A NOTE ON THE LIOUVILLE FUNCTION IN SHORT INTERVALS

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Abstract. In this note we give a short and self-contained proof that, for any $\delta > 0$, $\sum_{x \le n \le x + x^{\delta}} \lambda(n) = o(x^{\delta})$ for almost all $x \in [X, 2X]$. We also sketch a proof of a generalization of such a result to general real-valued multiplicative functions. Both results are special cases of results in our more involved and lengthy recent pre-print.

1. Introduction

In our recent pre-print [4] we have proved (among other things) the following theorem.

Theorem 1. *Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function, and let $h = h(X) \to \infty$, arbitrarily slowly with $X \to \infty$. Then, for almost all $X \le x \le 2X$,*

$$\frac{1}{h} \sum_{x \le n \le x+h} f(n) = \frac{1}{X} \sum_{X \le n \le 2X} f(n) + o(1)$$

with $o(1)$ not depending on f.

In particular for the Liouville function this result implies that, for any $\psi(X) \to \infty$ with $X \to \infty$, we have

$$\sum_{x \le n \le x + \psi(X)} \lambda(n) = o(\psi(X)) \tag{1}$$

for almost all $X \le x \le 2X$. Previously this was known unconditionally only when $\psi(X) \ge X^{1/6}$ (using zero-density theorems), and under the density hypothesis for $\psi(X) \ge X^{\delta}$ for any $\delta > 0$.

The proof of Theorem 1 is complicated for two reasons. First of all, in order to achieve the result for a specific function such as $\lambda(n)$ with h growing arbitrarily slowly we need to perform a messy decomposition of the Dirichlet polynomial

$$\sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}}$$

according to the size of $\sum_{P<p<Q} \lambda(p)p^{-1-it}$ for suitable intervals $[P, Q]$. Secondly, to obtain the result for arbitrary f, we need to input some additional ideas dealing with large values of Dirichlet polynomials. We realized recently that in the special case of the Liouville function and intervals of length X^δ neither is necessary.

Our goal in this short note is to give a short and self-contained proof of the following special case of Theorem 1.

Theorem 2. *Let $\delta > 0$ be given. Then, for almost all $X \le x \le 2X$, we have*

$$\sum_{x \le n \le x + X^\delta} \lambda(n) = o(X^\delta).$$

We have not tried to optimize any of the bounds for the amount of cancellations or for the size of the exceptional set. With a bit additional effort this can be done (but we refer the reader to our paper [4]).

For the convenience of the reader we have also indicated in the appendix how to generalize this result to arbitrary multiplicative f. This is more intricate and depends on a number of lemmas which are proven in our paper [4]. We will invoke these lemmas freely throughout the proof of the following theorem.

Theorem 3. *Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function. Let $\delta > 0$. Then, for almost all $X \le x \le 2X$, we have*

$$\frac{1}{X^\delta} \sum_{x \le n \le x + X^\delta} f(n) = \frac{1}{X} \sum_{X \le n \le 2X} f(n) + o(1)$$

with $o(1)$ not depending on f.

It is worthwhile to point out that essentially the only non-standard idea from [4] that is needed in the proof of Theorem 2 is the use of Ramaré type identity (see (3) below).

2. The main propositions

Theorem 2 follows immediately from the following proposition.

Proposition 1. *Let $\delta > 0$ be given. Then, for any $\varepsilon > 0$,*

$$\int_X^{2X} \Big| \frac{1}{X^\delta} \sum_{x \le n \le x + X^\delta} \lambda(n) \Big|^2 dx \ll_\varepsilon \frac{X}{(\log X)^{1/3 - \varepsilon}}.$$

Deduction of Theorem 2 from Proposition 1. By Chebyschev's inequality the number of exceptional $x \in [X, 2X]$ for which

$$\Big| \frac{1}{X^\delta} \sum_{x \le n \le x + X^\delta} \lambda(n) \Big| \ge \frac{1}{(\log X)^{1/9}}.$$

is at most

$$(\log X)^{2/9} \int_X^{2X} \Big| \frac{1}{X^\delta} \sum_{x \le n \le x + X^\delta} \lambda(n) \Big|^2 dx \ll_\varepsilon \frac{X}{(\log X)^{1/9 - \varepsilon}} = o(X)$$

as claimed. □

In order to prove Theorem 3 we will sketch the proof of the following proposition in the Appendix.

Proposition 2. *Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function. Let $\delta > 0$ be given. Then*

$$\int_X^{2X} \Big| \frac{1}{X^\delta} \sum_{x \le n \le x + X^\delta} f(n) - \frac{1}{X} \sum_{X \le n \le 2X} f(n) \Big|^2 dx \ll \frac{X}{(\log X)^{1/48}}.$$

3. Lemmas

In Lemma 4 below we relate the integral in Proposition 1 to a mean square of a Dirichlet polynomial. To deal with this, we use the following three standard lemmas.

Lemma 1. *Let $A > 0$ be given. We have, uniformly in $|t| \le (\log X)^A$,*

$$\sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \ll (\log X)^{-A}.$$

Proof. By the prime number theorem for any $A > 0$, we have,

$$\sum_{X \le n \le u} \frac{\lambda(n)}{n} \ll (\log X)^{-2A}$$

for any $u \in [X, 2X]$. Therefore, integrating by parts we find

$$\begin{aligned} \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} &= \int_X^{2X} u^{-it} d \sum_{X \le n \le u} \frac{\lambda(n)}{n} \\ &\ll \frac{|t|}{X} \int_X^{2X} \Big| \sum_{X \le n \le u} \frac{\lambda(n)}{n} \Big| du + (\log X)^{-2A} \\ &\ll (\log X)^A \cdot (\log X)^{-2A} = (\log X)^{-A} \end{aligned}$$

which gives the claim. □

Lemma 2. *Let $A > 0$ be given and $X \ge 1$. Assume that* $\exp((\log X)^\theta) \le P \le Q \le X$ *for some $\theta > 2/3$ and let*

$$\mathcal{P}(1 + it) = \sum_{P \le p \le Q} \frac{1}{p^{1+it}}.$$

Then, for any $|t| \le X$,

$$|\mathcal{P}(1+it)| \ll \frac{\log X}{1+|t|} + (\log X)^{-A}.$$

Proof. In case $|t| \le 10$, the claim follows immediately from the prime number theorem, so we can assume $|t| > 10$. We can also assume that fractional parts of P and Q are $1/2$ each. Perron's formula says that, for any $\kappa > 0$ and $y > 0$, we have

$$\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} y^s \cdot \frac{ds}{s} = \begin{cases} 1 & \text{if } y > 1 \\ 0 & \text{if } y < 1 \end{cases} + O\Big(\frac{y^\kappa}{\max(1, T|\log y|)}\Big).$$

Therefore, letting $\kappa = 1/\log X$, and $T = (|t|+1)/2 < |t| - 1$, we have

$$\mathcal{P}(1+it) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \log\zeta(s+1+it) \cdot \frac{Q^s - P^s}{s} \cdot ds + O\left(\frac{\log X}{|t|+1} + \frac{1}{P^{1/2}}\right). \tag{2}$$

Using Vinogradov's zero-free region, we see that $\log\zeta(s+1+it)$ is well defined in the region

$$\mathcal{R} : 1 \le |\Im s + t| \le 2X \ , \ \Re s \ge -\sigma_0 := -\frac{1}{(\log X)^{2/3}(\log\log X)}.$$

In addition for $s \in \mathcal{R}$ we have $|\log\zeta(s+1+it)| \ll (\log X)^2$. Therefore shifting the contour in (2) to the edge of this region, we see that

$$\begin{aligned}\mathcal{P}(1+it) &= \frac{1}{2\pi i}\int_{-T}^{T} \log\zeta(1-\sigma_0+iu+it) \cdot \frac{Q^{-\sigma_0+iu} - P^{-\sigma_0+iu}}{-\sigma_0+iu} du + O\Big(\frac{\log X}{|t|+1} + \frac{1}{P^{1/2}}\Big) \\ &\ll (\log X)^{-A} + \frac{\log X}{|t|+1}.\end{aligned}$$

as claimed. □

Lemma 3. *One has*

$$\int_{-T}^{T} \Big|\sum_{n\sim X} \frac{a_n}{n^{1+it}}\Big|^2 \ll (T+X)\sum_{n\sim X}\frac{|a_n|^2}{n^2}.$$

Proof. See [3, Theorem 9.1]. □

4. Proof of Proposition 1

We start with the following lemma which is in the spirit of previous work on primes in almost all intervals, see for instance [2, Lemma 9.3].

Lemma 4. *Let $\delta > 0$ be given. Then*

$$\frac{1}{X}\int_X^{2X} \Big|\frac{1}{X^\delta}\sum_{x\le n\le x+X^\delta}\lambda(n)\Big|^2 dx$$
$$\ll \int_0^{X^{1-\delta}} \Big|\sum_{n\sim X}\frac{\lambda(n)}{n^{1+it}}\Big|^2 dt + \max_{T>X^{1-\delta}}\frac{X^{1-\delta}}{T}\int_T^{2T}\Big|\sum_{n\sim X}\frac{\lambda(n)}{n^{1+it}}\Big|^2 dt.$$

Proof. Write $h := X^\delta$. By Perron's formula

$$\frac{1}{h}\sum_{x\le n\le x+h}\lambda(n) = \frac{1}{h}\cdot\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\Big(\sum_{n\sim X}\frac{\lambda(n)}{n^s}\Big)\cdot\frac{(x+h)^s-x^s}{s}ds.$$

Hence it is enough to bound

$$V := \frac{1}{h^2X}\int_X^{2X}\left|\int_1^{1+i\infty}F(s)\frac{(x+h)^s-x^s}{s}ds\right|^2 dx,$$

where $F(s)=\sum_{n\sim X}\lambda(n)n^{-s}$. We would like to add a smoothing, take out a factor x^s, expand the square, exchange the order of integration and integrate over x. However, the term $(x+h)^s$ prevents us from doing this and we overcome this problem in a similar way to [5, Page 25]. We write

$$\frac{(x+h)^s-x^s}{s} = \frac{1}{2h}\left(\int_h^{3h}\frac{(x+w)^s-x^s}{s}dw - \int_h^{3h}\frac{(x+w)^s-(x+h)^s}{s}dw\right)$$
$$= \frac{x}{2h}\int_{h/x}^{3h/x}x^s\frac{(1+u)^s-1}{s}du - \frac{x+h}{2h}\int_0^{2h/(x+h)}(x+h)^s\frac{(1+u)^s-1}{s}du.$$

where we have substituted $w = x\cdot u$ in the first integral and $w = h+(x+h)u$ in the second integral. Let us only study the first summand, the second one being handled completely similarly. Thus we assume that

$$V \ll \frac{X}{h^4}\int_X^{2X}\left|\int_{h/x}^{3h/x}\int_1^{1+i\infty}F(s)x^s\frac{(1+u)^s-1}{s}dsdu\right|^2 dx$$
$$\ll \frac{1}{h^3}\int_{h/(2X)}^{3h/X}\int_X^{2X}\left|\int_1^{1+i\infty}F(s)x^s\frac{(1+u)^s-1}{s}ds\right|^2 dxdu$$
$$\ll \frac{1}{h^2X}\int_X^{2X}\left|\int_1^{1+i\infty}F(s)x^s\frac{(1+u)^s-1}{s}ds\right|^2 dx$$

for some $u \ll h/X$.

Let us introduce a smooth function $g(x)$ supported on $[1/2, 4]$ and equal to 1 on $[1, 2]$. We obtain

$$\begin{aligned}
V &\ll \frac{1}{h^2 X} \int g\Big(\frac{x}{X}\Big) \Big| \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds \Big|^2 dx \\
&\le \frac{1}{h^2 X} \int_1^{1+i\infty} \int_1^{1+i\infty} \Big| F(s_1) F(s_2) \frac{(1+u)^{s_1} - 1}{s_1} \frac{(1+u)^{s_2} - 1}{s_2} \Big| \Big| \int g\Big(\frac{x}{X}\Big) x^{s_1 + \overline{s_2}} dx \Big| |ds_1 ds_2| \\
&\ll \frac{1}{h^2 X} \int_1^{1+i\infty} \int_1^{1+i\infty} |F(s_1) F(s_2)| \min\Big\{ \frac{h}{X}, \frac{1}{|t_1|} \Big\} \min\Big\{ \frac{h}{X}, \frac{1}{|t_2|} \Big\} \frac{X^3}{|t_1 - t_2|^2 + 1} |ds_1 ds_2| \\
&\ll \frac{X^2}{h^2} \int_1^{1+i\infty} \int_1^{1+i\infty} \frac{|F(s_1)|^2 \min\{(h/X)^2, |t_1|^{-2}\} + |F(s_2)|^2 \min\{(h/X)^2, |t_2|^{-2}\}}{|t_1 - t_2|^2 + 1} |ds_1 ds_2| \\
&\ll \int_1^{1+iX/h} |F(s)|^2 |ds| + \frac{X^2}{h^2} \int_{1+iX/h}^{1+i\infty} \frac{|F(s)|^2}{|t|^2} |ds|.
\end{aligned}$$

The second summand is

$$\ll \frac{X^2}{h^2} \int_{1+iX/(2h)}^{1+i\infty} \frac{1}{T^3} \int_{1+iT}^{1+i2T} |F(s)|^2 |ds| dT \ll \frac{X^2}{h^2} \cdot \frac{1}{X/h} \max_{T \ge X/(2h)} \frac{1}{T} \int_{1+iT}^{1+i2T} |F(s)|^2 |ds|$$

and the claim follows. □

Proposition 1 will follow from combining Lemma 4 with the following lemma.

Lemma 5. *Let $\delta > 0$ be given. Then,*

$$\int_0^T \Big| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \Big|^2 dt \ll_\varepsilon \frac{1}{(\log X)^{1/3-\varepsilon}} \cdot \Big(\frac{T}{X} + 1 \Big) + \frac{T}{X^{1-\delta/2}}.$$

Proof. Since the mean value theorem (Lemma 3) gives the bound $O(\frac{T}{X} + 1)$, we can assume $T \le X$. Furthermore, by Lemma 1, the part of the integral with $t \le T_0 := (\log X)^{10}$ contributes $O((\log X)^{-10})$.

Let us now concentrate to the integral over $[T_0, T]$ with $T \le X$. Let $P = \exp((\log X)^{2/3+\varepsilon})$ and $Q = X^{\delta/3}$. We use the decomposition

(3)
$$\sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} = \sum_{P \le p \le Q} \frac{\lambda(p)}{p^{1+it}} \sum_{m \sim X/p} \frac{\lambda(m)}{(\#\{p \in [P, Q] \colon p \mid m\} + 1) m^{1+it}} + \sum_{\substack{n \sim X \\ p \mid n \implies p \notin [P, Q]}} \frac{\lambda(n)}{n^{1+it}},$$

which is a variant of Ramaré's identity [1, Section 17.3]. Writing $a_m = \lambda(m)/(\#\{p \in [P,Q]\colon p \mid m\}+1)$, we obtain

$$\int_{T_0}^{T} \Big|\sum_{n\sim X} \frac{\lambda(n)}{n^{1+it}}\Big|^2 dt \ll \int_{T_0}^{T} \Big|\sum_{P\le p\le Q} \frac{1}{p^{1+it}} \sum_{m\sim X/p} \frac{a_m}{m^{1+it}}\Big|^2 dt + \tag{4}$$
$$+ \int_{T_0}^{T} \Big| \sum_{\substack{n\sim X\\ p\mid n \implies p\notin[P,Q]}} \frac{\lambda(n)}{n^{1+it}}\Big|^2 dt.$$

We estimate the second term by completing the integral to $|t|\le T$ and by applying the mean-value theorem (Lemma 3). This shows that the second term is bounded by

$$\ll (T+X)\frac{1}{X^2}\sum_{\substack{n\sim X\\ p\mid n \implies p\notin[P,Q]}} 1 \ll \Big(\frac{T}{X}+1\Big)\cdot\frac{\log P}{\log Q} \ll \frac{1}{(\log X)^{1/3-\varepsilon}}\cdot\Big(\frac{T}{X}+1\Big)$$

by the fundamental lemma of the sieve. To deal with the first term in (4), we would like to dispose of the condition $mp\sim x$, so that we can use lemmas in Section 3 to Dirichlet polynomials over p and m separately. To do this, we let $H=(\log X)^5$ and split the summations in the appearing Dirichlet polynomial into short ranges, getting

$$\sum_{P\le p\le Q}\frac{1}{p^s}\sum_{m\sim X/p}\frac{a_m}{m^s} = \sum_{\lfloor H\log P\rfloor \le j\le H\log Q}\ \sum_{\substack{e^{j/H}\le p< e^{(j+1)/H}\\ P\le p\le Q}}\frac{1}{p^s}\sum_{\substack{Xe^{-(j+1)/H}\le m\le 2Xe^{-j/H}\\ X\le mp\le 2X}}\frac{a_m}{m^s}. \tag{5}$$

Now we can remove the condition $X\le mp\le 2X$ over-counting at most by the integers mp in the ranges $[Xe^{-1/H}, X]$ and $[2X, 2Xe^{1/H}]$. Therefore we can, for some bounded d_m, rewrite (5) as

$$\sum_{\lfloor H\log P\rfloor\le j\le H\log Q} Q_{j,H}(s)F_{j,H}(s) + \sum_{Xe^{-1/H}\le m\le X}\frac{d_m}{m^s} + \sum_{2X\le m\le 2Xe^{1/H}}\frac{d_m}{m^s}$$

where

$$Q_{j,H}(s) := \sum_{e^{j/H}\le p\le e^{(j+1)/H}}\frac{1}{p^s} \quad\text{and}\quad F_{j,H}(s) := \sum_{Xe^{-(j+1)/H}\le m\le 2Xe^{-j/H}}\frac{a_m}{m^s}.$$

Using this decomposition, applying Cauchy-Schwarz and then taking the maximal term in the resulting sum, we get

$$\int_{T_0}^{T} \Big| \sum_{P \le p \le Q} \frac{1}{p^{1+it}} \sum_{m \sim X/p} \frac{a_m}{m^{1+it}} \Big|^2 dt \ll (H \log(Q/P))^2 \int_{T_0}^{T} \Big| Q_{j,H}(1+it) F_{j,H}(1+it) \Big|^2 dt +$$

$$+ \int_{T_0}^{T} \Big| \sum_{Xe^{-1/H} \le m \le X} \frac{d_m}{m^{1+it}} \Big|^2 dt + \int_{T_0}^{T} \Big| \sum_{2X \le m \le 2Xe^{1/H}} \frac{d_m}{m^{1+it}} \Big|^2 dt.$$

for some $j \in [\lfloor H \log P \rfloor, H \log Q]$ depending at most on X and T. We compute the last two integrals by completing the integral to $|t| \le T$, and applying the mean value theorem (Lemma 3). This way we see that they are bounded by

$$\ll (T + X) \frac{1}{X^2} \cdot (Xe^{1/H} - X) \ll \Big(\frac{T}{X} + 1 \Big) \frac{1}{H} = \frac{1}{(\log X)^5} \Big(\frac{T}{X} + 1 \Big).$$

Finally, since $X^{\delta/3} = Q \ge e^{j/H} \ge P/e > \exp((\log X)^{2/3+\varepsilon/2})$, using Lemma 2 we have, for $T_0 \le t \le X$,

$$|Q_{j,H}(1+it)| \ll (\log X)^{-9}.$$

Therefore, by the mean value theorem (Lemma 3),

$$\begin{aligned} \int_{T_0}^{T} |Q_{j,H}(1+it) F_{j,H}(1+it)|^2 dt &\ll (\log X)^{-18} \int_{T_0}^{T} |F_{j,H}(1+it)|^2 dt \\ &\ll (\log X)^{-18} \cdot (T + Xe^{-j/H}) \frac{1}{Xe^{-j/H}} \\ &\ll (\log X)^{-18} \cdot \Big(\frac{QT}{X} + 1 \Big) \ll \frac{T}{X^{1-\delta/3}} + \frac{1}{(\log X)^{18}} \end{aligned}$$

since $e^{j/H} \le Q = X^{\delta/3}$. Combining everything together we get the following bound

$$\int_{0}^{T} \Big| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \Big|^2 dt \ll \frac{1}{(\log X)^{1/3-\varepsilon}} \cdot \Big(\frac{T}{X} + 1 \Big) + (\log X)^{12} \cdot \Big(\frac{T}{X^{1-\delta/3}} + \frac{1}{(\log X)^{18}} \Big)$$

which implies the required result. □

We are now ready to prove Proposition 1.

Proof of Proposition 1. Using Lemma 5 we get

$$\int_{0}^{X^{1-\delta}} \Big| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \Big|^2 dt \ll \frac{1}{(\log X)^{1/3-\varepsilon}}$$

and similarly

$$\max_{T>X^{1-\delta}} \frac{X^{1-\delta}}{T} \int_T^{2T} \Big| \sum_{n\sim X} \frac{\lambda(n)}{n^{1+it}} \Big|^2 dt \ll \frac{1}{(\log X)^{1/3-\varepsilon}}.$$

We conclude therefore using Lemma 4, that,

$$\frac{1}{X} \int_X^{2X} \Big| \frac{1}{X^\delta} \sum_{x\le n\le x+X^\delta} \lambda(n) \Big|^2 dx \ll \frac{1}{(\log X)^{1/3-\varepsilon}}$$

as claimed. □

5. Appendix: Proof of Proposition 2

The proof of Proposition 2 is more involved and involves more tools. We will therefore freely make appeal to [4] whenever necessary. First, [4, Lemma 14] (a variant of Lemma 4 here), implies that in order to establish Proposition 2, we need to bound

$$\int_{(\log X)^{1/15}}^T \Big| \sum_{n\sim X} \frac{f(n)}{n^{1+it}} \Big|^2 dt.$$

and perform a minor cosmetic operation. The main ingredient in the proof of Proposition 2 is thus the following lemma.

Lemma 6. *We have,*

$$\int_{(\log X)^{1/15}}^T \Big| \sum_{n\sim X} \frac{f(n)}{n^{1+it}} \Big|^2 dt \ll \frac{1}{(\log X)^{1/48}} \cdot \Big(\frac{T}{X} + 1\Big) + \frac{TX^{o(1)}}{X}.$$

Proof. In view of the trivial bound $O(T/X+1)$ from the mean value theorem (Lemma 3) we can assume that $T \le X$.

Let

$$H = (\log X)^{1/48}, \quad P = \exp((\log X)^{1-1/48}), \quad Q = \exp(\log X/\log\log X),$$

and let

$$Q_{j,H}(s) := \sum_{e^{j/H}\le p\le e^{(j+1)/H}} \frac{f(p)}{p^s} \quad \text{and} \quad F_{j,H}(s) := \sum_{Xe^{-(j+1)/H}\le m\le 2Xe^{-j/H}} \frac{f(m)}{m^s}.$$

Then using [4, Lemma 12] (which is a slightly more involved version of some of the arguments in proof of Lemma 5) we find the following bound,

$$\int_{(\log X)^{1/15}}^{T} \Big| \sum_{n\sim X} \frac{f(n)}{n^{1+it}} \Big|^2 dt \ll$$
$$\ll (\log X)^{2+1/24} \int_{(\log X)^{1/15}}^{T} |Q_{j,H}(1+it) F_{j,H}(1+it)|^2 dt + \frac{1}{(\log X)^{1/48}} \cdot \Big(\frac{T}{X} + 1 \Big)$$

for some $\lfloor H \log P \rfloor \le j \le H \log Q$ depending at most on T and X.

Let us define

$$\mathcal{T}_S = \{ t \in [(\log X)^{1/15}, T] : |Q_{j,H}(1+it)| \le (\log X)^{-100} \}$$
$$\text{and} \quad \mathcal{T}_L = \{ t \in [(\log X)^{1/15}, T] : |Q_{j,H}(1+it)| > (\log X)^{-100} \}.$$

On $\mathcal{T}_S$ we have by definition and the mean value theorem (Lemma 3)

$$\begin{aligned} \int_{\mathcal{T}_S} \Big| Q_{j,H}(1+it) F_{j,H}(1+it) |^2 dt &\ll (\log X)^{-200} \int_0^T |F_{j,H}(1+it)|^2 dt \\ &\ll (\log X)^{-200} \cdot (T + X e^{-j/H}) \cdot \frac{1}{X e^{-j/H}} \\ &\ll (\log X)^{-200} \cdot \Big(\frac{T X^{o(1)}}{X} + 1 \Big) \end{aligned}$$

since $e^{j/H} \le Q = X^{o(1)}$, which is a sufficient saving in the logarithm since we need to beat $(\log X)^{2+1/24}$ by at least $(\log X)^{1/48}$.

Let us now turn to $\mathcal{T}_L$. We can find a well-spaced subset $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{L}}$ such that

$$\int_{\mathcal{T}_L} \Big| Q_{j,H}(1+it) F_{j,H}(1+it) |^2 dt \ll \sum_{t \in \mathcal{T}} \Big| Q_{j,H}(1+it) F_{j,H}(1+it) |^2 dt$$

Using [4, Lemma 8], we see that

$$\begin{aligned} |\mathcal{T}| &\ll \exp\Big(2 \frac{\log (\log X)^{100}}{j/H} \log T + 2 \log (\log X)^{100} + 2 \frac{\log T}{j/H} \log\log T \Big) \\ &\ll \exp\Big(\frac{(\log X)^{1+o(1)}}{\log P} \Big) \ll \exp((\log X)^{1/48 + o(1)}). \end{aligned}$$

In addition, using [4, Lemma 3] (a consequence of Halász's theorem), we find that

$$\sup_{(\log X)^{1/15} \le |t| \le T} |F_{j,H}(1+it)| \ll (\log X)^{-1/16} \cdot \frac{\log Q}{\log P} \ll (\log X)^{-1/24}.$$

Therefore using [4, Lemma 11] (a large value result for Dirichlet polynomials over primes) this time, we get

$$\sum_{t\in\mathcal{T}} |Q_{j,H}(1+it)F_{j,H}(1+it)|^2 \ll (\log X)^{-1/12} \sum_{t\in\mathcal{T}} |Q_{j,H}(1+it)|^2$$
$$\ll (\log X)^{-1/12} \cdot \Big(e^{j/H} + |\mathcal{T}| e^{j/H} \exp(-(\log X)^{1/5})\Big) \sum_{e^{j/H} < p < e^{(j+1)/H}} \frac{1}{p^2 \log p}$$
$$\ll (\log X)^{-1/12} \cdot \frac{e^{(j+1)/H} - e^{j/H}}{e^{j/H}(\log e^{j/H})^2} \ll \frac{1}{H(\log X)^{1/12}(\log P)^2} \ll (\log X)^{-2-1/16}.$$

Therefore combining everything together we get

$$\int_{(\log X)^{1/15}}^{T} \Big| \sum_{n\sim X} \frac{f(n)}{n^{1+it}} \Big|^2 dt \ll (\log X)^{-100} \Big(\frac{T X^{o(1)}}{X} + 1 \Big) + (\log X)^{-1/48} \Big(\frac{T}{X} + 1 \Big),$$

and the claim follows. □

We are finally ready to prove the Proposition.

Proof of Proposition 1. Now, by [4, Lemma 14] and Lemma 6, we get, for $X^\delta = h_1 \leq h_2 = X/(\log X)^{1/5}$,

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h_1} \sum_{x \leq m \leq x+h_1} f(m) - \frac{1}{h_2} \sum_{x\leq m \leq x+h_2} f(m) \right|^2 dx \ll (\log X)^{-1/48}.$$

The claim follows since by [4, Lemma 4]

$$\frac{1}{h_2} \sum_{x\leq n\leq x+h_2} f(n) = \frac{1}{x} \sum_{X \leq n \leq 2X} f(n) + O((\log X)^{-1/20}).$$

□

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