt{x/(8\ell)}, \sqrt{x/\ell}\right)} |a_P||a_Q| \\ &\ll |a_\ell| \pi \left(2\left(\frac{x}{\ell}\right)^{1/2}\right)^2 \ll \frac{x|a_\ell|}{\ell(\log(x/\ell))^2} \ll \frac{x}{(\log y)^2} \frac{|a_\ell|}{\ell}.\end{aligned}$$

Summing up over all the possible values of ℓ and invoking Lemma 5 (i) and estimate (2), we get that

$$\begin{aligned}\left|\sum_{n \in \mathcal{N}_3(x)} a_n\right| &\leq \sum_{\ell \in \mathcal{N}(x)} \sum_{n \in \mathcal{N}_{3,\ell}(x)(x)} |a_n| \ll \frac{x}{(\log y)^2} \sum_{\ell \leq x} \frac{|a_\ell|}{\ell} \\ &\ll \frac{x(\log x)^{0.85}(\log_2 x)^2}{(\log x)^2(\log_3 x)^2} \ll \frac{x}{(\log x)^{1.1}} = o(T)\end{aligned}$$

as $x \to \infty$.

4.3 Contributions to S and T from typical integers

Define

$$\mathcal{N}(x) \in [x/2, x] \setminus (\mathcal{N}_1(x) \cup \mathcal{N}_2(x) \cup \mathcal{N}_3(x)).$$

For each such $n \in \mathcal{N}(x)$, we write as before $n = Pm$ and let $\mathcal{M}(x)$ be the set of all possible values of m. Fix $m \in \mathcal{M}(x)$. We compare

$$S_m = \sum_{\substack{n \in \mathcal{N}(x) \\ n = P(n)m}} a_n = a_m \sum_{P \in [x/2m, x/m]} a_P$$

and

$$T_m = \sum_{\substack{n \in \mathcal{N}(x) \\ n = P(n)m}} |a_n| = |a_m| \sum_{P \in [x/(2m), x/m]} |a_P|.$$

Note that

$$\int_0^\pi \cos\vartheta \sin^2\vartheta d\vartheta = 0 \qquad \text{and} \qquad \int_0^\pi |\cos\vartheta| \sin^2\vartheta d\vartheta = \frac{1}{3}.$$

Clearly, since $\{a_n\}_{n\geq 1} \in \mathcal{A}_{\mathrm{ST}}$, we have

$$\begin{aligned} \sum_{P \in [x/(2m), x/m]} a_P &= 2 \sum_{P \in [x/(2m), x/m]} \cos\vartheta_P \\ &= \left(\frac{4}{\pi} \int_0^\pi \cos\vartheta \sin^2\vartheta d\vartheta + o(1) \right) (\pi(x/m) - \pi(x/2m)) \\ &= o(\pi(x/m)), \end{aligned} \tag{3}$$

whereas

$$\begin{aligned} \sum_{P \in [x/(2m), x/m]} |a_P| &= 2 \sum_{P \in [x/(2m), x/m]} |\cos\vartheta_P| \\ &= \left(\frac{4}{\pi} \int_0^\pi |\cos\vartheta| \sin^2\vartheta d\vartheta + o(1) \right) (\pi(x/m) - \pi(x/2m)) \\ &\gg \pi(x/m). \end{aligned} \tag{4}$$

We see from the definition of S_m and T_m that if $a_m = 0$ then $S_m = T_m = 0$. Comparing (3) and (4), we obtain that if $a_m \neq 0$ then $S_m = o(T_m)$. Therefore,

$$\begin{aligned} S = \sum_{n \in [x/2, x]} a_n &= \sum_{m \in \mathcal{M}(x)} S_m + o(T) = \sum_{\substack{m \in \mathcal{M}(x) \\ a_m \neq 0}} S_m + o(T) \\ &= \sum_{\substack{m \in \mathcal{M}(x) \\ a_m \neq 0}} o\left(T_m\right) + o(T) = \sum_{m \in \mathcal{M}(x)} o\left(T_m\right) + o(T) \\ &= o\left(\sum_{m \in \mathcal{M}(x)} T_m \right) + o(T) = o(T) \qquad (x \to \infty), \end{aligned}$$

as desired.

5 Proof of Theorem 3

5.1 The sets $\mathcal{N}(x)$ and $\mathcal{A}(x)$

As in the statement of Theorem 3, let $\mathcal{N}(x) = \{n \le x \ : \ a_n \ne 0\}$.

We recall that all implied constants may depend on the parameter A.

Lemma 6. *For a fixed $A > 1$, and $y \ge 2$, then*

$$\sum_{\substack{p \ge y \\ |a_p| < (\log_2 p)^{-A}}} \frac{1}{p} \ll \frac{1}{(\log_2 y)^{A-1}}.$$

Proof. We may assume that $y \ge 10$. Let $k_y \ge 3$ be the minimal positive integer such that $e^{k_y} \ge y$. Splitting the summation range into intervals of the form $[e^k, e^{k+1}]$ for integers $k \ge k_y - 1$, and using Assumption A2, we get

$$\begin{aligned} \sum_{\substack{p \ge y \\ |a_p| < (\log_2 p)^{-A}}} \frac{1}{p} &\le \sum_{k \ge k_y - 1}^{\infty} \sum_{\substack{e^k < p \le e^{k+1} \\ |a_p| < (\log k)^{-A}}} \frac{1}{p} \ll \sum_{k \ge k_y - 1}^{\infty} \frac{1}{e^k} \sum_{\substack{p < e^{k+1} \\ |a_p| < (\log k)^{-A}}} 1 \\ &\ll \sum_{k \ge k_y - 1} \frac{1}{e^k} \cdot \frac{e^{k+1}}{k(\log k)^A} \ll \sum_{k \ge k_y - 1} \frac{1}{k(\log k)^A} \\ &\ll \frac{1}{(\log k_y)^{A-1}} \ll \frac{1}{(\log_2 y)^{A-1}}, \end{aligned}$$

as claimed. ◻

Put

$$\kappa = \prod_{\substack{p \ge 2 \\ a_p = 0}} \left(1 - \frac{1}{p}\right).$$

Lemma 6 with any $A > 1$ shows that $\kappa > 0$.

Lemma 7. *For a fixed $A > 1$, we have*

$$\#\mathcal{N}(x) = \kappa x \left(1 + O\left(\frac{1}{(\log_2 x)^{A-1}}\right)\right).$$

Proof. Let f be a multiplicative function such that $f(p^\alpha) = f(p) = 1$ if $a_p \neq 0$ and $f(p) = 0$ otherwise. Write

$$f(n) = \sum_{d|n} g(d)$$

with g a multiplicative function such that $g(p^\alpha) = f(p) - 1$ if $\alpha = 1$ and $p \leq x$ and $g(p^\alpha) = 0$ otherwise. Then,

$$\#\mathcal{N}(x) = \sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d|n} g(d) = \sum_{d \leq x} g(d) \left(\frac{x}{d} + O(1) \right). \tag{5}$$

By Lemma 4,

$$\sum_{n \leq x} |g(n)| \ll \frac{x}{\log x} \prod_{\substack{p \leq x \\ p \notin \mathcal{N}(x)}} \left(1 + \frac{1}{p} \right).$$

Since $p \notin \mathcal{N}(x)$ implies, in particular, that $|a_p| < (\log_2 p)^{-2}$, we have

$$\sum_{\substack{p \leq x \\ p \notin \mathcal{N}(x)}} \frac{1}{p} = O(1)$$

by Lemma 6 with $A = 2$ and $y = 2$. Hence,

$$\prod_{\substack{p \leq x \\ p \notin \mathcal{N}(x)}} \left(1 + \frac{1}{p} \right) = O(1).$$

Furthermore, observe that

$$\sum_{d > x} \frac{|g(d)|}{d} = \sum_{\substack{d > x \\ p|d \Rightarrow p \notin \mathcal{N}(x)}} \frac{\mu^2(d)}{d}.$$

To evaluate the last sum above, we split it at $y = \exp((\log x)^{1/3})$. In the lower range, the sum is bounded by

$$S_1 = \sum_{\substack{d > x \\ P(d) < y}} \frac{\mu(d)^2}{d}.$$

Putting $u = \log x / \log y = (\log x)^{2/3}$, we have, by known estimates concerning smooth numbers,

$$S_1 \ll \frac{1}{\exp(u)} \sum_{P(n) \le y} \frac{1}{n} \ll \frac{\log y}{e^u} \ll \frac{1}{e^{u/2}} \ll \frac{1}{(\log_2 x)^{A-1}}.$$

In the upper range, we have $P(d) \ge y$, so writing $d = pm$, where $p = P(d)$, we get that the sum in this range is bounded by

$$\begin{aligned} S_2 = \sum_{\substack{p \notin \mathcal{N}(x) \\ p > y}} \frac{1}{p} \sum_{\substack{m \\ q \mid m \Rightarrow q \notin \mathcal{N}(x)}} \frac{1}{m} &\ll \prod_{q \notin \mathcal{N}(x)} \left(1 + \frac{1}{q}\right) \sum_{\substack{p \notin \mathcal{N}(x) \\ p > y}} \frac{1}{p} \\ &\ll \frac{1}{(\log_2 y)^{A-1}} \ll \frac{1}{(\log_2 x)^{A-1}}, \end{aligned}$$

by Lemma 6. Hence, the equation (5) becomes

$$\#\mathcal{N}(x) = \prod_{p \le x} \left(1 + \frac{g(p)}{p}\right) x \left(1 + O\left(\frac{1}{(\log_2 x)^{A-1}}\right)\right).$$

It remains to show that

$$\kappa = \prod_{p \le x} \left(1 + \frac{g(p)}{p}\right) \left(1 + O\left(\frac{1}{(\log_2 x)^{A-1}}\right)\right).$$

However this is equivalent to

$$\prod_{\substack{p > x \\ a_p = 0}} \left(1 - \frac{1}{p}\right) = 1 + O\left(\frac{1}{(\log_2 x)^{A-1}}\right),$$

and this follows immediately from Lemma 6 for any $A > 1$ and $y = x$. $\square$

Note that by Assumption A2, for every $A > 0$, we have

$$\frac{1}{\pi(x)} \cdot \# \left\{ p \le x \ : \ |a_p| < (\log_2 x)^{-A} \right\} \ll (\log_2 x)^{-A}.$$

Instead of working on the set $\mathcal{N}(x)$ we work on the more convenient set

$$\mathcal{N}_A(x) = \left\{ n \le \mathcal{N}(x) \ : \ p \mid n \implies |a_p| > (\log_2 x)^{-A} \right\}.$$

The following result justifies this change.

Lemma 8. *For a fixed $A > 1$, uniformly for sets $\mathcal{A} \subset \mathbb{R}$, we have*

$$\frac{\#\{n \in \mathcal{N}(x) \ : \ \log|a_n| \in \mathcal{A}\}}{\#\mathcal{N}(x)} = \frac{\#\{n \in \mathcal{N}_A(x) : \log|a_n| \in \mathcal{A}\}}{\#\mathcal{N}_A(x)} + O\left(\frac{1}{(\log_4 x)^{A-1}}\right)$$

as $x \to \infty$.

Proof. Let $\mathcal{E}_A(x)$ be the set of those $n \in \mathcal{N}(x)$ which are divisible by a prime $p \le x$ such that $|a_p| < (\log_2 x)^{-A}$. Since $a_p \neq 0$, by Assumption A1, we have $|a_p| \gg p^{-C}$ for some absolute constant $C > 0$. Hence, $|a_p| < (\log_2 x)^{-A}$ implies that $p > (\log_2 x)^{A/(2C)}$ provided that x is sufficiently large. Thus, using Lemma 6 with $y = (\log_2 x)^{A/(2C)}$, we have

$$\#\mathcal{E}_A(x) \ll \sum_{\substack{p \le x \\ p \notin \mathcal{N}_A(x)}} \frac{x}{p} \le \sum_{\substack{p > (\log_2 x)^{A/(2C)} \\ |a_p| < (\log_2 p)^{-A}}} \frac{x}{p} \ll \frac{x}{(\log_4 x)^{A-1}}. \tag{6}$$

Hence,

$$\#\mathcal{N}(x) = \#\mathcal{N}_A(x) + O\left(\frac{x}{(\log_4 x)^{A-1}}\right),$$

and since

$$\#\mathcal{N}(x) = \kappa x \left(1 + O\left(\frac{1}{(\log_2 x)^{A-1}}\right)\right),$$

by Lemma 7, we have that

$$\#\mathcal{N}(x) = \left(1 + O\left(\frac{1}{(\log_4 x)^{A-1}}\right)\right) \#\mathcal{N}_A(x). \tag{7}$$

In addition,

$$\#\{n \in \mathcal{N}(x) : \log|a_n| \in \mathcal{A}\} = \#\{n \in \mathcal{N}_A(x) : \log|a_n| \in \mathcal{A}\} + O(\#\mathcal{E}_A(x)).$$

Dividing the last relation above by $\#\mathcal{N}(x)$ and using (6) and (7), we obtain the claim. ∎

5.2 Approximation to a_n

We now define a strongly multiplicative (that is, $h(p^k) = h(p)$) function h such that $h(p) = a_p$. Next we show that $h(n)$ is a good approximation to a_n.

Lemma 9. *Let $A > 1$ be fixed and let $\psi(x) \to \infty$ as $x \to \infty$. For all but $o(x)$ integers $n \in \mathcal{N}_A(x)$ we have*

$$\log |h(n)| = \log |a_n| + O(\psi(x)).$$

Proof. Let $c(n) = \log |h(n)| - \log |a_n|$. Notice that for $n \in \mathcal{N}_A(x)$ by Assumption A1 and our standard assumption that $|a_{p^\alpha}| < p^{(\alpha-1)/2-\delta}$, we have

$$|c(p^\alpha)| \ll \alpha \log p.$$

Consider,

$$\sum_{n \in \mathcal{N}_A(x)} |c(n)|^2 = \sum_{n \in \mathcal{N}_A(x)} \left| \sum_{\substack{p^\alpha \| n \\ \alpha \geq 2}} c(p^\alpha) \right|^2 = \sum_{\substack{p^\alpha, q^\beta \leq x \\ \alpha, \beta \geq 2}} |c(p^\alpha)| \cdot |c(q^\beta)| \sum_{\substack{n \in \mathcal{N}_A(x) \\ \mathrm{lcm}\,[p^\alpha, q^\beta] | n}} 1$$
$$\ll x \sum_{\substack{p^\alpha, q^\beta \leq x \\ \alpha, \beta \geq 2}} \frac{\alpha \log p \cdot \beta \log q}{\mathrm{lcm}\,[p^\alpha, q^\beta]} = O(x).$$

It is now obvious that for all but $O(x/\psi^2(x))$ integers $n \leq x$ we have $|c(n)| = O(\psi(x))$. □

5.3 Some properties of the set $\mathcal{N}_A(x)$

We are almost ready to prove Theorem 3, but before hand we need the following three results about the set $\mathcal{N}_A(x)$.

Lemma 10. *Let $A > 1$ be fixed and let $f_{A,x}$ be the multiplicative function such that $f_{A,x}(p^k) = f_{A,x}(p)$ and $f_{A,x}(p) = 1$ if $p \in \mathcal{N}_A(x)$ and $f_{A,x}(p) = 0$ otherwise. Then, uniformly over integers $d \geq 1$,*

$$\sum_{\substack{d | n \\ n \in \mathcal{N}_A(x)}} 1 = \frac{f_{A,x}(d)}{d} \cdot \#\mathcal{N}_A(x) + r_{A,x,d},$$

where

$$|r_{A,x,d}| \ll \frac{x}{d(\log_2(x/d))^{A-1}}.$$

Proof. Notice that $f_{A,x}$ is completely multiplicative, so that

$$f_{A,x}(dn) = f_{A,x}(d) f_{A,x}(n).$$

Thus,

$$\sum_{\substack{d|n \\ n\in\mathcal{N}_A(x)}} 1 = \sum_{n\le x/d} f_{A,x}(dn) = f_{A,x}(d) \sum_{n\le x/d} f_{A,x}(n).$$

Write

$$f_{A,x}(n) = \sum_{d|n} g_{A,x}(d),$$

with $g_{A,x}(p^k) = f_{A,x}(p) - 1$ for $k = 1$ and $p \le x$ and $g_{A,x}(p^k) = 0$ otherwise. Clearly,

$$\sum_{n\le x/d} f_{A,x}(n) = \sum_{n\le x/d} \sum_{d|n} g_{A,x}(d) = \sum_{e\le x/d} g_{A,x}(e) \left(\frac{x}{de} + O(1)\right). \tag{8}$$

Since

$$\sum_{e=1}^{\infty} \frac{g_{A,x}(e)}{e} = \prod_p \left(1 - \frac{g_{A,x}(p)}{p}\right), \tag{9}$$

it remains to estimate the sums

$$\sum_{e\le x/d} |g_{A,x}(e)| \qquad \text{and} \qquad \sum_{n>x/d} \frac{|g_{A,x}(n)|}{n}.$$

However, it is easy to see that

$$\sum_{e\le x/d} |g_{A,x}(e)| \ll \frac{x}{d\log(x/d)}, \tag{10}$$

and

$$\sum_{n>x/d} \frac{|g_{A,x}(n)|}{n} \ll \frac{x}{d(\log_2(x/d))^{A-1}}, \tag{11}$$

by Lemmas 4 and 6, and the argument from the proof of Lemma 7, respectively. Making $d = 1$ in (8), we see that

$$\begin{aligned}\#\mathcal{N}_A(x) &= \sum_{n\in\mathcal{N}_A(x)} 1 = x\sum_{e=1}^{\infty}\frac{g_{A,x}(e)}{e} + O\left(\frac{x}{(\log_2 x)^{A-1}}\right)\\ &= x\prod_p\left(1-\frac{g_{A,x}(p)}{p}\right) + O\left(\frac{x}{(\log_2 x)^{A-1}}\right),\end{aligned} \tag{12}$$

and we derive the desired conclusion from (8) and (9), error estimates (10) and (11), and the main term (12). □

We define

$$\mu_{A,x} = \sum_{p\in\mathcal{N}_A(x)}\frac{\log|a_p|}{p} \qquad \text{and} \qquad \sigma_{A,x}^2 = \sum_{p\in\mathcal{N}_A(x)}\frac{(\log|a_p|)^2}{p}\left(1-\frac{1}{p}\right).$$

Lemma 11. *For a fixed $A > 1$, we have*

$$\begin{aligned}\mu_{A,x} &= -\frac{1}{2}\log_2 x + O(\log_3 x),\\ \sigma_{A,x}^2 &= \left(\frac{1}{2}+\frac{\pi^2}{12}\right)\log_2 x + O\left((\log_3 x)^2\right).\end{aligned}$$

Proof. By Lemma 6, applied with $A = 2$ and $y = 3$ we see that the primes p with $|a_p| \le (\log_2 x)^{-2}$ contribute $O(\log_3 x)$ to $\mu_{A,x}$ and $O((\log_3 x)^2)$ to $\sigma_{A,x}^2$.

The contribution of the remaining p to $\mu_{A,x}$, by integration by parts, can be estimated as

$$\begin{aligned}&\frac{2\log_2 x}{\pi}\int_{(\log_2 x)^{-2}}^{2}\log u\cdot\sqrt{1-(u/2)^2}du + O(\log_3 x)\\ &\qquad= \frac{2\log_2 x}{\pi}\int_0^2 \log u\cdot\sqrt{1-(u/2)^2}du + O(\log_3 x)\\ &\qquad= -\frac{1}{2}\log_2 x + O(\log_3 x)\end{aligned}$$

which gives the first part of the claim.

The proof of the second part of the claim uses the fact that

$$\frac{2}{\pi}\int_0^2(\log u)^2\sqrt{1-(u/2)^2}du = \frac{1}{2}+\frac{\pi^2}{12},$$

and is completely similar. □

Finally using the work of Granville and Soundararajan [12] and Lemmas 10 and 11, we are able to conclude.

Lemma 12. *For a fixed $A > 1$, we have,*

$$\frac{1}{\#\mathcal{N}_A(x)} \# \left\{ n \in \mathcal{N}_A(x) : \frac{\log|g(n)| + \frac{1}{2}\log_2 x}{\sqrt{c \log_2 x}} \in [\alpha, \beta] \right\} \to \int_\alpha^\beta e^{-u^2/2} \frac{du}{\sqrt{2\pi}}$$

as $x \to \infty$, with

$$c = \frac{1}{2} + \frac{\pi^2}{12}.$$

Proof. Let $\mathcal{P}$ be the set of primes $p \in \mathcal{N}_A(x)$ which are less than $y = x^{1/\log_3 x}$. Let $M(x) = \log_3 x$. Note that $|\log|g(p)|| \ll M(x)$ for every $p \in \mathcal{P}$. Applying a result of Granville and Soundararajan [12, Proposition 4], in the notation of Lemmas 10 and 11, we get for every fixed even integer k,

$$\sum_{a \in \mathcal{N}_A(x)} \left(\sum_{\substack{p|a \\ p \in \mathcal{P}}} \log|g(p)| - \mu_{A,y} \right)^k = C_k \# \mathcal{N}_A(x) \sigma_{A,y}^k \left(1 + O\left(\frac{M(x)}{\sigma_{A,x}^2} \right) \right)$$
$$+ O\left(M(x)^k (\log_2 x)^k \sum_{d \in \mathcal{D}_k(\mathcal{P})} |r_{A,x,d}| \right),$$

where the implied constant may depend on both A and k and

$$C_k = \frac{\Gamma(k+1)}{2^{k/2}\Gamma(k/2+1)},$$

is the k-th Gaussian moment and $\mathcal{D}_k(\mathcal{P})$ is the set of integers formed out of k prime factors from the set $\mathcal{P}$. For k odd, we have,

$$\sum_{a \in \mathcal{N}_A(x)} \left(\sum_{\substack{p|a \\ p \in \mathcal{P}}} \log|g(p)| - \mu_y \right)^k$$
$$\ll \# \mathcal{N}_A(x) \sigma_y^k \frac{k^{3/2} M(x)}{\sigma_{A,y}} + M(x)^k (\log_2 x)^k \sum_{d \in \mathcal{D}_k(\mathcal{P})} |r_{A,x,d}|,$$

where the implied constants may also depend on k. Using Lemma 10, we see that

$$\sum_{d\in\mathcal{D}_k(\mathcal{P})} |r_{A,x,d}| \ll \sum_{d\in\mathcal{D}_k(\mathcal{P})} \frac{x}{d(\log_2(x^{1-k/\log_3 x}))^{3k}} \ll x(\log_2 x)^{-2k},$$

provided $A > 3k+1$. Therefore, we conclude that

$$\sum_{a\in\mathcal{N}_A(x)} \left(\sum_{\substack{p|a\\ p\in\mathcal{P}}} \log|g(p)| - \mu_y \right)^k = (\eta_k + o(1))\, C_k \#\mathcal{N}_A(x)\sigma_{A,y}^k$$

as $x\to\infty$, where

$$\eta_k = \begin{cases} 1 & \text{if } k \text{ even,} \\ 0 & \text{if } k \text{ odd.} \end{cases}$$

By the method of moments the above estimates imply that

$$\frac{1}{\#\mathcal{N}_A(x)} \# \left\{ n\in\mathcal{N}_A(x) \;:\; \frac{\sum_{\substack{p|n\\ p\in\mathcal{P}}} \log|g(p)| - \mu_{A,y}}{\sigma_{A,y}} \in [\alpha,\beta] \right\} \to \int_\alpha^\beta e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \tag{13}$$

as $x\to\infty$. Notice that since $p\in\mathcal{N}_A(x)$ we have $|\log|g(p)|| \ll M(x)$. Hence, for $n\in\mathcal{N}_A(x)$,

$$\left| \sum_{\substack{p|n\\ p\in\mathcal{P}}} \log|g(p)| - \log|g(n)| \right| \ll M(x)\log_3 x \ll (\log_3 x)^2.$$

In addition, by Lemma 11, for the above choice of y, after simple calculations, we derive

$$\mu_{A,y} = -\frac{1}{2}\log_2 x + O(\log_3 x)$$

and similarly,

$$\sigma_{A,y}^2 = \left(\frac{1}{2} + \frac{\pi^2}{12}\right)\log_2 x + O(\log_3 x)^2).$$

Combining this with (13), we obtain the claim. □

5.4 Concluding the proof

The result follows upon combining Lemmas 8, 9 and 12.

6 Comments

Let $\tau(n)$ by the Ramanujan τ function defined by

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{k\geq 1} (1-q^k)^{24},$$

introduced in [21]. By the celebrated Deligne [5] bound, we can write

$$\tau(p) = 2p^{11/2} \cos \vartheta_p$$

with $\vartheta_p \in [0, \pi]$. It is also known that

$$\tau(n) \leq d(n) n^{11/2},$$

where $d(n)$ denotes the number of divisors of n (see [15, Equation (3.32)]). Recall that $d(n) \leq (\log n)^{\log 2 + o(1)}$ for almost all n (see [17, Equation (7.14) and Problem 7.3] or [27, Chapter I.5, Equation (9) and Theorem 5]). In particular, the inequality

$$\tau(n) < n^{11/2} (\log n)^{\log 2 + o(1)}$$

holds for almost all n. Recalling that as $\tau(n)$ is multiplicative (see [15, Equation (3.34)]), and by [2, 4, 14] we know that $\tau(p)/p^{11/2}$ satisfies the Sato–Tate distribution, we derive from Theorem 1 that in fact the inequality

$$\tau(n) \leq n^{11/2} (\log n)^{-0.5 + o(1)}$$

holds as n tends to infinity in a density 1 subset of $\mathbb{N}$.

It is certainly interesting to have a more quantiative version of Theorem 2 with an explicit saving on the right hand side. Such a general result seems impossible as one needs concrete estimates on the rate of convergency to the Sato–Tate distribution in the definition of the sequence $\{a_n\}_{n\geq 1}$. However, for the above concrete examples our method can lead to such a result.

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