LIMITING DISTRIBUTION OF EIGENVALUES IN THE LARGE SIEVE MATRIX

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Abstract. The large sieve inequality is equivalent to the bound $\lambda_1 \leqslant N + Q^2 - 1$ for the largest eigenvalue λ_1 of the N by N matrix $A^\star A$, naturally associated to the positive definite quadratic form arising in the inequality. For arithmetic applications the most interesting range is $N \asymp Q^2$. Based on his numerical data Ramaré conjectured that when $N \sim \alpha Q^2$ as $Q \to \infty$ for some finite positive constant α, the limiting distribution of the eigenvalues of $A^\star A$, scaled by $1/N$, exists and is non-degenerate. In this paper we prove this conjecture by establishing the convergence of all moments of the eigenvalues of $A^\star A$ as $Q \to \infty$. Previously only the second moment was known, due to Ramaré. Furthermore, we obtain an explicit description of the moments of the limiting distribution, and establish that they vary continuously with α. Some of the main ingredients in our proof include the large-sieve inequality and results on n-correlations of Farey fractions.

1. Introduction

Let $\mathcal{F}_Q$ denote the set of Farey fractions of order Q, that is the set of reduced fractions $\frac{a}{q}$ with $0 < a \leqslant q \leqslant Q$. In particular $|\mathcal{F}_Q| = \sum_{q \leqslant Q} \varphi(q) \sim \frac{3}{\pi^2} Q^2$ as $Q \to \infty$. The large sieve inequality states that, for any sequence of complex numbers $a(n)$,

$$\sum_{\theta \in \mathcal{F}_Q} \Big| \sum_{n \leqslant N} a(n) e(n\theta) \Big|^2 \leqslant (N + Q^2 - 1) \sum_{n \leqslant N} |a(n)|^2. \tag{1.1}$$

The large sieve was first discovered by Linnik [11], who applied it to bound the number of moduli q for which the least quadratic non-residue exceeds q^ε. Since its inception the large sieve fascinated analytic number theorists, not the least because of the variety of its incarnations (probabilistic [16], arithmetic [11], analytic [4]). The form (1.1) is the outcome of a long chain of improvements, due among others to Bombieri [5], Bombieri-Davenport [4], Gallagher [9], Montgomery [13], Montgomery-Vaughan [12], Rényi [16], Roth [17], Selberg [18], ... One of the major applications of (1.1) is the Bombieri-Vinogradov [3] theorem on primes in arithmetic progressions.

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A fruitful point of view is to interpret (1.1) in terms of the eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0$ of the $N \times N$ symmetric positive definite matrix,

$$A^\star A = \Big(\sum_{\theta \in \mathcal{F}_Q} e\big((n_1 - n_2)\theta\big)\Big)_{1 \leqslant n_1, n_2 \leqslant N} \text{ where } A = \Big(e(n\theta)\Big)_{\substack{\theta \in \mathcal{F}_Q \\ 1 \leqslant n \leqslant N}}.$$

Note that $\sqrt{\lambda_1} \geqslant \sqrt{\lambda_2} \geqslant \ldots \geqslant \sqrt{\lambda_N} \geqslant 0$ are the singular values of A and the following idenity holds trivially:

$$\sum_{i \leqslant N} \lambda_i = \mathrm{Tr}(A^\star A) = |\mathcal{F}_Q| N.$$

Since $\|A\mathbf{v}\|^2$ is equal to (1.1) when $\mathbf{v} = (a(1), \ldots, a(n))$ and $\lambda_1 = \|A\|^2 = \|A^\star A\| = \|AA^\star\|$ the large sieve inequality (1.1) is equivalent to $\lambda_1 \leqslant N + Q^2 - 1$. It is very desirable, from the point of view of applications, to replace the inequality (1.1) by an asymptotic equality. In the range $N < Q^{2-\varepsilon}$ one can adapt the results of Conrey-Iwaniec-Soundararajan [6] to obtain an asymptotic for a class of sequences $a(n)$.

We would like to investigate the problem of refining the large sieve inequality to an asymptotic equality in wide generality, and in particular in the range $N \asymp Q^2$. This range is particularly interesting from an arithmetic point of view; for example it comes up naturally in the proof of the explicit Brun-Titchmarsh theorem. As a first step in this direction, one would like to understand the limiting distribution of the eigenvalues of $A^\star A$, that is the limiting distribution of the sequence of probability measures on $[0, \infty)$ given by

$$\mu_{Q,N} = \frac{1}{N} \sum_{i \leqslant N} \delta_{\lambda_i / N},$$

where δ_λ denotes the Dirac probability measure supported at $\lambda \in \mathbb{R}$. It turns out that this is relatively easy when the ratio Q^2/N either tends to infinity or to zero. When $N/Q^2 \to \infty$ as $Q \to \infty$, then since the rank of $A^\star A$ is $\leqslant Q^2$, it follows that most eigenvalues are zero, therefore $\mu_{Q,N} \to \delta_0$. On the other hand, when $N/Q^2 \to 0$ as $Q \to \infty$, then according to a deeper result of Ramaré [14] concerning the asymptotic behaviour of $\sum_{i \leqslant N} \lambda_i^2$, one concludes that when $N/Q^2 \to 0$ all but $o(N)$ of the eigenvalues cluster close to $|\mathcal{F}_Q|$. We will be concerned with the remaining regime $N \asymp Q^2$.

In [14, 15] Ramaré conducted several numerical experiments that suggested the existence of a *non-degenerate* limiting distribution function as soon as the ratio $\frac{N}{Q^2}$ tends to a finite limit with $Q \to \infty$. In support of the numerical data Ramaré established in [14] the convergence of the second moment

$$\mathfrak{M}_Q(2) := \int_0^\infty t^2 d\mu_{Q,N}(t) = \frac{1}{N} \sum_{i \leqslant N} \Big(\frac{\lambda_i}{N}\Big)^2$$

as $Q \to \infty$. The form of the second moment (in particular its variation with $\frac{N}{Q^2}$) ruled out the possibility of convergence to any standard probability law.

In this paper we estimate all moments of $\mu_{Q,N}$,

$$\begin{aligned} \mathfrak{M}_Q(\ell) &:= \int_0^\infty t^\ell d\mu_{Q,N}(t) = \frac{1}{N}\sum_{i\leqslant N}\Big(\frac{\lambda_i}{N}\Big)^\ell = \frac{1}{N}\mathrm{Tr}\Big(\frac{A^\star A}{N}\Big)^\ell \\ &= \frac{1}{N^{\ell+1}}\sum_{\substack{\theta_1,\ldots,\theta_\ell\in\mathcal{F}_Q\\ 1\leqslant n_1,\ldots,n_\ell\leqslant N}} e\big((n_1-n_2)\theta_1+(n_2-n_3)\theta_2+\ldots+(n_\ell-n_1)\theta_\ell\big), \end{aligned} \tag{1.2}$$

and prove Ramaré's conjecture, that is the weak*-convergence of $\mu_{Q,N}$ to a limiting distribution when $N \sim \alpha Q^2$:

Corollary 1. *Suppose that $N \sim \alpha Q^2$ as $Q \to \infty$ for some fixed constant $\alpha \in (0,\infty)$. There exists a non-degenerate probability measure μ_α on $[0,\infty)$ such that*

$$\mu_{Q,N} \xrightarrow{\text{w}^*} \mu_\alpha$$

as $Q \to \infty$. Moreover μ_α is determined by its moments

$$M_\ell(\alpha) = \int_0^\infty t^\ell d\mu_\alpha(t),$$

explicitly described in Theorem 1 below.

Remark 2. In principle our proof delivers a rate of convergence. For example, when $N = |\mathcal{F}_Q|$ our approach shows that convergence of the ℓ^{th} moment occurs at a rate of at least $\ll Q^{-\delta_\ell}$ for some exponent $\delta_\ell > 0$. Since our proof reveals that $\delta_\ell \ll \ell^A$ for some absolute constant $A > 0$, it is possible to show by Fourier analytic techniques that when $N = |\mathcal{F}_Q|$ (and thus $\alpha = 3/\pi^2$) there exists a $\delta > 0$, such that for any fixed smooth function f,

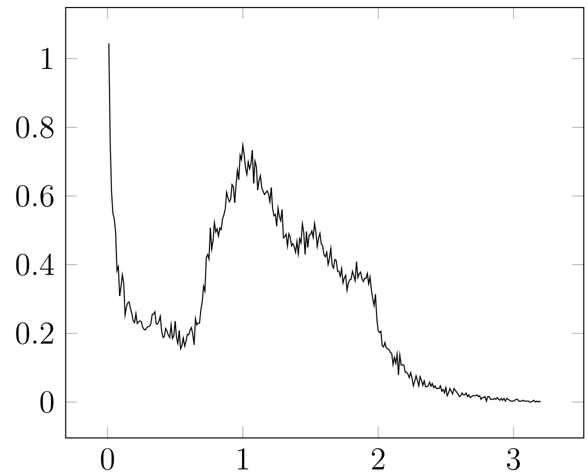
$$\int_0^\infty f(t)d\mu_{Q,N}(t) = \int_0^\infty f(t)d\mu_\alpha(t) + O((\log Q)^{-\delta})$$

as $Q \to \infty$. More generally this holds whenever $N = \alpha Q^2 + O(Q^{2-\eta})$ for some $\eta > 0$.

We include below an empirical approximation for the probability density function of μ_{3/π^2}, based on an approximation with $Q = 500$ and $N = |\mathcal{F}_{500}| = 76116$ [1]. This resembles the previous data obtained by Ramaré [15]. There is a large number of eigenvalues in $[0, 0.01]$ (roughly 10%) and we have omitted them from the graph (they contribute a disproportionate 6 on the scale of the graph). Note that when $N = |\mathcal{F}_Q|$ there are no eigenvalues that are equal to 0, since in this case $\det A$ is just a non-zero Vandermonde determinant.

[1]The computation took less than a day, requiring 8 cores and 44GB of RAM memory. We used a custom C program invoking LAPACK linear algebra routines. The code is available on request.

Approximation for the p.d.f of μ_{3/π^2}



The description of the moments $M_\ell(\alpha)$ is rather complicated, so we start with some preliminary remarks. Since $\lambda_1 \leqslant 1 + \frac{Q^2}{N} \leqslant 1 + \alpha^{-1} + o(1)$, all probability measures $\mu_{Q,N}$ are supported in $[0, C(\alpha)]$ for some constant $C(\alpha) > 0$, so we have for free that

$$0 \leqslant \mathfrak{M}_Q(\ell) = \int_0^\infty t^\ell d\mu_{Q,N}(t) \leqslant C(\alpha)^\ell.$$

In particular, if each $\mathfrak{M}_Q(\ell)$ converges to a limit C_ℓ as $Q \to \infty$, then $\mu_{Q,N} \xrightarrow{\mathrm{w}^*} \mu$ for some probability measure μ supported in $[0, 1 + \alpha^{-1}]$ with moments C_ℓ . We also notice that the first moment is trivial:

$$\mathfrak{M}_Q(1) = \frac{1}{N} \operatorname{Tr}\left(\frac{A^\star A}{N}\right) = \frac{|\mathcal{F}_Q|}{N} \sim \frac{3}{\pi^2 \alpha}.$$

For $\ell \geqslant 2$, our starting point will be the analysis of the exponential sum in (1.2). This is a sum of $\asymp N^{2\ell}$ oscillating terms, which by the large sieve are bounded by $\ll N^{\ell+1}$, so close to square-root cancellation. Our task is to refine this bound to an asymptotic equality. We accomplish this in the theorem below.

Theorem 1. (i) *For each $\ell \geqslant 2$, there exists a continuous function $M_\ell(\alpha)$ on $(0, \infty)$ and some explicit exponent $\theta_\ell > 0$ such that, given $0 < \gamma_1 < \gamma_2$, one has*

$$\mathfrak{M}_Q(\ell) = M_\ell\Big(\frac{N}{Q^2}\Big) + O_{\ell,\gamma_1,\gamma_2}(Q^{-\theta_\ell})$$

whenever $\gamma_1 Q^2 \leqslant N \leqslant \gamma_2 Q^2$ *as* $Q \to \infty$*. Precisely, taking* $(A, 0) = |A|$, $\mathbf{A} = (A_1, \ldots, A_{\ell-1})$, $\mathbf{B} = (B_1, \ldots, B_{\ell-1})$, $\operatorname{sinc}(x) := \frac{\sin x}{x}$ *if* $x \neq 0$, $\operatorname{sinc}(0) = 1$, *and*

$$h_{A,B}(x,y) := \frac{B}{y(Ay - Bx)},$$

$$\mathcal{D}_{\mathbf{A},\mathbf{B}} := \{(x,y) \in [0,1]^2 : x \leqslant y, 0 < A_i y - B_i x \leqslant 1, \forall i \in [1, \ell-1]\}, \tag{1.3}$$

we have

$$\begin{aligned} M_\ell(\alpha) = \frac{6}{\pi^2\alpha} \sum_{\substack{\mathbf{A},\mathbf{B} \in \mathbb{Z}^{\ell-1} \\ (A_i,B_i)=1, \forall i \\ A_i^2 + B_i^2 \neq 0, \forall i}} \iint_{\mathcal{D}_{\mathbf{A},\mathbf{B}}} & \operatorname{sinc}\big(\pi\alpha h_{A_1,B_1}(x,y)\big) \operatorname{sinc}\big(\pi\alpha h_{A_{\ell-1},B_{\ell-1}}(x,y)\big) \\ & \times \prod_{i=1}^{\ell-2} \operatorname{sinc}\big(\pi\alpha h_{A_i,B_i}(x,y) - \pi\alpha h_{A_{i+1},B_{i+1}}(x,y)\big) dx dy \in [0, \infty). \end{aligned} \tag{1.4}$$

(ii) *The expression defining* $M_\ell(\alpha)$ *above is absolutely convergent.*

(iii) *For each* $\ell \geqslant 2$*, there exists* $\kappa_\ell > 0$ *such that, given* $0 < \gamma_1 < \gamma_2$*, one has*

$$|M_\ell(\alpha) - M_\ell(\beta)| \ll_{\gamma_1,\gamma_2} |\alpha - \beta|^{\kappa_\ell}$$

for all $\alpha, \beta \in [\gamma_1, \gamma_2]$*.*

We highlight that both continuity and absolute convergence of the expression defining $M_\ell(\alpha)$ are non-trivial. Parts (ii) and (iii) in Theorem 1 will be proved in Section 4, while part (i) will be proved in Section 5.

With Theorem 1 at hand the deduction of our main result Corollary 1 is immediate.

Deduction of Corollary 1. The large sieve inequality yields $\operatorname{supp} \mu_{Q,N} \subseteq [0, 1 + \frac{Q^2}{N}]$ for all Q. Hence there exists a positive constant K such that $\operatorname{supp} \mu_{Q,N} \subseteq [0, K]$ for all Q. Banach-Alaoglu's theorem shows that the sequence of probability measures $(\mu_{Q,N})$ on the compact set $[0, K]$ has at least one cluster point μ_α in the weak* topology on $C([0,K])^*$, and μ_α is a probability measure on $[0, K]$. Theorem 1 shows that any two such cluster points μ, μ_α have the same moments $M_\ell(\alpha)$, $\ell \in \mathbb{N}$, thus $\mu = \mu_\alpha$ by Stone-Weierstrass.

It remains to show that the limiting distribution μ_α is not degenerate, that is its variance $M_\ell(2) - M_\ell(1)^2 = M_\ell(2) - (\frac{3}{\pi^2\alpha})^2$ is non-zero. For $\alpha \geqslant 1$ this is impossible, since by Proposition 4 from Section 2, the second moment is $> \frac{3}{\pi^2\alpha} \geqslant (\frac{3}{\pi^2\alpha})^2$. For $\alpha < 1$ according to Ramaré's formula or the proof of Proposition 4 below,

$$\int_0^\infty t^2 d\mu_\alpha(t) = \frac{3}{\pi^2\alpha} + \frac{6}{\pi^2\alpha} \cdot \frac{1}{\pi i} \int_{-\frac{1}{8}+i\infty}^{-\frac{1}{8}+i\infty} \frac{\alpha^{s-1}\zeta(s)}{s(s+1)(2-s)^2\zeta(2-s)} ds.$$

Shifting the contour of integration to $\Re s = 1$, we collect a pole at $s = 0$ that contributes $(\frac{3}{\pi^2\alpha})^2$. We conclude that

$$\left| \int_0^\infty t^2 d\mu_\alpha(t) - \Big(\frac{3}{\pi^2\alpha}\Big)^2 \right| > \frac{3}{\pi^2\alpha} - \left| \frac{6}{\pi^2\alpha} \cdot \frac{1}{\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\alpha^{s-1}\zeta(s)}{s(s+1)(2-s)^2\zeta(2-s)} ds \right|. \quad (1.5)$$

Notice that the rightmost term is **strictly** less than

$$\frac{6}{\pi^2\alpha} \cdot \frac{1}{\pi} \int_{\mathbb{R}} \frac{dt}{|1+it|^4} = \frac{3}{\pi^2\alpha}.$$

It follows that the left-hand side of (1.5) is > 0 and hence the distribution $\mu_\alpha(t)$ is not degenerate. □

One wonders if μ_α is absolutely continuous with respect to the Lebesgue measure, except for possible atoms at 0 (which arise naturally when $N > (1+\varepsilon)|\mathcal{F}_Q|$, since A is not of full rank as soon as $N > |\mathcal{F}_Q|$).

The result in [2] shows (after a small modification) that when $N = |\mathcal{F}_Q|$, there exists a positive measure g_ℓ, supported on $[\frac{3}{\pi^2}, \infty)$ when $\ell = 2$ and on a countable union of surfaces in $\mathbb{R}^{\ell-1}$ when $\ell > 2$ (thus in particular having Lebesgue measure zero support when $\ell > 3$), such that,

$$S_\ell(Q; f) := \frac{1}{N} \sum_{\substack{\theta_1,\ldots,\theta_\ell \in \mathcal{F}_Q \\ \text{distinct}}} F_Q(\theta_1 - \theta_2, \theta_2 - \theta_3, \ldots, \theta_{\ell-1} - \theta_\ell) = 2 \int_{[0,\infty)^{\ell-1}} f \, dg_\ell + o(1),$$

as $Q \to \infty$, for smooth functions f compactly supported in $(0,\infty)^{\ell-1}$, where F_Q denotes the $\mathbb{Z}^{\ell-1}$-periodization of f given by

$$F_Q(\mathbf{y}) := \sum_{\mathbf{m} \in \mathbb{Z}^{\ell-1}} f\big(N(\mathbf{m} + \mathbf{y})\big), \quad \mathbf{y} \in \mathbb{R}^{\ell-1}/\mathbb{Z}^{\ell-1}.$$

Concretely, the measure g_ℓ is supported on the union of all surfaces $\Phi_{\mathbf{A},\mathbf{B}}(\mathcal{D}_{\mathbf{A},\mathbf{B}})$ with $\mathbf{A}, \mathbf{B} \in \mathbb{N}^{\ell-1}$, $(A_i, B_i) = 1, \forall i$, $\Phi_{\mathbf{A},\mathbf{B}} = T \circ T_{\mathbf{A},\mathbf{B}}$, $T_{\mathbf{A},\mathbf{B}} = \frac{3}{\pi^2}(h_{A_1,B_1}, \ldots, h_{A_{\ell-1},B_{\ell-1}})$ and $T(x_1, \ldots, x_{\ell-1}) = (x_1 - x_2, x_2 - x_3, \ldots, x_{\ell-2} - x_{\ell-1}, x_{\ell-1})$. It is important here that the support of f is compact, as it implies that the number of $(2\ell-2)$-tuples $(\mathbf{A}, \mathbf{B})$ that produce non-zero terms in $\int_{[0,\infty)^{\ell-1}} f \, dg_\ell$ is finite. However, when the support of f contains 0 or when f is not compactly supported, the question of convergence of such an expression becomes delicate, in particular since we do not have non-trivial point-wise bounds for g_ℓ as soon as $\ell > 2$.

There is a close relationship between (1.4) and the density g_ℓ. Using the absolute convergence of (1.4) and an explicit formula for g_ℓ, provided by instance by formula

(1.4) in [2], it is possible to re-write $M_\ell(\alpha)$ in (1.4) as

$$\frac{6}{\pi^2\alpha}\int_{[0,\infty)^{\ell-1}} \operatorname{sinc}\left(\frac{\pi^3\alpha(x_1+\ldots+x_{\ell-1})}{3}\right)\prod_{i=1}^{\ell-1}\operatorname{sinc}\left(\frac{\pi^3\alpha x_i}{3}\right)d\widetilde{g}_\ell(x_1,\ldots,x_{\ell-1}), \quad (1.6)$$

where the measure $\widetilde{g}_\ell$ is defined in a similar way as g_ℓ, but summing over the larger range $(\mathbf{A},\mathbf{B}) \in \mathbb{Z}^{2\ell-2}$ with $(A_i, B_i) = 1$ and $A_i^2 + B_i^2 \neq 0$ for all i (in particular the support of $\widetilde{g}_\ell$ still has zero Lebesgue measure in $\mathbb{R}^{\ell-1}$). From this we see that the proof of absolute convergence of the expression defining $M_\ell(\alpha)$ amounts to establishing bounds for the decay rate of $\widetilde{g}_\ell$ in an averaged sense.

We close by mentioning that two of the remaining challenges are to determine finer properties of the distribution function of the limiting probability measure μ_α and to obtain information about the limiting eigenvectors of the large sieve matrix $A^\star A$. We hope to come back to these questions in a later paper.

1.1. **Outline of the argument and plan of the paper.** We now highlight the main steps in our proof. We first address in Section 2 the case $\ell = 2$, adapting techniques from [2]. This recovers Ramaré's initial result. It is not clear how to proceed when $\ell \geqslant 3$ without introducing a smoothing on the $n_1, \ldots, n_\ell$ variables. Since our sum is highly oscillating, it is also not immediately clear that a smoothing can be efficiently introduced. Ramaré remarks in his paper [14] that this is a significant stumbling block. In Section 3 we show that one can introduce a substantial smoothing by using the large sieve inequality. After having smoothed, we would like to relate the question of computing the moments to the n-correlation function of Farey fractions which was computed in [2]. Here an initial obstacle is that the variables θ_i are chained in a circular manner, requiring us to control simultaneously $N(\theta_1-\theta_2), N(\theta_2-\theta_3), \ldots, N(\theta_\ell-\theta_1)$. We resolve this problem by using a Fourier analytic trick, which reduces us to the case where we need to understand $N(\theta_1-\theta_2), \ldots, N(\theta_{\ell-1}-\theta_\ell)$, that is, without the circular chaining. We then adapt in Section 5 the argument from a paper by Zaharescu and the first author [2] where the higher correlation measures of Farey fractions are computed. One of the key arguments from [2] relies on the divisor switching technique. It is interesting to notice that this is also the crucial ingredient in the recent work on the "asymptotic large sieve" by Conrey-Iwaniec-Soundararajan [6]. Finally, in order to conclude the computation carried out in Section 5, we need to establish the absolute convergence of the expression defining $M_\ell(\alpha)$. This requires a rather substantial elementary argument that splits into several cases. We perform this analysis in Section 4. The main ingredient is a counting lemma for simultaneous solutions to a system of equations of the form $A_iB_{i+1} - A_{i+1}B_i = \Delta_i$. In a subsequent paper we hope to apply this argument to analyze the behavior at infinity of higher correlation functions of Farey fractions, which was only worked out for the pair correlation.

2. Moments of second order

We first consider in detail the case $\ell = 2$. An asymptotic formula for $\mathfrak{M}_Q(2)$ was previously established in [14]. Here we follow a different approach in the spirit of [2].

Denote by H_N the characteristic function of the interval $[\frac{1}{N}, 1]$. We can write

$$\begin{aligned} \mathfrak{M}_Q(2) &= \frac{1}{N^3} \sum_{\theta_1,\theta_2 \in \mathcal{F}_Q} \sum_{n_1,n_2 \in \mathbb{Z}} e\big((n_1-n_2)(\theta_1-\theta_2)\big) H_N\Big(\frac{n_1}{N}\Big) H_N\Big(\frac{n_2}{N}\Big) \\ &= \frac{1}{N^3} \sum_{\theta_1,\theta_2 \in \mathcal{F}_Q} \sum_{n \in \mathbb{Z}} e\big(n(\theta_1-\theta_2)\big) \sum_{n_2 \in \mathbb{Z}} H_N\Big(\frac{n_2}{N}\Big) H_N\Big(\frac{n_2+n}{N}\Big). \end{aligned} \tag{2.1}$$

The inner sum in (2.1) is seen to coincide with $(N-|n|)\chi_{[-N,N]}(n)$, and so

$$\mathfrak{M}_Q(2) = \frac{1}{N^2} \sum_{n=-N}^{N} \phi\Big(\frac{n}{N}\Big) \sum_{\theta_1,\theta_2 \in \mathcal{F}_Q} e\big(n(\theta_1-\theta_2)\big), \tag{2.2}$$

where

$$\phi(x) := (1-|x|)\chi_{[-1,1]}(x) = (\chi_{[0,1]} * \chi_{[-1,0]})(x).$$

Using also

$$\widehat{\chi}_{[0,1]}(x) = e^{-\pi i x}\,\mathrm{sinc}(\pi x),$$

we find

$$\psi(x) := \widehat{\phi}(x) = \widehat{\chi}_{[0,1]}(x)\widehat{\chi}_{[-1,0]}(x) = \mathrm{sinc}^2(\pi x) = \psi(-x). \tag{2.3}$$

From (2.2) and (2.3) we infer

$$\mathfrak{M}_Q(2) = \frac{1}{N} \sum_{n \in \mathbb{Z}} \frac{1}{N} \widehat{\psi}\Big(\frac{n}{N}\Big) \sum_{\theta_1,\theta_2 \in \mathcal{F}_Q} e\big(n(\theta_1-\theta_2)\big), \tag{2.4}$$

thus it suffices to reprove an analogue of [2, Theorem 2] with the compactly supported smooth function H there being replaced by ψ here.

2.1. An asymptotic formula for $\mathfrak{M}_Q(2)$.

We follow closely Sections 2 and 4 in [2] with

$$c_n = \frac{1}{N} \widehat{\psi}\Big(\frac{n}{N}\Big).$$

Consider the Möbius function μ and the summation function

$$M(X) := \sum_{n \leqslant X} \mu(n).$$

An application of Möbius inversion shows that (see, e.g., formula (1) in Section 12.2 of [7]), for every function $f : \mathbb{Q} \cap [0,1] \to \mathbb{C}$,

$$\sum_{\theta \in \mathcal{F}_Q} f(\theta) = \sum_{k \geqslant 1} M\Big(\frac{Q}{k}\Big) \sum_{j=1}^{k} f\Big(\frac{j}{k}\Big).$$

In particular this provides the following well-known identity:

$$\sum_{\theta \in \mathcal{F}_Q} e(n\theta) = \sum_{d \mid n} dM\Big(\frac{Q}{d}\Big), \qquad n \in \mathbb{Z}, Q \in \mathbb{N}. \tag{2.5}$$

For $n = 0$ this corresponds to $|\mathcal{F}_Q| = \sum_{d \geqslant 1} M(\frac{Q}{d})$.

Poisson's summation formula [8, Theorem 8.36] holds true when applied to a pair $(\psi_h, \widehat{\psi}_h)$, where $\psi_h(x) := \psi(hx)$, $h > 0$, because $|\psi(x)| \leqslant \frac{1}{(1+|x|)^2}$ and $\widehat{\psi}$ has compact support. Proceeding exactly as in [2], we arrive at the following closed form analogue of formulas (4.4) and (4.5) in [2]:

$$\begin{aligned}
\mathfrak{M}_Q(2) &= \frac{1}{N} \sum_{r_1, r_2 \in [1,Q]} \mu(r_1)\mu(r_2) \sum_{\substack{d_1 \in [1, \frac{Q}{r_1}] \\ d_2 \in [1, \frac{Q}{r_2}]}} (d_1, d_2) \sum_{n \in \mathbb{Z}} \psi\Big(\frac{nN}{[d_1, d_2]}\Big) \\
&= \frac{1}{N} \sum_{r_1, r_2 \in [1,Q]} \mu(r_1)\mu(r_2) \sum_{\delta \in [1, \min\{\frac{Q}{r_1}, \frac{Q}{r_2}\}]} \delta \sum_{n \in \mathbb{Z}} \sum_{\substack{q_1 \in [1, \frac{Q}{r_1 \delta}] \\ q_2 \in [1, \frac{Q}{r_2 \delta}] \\ (q_1, q_2) = 1}} \psi\Big(\frac{nN}{q_1 q_2 \delta}\Big).
\end{aligned} \tag{2.6}$$

This sum is split as $\mathfrak{M}_Q^I(2) + \mathfrak{M}_Q^{II}(2) + \mathfrak{M}_Q^{III}(2)$, with terms arising from the contribution of $n = 0$, $\psi_\Lambda(x) := \psi(x)\chi_{\{0 < |x| \leqslant \Lambda\}}$, and respectively $\psi(x)\chi_{\{|x| > \Lambda\}}$, where we take $\Lambda := N^{1/2} \asymp Q$.

The contribution of $n = 0$ is given by

$$\begin{aligned}
\mathfrak{M}_Q^I(2) &= \frac{1}{N} \sum_{r_1, r_2 \in [1,Q]} \mu(r_1)\mu(r_2) \sum_{\substack{d_1 \in [1, \frac{Q}{r_1}] \\ d_2 \in [1, \frac{Q}{r_2}]}} (d_1, d_2) \\
&= \frac{1}{N} \sum_{d_1, d_2 \in [1,Q]} (d_1, d_2) M\Big(\frac{Q}{d_1}\Big) M\Big(\frac{Q}{d_2}\Big) = \frac{|\mathcal{F}_Q|}{N},
\end{aligned} \tag{2.7}$$

the last equality being noticed at the top of page 420 in [14].

The contribution of n with $\frac{|n|N}{q_1q_2\delta} > \Lambda$ to the inner sum in (2.6) is

$$\ll \frac{q_1^2q_2^2\delta^2}{N^2} \sum_{n > \frac{\Lambda q_1 q_2 \delta}{N}} \frac{1}{n^2} \ll \frac{q_1^2q_2^2\delta^2}{N^2} \cdot \frac{N}{\Lambda q_1 q_2 \delta} = \frac{q_1q_2\delta}{\Lambda N},$$

hence

$$\mathfrak{M}_Q^{III}(2) \ll \frac{1}{\Lambda N^2} \sum_{r_1,r_2 \in [1,Q]} \sum_{\delta \in [1, \frac{Q}{\max\{r_1,r_2\}}]} \delta^2 \Big(\frac{Q}{r_1\delta}\Big)^2 \Big(\frac{Q}{r_2\delta}\Big)^2 \ll \frac{Q^4}{\Lambda N^2} \ll Q^{-1}. \tag{2.8}$$

Finally we estimate $\mathfrak{M}_Q^{II}(2)$. In this situation we have $0 < \frac{|n|N}{q_1q_2\delta} \leqslant \Lambda$, leading to $N \leqslant \Lambda q_1 q_2 \delta \leqslant \Lambda Q \min\{q_1,q_2\}$ and thus $\min\{q_1,q_2\} \geqslant \frac{N}{\Lambda Q}$. We also have

$$|n| r_1 r_2 \delta \leqslant r_1 r_2 \delta \cdot \frac{q_1 q_2 \delta \Lambda}{N} \leqslant \frac{\Lambda Q^2}{N} \ll \Lambda.$$

To estimate

$$S_{r_1,r_2,\delta,n}(Q) := \sum_{\substack{\min\{q_1,q_2\} > \frac{N}{\Lambda Q} \\ q_1 \in [1,\frac{Q}{r_1\delta}], q_2 \in [1,\frac{Q}{r_2\delta}] \\ (q_1,q_2)=1}} \psi\Big(\frac{nN}{q_1q_2\delta}\Big),$$

we take $f(x,y) := \psi(\frac{nN}{\delta xy})$, $\Omega := \{(x,y) : x \leqslant \frac{Q}{r_1\delta}, y \leqslant \frac{Q}{r_2\delta}, \min\{x,y\} \geqslant \frac{N}{\Lambda Q}\}$, and apply the following:

Lemma 3 (Lemma 2 and Corollary 1 in [1])**.** *Suppose that $\Omega \subseteq [1,R]^2$ is a region with rectifiable boundary and $f \in C^1(\Omega)$ with $Df = \big|\frac{\partial f}{\partial x}\big| + \big|\frac{\partial f}{\partial y}\big|$ and $\| \ \|_\infty$ denoting the sup norm on Ω. Then*

$$\sum_{\substack{(m,n)\in\Omega \\ (m,n)=1}} f(m,n) = \frac{6}{\pi^2} \iint_\Omega f(x,y)\,dxdy + \mathcal{E}_{f,\Omega,R},$$

with

$$\mathcal{E}_{f,\Omega,R} \ll \|Df\|_\infty \operatorname{Area}(\Omega) \log R + \|f\|_\infty \big(R + \operatorname{length}(\partial\Omega)\log R\big).$$

Furthermore, if Ω is also convex, then

$$\mathcal{E}_{f,\Omega,R} \ll \|Df\|_\infty \operatorname{Area}(\Omega) \log R + \|f\|_\infty R \log R.$$

It is plain that $|\psi(x)| \leqslant \frac{1}{\pi^2 x^2}$ and $|\psi'(x)| \leqslant \frac{4}{\pi x^2}$, thus on Ω we have $|f(x,y)| \leqslant \frac{\delta^2 x^2 y^2}{n^2 N^2} \leqslant \frac{Q^4}{N^2} \cdot \frac{1}{r_1^2 r_2^2 \delta^2 n^2}$ and $|(Df)(x,y)| \leqslant \frac{2\delta^2x^2y^2}{n^2N^2} \cdot \frac{|n|N}{\delta}\big(\frac{1}{x^2y} + \frac{1}{xy^2}\big) \ll \frac{\delta(x+y)}{|n|N} \ll \frac{1}{|n|Q}$.

This yields

$$S_{r_1,r_2,\delta,n}(Q) = \frac{6}{\pi^2} \iint\limits_{\substack{x\leqslant \frac{Q}{r_1\delta},\, y\leqslant \frac{Q}{r_2\delta} \\ \min\{x,y\}\geqslant \frac{N}{\Lambda Q}}} \psi\Big(\frac{nN}{\delta xy}\Big) dxdy + E_{r_1,r_2,\delta,n}(Q), \tag{2.9}$$

with error terms

$$E_{r_1,r_2,\delta,n}(Q) \ll \frac{1}{|n|Q} \cdot \frac{Q^2}{r_1^2 r_2^2 \delta^2} \log Q + \frac{Q\log Q}{r_1^2 r_2^2 \delta^2 n^2},$$

summing up in $\mathfrak{M}_Q^{II}(2)$ to

$$\mathcal{E}(Q) = \frac{1}{N} \sum_{\substack{|n|,r_1,r_2,\delta\geqslant 1 \\ |n|r_1r_2\delta\ll\Lambda}} \delta E_{r_1,r_2,\delta,n}(Q) \ll \frac{\log Q}{Q} \sum_{\substack{n,r_1,r_2,\delta\geqslant 1 \\ nr_1r_2\delta\ll\Lambda}} \frac{1}{r_1^2 r_2^2 \delta n} \ll \frac{(\log Q)^3}{Q}. \tag{2.10}$$

With the change of variables $(x, y) = (Qu, Qv)$ the main term in (2.9) becomes

$$\frac{6Q^2}{\pi^2} \iint\limits_{[\frac{N}{\Lambda Q^2}, \frac{1}{r_1\delta}]\times[\frac{N}{\Lambda Q^2}, \frac{1}{r_2\delta}]} \psi\Big(\frac{nN}{Q^2\delta uv}\Big) dudv. \tag{2.11}$$

When $0 < \min\{u, v\} < \frac{N}{\Lambda Q^2}$ and $\max\{\delta u, \delta v\} \leqslant 1$ we have $\frac{|n|N}{Q^2 uv\delta} \geqslant \frac{N}{Q^2} \cdot \frac{1}{uv\delta} \geqslant \frac{N}{Q^2} \cdot \frac{1}{\min\{u,v\}} > \Lambda$, so $\psi_\Lambda(\frac{nN}{Q^2\delta uv}) = 0$. Thus the expression in (2.11) amounts to

$$\frac{6Q^2}{\pi^2} \iint\limits_{[0,\frac{1}{r_1\delta}]\times[0,\frac{1}{r_2\delta}]} \psi\Big(\frac{nN}{Q^2\delta uv}\Big) dudv + O\Big(Q^2 \cdot \frac{1}{\Lambda^2} \cdot \frac{1}{r_1 r_2 \delta^2}\Big), \tag{2.12}$$

with the total contribution of the error term to $\mathfrak{M}_Q^{II}(2)$ being

$$\ll \frac{1}{N} \sum_{\substack{n,r_1,r_2,\delta\geqslant 1 \\ nr_1r_2\delta\ll\Lambda}} \frac{1}{r_1 r_2 \delta^2} \ll \frac{Q\log^2 Q}{N} \ll \mathcal{E}(Q).$$

Using (2.11), (2.12), $\psi(x) = \psi(-x)$ and the change of variable $(u, v) \mapsto (u, \lambda)$ with $\lambda = \frac{N}{Q^2} \cdot \frac{|n|}{uv\delta}$, the main term in (2.9) becomes, up to an additive error of order $O(\frac{1}{r_1 r_2 \delta^2})$,

$$\frac{6N}{\pi^2} \cdot \frac{|n|}{\delta} \int_0^{\frac{1}{r_1\delta}} du \int_{\frac{N}{Q^2}\cdot\frac{r_2|n|}{u}}^{\Lambda} \psi(\lambda)\, \frac{d\lambda}{u\lambda^2} = \frac{6N|n|}{\pi^2\delta} \cdot \int_{\frac{N}{Q^2}\cdot|n|r_1r_2\delta}^{\Lambda} \int_{\frac{N}{Q^2}\cdot\frac{r_2|n|}{\lambda}}^{\frac{1}{r_1\delta}} \frac{\psi(\lambda)}{\lambda^2 u}\, dud\lambda.$$

We infer that

$$\mathfrak{M}_Q^{II}(2) = \frac{12}{\pi^2} \sum_{\substack{n,r_1,r_2,\delta \geqslant 1 \\ nr_1r_2\delta \leqslant \frac{\Lambda Q^2}{N}}} \mu(r_1)\mu(r_2) n \int_{\frac{N}{Q^2}\cdot nr_1r_2\delta}^{\Lambda} \frac{\psi(\lambda)}{\lambda^2} \log\Big(\frac{Q^2}{N}\cdot\frac{\lambda}{nr_1r_2\delta}\Big) d\lambda + \mathcal{E}(Q).$$

Taking $K = nr_1r_2\delta \in [1, \frac{\Lambda Q^2}{N}]$ and using

$$\sum_{\substack{n,r_1,r_2 \geqslant 1 \\ nr_1r_2 | K}} \mu(r_1)\mu(r_2) n = \varphi(K)$$

and $|\psi(x)| \leqslant \frac{1}{x^2}$, we infer

$$\begin{aligned}
\mathfrak{M}_Q^{II}(2) &= \frac{12}{\pi^2} \sum_{K \in [1, \frac{\Lambda Q^2}{N}]} \varphi(K) \int_{\frac{N}{Q^2}K}^{\Lambda} \frac{\psi(\lambda)}{\lambda^2} \log\Big(\frac{Q^2}{N}\cdot\frac{\lambda}{K}\Big) d\lambda + \mathcal{E}(Q) \\
&= \frac{12}{\pi^2} \int_0^{\Lambda} \frac{\psi(\lambda)}{\lambda^2} \sum_{K\in[1,\frac{\lambda Q^2}{N}]} \varphi(K) \max\Big\{0, \log\Big(\frac{Q^2}{N}\cdot\frac{\lambda}{K}\Big)\Big\} d\lambda + \mathcal{E}(Q) \\
&= \frac{18}{\pi^4}\cdot\frac{Q^4}{N^2} \int_0^{\Lambda} \psi(\lambda) g_2\Big(\frac{3}{\pi^2}\cdot\frac{Q^2}{N}\lambda\Big) d\lambda + \mathcal{E}(Q) \\
&= \frac{18}{\pi^4}\cdot\frac{Q^4}{N^2} \int_0^{\infty} \psi(\lambda) g_2\Big(\frac{3}{\pi^2}\cdot\frac{Q^2}{N}\lambda\Big) d\lambda + O(\Lambda^{-1}) + \mathcal{E}(Q) \\
&= \frac{18}{\pi^4}\cdot\frac{Q^2}{N} \int_0^{\infty} \psi\Big(\frac{N}{Q^2}x\Big) g_2\Big(\frac{3}{\pi^2}x\Big) dx + O\Big(\frac{(\log Q)^3}{Q}\Big),
\end{aligned} \tag{2.13}$$

with the function g_2 defined as in [2] by

$$g_2\Big(\frac{3}{\pi^2}u\Big) = \frac{2\pi^2}{3u^2} \sum_{K\in[1,u)} \varphi(K) \log\Big(\frac{u}{K}\Big), \tag{2.14}$$

being continuous, supported on $[\frac{3}{\pi^2}, \infty)$, with $\|g_2'\|_\infty < \infty$ and $g_2(x) = 1 + O(\frac{1}{x})$ as $x \to \infty$.

Using the dominated convergence theorem we conclude that,

Proposition 4. *If $N \sim \alpha Q^2$ for some $\alpha > 0$ as $Q \to \infty$, then*

$$\lim_Q \mathfrak{M}_Q(2) = M_2(\alpha) := \frac{3}{\pi^2\alpha} + \Big(\frac{3}{\pi^2}\Big)^2 \cdot \frac{2}{\alpha} \int_0^\infty \operatorname{sinc}^2(\pi\alpha u) g_2\Big(\frac{3}{\pi^2}u\Big) du.$$

Remark 5. Since $|\psi(x)| \leqslant \frac{1}{x^2}$, $\psi \in C^1([0,\infty))$ and $\|g_2\|_\infty < 1$, it is easily seen, by truncating the integral in Proposition 4 at $Q^{\beta/2}$, that if

$$N = \alpha Q^2\big(1 + O(Q^{-\beta})\big)$$

for some $\beta > 0$, then

$$\mathfrak{M}_Q(2) = M_2(\alpha) + O(Q^{-\beta/2}).$$

Using a different description of $g_2(x)$, due to R. R. Hall and presented in [2], we also see that this main term matches the expression given in Theorem 1 (we however do not need this, since we reprove Proposition 4 in Section 5, when dealing with the general case of all $\ell \geqslant 2$).

2.2. **Comparison with Ramaré's main term.** Ramaré's estimate of $\mathfrak{M}_Q(2)$ produced the following main term (see the formula between (46) and (47) and formula (47) in [14]):

$$\mathfrak{M}_Q(2) \sim \frac{|\mathcal{F}_Q|}{N} + \frac{Q^4}{N^2} \cdot \mathfrak{h}\Big(\frac{N}{Q^2}\Big), \tag{2.15}$$

where

$$\mathfrak{h}(x) = \frac{6}{\pi^3 i} \int_{-\frac{1}{8}-i\infty}^{-\frac{1}{8}+i\infty} \frac{x^s}{s(s+1)(2-s)^2} \cdot \frac{\zeta(s)}{\zeta(2-s)}\, ds. \tag{2.16}$$

Employing formula (4.15) in [2] we can write

$$\begin{aligned} g_2\Big(\frac{3}{\pi^2}u\Big) &= \frac{2\pi^2}{3u^2} \cdot \frac{1}{2\pi i} \int_{\frac{17}{8}-i\infty}^{\frac{17}{8}+i\infty} \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{u^s}{s^2}\, ds \\ &= \frac{2\pi^2}{3u^2} \cdot \frac{1}{2\pi i} \int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(s)}{\zeta(1+s)} \cdot \frac{u^{1+s}}{(1+s)^2}\, ds. \end{aligned} \tag{2.17}$$

Employing also Fubini we infer

$$\begin{aligned} I_\alpha &:= \frac{3}{\pi^2} \int_0^\infty \operatorname{sinc}^2(\alpha u) g_2\Big(\frac{3}{\pi^2}u\Big) du \\ &= \frac{1}{\pi i} \int_0^\infty \operatorname{sinc}^2(\alpha u) \int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{1}{u^2} \cdot \frac{u^{1+s}}{(1+s)^2} \cdot \frac{\zeta(s)}{\zeta(1+s)}\, ds du \\ &= \frac{1}{\pi i \alpha^2} \int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(s)}{(1+s)^2 \zeta(1+s)} \int_0^\infty \frac{\sin^2(\alpha u)}{u^{3-s}}\, du ds. \end{aligned} \tag{2.18}$$

Employing the identity (cf. formula 3.823 page 454 in [10])

$$\int_0^\infty \frac{\sin^2 x}{x^\nu}\, dx = -2^{\nu-2}\Gamma(1-\nu) \cos\Big(\frac{(1-\nu)\pi}{2}\Big) \qquad \text{if } 1 < \operatorname{Re}\nu < 3, \tag{2.19}$$

we find

$$\int_0^\infty \frac{\sin^2(\alpha u)}{u^{3-s}}\, du = \alpha^{2-s} 2^{1-s} \Gamma(s-2) \cos\Big(\frac{\pi s}{2}\Big) = \frac{2\alpha^2}{(2\alpha)^s} \cdot \frac{\Gamma(s) \cos\big(\frac{\pi s}{2}\big)}{(s-2)(s-1)},$$

which we insert into (2.18) to derive

$$I_\alpha = \frac{2}{\pi i}\int_{9/8-i\infty}^{9/8+i\infty} \frac{(2\alpha)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}{(2-s)(1-s)(1+s)^2}\cdot\frac{\zeta(s)}{\zeta(1+s)}\,ds.$$

The functional equation

$$\zeta(s) = \frac{\pi\zeta(1-s)}{(2\pi)^{1-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)}$$

and the change of variables $s \mapsto 1-s$ provide

$$\begin{aligned} I_\alpha &= \frac{1}{\pi i}\int_{9/8-i\infty}^{9/8+i\infty} \left(\frac{\pi}{\alpha}\right)^s \frac{1}{(2-s)(1-s)(1+s)^2}\cdot\frac{\zeta(1-s)}{\zeta(1+s)}\,ds \\ &= \frac{1}{\pi i}\int_{-1/8-i\infty}^{-1/8+i\infty} \left(\frac{\pi}{\alpha}\right)^{1-s} \frac{1}{s(1+s)(2-s)^2}\cdot\frac{\zeta(s)}{\zeta(2-s)}\,ds. \end{aligned}$$

Finally, inserting this into (2.13) we infer

$$\begin{aligned} \mathfrak{M}_Q^{II}(2) &\sim \frac{Q^2}{N}\cdot\frac{6}{\pi^2} I_{\pi N/Q^2} \\ &= \frac{Q^2}{N}\cdot\frac{6}{\pi^2}\cdot\frac{1}{\pi i}\int_{-1/8-i\infty}^{-1/8+i\infty} \frac{(Q^2/N)^{1-s}}{s(s+1)(2-s)^2}\cdot\frac{\zeta(s)}{\zeta(2-s)}\,ds \\ &= \frac{Q^4}{N^2}\cdot\frac{6}{\pi^3 i}\int_{-1/8-i\infty}^{-1/8+i\infty} \frac{(N/Q^2)^s}{s(s+1)(2-s)^2}\cdot\frac{\zeta(s)}{\zeta(2-s)}\,ds \\ &= \frac{Q^4}{N^2}\cdot\mathfrak{h}\Big(\frac{N}{Q^2}\Big). \end{aligned} \tag{2.20}$$

From (2.7) and (2.20) we notice that our main term $M_2(\alpha)$ in the asymptotic formula for $\mathfrak{M}_Q(2)$ given in Proposition 4 coincides with the one in [14, Theorem 1.1].

3. Smoothing of $\mathfrak{M}_Q(\ell)$

As seen in the previous section, when dealing with $\mathfrak{M}_Q(2)$ it is possible to proceed directly without smoothing the characteristic function H_N. However, smoothing becomes necessary for $\ell \geqslant 3$, due to the accumulations of terms. In this section we show that this can be efficiently achieved employing the large sieve inequality.

Let $\delta \in (0,1)$. We pick a function $f_\delta \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant f_\delta \leqslant 1$, $f_\delta \equiv 1$ on the interval $[\delta, 1]$, and $\operatorname{supp} f_\delta = [0, 1+\delta]$. Consider $\Theta = (\theta_1, \ldots, \theta_\ell) \in \mathcal{F}_Q^\ell$, the function

$$\begin{aligned} h_\Theta(x_1,\ldots,x_\ell) &:= e\big(x_1(\theta_1-\theta_\ell) + x_2(\theta_2-\theta_1) + \cdots + x_\ell(\theta_\ell-\theta_{\ell-1})\big) \\ &= e\big((x_1-x_2)\theta_1 + (x_2-x_3)\theta_2 + \cdots + (x_\ell - x_1)\theta_\ell\big), \end{aligned} \tag{3.1}$$

and its smoothed form

$$h_{\delta;\Theta}(x_1,\ldots,x_\ell) := h_\Theta(x_1,\ldots,x_\ell) f_\delta\Big(\frac{x_1}{N}\Big)\cdots f_\delta\Big(\frac{x_\ell}{N}\Big). \tag{3.2}$$

In this section we will show that the large sieve inequality allows us to replace $\mathfrak{M}_Q(\ell)$ by its smoothed version

$$\begin{aligned}\mathfrak{M}_{Q;\delta}(\ell) &:= \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1,\ldots,\theta_\ell\in\mathcal{F}_Q\\ n_1,\ldots,n_\ell\in\mathbb{Z}}} h_{\delta;\Theta}(n_1,\ldots,n_\ell)\\ &= \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1,\ldots,\theta_\ell\in\mathcal{F}_Q\\ 0<n_1,\ldots,n_\ell<(1+\delta)N}} h_{\delta;\Theta}(n_1,\ldots,n_\ell).\end{aligned} \tag{3.3}$$

On the other hand we have

$$\mathfrak{M}_Q(\ell) = \frac{1}{N^{\ell+1}} \sum_{\substack{\theta_1,\ldots,\theta_\ell\in\mathcal{F}_Q\\ 0<n_1,\ldots,n_\ell<(1+\delta)N}} h_\Theta(n_1,\ldots,n_\ell)\mathbf{1}_{(0,1]}\Big(\frac{n_1}{N}\Big)\cdots\mathbf{1}_{(0,1]}\Big(\frac{n_\ell}{N}\Big),$$

where $\mathbf{1}_{\mathcal{S}}$ denotes the characteristic function of a set $\mathcal{S}$.

For disjoint subsets $\mathcal{M},\mathcal{A},\mathcal{B}$ of $[1,\ell]$ consider

$$\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}} := \frac{1}{N^{\ell+1}} \sum_{\theta_1,\ldots,\theta_\ell\in\mathcal{F}_Q} \sum_{\substack{n_1,\ldots,n_\ell\\ \{j:0<n_j<\delta N\}=\mathcal{A}\\ \{k:N<n_k<(1+\delta)N\}=\mathcal{B}\\ \{i:\delta N\leqslant n_i\leqslant N\}=\mathcal{M}}} h_{\delta;\Theta}(n_1,\ldots,n_\ell),$$

$$\mathcal{I}^{(2)}_{\mathcal{M},\mathcal{A},\mathcal{B}} := \frac{1}{N^{\ell+1}} \sum_{\theta_1,\ldots,\theta_\ell\in\mathcal{F}_Q} \sum_{\substack{n_1,\ldots,n_\ell\\ \{j:0<n_j<\delta N\}=\mathcal{A}\\ \{k:N<n_k<(1+\delta)N\}=\mathcal{B}\\ \{i:\delta N\leqslant n_i\leqslant N\}=\mathcal{M}}} h_\Theta(n_1,\ldots,n_\ell)\mathbf{1}_{(0,1]}\Big(\frac{n_1}{N}\Big)\cdots\mathbf{1}_{(0,1]}\Big(\frac{n_\ell}{N}\Big).$$

If $\mathcal{B}\neq\emptyset$, then $\mathcal{I}^{(2)}_{\mathcal{M},\mathcal{A},\mathcal{B}}=0$. We have

$$\mathfrak{M}_Q(\ell) = \sum_{\mathcal{A}\sqcup\mathcal{B}\sqcup\mathcal{M}=[1,\ell]} \mathcal{I}^{(2)}_{\mathcal{M},\mathcal{A},\mathcal{B}}, \qquad \mathfrak{M}_{Q;\delta}(\ell) = \sum_{\mathcal{A}\sqcup\mathcal{B}\sqcup\mathcal{M}=[1,\ell]} \mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}}.$$

Employing $\mathcal{I}^{(1)}_{[1,\ell],\emptyset,\emptyset}=\mathcal{I}^{(2)}_{[1,\ell],\emptyset,\emptyset}$ we can write

$$\mathfrak{M}_Q(\ell)-\mathfrak{M}_{Q;\delta}(\ell) = \sum_{\substack{\mathcal{A}\sqcup\mathcal{B}\sqcup\mathcal{M}=[1,\ell]\\ \mathcal{M}\neq[1,\ell]}} \big(\mathcal{I}^{(2)}_{\mathcal{M},\mathcal{A},\mathcal{B}}-\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}}\big). \tag{3.4}$$

We will now bound the contribution of each $\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}}$ with $\mathcal{M}\neq[1,\ell]$ using the large sieve inequality. The contribution of $\mathcal{I}^{(2)}_{\mathcal{M},\mathcal{A},\mathcal{B}}$ will be dealt with in identical manner by taking $f_\delta(\frac{n}{N}) = \mathbf{1}_{(0,1]}(\frac{n}{N})$ in the argument that is about to follow. For this reason we will only write down the argument for $\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}}$. Consider

$$x_{n,\theta} := e(n\theta)\sqrt{f_\delta\Big(\frac{n}{N}\Big)}, \quad 0 < n < (1+\delta)N, \theta \in \mathcal{F}_Q,$$

and the rectangular matrices $A, B \in M_{[\delta N],|\mathcal{F}_Q|}(\mathbb{C})$, $M \in M_{N-[\delta N],|\mathcal{F}_Q|}(\mathbb{C})$ with entries $x_{n,\theta}$ where $\theta \in \mathcal{F}_Q$ and $0 < n < \delta N$ for A, $N < n < (1+\delta)N$ for B, and $\delta N \leqslant n \leqslant N$ for M, respectively. Clearly $A^\star A, B^\star B, M^\star M$ are $|\mathcal{F}_Q| \times |\mathcal{F}_Q|$ matrices with

$$\begin{aligned}
(A^\star A)_{\theta',\theta''} &= \sum_{0<n<\delta N} e\big(n(\theta''-\theta')\big) f_\delta\Big(\frac{n}{N}\Big), \\
(B^\star B)_{\theta',\theta''} &= \sum_{N<n<(1+\delta)N} e\big(n(\theta''-\theta')\big) f_\delta\Big(\frac{n}{N}\Big), \\
(M^\star M)_{\theta',\theta''} &= \sum_{\delta N\leqslant n\leqslant N} e\big(n(\theta''-\theta')\big) f_\delta\Big(\frac{n}{N}\Big).
\end{aligned} \tag{3.5}$$

Writing

$$\begin{aligned}
M &= \operatorname{diag}\bigg(\sqrt{f_\delta\Big(\frac{n}{N}\Big)}\bigg)_{\delta N\leqslant n\leqslant N} \cdot \Big(e(n\theta)\Big)_{\substack{\delta N\leqslant n\leqslant N \\ \theta\in\mathcal{F}_Q}}, \\
A &= \operatorname{diag}\bigg(\sqrt{f_\delta\Big(\frac{n}{N}\Big)}\bigg)_{0<n<\delta N} \cdot \Big(e(n\theta)\Big)_{\substack{0<n<\delta N \\ \theta\in\mathcal{F}_Q}}, \\
B &= \operatorname{diag}\bigg(\sqrt{f_\delta\Big(\frac{n}{N}\Big)}\bigg)_{N<n<(1+\delta)N} \cdot \Big(e(n\theta)\Big)_{\substack{N<n<(1+\delta)N \\ \theta\in\mathcal{F}_Q}}
\end{aligned}$$

and employing $0 \leqslant f_\delta \leqslant 1$, the large sieve inequality provides

$$\|M^\star M\| \leqslant N + Q^2 \quad \text{and} \quad \max\big\{\|A^\star A\|, \|B^\star B\|\big\} \leqslant \delta N + Q^2. \tag{3.6}$$

Since $\max\{\operatorname{rank}(A), \operatorname{rank}(B)\} \leqslant \delta N$, we have $\max\{\operatorname{rank}(A^* A), \operatorname{rank}(B^* B)\} \leqslant \delta N$. Since $\operatorname{rank}(X_1 \cdots X_\ell) \leqslant \min\{\operatorname{rank}(X_1), \ldots, \operatorname{rank}(X_\ell)\}$, we infer

$$\operatorname{rank}\left(\prod_{r=1}^{\ell} (M^\star M)^{\alpha_r} (A^\star A)^{\beta_r} (B^\star B)^{\gamma_r}\right) \leqslant \delta N \tag{3.7}$$

whenever $\alpha_r, \beta_r, \gamma_r \in \{0, 1\}$ and there exists $r_0 \in [1, \ell]$ such that $\beta_{r_0} > 0$ or $\gamma_{r_0} > 0$.

On the other hand, setting $S(r) := \mathcal{A}$, $\mathcal{B}$ or $\mathcal{M}$ according to whether $r \in \mathcal{A}$, $r \in \mathcal{B}$ or $r \in \mathcal{M}$ and using (3.5), we see that the (θ', θ'')-entry of the product $\prod_{r=1}^{\ell} (M^\star M)^{\mathbf{1}_{\mathcal{M}}(r)} (A^\star A)^{\mathbf{1}_{\mathcal{A}}(r)} (B^\star B)^{\mathbf{1}_{\mathcal{B}}(r)}$ of ℓ matrices of the form $M^\star M$, $A^\star A$ or $B^\star B$

is given by

$$\sum_{\substack{\theta_1,\dots,\theta_{\ell-1}\in\mathcal{F}_Q\\ 0<n_1,\dots,n_\ell<(1+\delta)N\\ n_r\in S(r),\forall r\in[1,\ell]}} e\Big(n_1(\theta_1-\theta')\Big)f_\delta\Big(\frac{n_1}{N}\Big)e\Big(n_2(\theta_2-\theta_1)\Big)f_\delta\Big(\frac{n_2}{N}\Big)$$
$$\cdots e\Big(n_{\ell-1}(\theta_{\ell-1}-\theta_{\ell-2})\Big)f_\delta\Big(\frac{n_{\ell-1}}{N}\Big)e\Big(n_\ell(\theta''-\theta_{\ell-1})\Big)f_\delta\Big(\frac{n_\ell}{N}\Big).$$

In conjunction with the definition of $\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}}$, (3.1) and (3.2) and setting $\theta_0=\theta_\ell=\theta'=\theta''$, this further leads to

$$\begin{aligned}\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}} &= \frac{1}{N^{\ell+1}}\sum_{\theta_1,\dots,\theta_\ell\in\mathcal{F}_Q}\ \sum_{\substack{\{j:0<n_j<\delta N\}=\mathcal{A}\\ \{k:N<n_k<(1+\delta)N\}=\mathcal{B}\\ \{i:\delta N\leqslant n_i\leqslant N\}=\mathcal{M}}}\ \prod_{r=1}^{\ell} e\Big(n_r(\theta_r-\theta_{r-1})\Big)f_\delta\Big(\frac{n_r}{N}\Big)\\ &= \frac{1}{N^{\ell+1}}\operatorname{Tr}\left(\prod_{r=1}^{\ell}(M^\star M)^{\mathbf{1}_{\mathcal{M}}(r)}(A^\star A)^{\mathbf{1}_{\mathcal{A}}(r)}(B^\star B)^{\mathbf{1}_{\mathcal{B}}(r)}\right).\end{aligned} \tag{3.8}$$

Employing (3.8), the inequality $\operatorname{Tr}(X)\leqslant\operatorname{rank}(X)\|X\|$ for any square matrix X, and inequalities (3.6) and (3.7), we infer

$$|\mathcal{I}^{(1)}_{\mathcal{M},\mathcal{A},\mathcal{B}}|\leqslant\frac{1}{N^{\ell+1}}\cdot\delta N(N+Q^2)^\ell\ll_\ell\delta\quad\text{whenever }\mathcal{M}\neq[1,\ell]. \tag{3.9}$$

A similar bound holds on $\mathcal{I}^{(2)}_{\mathcal{M},\mathcal{A},\mathcal{B}}$, with $f_\delta(\frac{n}{N})=\mathbf{1}_{(0,1]}(\frac{n}{N})$ above, hence (3.4) and (3.9) yield

$$\mathfrak{M}_Q(\ell)=\mathfrak{M}_{Q;\delta}(\ell)+O_\ell(\delta). \tag{3.10}$$

4. Analysis of the main term $M_\ell(\alpha)$

Fix $k=\ell-1\geqslant 1$ and a constant $\alpha>0$. For every $A,B\in\mathbb{Z}$, $A^2+B^2\neq 0$, consider the function $\beta_{A,B,\alpha}$ defined by

$$\beta_{A,B,\alpha}(x,y):=\frac{\alpha B}{y(Ay-Bx)}. \tag{4.1}$$

Let $\mathfrak{F}$ denote the set of functions $F:\mathbb{R}\to\mathbb{C}$ that satisfy

$$F(0)=1,\qquad F(-u)=F(u),\qquad |F(u)|\leqslant\min\Big\{1,\frac{1}{|u|}\Big\},\qquad\forall u\in\mathbb{R}.$$

Denote

$$\psi_F(x_1,\dots,x_k):=\begin{cases}F(-x_1)\prod_{i=1}^{k-1}F(x_i-x_{i+1})F(x_k) & \text{if } k\geqslant 2,\\ F(-x_1)F(x_1) & \text{if } k=1.\end{cases} \tag{4.2}$$

For every $\mathbf{A} = (A_1, \dots, A_k), \mathbf{B} = (B_1, \dots, B_k) \in \mathbb{Z}^k$, consider the function in two variables

$$\Psi_{F;\mathbf{A},\mathbf{B},\alpha}(x,y) := \Psi_F\big(\beta_{A_1,B_1,\alpha}(x,y), \dots, \beta_{A_k,B_k,\alpha}(x,y)\big), \tag{4.3}$$

and the set $\mathcal{D}_{\mathbf{A},\mathbf{B}}$ defined in (1.3). Consider also

$$I_{k,\delta,\alpha}(F) := \sum_{\substack{\mathbf{A},\mathbf{B}\in\mathbb{Z}^k \\ (A_i,B_i)=1,\forall i \\ A_i^2+B_i^2\neq 0,\forall i}} \max_{i\in[1,k]} \{|A_i|^\delta, |B_i|^\delta\} \iint_{\mathcal{D}_{\mathbf{A},\mathbf{B}}} |\Psi_{F;\mathbf{A},\mathbf{B},\alpha}(x,y)|\, dxdy \in [0,\infty].$$

Recall that we take $(A,0) = |A|$, so if $B_i = 0$ for some i in some non-zero term of $I_{k,\delta,\alpha}(F)$, then $|A_i| \geqslant 1$.

The aim of this section is to prove the following:

Proposition 6. *There exists $\delta = \delta_\ell > 0$ such that for every $\alpha > 0$ we have*

$$\sup_{F\in\mathfrak{F}} I_{k,\delta,\alpha}(F) \ll_k \alpha^{-k-1} + 1 < \infty.$$

In particular this establishes part (ii) in Theorem 1. Before starting the proof of Proposition 6, we note its subsequent important consequence, which gives part (iii) in Theorem 1.

Corollary 7. *Let $0 < \gamma_1 < \gamma_2$ be given. With $M_\ell(\alpha)$ as in (1.4), and uniformly in $\alpha, \beta \in [\gamma_1, \gamma_2]$, we have*

$$|M_\ell(\alpha) - M_\ell(\beta)| \ll_{\gamma_1,\gamma_2} |\alpha - \beta|^{\kappa_\ell}$$

for some exponent $\kappa_\ell \in (0,1)$.

Proof. Consider

$$G_F(\alpha,\ell) = \sum_{\substack{\mathbf{A},\mathbf{B}\in\mathbb{Z}^k \\ (A_i,B_i)=1,\forall i}} \iint_{\mathcal{D}_{\mathbf{A},\mathbf{B}}} \Psi_{F;\mathbf{A},\mathbf{B},\alpha}(x,y)dxdy.$$

If F is continuous, then Proposition 6 and the dominated convergence theorem show that $G_F(\alpha,\ell)$ is continuous in α. Note that $M_\ell(\alpha) = \frac{6}{\pi^2\alpha} G_{\widehat{\chi}_{[0,1]}}(\alpha,\ell)$. Since $\widehat{\chi}_{[0,1]}$ is continuous it follows that $M_\ell(\alpha)$ is also continuous. In order to establish the stronger bound note that for arbitrary functions f and g we have,

$$|(f\cdot g)(\alpha) - (f\cdot g)(\beta)| \leqslant |f(\alpha) - f(\beta)| \cdot \|g\|_\infty + |g(\alpha) - g(\beta)| \cdot \|f\|_\infty. \tag{4.4}$$

Since $\alpha \mapsto \frac{1}{\alpha}$ is Lipschitz continuous on the interval $[\gamma_1, \gamma_2]$, it is therefore enough to show that $G_{\widehat{\chi}_{[0,1]}}(\alpha,\ell)$ satisfies the bound

$$|G_{\widehat{\chi}_{[0,1]}}(\alpha,\ell) - G_{\widehat{\chi}_{[0,1]}}(\beta,\ell)| \ll_{\gamma_1,\gamma_2} |\alpha - \beta|^{\kappa_\ell}$$

for some exponent $\kappa_\ell > 0$.

Using Proposition 6 we can truncate the expressions defining $G_{\widehat{\chi}_{[0,1]}}(\alpha,\ell)$ and $G_{\widehat{\chi}_{[0,1]}}(\beta,\ell)$ at $\max_i\{|A_i|,|B_i|\} \leqslant |\alpha-\beta|^{-\eta}$ at the price of an error term $\ll |\alpha-\beta|^{\eta\delta_\ell}$. That is, $G_{\widehat{\chi}_{[0,1]}}(\alpha,\ell) - G_{\widehat{\chi}_{[0,1]}}(\beta,\ell)$ is equal to

$$\sum_{\substack{|A_i|\leqslant|\alpha-\beta|^{-\eta}\\ 0<|B_i|\leqslant|\alpha-\beta|^{-\eta}\\ (A_i,B_i)=1,\forall i}} \iint_{\mathcal{D}_{\mathbf{A},\mathbf{B}}} \big(\Psi_{\widehat{\chi}_{[0,1]},\mathbf{A},\mathbf{B},\alpha}(x,y) - \Psi_{\widehat{\chi}_{[0,1]},\mathbf{A},\mathbf{B},\beta}(x,y)\big)dxdy + O(|\alpha-\beta|^{\eta\delta_\ell}),$$

where

$$\Psi_{\widehat{\chi}_{[0,1]},\mathbf{A},\mathbf{B},\alpha}(x,y) := \Psi_{\widehat{\chi}_{[0,1]}}\big(\beta_{A_1,B_1,\alpha}(x,y),\dots,\beta_{A_{\ell-1},B_{\ell-1},\alpha}(x,y)\big).$$

The product of bounded Lipschitz continuous functions is Lipschitz continuous by (4.4) and therefore,

$$|\Psi_{\widehat{\chi}_{[0,1]},\mathbf{A},\mathbf{B},\alpha}(x,y) - \Psi_{\widehat{\chi}_{[0,1]},\mathbf{A},\mathbf{B},\beta}(x,y)| \ll_{\gamma_1,\gamma_2} |\alpha-\beta|.$$

Combining the previous three equations we conclude that,

$$|G_{\widehat{\chi}_{[0,1]}}(\alpha,\ell) - G_{\widehat{\chi}_{[0,1]}}(\beta,\ell)| \ll_{\gamma_1,\gamma_2} |\alpha-\beta|^{\kappa_\ell},$$

where $\kappa_\ell := \min(1-2(2\ell-2)\eta, \eta\delta_\ell) \in (0,1)$. Taking $\eta>0$ sufficiently small shows that $\kappa_\ell \in (0,1)$. □

We will require two lemmas for the proof of Proposition 6. First we record a simple bound for $\Psi_{F;\mathbf{A},\mathbf{B}}$:

Lemma 8. *Let $I := \{i \in [1,k-1] : A_iB_{i+1} - A_{i+1}B_i \neq 0\}$. Suppose that $B_1 \neq 0$ and $B_k \neq 0$. Then, for every $\alpha > 0$ and $(x,y) \in \mathcal{D}_{\mathbf{A},\mathbf{B}}$ we have*

$$\sup_{F\in\mathfrak{F}}|\Psi_{F;\mathbf{A},\mathbf{B}}(x,y)| \leqslant \frac{y^2}{\alpha^{|I|+2}|B_1B_k|}\prod_{i\in I}\frac{|A_iy-B_ix|}{|A_iB_{i+1}-A_{i+1}B_i|}.$$

Proof. The first inequality follows from the bounds

$$|\psi_F(x_1,\dots,x_k)| \leqslant \frac{1}{|x_1x_k|}\prod_{i\in I}\frac{1}{|x_i-x_{i+1}|}, \qquad |A_iy-B_ix| \leqslant 1,$$

and from equality

$$\beta_{A_i,B_i,\alpha}(x,y) - \beta_{A_{i+1},B_{i+1},\alpha}(x,y) = \frac{\alpha(A_{i+1}B_i - A_iB_{i+1})}{(A_iy-B_ix)(A_{i+1}y-B_{i+1}x)}. \tag{4.5}$$

□

Our argument will rely crucially on the (non-disjoint) dy-adic set equality $\mathbb{N} = \bigcup_{a\in\mathbb{N}_0}[2^a-1, 2^{a+1}]$ and on the following "counting lemma":

Lemma 9. *Given* $\mathbf{D} = (D_1, \ldots, D_{k-1}) \in \mathbb{Z}^{2k-2}$, $\mathbf{a} = (a_1, \ldots, a_k), \mathbf{b} = (b_1, \ldots, b_k) \in \mathbb{N}_0^k$, *consider the set*

$$\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}} := \left\{ (\mathbf{A}, \mathbf{B}) \in \mathbb{Z}^{2k} : \begin{array}{c} 2^{a_i} - 1 \leqslant |A_i| \leqslant 2^{a_i+1}, \ 2^{b_i} - 1 \leqslant |B_i| \leqslant 2^{b_i+1}, \\ A_i B_{i+1} - A_{i+1} B_i = D_i, \ (A_i, B_i) = 1, \forall i \end{array} \right\},$$

with $\mathbf{A} = (A_1, \ldots, A_k)$, $\mathbf{B} = (B_1, \ldots, B_k)$.

For every $x, y \in [0,1]^2$ *with* $x \leqslant y$ *we have*

(i) $$\sum_{(\mathbf{A},\mathbf{B}) \in \mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}} \prod_{i=1}^{k} \mathbf{1}_{0 < A_i y - B_i x \leqslant 1} \ll \frac{2^{b_1}}{y} \prod_{i=1}^{k-1} \min\left\{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \right\}.$$

(ii) $$\sum_{(\mathbf{A},\mathbf{B}) \in \mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}} 1 \ll 2^{a_1} 2^{b_1} \prod_{i=1}^{k-1} \min\{2^{b_{i+1} - b_i} + 1, 2^{a_{i+1} - a_i} + 1\}.$$

Proof. To prove (i), notice that given B_1 and A_1, the condition $A_1 B_2 - A_2 B_1 = D_1$ implies that

$$A_2 = x_1 + k_1 A_1 \quad \text{and} \quad B_2 = y_1 + k_1 B_1, \tag{4.6}$$

with (x_1, y_1) particular solution of $A_1 y - B_1 x = D_1$ (so $x_1 \equiv -\overline{B_1} D_1 \pmod{A_1}$ and $y_1 \equiv \overline{A_1} D_1 \pmod{B_1}$) and $k_1 \in \mathbb{Z}$. In addition, since $\frac{B_2 x}{y} \leqslant A_2 \leqslant \frac{B_2 x}{y} + \frac{1}{y}$ has to fit into an interval of length $\leqslant \frac{1}{y}$ and $|A_1| \geqslant 2^{a_1}$, the number of choices of k_1 for fixed (A_1, B_1) is

$$\ll (1/y) 2^{-a_1} + 1,$$

regardless of the choice of (x, y). Repeating the same reasoning with B_2 in place of A_2 we see that B_2 is also required to be contained in a short interval of length $\leqslant \frac{1}{x}$, since $-\frac{A_2 y}{x} \leqslant -B_2 \leqslant -\frac{A_2 y}{x} + \frac{1}{x}$. By the same argument it follows that the number of admissible choices for k_1 is also

$$\ll (1/x) 2^{-b_1} + 1.$$

Therefore, regardless on (x, y), the number of choices for k_1 is

$$\ll \min\left\{ (1/y) 2^{-a_1} + 1, (1/x) 2^{-b_1} + 1 \right\}.$$

Continuing, we see that in general, given (A_i, B_i), we have $A_i B_{i+1} - A_{i+1} B_i = D_i$ and therefore (A_{i+1}, B_{i+1}) is parameterized as

$$A_{i+1} = x_i + k_i A_i \quad \text{and} \quad B_{i+1} = y_i + k_i B_i,$$

with (x_i, y_i) fixed solution of $A_i y - B_i x = D_i$ and $k_i \in \mathbb{Z}$. It follows that if we are given A_1 and B_1, then the number of admissible choices for $A_2, B_2, \ldots, A_k, B_k$ is

$$\ll \prod_{i=1}^{k-1} \min\left\{ (1/y) 2^{-a_i} + 1, (1/x) 2^{-b_i} + 1 \right\}.$$

Finally, the number of choices for (A_1, B_1) is $\leqslant 2^{b_1+1}(1+\frac{1}{y}) \ll \frac{2^{b_1}}{y}$ regardless of (x, y) since $|B_1| \leqslant 2^{b_1+1}$ and $\frac{B_1 x}{y} \leqslant A_1 \leqslant \frac{B_1 x}{y} + \frac{1}{y}$. This proves (i).

Part (ii) is proved in a similar way by first selecting (A_1, B_1) in at most $2^{a_1+1}2^{b_1+1}$ ways, and then parameterizing as in (4.6). Since we require $|B_2| \leqslant 2^{b_2+1}$ and $|B_1| \geqslant 2^{b_1+1}$, the number of choices for k_1 is $\ll 2^{b_2-b_1} + 1$, and therefore the number of choices for (A_2, B_2) is $\ll 2^{b_2-b_1} + 1$ as well. Now that we fixed (A_2, B_2), it is seen in a similar way that the number of choices for (A_3, B_3) is $\ll 2^{b_3-b_2} + 1$, and so on, showing that the left hand side in (ii) is $\ll 2^{a_1+b_1}\prod_{i=1}^{k-1}(2^{b_{i+1}-b_i} + 1)$. Finally the roles of A_i and B_i can be interchanged to prove that the left hand side in (ii) is $\ll 2^{a_1+b_1}\prod_{i=1}^{k-1}(2^{a_{i+1}-a_i} + 1)$. □

With these two lemmas at hand, we are ready to start the proof of Proposition 6.

First we dispose of the easy case, $k = 1$. The constraint $0 < A_1 y - B_1 x \leqslant 1$ gives $\frac{B_1 x}{y} \leqslant A_1 \leqslant \frac{B_1 x}{y} + \frac{1}{y}$, and so for fixed B_1 the number of admissible A_1's is $\ll \frac{1}{y}$ and $|A_1| \leqslant \frac{|B_1|+1}{y}$. On the other hand, if $B_1 \neq 0$, then

$$|\Psi_{F;A_1,B_1,\alpha}(x, y)| \leqslant \frac{y^2}{\alpha^2|B_1|^2},$$

providing, for every $\delta \in (0, 1)$,

$$\begin{aligned}|I_{1,\delta,\alpha}(F)| \ll \sum_{B_1\in\mathbb{Z}^*}\int_0^1 \frac{y^2}{\alpha^2|B_1|^2}\cdot\frac{1}{y}\cdot\frac{(|B_1|+1)^\delta}{y^\delta}\int_0^y dxdy \\ + \sum_{A_1\in\mathbb{N}}\iint_{0\leqslant x\leqslant y\leqslant 1/A_1} dxdy \ll_\delta \alpha^{-2} + 1.\end{aligned}$$

Secondly, proceeding by induction on k allows us to reduce ourselves to the case when $B_1 \neq 0$, $B_k \neq 0$, and $A_i B_{i+1} - A_{i+1} B_i \neq 0$ for all $1 \leqslant i < k$. Indeed, notice that equality (4.5) shows that when there exists an i such that $A_{i+1}B_i - A_i B_{i+1} = 0$, the requirements $(A_i, B_i) = (A_{i+1}, B_{i+1}) = 1$ lead to $A_{i+1} = A_i$ and $B_{i+1} = B_i$, thus Proposition 6 follows from the situation where k is replaced by $k - 1$ (see (4.2)). Similarly if $B_1 = 0$ then $\beta_{A_1,B_1,\alpha}(x, y) = 0$, therefore

$$\beta_{A_1,B_1,\alpha}(x, y) - \beta_{A_2,B_2,\alpha}(x, y) = -\beta_{A_2,B_2,\alpha}(x, y).$$

Since $F(0) = 1$ and $F(x) = F(-x)$, Proposition 6 once again reduces to the case of $k - 1$ variables. The same argument allows us to assume that $B_k \neq 0$.

According to Lemma 8, and the previous remark, it will suffice to establish that the following expression converges for some $\delta = \delta_\ell > 0$,

$$\sum_{\substack{A_1,\dots,A_k,B_1,\dots,B_k\in\mathbb{Z}\\ A_iB_{i+1}-A_{i+1}B_i\neq 0\\ (A_i,B_i)=1,\forall i\\ B_1\neq 0, B_k\neq 0}} \max_{i\in[1,k]}\{|A_i|^\delta,|B_i|^\delta\}\frac{1}{|B_1B_k|}\prod_{i=1}^{k-1}\frac{1}{|A_iB_{i+1}-A_{i+1}B_i|} \times \iint_{0\leqslant x\leqslant y\leqslant 1} y^2\prod_{i=1}^{k}(A_iy-B_ix)\mathbf{1}_{0<A_iy-B_ix\leqslant 1}\,dxdy. \tag{4.7}$$

Fix some $\boxed{\varepsilon\in(0,\frac{1}{1000k})}$. We start by making several reductions, the outcome of which is that we can focus on the scenario in which both of the following conditions hold:

(I) The range of integration over y is restricted to $y>\max_{i\in[1,k]}|B_i|^{-\varepsilon^2}$.

(II) For all $i\in[1,k-1]$ we have $|A_iB_{i+1}-A_{i+1}B_i|\ll\max_{i\in[1,k]}|B_i|^{\varepsilon^2}$.

We then use two different arguments to handle the total contributions of the integers $A_1,\dots,A_k,B_1,\dots,B_k$ for which there exists an index $j\in[1,k-1]$ such that $|B_{j+1}|>|B_j|\cdot\max_{i\in[1,k]}|B_i|^\varepsilon$, and respectively the contributions of the integers for which there is no such index. Finally, we set $\delta=\delta_\ell=\varepsilon^3$.

In the remaining part of this section we will group the integers A_i and B_i into dy-adic ranges $2^{a_i}-1\leqslant|A_i|\leqslant 2^{a_i+1}$ and $2^{b_i}-1\leqslant|B_i|\leqslant 2^{b_i+1}$,with a_i and b_i running through the non-negative integers. Note that the intervals $[2^a-1,2^{a+1}]$ are overlapping, but this is not a problem because in this section we only add or integrate non-negative quantities.

4.1. **Disposing of the y's for which $y<\max_{i\in[1,k]}|B_i|^{-\varepsilon^2}$.** With the grouping described above we can re-phrase the condition $y<\max|B_i|^{-\varepsilon^2}$ as $y\ll 2^{-\varepsilon^2\max(b_i)}$. Notice also that

$$\begin{aligned}(A_iy-B_ix)\mathbf{1}_{0<A_iy-B_ix\leqslant 1}&\leqslant 2\min\big\{\max\{|A_i|y,|B_i|x\},1\big\}\\ &\ll\min\big\{\max\{2^{a_i}y,2^{b_i}x\},1\big\}\end{aligned}\tag{4.8}$$

and

$$\min\big\{(1/y)2^{-a_i}+1,(1/x)2^{-b_i}+1\big\}\min\big\{\max\{2^{a_i}y,2^{b_i}x\},1\big\}\leqslant 1. \tag{4.9}$$

The conditions $0<A_iy-B_ix\leqslant 1$ and $0\leqslant x\leqslant y\leqslant 1$ imply that $|A_i|\leqslant|B_i|+\frac{1}{y}$, so that if we assign $A_iB_{i+1}-A_{i+1}B_i=D_i$, then we have

$$\prod_{i=1}^{k-1}|D_i|\leqslant 2^{2(b_1+\dots+b_k)}(1+1/y)^k=:L(\mathbf{b},y).$$

Therefore, using also (4.8), and $\max_i\{|A_i|^\delta, |B_i|^\delta\} \leqslant 2^{\varepsilon^3 \max(a_i)} 2^{\varepsilon^3 \max(b_i)}$, we see that the expression in (4.7) is

$$\ll \sum_{b_1,\dots,b_k \geqslant 0} \frac{2^{\varepsilon^3 \max(b_i)}}{2^{b_1+b_k}} \int_0^{2^{-\varepsilon^2 \max(b_i)}} y^2 \sum_{1\leqslant |D_1\cdots D_{k-1}|\leqslant L(\mathbf{b},y)} \frac{1}{|D_1|\cdots|D_{k-1}|} \sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(a_i)}$$
$$\times \int_0^y \sum_{\mathbf{A},\mathbf{B}\in\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}} \prod_{i=1}^{k-1} \min\Big\{ \max\{2^{a_i}y, 2^{b_i}x\}, 1\Big\} \mathbf{1}_{0<A_iy-B_ix\leqslant 1}\, dxdy. \tag{4.10}$$

According to Lemma 9 and (4.9) the expression after the innermost integral is

$$\ll \frac{2^{b_1}}{y} \prod_{i=1}^{k-1} \min\big\{(1/y)2^{-a_i}+1, (1/x)2^{-b_i}+1\big\} \min\Big\{ \max\{2^{a_i}y, 2^{b_i}x\}, 1\Big\} \ll \frac{2^{b_1}}{y}.$$

Moreover, uniformly in $y\in(0,1]$,

$$\sum_{1\leqslant |D_1\cdots D_{k-1}|\leqslant L(\mathbf{b},y)} \frac{1}{|D_1|\cdots|D_{k-1}|} \ll \big(\log L(\mathbf{b},y)\big)^k$$
$$\ll_k \Big(b_1+\cdots+b_k+\log(1+1/y)\Big)^k \ll_k (b_1+\cdots+b_k)^k \max_{j\in[0,k]} \big(\log(1+1/y)\big)^j$$

and

$$\sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 1 \ll \prod_{i=1}^{k} \log(2^{b_i+1}+1/y) \ll b_1\cdots b_k \big(\log(1+1/y)\big)^k,$$

so

$$\sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(a_i)} \ll \sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} \big(2\cdot 2^{\max(b_i)}+1/y\big)^{\varepsilon^3}$$
$$\ll \sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(b_i)} y^{-\varepsilon^3} \ll b_1\cdots b_k 2^{\varepsilon^3\max(b_i)} y^{-\varepsilon^3} \big(\log(1+1/y)\big)^k.$$

It follows that the whole expression from (4.10) is bounded by

$$\sum_{b_1,\ldots,b_k\geqslant 0} \frac{2^{2\varepsilon^3\max(b_i)}(b_1+\cdots+b_k)^k}{2^{b_1+b_k}} 2^{b_1} \int_0^{2^{-\varepsilon^2\max(b_i)}} y^{1-\varepsilon^2} \max_{j\in[0,k]} \big(\log(1+1/y)\big)^{j+k}\,dy$$

$$= \sum_{b_1,\ldots,b_k\geqslant 0} 2^{2\varepsilon^3\max(b_i)}(b_1+\cdots+b_k)^k \int_{2^{\varepsilon^2\max(b_i)}}^{\infty} \frac{\max_{j\in[k,2k]}\big(\log(1+u)\big)^j}{u^{3-\varepsilon^2}}\,du$$

$$\ll \sum_{b_1,\ldots,b_k\geqslant 0} \frac{(b_1+\cdots+b_k)^k}{2^{(\varepsilon^2-2\varepsilon^3)\max(b_i)}} \ll_k \Big(\sum_{b=1}^{\infty}\frac{b^{2k}}{2^{(\varepsilon^2-2\varepsilon^3)b/k}}\Big)^k \ll_k 1.$$

4.2. **Disposing of k-tuples of integers with $|A_jB_{j+1}-A_{j+1}B_j| > \max_{i\in[1,k]}|B_i|^{\varepsilon^2}$ for some $j\in[1,k-1]$.** By the union bound and the bound provided by Lemma 8 the contribution of such integers is

$$\ll \sum_{j=1}^{k} \sum_{\substack{A_1,\ldots,A_k,B_1,\ldots,B_k\in\mathbb{Z}\\ A_iB_{i+1}-A_{i+1}B_i\neq 0\\ (A_i,B_i)=1,\forall i}} \frac{\max_{i\in[1,k]}\{|A_i|^{\varepsilon^3},|B_i|^{\varepsilon^3}\}}{|B_1B_k|}\cdot\frac{1}{\max_{i\in[1,k]}|B_i|^{\varepsilon^2}} \prod_{i\neq j}\frac{1}{|A_iB_{i+1}-A_{i+1}B_i|}$$

$$\times \iint_{0\leqslant x\leqslant y\leqslant 1} y^2 \prod_{i=1}^{k}(A_iy-B_ix)\mathbf{1}_{0<A_iy-B_ix\leqslant 1}\,dxdy.$$

It is enough to show that each of the inner expressions is convergent. We fix, for all $i\neq j$, values $D_i = A_iB_{i+1}-A_{i+1}B_i$. As before, we have $\prod_{i\neq j}|D_i|\leqslant L(\mathbf{b},y)$ with $L(\mathbf{b},y)=2^{2(b_1+\cdots+b_k)}(1+\frac{1}{y})^k$. We are thus led to the following expression:

$$\sum_{b_1,\ldots,b_k\geqslant 0}\frac{2^{(\varepsilon^3-\varepsilon^2)\max(b_i)}}{2^{b_1+b_k}}\int_0^1 y^2 \sum_{\substack{D_1,\ldots,D_{j-1},D_{j+1},\ldots,D_{\ell-1}\\ \prod_{i\neq j}|D_i|\leqslant L(\mathbf{b},y)}} \prod_{\substack{1\leqslant i\leqslant k-1\\ i\neq j}}\frac{1}{|D_i|} \sum_{\substack{a_1,\ldots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(a_i)}$$

$$\times\int_0^y \sum_{\substack{A_1,\ldots,A_k,B_1,\ldots,A_k\in\mathbb{Z}\\ 2^{a_i}-1\leqslant|A_i|\leqslant 2^{a_i+1}\\ 2^{b_i}-1\leqslant|B_i|\leqslant 2^{b_i+1}\\ A_iB_{i+1}-A_{i+1}B_i=D_i,\forall i\neq j\\ (A_i,B_i)=1,\forall i}} \prod_{i=1}^{k}\min\big\{\max\{2^{a_i}y,2^{b_i}x\},1\big\}\mathbf{1}_{0<A_iy-B_ix\leqslant 1}\,dxdy. \tag{4.11}$$

We now apply Lemma 9 twice, first to the variable (A_i,B_i) with $i\leqslant j$, and then to the variables (A_ℓ,B_ℓ) with $j+1\leqslant\ell\leqslant k$ (in particular in the second application we reverse the order of the variables and identify A_{k-i},B_{k-i} with A_{i+1},B_{i+1} for

$i = 1, \dots, k-j-1$). This gives,

$$\sum_{\substack{A_1,\dots,A_j,B_1,\dots,B_j\in\mathbb{Z}\\ 2^{a_i}-1\leqslant|A_i|\leqslant 2^{a_i+1},\forall i\leqslant j\\ 2^{b_i}-1\leqslant|B_i|\leqslant 2^{b_i+1},\forall i\leqslant j\\ A_iB_{i+1}-A_{i+1}B_i=D_i,\forall i<j\\ (A_i,B_i)=1,\forall i\leqslant j}} \prod_{i=1}^{j} \mathbf{1}_{0<A_iy-B_ix\leqslant 1} \ll \frac{2^{b_1}}{y}\prod_{i=1}^{j-1}\min\big\{(1/y)2^{-a_i}+1,(1/x)2^{-b_i}+1\big\},$$

$$\sum_{\substack{A_{j+1},\dots,A_k,B_{j+1},\dots,B_k\in\mathbb{Z}\\ 2^{a_i}-1\leqslant|A_i|\leqslant 2^{a_i+1},\forall i\geqslant j+1\\ 2^{b_i}-1\leqslant|B_i|\leqslant 2^{b_i+1},\forall i\geqslant j+1\\ A_iB_{i+1}-A_{i+1}B_i=D_i,\forall i>j+1\\ (A_i,B_i)=1,\forall i\geqslant j+1}} \prod_{i=j+1}^{k} \mathbf{1}_{0<A_iy-B_ix\leqslant 1} \ll \frac{2^{b_k}}{y}\prod_{i=j+2}^{k}\min\big\{(1/y)2^{-a_i}+1,(1/x)2^{-b_i}+1\big\}.$$

In conjunction with (4.9) this shows that the expression inside the innermost integral in (4.11) is

$$\ll \frac{2^{b_1+b_k}}{y^2}\prod_{i\neq j,j+1}\min\{(1/y)2^{-a_i}+1,(1/x)2^{-b_i}+1\}\min\big\{\max\{2^{a_i}y,2^{b_i}x\},1\big\} \ll \frac{2^{b_1+b_k}}{y^2}.$$

Using this bound and proceeding as in the previous case for the other sums, we conclude that (4.11) is

$$\ll \sum_{b_1,\dots,b_k\geqslant 0} 2^{(\varepsilon^3-\varepsilon^2)\max(b_i)}\int_0^1 y \sum_{\substack{D_1,\dots,D_{j-1},D_{j+1},\dots,D_{k-1}\\ 1\leqslant\prod_{i\neq j}|D_i|\leqslant L(\mathbf{b},y)}} \prod_{\substack{1\leqslant i\leqslant k-1\\ i\neq j}}\frac{1}{|D_i|}\sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(a_i)}dy.$$

Using also $2^{\varepsilon^3\max(a_i)} \ll_\varepsilon 2^{\varepsilon^3\max(b_i)}y^{-\varepsilon^3}$ and other estimates from the previous subsection we see that this is

$$\ll_{k,\varepsilon} \sum_{b_1,\dots,b_k\geqslant 0}\frac{(b_1+\cdots+b_k)^{2k}}{2^{(\varepsilon^2-2\varepsilon^3)\max(b_i)}}\int_1^\infty \frac{\max_{j\in[k,2k]}\big(\log(1+u)\big)^j}{u^{2-\varepsilon^3}}\,du \ll_k 1. \tag{4.12}$$

4.3. **(II) is fulfilled and $b_{j+1}-b_j \leqslant \varepsilon\max_{i\in[1,k]} b_i$ for all $j\in[1,k-1]$.** Since (II) is fulfilled we have $|D_i| \leqslant 2^{\varepsilon\max_{i\in[1,k]} b_i}$ for all $i\in[1,k-1]$, so it suffices to bound above the expression

$$\begin{aligned}\sum_{b_1,\dots,b_k\geqslant 0}\frac{2^{\varepsilon^3\max(b_i)}}{2^{b_1+b_k}}\sum_{\substack{D_1,\dots,D_{k-1}\\ 1\leqslant|D_i|\leqslant 2^{\varepsilon\max(b_i)}}}\frac{1}{|D_1|\cdots|D_{k-1}|}\\ \times\int_0^1 y^2\sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}}2^{\varepsilon^3\max(a_i)}\int_0^y\sum_{(\mathbf{A},\mathbf{B})\in\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}}\prod_{i=1}^k\mathbf{1}_{0<A_iy-B_ix\leqslant 1}\,dxdy.\end{aligned}\tag{4.13}$$

Next notice that $0<A_iy-B_ix\leqslant 1$ implies that $|x-\frac{A_i}{B_i}y|\leqslant\frac{1}{|B_i|}$. Therefore the contribution of the integral over x is $\ll\min_{i\in[1,k]}\frac{1}{|B_i|}\ll 2^{-\max_{i\in[1,k]}b_i}$.

Using that $b_{j+1}-b_j\leqslant\varepsilon\max_{i\in[1,k]}b_i$ for all $j\leqslant k-1$ and Lemma 9, we infer

$$\sum_{(\mathbf{A},\mathbf{B})\in\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}}1\ll 2^{a_1}2^{b_1}\prod_{i=1}^{k-1}(2^{b_{i+1}-b_i}+1)\ll 2^{a_1+b_1}\cdot 2^{k\varepsilon\max(b_i)}.\tag{4.14}$$

It further follows from (4.14) that the expression in (4.13) is

$$\begin{aligned}\ll\sum_{b_1,\dots,b_k\geqslant 0}\frac{2^{\varepsilon^3\max(b_i)}}{2^{b_1+b_k}}\sum_{\substack{D_1,\dots,D_{k-1}\\ 1\leqslant|D_i|\leqslant 2^{\varepsilon\max(b_i)}}}\frac{1}{|D_1|\cdots|D_{k-1}|}\\ \times\int_0^1 y^2\sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}}2^{b_1}\cdot 2^{\varepsilon^3\max(a_i)}\cdot 2^{k\varepsilon\max(b_i)}\cdot 2^{-\max(b_i)}\,dy.\end{aligned}$$

Proceeding as in the previous sections to handle the sums over a_i and the integral, we conclude that this is bounded above by the quantity in (4.12).

4.4. **(I) and (II) are fulfilled and there exists an index $j\in[1,k-1]$ such that $b_{j+1}-b_j>\varepsilon\max_{i\in[1,k]}b_i$.** In this case because of (I) the range of integration is restricted to $y>2^{-\varepsilon^2\max_{i\in[1,k]}b_i}$ and we take $|D_i|\leqslant 2^{\varepsilon^2\max_{i\in[1,k]}b_i}$ due to (II). Note that the ε^2 in the bound for $|D_i|$ is important because will be matched against the larger ε in $b_{j+1}-b_j\geqslant\varepsilon\max_{i\in[1,k]}b_i$ at a crucial point in the argument. Therefore in this case it is enough to bound

$$\begin{aligned}\sum_{\substack{b_1,\dots,b_k\geqslant 0\\ \exists j, b_{j+1}-b_j\geqslant \varepsilon\max(b_i)}} \frac{2^{\varepsilon^3\max(b_i)}}{2^{b_1+b_k}} \sum_{\substack{D_1,\dots,D_{k-1}\\ 1\leqslant |D_i|\leqslant 2^{\varepsilon^2\max(b_i)}}} \frac{1}{|D_1|\cdots|D_{k-1}|}\int_{2^{-\varepsilon^2\max(b_i)}}^1 y^2\\ \times \sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(a_i)}\int_0^y \sum_{(\mathbf{A},\mathbf{B})\in\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}}\prod_{i=1}^k (A_iy-B_ix)\mathbf{1}_{0<A_iy-B_ix\leqslant 1}\,dxdy.\end{aligned} \tag{4.15}$$

Using the union bound we fix an index j such that $b_{j+1}-b_j\geqslant\varepsilon\max(b_i)$. In the integral above we are requiring $0<A_jy-B_jx\leqslant 1$ and $0<A_{j+1}y-B_{j+1}x\leqslant 1$. Set

$$\xi_1 := A_jy-B_jx\in[0,1],\quad \xi_2 = A_{j+1}y-B_{j+1}x\in[0,1].$$

Solving this system of equations we see that

$$y=\frac{B_{j+1}\xi_1-B_j\xi_2}{D_j}.$$

This leads to

$$B_{j+1}\xi_1 = O(y|D_j|+|B_j|) = O(2^{\varepsilon^2\max(b_i)}+2^{b_j}).$$

Using also $b_{j+1}-b_j\geqslant\varepsilon\max(b_i)$ and $b_j\geqslant 0$ we infer

$$\xi_1\ll 2^{\varepsilon^2\max(b_i)-b_{j+1}}+2^{b_j-b_{j+1}}\leqslant 2^{(\varepsilon^2-\varepsilon)\max(b_i)}+2^{-\varepsilon\max(b_i)}\leqslant 2\cdot 2^{-(\varepsilon/2)\max(b_i)}.$$

Therefore $\xi_1=|A_jy-B_jx|\ll 2^{(-\varepsilon/2)\max_i b_i}$.

It follows therefore that the expression in (4.15) is

$$\begin{aligned}\ll \sum_{b_1,\dots,b_k\geqslant 0} \frac{2^{\varepsilon^3\max(b_i)}}{2^{b_1+b_k}} \sum_{\substack{D_1,\dots,D_{k-1}\\ 1\leqslant |D_i|\leqslant 2^{\varepsilon^2\max(b_i)}}} \frac{1}{|D_1|\cdots|D_{k-1}|}\int_{2^{-\varepsilon^2\max(b_i)}}^1 y^2\\ \times \sum_{\substack{a_1,\dots,a_k\\ 2^{a_i}-1\leqslant 2^{b_i+1}+1/y}} 2^{\varepsilon^3\max(a_i)}\int_0^y 2^{-(\varepsilon/2)\max(b_i)} \sum_{(\mathbf{A},\mathbf{B})\in\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}}\prod_{i=1}^k \mathbf{1}_{0<A_iy-B_ix\leqslant 1}\,dxdy.\end{aligned} \tag{4.16}$$

Using Lemma 9 and our assumption that $y>2^{-\varepsilon^2\max(b_i)}$, we see that

$$\sum_{(\mathbf{A},\mathbf{B})\in\mathcal{S}_{\mathbf{D},\mathbf{a},\mathbf{b}}}\mathbf{1}_{0<A_iy-B_ix\leqslant 1}\ll 2^{b_1}2^{k\varepsilon^2\max(b_i)}.$$

Combining everything together we conclude that the expression in (4.16) is

$$\ll_{k,\varepsilon}\sum_{b_1,\dots,b_k\geqslant 0}\frac{(b_1+\cdots+b_k)^{2k}}{2^{(\varepsilon/2-k\varepsilon^2+2\varepsilon^3)\max(b_i)}}\ll_{k,\varepsilon} 1.$$

5. Asymptotic formulas for the moments of the large sieve matrix

Fix a non-decreasing C^∞ function $\Xi : \mathbb{R} \to [0,1]$ with $\Xi \equiv 0$ on $(-\infty, 0]$, $\Xi \equiv 1$ on $[1,\infty)$, and

$$\Xi^{(k)}(0) = \Xi^{(k)}(1) = 0, \quad \forall k \in \mathbb{N}, \qquad \Xi(x) + \Xi(1-x) = 1, \quad \forall x \in [0,1].$$

Fix $\boxed{c \in (0,1)}$ and let $\boxed{\delta = N^{-1+c} > 0}$. Consider the function $f_\delta \in C_c^\infty(\mathbb{R})$ defined by $f_\delta(x) \equiv 0$ on $(-\infty, 0] \cup [1+\delta, \infty)$, $f_\delta(x) \equiv 1$ on $[\delta, 1]$, $f_\delta(x) = \Xi\big(\frac{x}{\delta}\big)$ if $x \in [0,\delta]$, and $f_\delta(x) = \Xi\big(\frac{1+\delta-x}{\delta}\big)$ if $x \in [1, 1+\delta]$. Consider also the function defined by

$$\phi(u) = \int_0^1 \Xi'(y) e(-uy)\, dy.$$

A direct calculation provides $f_\delta(x) + f_\delta(x+1) = 1$ for all $x \in [0,1]$, and

$$\widehat{f_\delta}(u) = \frac{1 - e(-u)}{2\pi i u} \phi(\delta u) = e^{-\pi i u} \operatorname{sinc}(\pi u) \phi(\delta u) = \widehat{\chi}_{[0,1]}(u) \big(1 - \phi(\delta u)\big).$$

It is clear that $\|\phi\|_\infty \leqslant \phi(0) = 1$, $\phi(u) = 1 + O(|u|)$, and

$$\phi(u) = O_A\big(|u|^{-A}\big), \quad \forall A > 0.$$

Since $|\operatorname{sinc}(\pi u)| \leqslant \frac{1}{|u|}$ and $\|\widehat{f_\delta}\|_\infty \leqslant 1$, we also infer

$$\|\widehat{f_\delta} - \widehat{\chi}_{[0,1]}\|_\infty = \sup_{u \in \mathbb{R}\setminus\{0\}} \big(|\operatorname{sinc}(\pi u)| \cdot |1 - \phi(\delta u)|\big) \ll \sup_{u \in \mathbb{R}\setminus\{0\}} \big((1/|u|)\delta |u|\big) = \delta,$$

and taking Ψ_F as in (4.2),

$$\big\|\Psi_{\widehat{f_\delta}} - \Psi_{\widehat{\chi}_{[0,1]}}\|_\infty \ll \delta. \tag{5.1}$$

It is also plain that

$$\widehat{f_\delta}\,'(x) = -2\pi i \int_{\mathbb{R}} \xi f_\delta(\xi) e(-x\xi)\, d\xi = O(1).$$

Integrating by parts and employing $\|f_\delta^{(A)}\|_\infty \ll_A \delta^{-A}$, we obtain

$$\widehat{f_\delta}(x) = \frac{1}{(2\pi i x)^A} \int_{\mathbb{R}} f_\delta^{(A)}(\xi) e(-x\xi)\, d\xi \ll_A |x|^{-A} \delta \cdot \delta^{-A} = \frac{N^{(1-c)(A-1)}}{|x|^A}. \tag{5.2}$$

As shown in (3.10), $\mathfrak{M}_Q(\ell)$ can be replaced by the smoothed sum $\mathfrak{M}_{Q;\delta}(\ell)$ defined in (3.3). Consider the associated $\mathbb{Z}$-periodic function defined by

$$F_Q(x) := \sum_{k \in \mathbb{Z}} \widehat{f_\delta}\big(N(x+k)\big) = \sum_{k \in \mathbb{Z}} c_k e(kx). \tag{5.3}$$

Its Fourier coefficients,

$$c_n = \int_0^1 F_Q(x) e(-nx)\, dx = \int_{\mathbb{R}} \widehat{f_\delta}(Nu) e(-nu)\, du = \frac{1}{N} f_\delta\Big(\frac{-n}{N}\Big), \tag{5.4}$$

satisfy $0 \leqslant c_n \leqslant \frac{1}{N}$ for all n, $c_n = 0$ unless $-(1+\delta)N < n < 0$, and

$$F_Q(0) = \sum_{k\in\mathbb{Z}} c_k = 1 + O(\delta). \tag{5.5}$$

We also have $\|F_Q\|_\infty \leqslant 2$.

With $\mathbf{x} = (x_1, \dots, x_\ell)$, $\xi = (\xi_1, \dots, \xi_\ell)$, the Fourier transform of the function $h_{\delta;\Theta}$ defined in (3.2) is given by

$$\begin{aligned} \widehat{h}_{\delta;\Theta}(\mathbf{x}) &= \int_{\mathbb{R}^\ell} e(-\mathbf{x}\cdot\xi) h_{\delta;\Theta}(\xi)d\xi \\ &= N^\ell \widehat{f}_\delta\big(N(x_1 + \theta_\ell - \theta_1)\big)\widehat{f}_\delta\big(N(x_2 + \theta_1 - \theta_2)\big)\cdots\widehat{f}_\delta\big(N(x_\ell + \theta_{\ell-1} - \theta_\ell)\big). \end{aligned}$$

Poisson summation, (3.3), and the above formula for $\widehat{h}_{\delta;\Theta}(n_1, \dots, n_\ell)$ provide

$$\mathfrak{M}_{Q;\delta}(\ell) = \frac{1}{N} \sum_{\theta_1,\dots,\theta_\ell \in \mathcal{F}_Q} F_Q(\theta_\ell - \theta_1) F_Q(\theta_1 - \theta_2) \cdots F_Q(\theta_{\ell-1} - \theta_\ell).$$

Taking

$$\begin{aligned} F_{Q;n}(x) &:= F_Q(x)e(-nx) = \sum_{k\in\mathbb{Z}} c_{n+k} e(kx), \\ S_{Q;n}(\ell) &:= \sum_{\theta_1,\dots,\theta_\ell\in\mathcal{F}_Q} F_{Q;n}(\theta_1 - \theta_2)F_{Q;n}(\theta_2 - \theta_3)\cdots F_{Q;n}(\theta_{\ell-1} - \theta_\ell), \end{aligned}$$

and employing (5.4) to express $F_Q(\theta_\ell - \theta_1)$ we can write

$$\begin{aligned} \mathfrak{M}_{Q;\delta}(\ell) &= \frac{1}{N} \sum_{n\in\mathbb{Z}} c_n \sum_{\theta_1,\dots,\theta_\ell\in\mathcal{F}_Q} e\big(n(\theta_\ell - \theta_1)\big) \prod_{i=1}^{\ell-1} F_{Q;n}(\theta_i - \theta_{i+1}) \\ &= \frac{1}{N} \sum_{-(1+\delta)N<n<0} c_n S_{Q;n}(\ell). \end{aligned} \tag{5.6}$$

Next we focus on $S_{Q;n}(\ell)$, which is expressed after replacing θ_ℓ by $-\theta_\ell$, as

$$\begin{aligned} &\sum_{\substack{\theta_1,\dots,\theta_\ell\in\mathcal{F}_Q \\ n_1,\dots,n_{\ell-1}\in\mathbb{Z}}} c_{n+n_1}\cdots c_{n+n_{\ell-1}} e\big(n_1(\theta_1-\theta_2) + n_2(\theta_2-\theta_3) + \cdots + n_{\ell-1}(\theta_{\ell-1}-\theta_\ell)\big) \\ &= \sum_{\substack{\theta_1,\dots,\theta_\ell\in\mathcal{F}_Q \\ n_1,\dots,n_{\ell-1}\in\mathbb{Z}}} c_{n+n_1}\cdots c_{n+n_{\ell-1}} e\big(n_1\theta_1 + (n_2-n_1)\theta_2 + \cdots + (n_{\ell-1}-n_{\ell-2})\theta_{\ell-1} + n_{\ell-1}\theta_\ell\big). \end{aligned}$$

Upon (2.5), with $\mathbf{r} := (r_1, \ldots, r_{\ell-1})$, $\mathbf{d} := (d_1, \ldots, d_{\ell-1})$, this can also be written as

$$\begin{aligned} S_{Q;n}(\ell) &= \sum_{n_1,\ldots,n_{\ell-1}\in\mathbb{Z}} c_{n+n_1}\cdots c_{n+n_{\ell-1}} \sum_{\substack{d_1\mid n_1\\ d_2\mid n_2-n_1\\ \cdots\\ d_{\ell-1}\mid n_{\ell-1}-n_{\ell-2}\\ d_\ell\mid n_{\ell-1}}} d_1\cdots d_\ell M\Big(\frac{Q}{d_1}\Big)\cdots M\Big(\frac{Q}{d_\ell}\Big) \\ &= \sum_{\mathbf{d}\in[1,Q]^{\ell-1}} M\Big(\frac{Q}{d_1}\Big)\cdots M\Big(\frac{Q}{d_{\ell-1}}\Big) \sum_{\mathbf{r}\in\mathbb{Z}^{\ell-1}} d_1\cdots d_{\ell-1} \\ &\quad \times c_{n+d_1r_1}c_{n+d_1r_1+d_2r_2}\cdots c_{n+d_1r_1+\cdots+d_{\ell-1}r_{\ell-1}} \sum_{d_\ell\mid d_1r_1+\cdots+d_{\ell-1}r_{\ell-1}} d_\ell M\Big(\frac{Q}{d_\ell}\Big). \end{aligned} \tag{5.7}$$

Taking into account (2.5) and (5.4) we can express the inner two sums in (5.7) as

$$\begin{aligned} &\sum_{\substack{\mathbf{r}\in\mathbb{Z}^{\ell-1}\\ \theta\in\mathcal{F}_Q}} c_{n+d_1r_1}c_{n+d_1r_1+d_2r_2}\cdots c_{n+d_1r_1+\cdots+d_{\ell-1}r_{\ell-1}} e\Big(-\theta\sum_{j=1}^{\ell-1} d_jr_j\Big) \\ &= \sum_{\substack{\mathbf{r}\in\mathbb{Z}^{\ell-1}\\ \theta\in\mathcal{F}_Q}} e\Big(-\theta\sum_{j=1}^{\ell-1} d_jr_j\Big)\int_{\mathbb{R}^{\ell-1}}\prod_{k=1}^{\ell-1}\widehat{f_\delta}(Nu_k)e\Big(-\sum_{j=1}^{\ell-1}\Big(n+\sum_{i=1}^{j} d_ir_i\Big)u_j\Big)d\mathbf{u} \\ &= \sum_{\substack{\mathbf{r}\in\mathbb{Z}^{\ell-1}\\ \theta\in\mathcal{F}_Q}} \int_{\mathbb{R}^{\ell-1}} e\Big(-\sum_{j=1}^{\ell-1} d_jr_j(u_j+\cdots+u_{\ell-1}+\theta)-n\sum_{j=1}^{\ell-1}u_j\Big)\prod_{k=1}^{\ell-1}\widehat{f_\delta}(Nu_k)\,d\mathbf{u} \\ &= \sum_{\substack{\mathbf{r}\in\mathbb{Z}^{\ell-1}\\ \theta\in\mathcal{F}_Q}} \int_{\mathbb{R}^{\ell-1}} e\Big(-\sum_{j=1}^{\ell-1} d_jr_j\sum_{i=j}^{\ell-1}u_i-n\sum_{j=1}^{\ell-1}u_j+n\theta\Big)\widehat{f_\delta}\big(N(u_{\ell-1}-\theta)\big)\prod_{k=1}^{\ell-2}\widehat{f_\delta}(Nu_k)\,d\mathbf{u}. \end{aligned}$$

With $\mathbf{y} = (y_1, \ldots, y_{\ell-1})$, denote

$$H_{Q;\mathbf{d},\theta,n}(\mathbf{y}) := e\Big(n\Big(\theta-\frac{y_1}{d_1}\Big)\Big)\prod_{k=1}^{\ell-2}\widehat{f_\delta}\Big(N\Big(\frac{y_k}{d_k}-\frac{y_{k+1}}{d_{k+1}}\Big)\Big)\widehat{f_\delta}\Big(N\Big(\frac{y_{\ell-1}}{d_{\ell-1}}-\theta\Big)\Big).$$

The change of variables $y_i = d_i(u_i + \cdots + u_{\ell-1})$, $i = 1, \ldots, \ell-1$, provides

$$\begin{aligned} &u_1 = \frac{y_1}{d_1}-\frac{y_2}{d_2},\quad u_2 = \frac{y_2}{d_2}-\frac{y_3}{d_3},\quad \ldots\quad , u_{\ell-2} = \frac{y_{\ell-2}}{d_{\ell-2}}-\frac{y_{\ell-1}}{d_{\ell-1}},\quad u_{\ell-1} = \frac{y_{\ell-1}}{d_{\ell-1}}, \\ &-\sum_{j=1}^{\ell-1} d_jr_j\sum_{i=j}^{\ell-1}u_i - n\sum_{j=1}^{\ell-1}u_j + n\theta = -\mathbf{r}\cdot\mathbf{y} + n\Big(\theta-\frac{y_1}{d_1}\Big), \end{aligned}$$

and the contribution of the two inner sums in (5.7) becomes

$$\sum_{\theta\in\mathcal{F}_Q}\sum_{\mathbf{r}\in\mathbb{Z}^{\ell-1}}\int_{\mathbb{R}^{\ell-1}} e(-\mathbf{r}\cdot\mathbf{y})H_{Q;\mathbf{d},\theta,n}(\mathbf{y})\,d\mathbf{y} = \sum_{\theta\in\mathcal{F}_Q}\sum_{\mathbf{r}\in\mathbb{Z}^{\ell-1}}\widehat{H}_{Q;\mathbf{d},\theta,n}(\mathbf{r})$$
$$= \sum_{\theta\in\mathcal{F}_Q}\sum_{\mathbf{r}\in\mathbb{Z}^{\ell-1}} H_{Q;\mathbf{d},\theta,n}(\mathbf{r}),$$

where Poisson summation was used in the last equality. Inserting this back into (5.7) and setting $\mu(\mathbf{k}) := \mu(k_1)\cdots\mu(k_{\ell-1})$, we find

$$\begin{aligned} S_{Q;n}(\ell) &= \sum_{\mathbf{d}\in[1,Q]^{\ell-1}} M\Big(\frac{Q}{d_1}\Big)\cdots M\Big(\frac{Q}{d_{\ell-1}}\Big)\sum_{\theta\in\mathcal{F}_Q}\sum_{\mathbf{r}\in\mathbb{Z}^{\ell-1}} H_{Q;\mathbf{d},\theta,n}(\mathbf{r}) \\ &= \sum_{\mathbf{k}\in[1,Q]^{\ell-1}} \mu(\mathbf{k})\sum_{\mathbf{r}\in\mathbb{Z}^{\ell-1}}\sum_{d_i\in[1,\frac{Q}{k_i}],\,\forall i}\sum_{\theta\in\mathcal{F}_Q} H_{Q;\mathbf{d},\theta,n}(\mathbf{r}). \end{aligned} \tag{5.8}$$

Consider the functions

$$\begin{aligned} \widetilde{F}_{Q;n}(x_1,\dots,x_{\ell-1}) &:= e\Big(-\frac{nx_1}{N}\Big)\prod_{i=1}^{\ell-2}\widehat{f}_\delta(x_i-x_{i+1})\widehat{f}_\delta(x_{\ell-1}), \\ h_{Q;e_i,\Delta_i}(b,q) &:= \beta_{e_i,\Delta_i,N}(b,q) = \frac{N\Delta_i}{q(e_iq-\Delta_i b)}, \\ G_{Q;\mathbf{e},\boldsymbol{\Delta}} &:= \sum_{n\in\mathbb{Z}} c_n\widetilde{F}_{Q;n}(h_{Q;e_1,\Delta_1},\dots,h_{Q;e_{\ell-1},\Delta_{\ell-1}}) \\ &= F_Q\Big(-\frac{1}{N}h_{Q;e_1,\Delta_1}\Big)\prod_{i=1}^{\ell-2}\widehat{f}_\delta(h_{Q;e_i,\Delta_i}-h_{Q;e_{i+1},\Delta_{i+1}})\widehat{f}_\delta(h_{Q;e_{\ell-1},\Delta_{\ell-1}}), \\ \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}} &:= \Psi_{\widehat{f}_\delta}(h_{Q;e_1,\Delta_1},\dots,h_{Q;e_{\ell-1},\Delta_{\ell-1}}). \end{aligned}$$

Denote $\theta = \frac{a}{q} \in \mathcal{F}_Q$ and let $b = \bar{a} \in [1,q]$ such that $a\bar{a} \equiv 1 \pmod{q}$. Setting $\Delta_i := r_iq - d_ia$, we have $d_i \equiv -\Delta_i b \pmod{q}$, $e_i := \frac{d_i+\Delta_i b}{q} \in \mathbb{Z}$, and

$$1 \leqslant d_i = e_iq - \Delta_i b \leqslant \frac{Q}{k_i}.$$

Note that if $\Delta_i = 0$ for some i, then $e_i \geqslant 1$. Employing also

$$\frac{r_i}{d_i} - \frac{r_{i+1}}{d_{i+1}} = \Big(\frac{r_i}{d_i}-\frac{a}{q}\Big) - \Big(\frac{r_{i+1}}{d_{i+1}}-\frac{a}{q}\Big) = \frac{\Delta_i}{qd_i} - \frac{\Delta_{i+1}}{qd_{i+1}},$$

we can rewrite

$$H_{Q;\mathbf{d},\theta,n}(\mathbf{r}) = \widetilde{F}_{Q;n}\Big(\frac{N\Delta_1}{qd_1},\dots,\frac{N\Delta_{\ell-1}}{qd_{\ell-1}}\Big).$$

Subsequently, with $\mathbf{\Delta}=(\Delta_1,\dots,\Delta_{\ell-1})$, $\mathbf{e}=(e_1,\dots,e_{\ell-1})$, using the second expression in (5.6) for $\mathfrak{M}_{Q;\delta}(\ell)$, equality (5.8) and the formulas for $H_{Q;\mathbf{d},\theta,n}$ and $G_{Q;\mathbf{e},\mathbf{\Delta}}$, we infer

$$\mathfrak{M}_{Q;\delta}(\ell)=\frac{1}{N}\sum_{\mathbf{k}\in[1,Q]^{\ell-1}}\mu(\mathbf{k})\sum_{\mathbf{e},\mathbf{\Delta}\in\mathbb{Z}^{\ell-1}}\sum_{\substack{1\leqslant b\leqslant q\leqslant Q,\,(q,b)=1\\ 1\leqslant d_i:=e_iq-\Delta_ib\leqslant\frac{Q}{k_i},\,\forall i}}G_{Q;\mathbf{e},\mathbf{\Delta}}(b,q). \tag{5.9}$$

Lemma 10. *If $|\beta|\leqslant\frac{N}{2}$, then for every integer $B>0$:*

$$F_Q\Big(\frac{\beta}{N}\Big)=\sum_{n\in\mathbb{Z}}\widehat{f_\delta}(\beta+Nn)=\widehat{f_\delta}(\beta)+O_B(N^{-B}).$$

Proof. An application of (5.2) gives, for every $A>0$,

$$\Big|\sum_{n\in\mathbb{Z}}\widehat{f_\delta}(\beta+Nn)-\widehat{f_\delta}(\beta)\Big|\ll_A\sum_{|n|\geqslant1}\frac{N^{(1-c)(A-1)}}{|\beta+Nn|^A}\ll_A N^{(1-c)(A-1)-A}.$$

The desired estimate follows choosing A with $A-(1-c)(A-1)=cA+1-c>B$. $\square$

Since $c_m=0$ for $|m|>2N$, the definitions of d_i,Δ_i,e_i and the condition

$$c_nc_{n+d_1r_1}\cdots c_{n+d_1r_1+\cdots+d_\ell r_\ell}\neq0$$

trivially imply $d_i\leqslant Q$, $|r_i|\ll N$, $|\Delta_i|\ll QN$. Since $\Delta_i\equiv-d_ia\pmod q$, for fixed d_i and q the number of admissible values for Δ_i is $\ll\frac{NQ}{q}$. This provides

$$\begin{aligned}\#\text{non-zero terms in (5.9)}&\ll\sum_{\mathbf{k}\in[1,Q]^{\ell-1}}\frac{Q^{\ell-1}}{k_1\cdots k_{\ell-1}}\sum_{q\in[1,Q]}q\Big(\frac{NQ}{q}\Big)^{\ell-1}\\&\ll N^{2\ell-2}(\log Q)^\ell.\end{aligned}\tag{5.10}$$

Let $\Lambda=N^d$ with $\boxed{0<1-c<d}$, where we think of $d>0$ small (to be indicated precisely later) and of c as being close to 1. Denote $\beta_i:=h_{Q;e_i,\Delta_i}(b,q)=\frac{N\Delta_i}{qd_i}$.

Lemma 11. *The contribution to $\mathfrak{M}_{Q;\delta}(\ell)$ in (5.9) from terms with $|\beta_i-\beta_{i+1}|>\Lambda$ for some $i\in[1,\ell-2]$, $|\beta_{\ell-1}|>\Lambda$, or $|\beta_1|>\Lambda$ is $\ll_{\ell,B}N^{-B}$ for every $B>0$.*

Proof. In (5.2) we choose $A>0$ such that $\boxed{(c+d-1)A>B+2\ell}$. In the first two cases (5.2) provides $|\widehat{f_\delta}(\beta_i-\beta_{i+1})|\ll_A N^{(1-c)(A-1)-dA}=N^{(1-c-d)A+c-1}$, and respectively $|\widehat{f_\delta}(\beta_{\ell-1})|\ll_A N^{(1-c-d)A+c-1}$. Combining this with (5.9), (5.10), $\|\widehat{f_\delta}\|_\infty\leqslant1$ and $\|F_{\delta,n}\|_\infty\leqslant2$, we infer that the contribution of these two cases to $\mathfrak{M}_{Q;\delta}(\ell)$ is $\ll_{\ell,A}N^{-1}N^{2\ell-1}N^{(1-c-d)A+c-1}\ll_B N^{-B}$. If this is not the case, then $|\beta_{\ell-1}|\leqslant\Lambda$ and $|\beta_i-\beta_{i+1}|\leqslant\Lambda$ for every $i\in[1,\ell-2]$, and so necessarily $|\beta_1|\leqslant(\ell-1)\Lambda<\frac{N}{2}$.

Applying Lemma 10 and proceeding as above with $\Lambda < |\beta_1| \leqslant \frac{N}{2}$ we conclude that the contribution of the case $|\beta_1| > \Lambda$ to $\mathfrak{M}_{Q;\delta}(\ell)$ is again $\ll_B N^{-B}$. $\square$

We now work only with $\max\{|\beta_1|, \dots, |\beta_{\ell-1}|\} \leqslant \Lambda$ and denote the resulting contributions to $\mathfrak{M}_{Q;\delta}(\ell)$ by $\mathfrak{M}^{(\Lambda)}_{Q;\delta}(\ell)$. Let us remark first that $|\beta_i| = \frac{N|\Delta_i|}{qd_i} \ll \Lambda$ yields $|\Delta_i| \ll \frac{\Lambda q d_i}{N} \ll \frac{\Lambda q}{Q^2} \cdot \frac{Q}{k_i} \ll \frac{\Lambda}{k_i}$. We also have $d_i \leqslant \frac{Q}{k_i}$, hence $\frac{N|\Delta_i|}{q} \ll \Lambda d_i \ll \frac{\Lambda Q}{k_i}$, leading to $k_i \ll \frac{\Lambda Q q}{N|\Delta_i|} \ll \frac{\Lambda}{|\Delta_i|} \leqslant \Lambda$. Notice also that $\min\{\frac{Q}{d_i}, \frac{Q|\Delta_i|}{q}\} \geqslant 1$ and $\frac{Q}{d_i} \cdot \frac{Q|\Delta_i|}{q} \ll \frac{N|\Delta_i|}{qd_i} \ll \Lambda$, and thus $\frac{Q}{d_i} \ll \Lambda$ and $\frac{Q}{\Lambda} \leqslant \frac{Q|\Delta_i|}{\Lambda} \ll q$, showing also that $|e_i| \ll \frac{Q}{q} + |\Delta_i| \ll \Lambda$.

If the region

$$\Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q) := \left\{ (b,q) \in [0,Q]^2 : b \leqslant q,\ \frac{N|\Delta_i|}{\Lambda q} < e_i q - \Delta_i b \leqslant \frac{Q}{k_i},\ \forall i \in [1, \ell-1] \right\}$$

is nonempty, then $k_i, |\Delta_i|, |e_i| \ll \Lambda$ (as above) together with Lemma 10 show that the price of replacing $\sum_{n\in\mathbb{Z}} c_n e(-\frac{n\beta_1}{N})$ by $\widehat{f}_\delta(-\beta_1)$ in $\mathfrak{M}^{(\Lambda)}_{Q;\delta}(\ell)$ as in (5.9) is $\ll_B \frac{1}{N}\Lambda^{3\ell} Q^2 N^{-B}$. Accordingly, for every $B > 0$:

$$\mathfrak{M}^{(\Lambda)}_{Q;\delta}(\ell) = \frac{1}{N} \sum_{\mathbf{k}\in[1,Q]^{\ell-1}} \mu(\mathbf{k}) \sum_{\mathbf{e},\boldsymbol{\Delta}\in\mathbb{Z}^{\ell-1}} \sum_{\substack{(b,q)\in\Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)\cap\mathbb{Z}^{\ell-1} \\ (q,b)=1}} \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}(b,q) + O_B(N^{-B}).$$

We wish to apply Lemma 3 to the function $f = \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}$ in the region $\Omega = \Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)$. Denote by $\|\ \|_\infty$ the sup norm on $\Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)$. It is plain that

$$\Big\| \frac{\partial h_{Q;e_i,\Delta_i}}{\partial b} \Big\|_\infty = N\Delta_i^2 \Big\| \frac{1}{q(e_i q - \Delta_i b)^2} \Big\|_\infty \ll N\Delta_i^2 \frac{\Lambda^2 Q}{N^2 \Delta_i^2} \ll \frac{\Lambda^2 Q}{N} \ll \frac{\Lambda^2}{Q},$$

$$\Big\| \frac{\partial h_{Q;e_i,\Delta_i}}{\partial q} \Big\|_\infty = N|\Delta_i| \Big\| \frac{2e_i q - \Delta_i b}{q^2(e_i q - \Delta_i b)^2} \Big\|_\infty \ll N|\Delta_i| \frac{(q+b)\Lambda}{q^2} \cdot \frac{\Lambda^2 q^2}{N^2 \Delta_i^2} \ll \frac{\Lambda^3 Q}{N} \ll \frac{\Lambda^3}{Q},$$

$$\|\Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}\|_\infty \ll_\ell 1, \qquad \|D\Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}\|_\infty \leqslant \Big\| \frac{\partial \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}}{\partial b} \Big\|_\infty + \Big\| \frac{\partial \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}}{\partial q} \Big\|_\infty \ll_\ell \frac{\Lambda^3}{Q}.$$

The boundary of $\Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)$ is the union of at most $2\ell+1$ line segments and parabola arcs, so Lemma 3 applies and yields

$$\sum_{\substack{(b,q)\in\Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)\cap\mathbb{Z}^2 \\ (q,b)=1}} \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}(b,q) = \frac{6}{\pi^2} \iint_{\Omega_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)} \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}(x,y)\, dx dy + \mathcal{E}_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q),$$

with error $\mathcal{E}_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q) \ll_\ell \Lambda^3 Q \log Q$. Due to the constraints $\max\{k_i, |e_i|\} \ll \Lambda$ and $|\Delta_i| \ll \frac{\Lambda}{k_i}$, this contributes to $\mathfrak{M}^{(\Lambda)}_{Q;\delta}(\ell)$ by a quantity that is

$$\ll_\ell N^{-1} (\Lambda^2 \log \Lambda)^{\ell-1} \Lambda^3 Q \log Q \ll Q^{-\alpha_\ell} (\log Q)^\ell,$$

where $\boxed{0 < \alpha_\ell := 1 - (4\ell + 2)d < 1}$.

Rescaling to $(b, q) = (Qx, Qy)$, we find

$$\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) = \frac{6Q^2}{\pi^2 N} \sum_{\mathbf{k} \in [1,Q]^{\ell-1}} \mu(\mathbf{k}) \sum_{\mathbf{e},\boldsymbol{\Delta} \in \mathbb{Z}^{\ell-1}} \iint_{\widetilde{\Omega}_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)} \Phi_{Q;\mathbf{e},\boldsymbol{\Delta}}(x, y)\, dxdy \\ + O_\ell\big(Q^{-\alpha_\ell} (\log Q)^\ell\big), \tag{5.11}$$

where

$$\widetilde{\Omega}_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q) := \bigg\{ (x, y) \in [0, 1]^2 : x \leqslant y,\ \frac{N|\Delta_i|}{\Lambda Q^2 y} < e_i y - \Delta_i x \leqslant \frac{1}{k_i},\ \forall i \in [1, \ell - 1] \bigg\}.$$

Note that $\widetilde{\Omega}_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q) \neq \emptyset$ produces $k_i, |\Delta_i|, |e_i| \ll \Lambda$. This is because $|\Delta_i| \leqslant \frac{\Lambda Q^2 y}{N} \ll \Lambda$, $k_i \leqslant \frac{1}{e_i y - \Delta_i x} \leqslant \frac{\Lambda Q^2 y}{N} \ll \Lambda$, $\frac{1}{y} \leqslant \frac{\Lambda Q^2}{N k_i} \ll \Lambda$, and $|e_i| \leqslant \frac{|e_i y - \Delta_i x|}{y} + \frac{|\Delta_i| y}{y} \leqslant \frac{1}{y} + |\Delta_i| \ll \Lambda$.

If $0 < e_i y - \Delta_i x \leqslant \frac{N|\Delta_i|}{\Lambda Q^2 y}$ for some $i \in [1, \ell - 1]$, then $|h_{Q;e_i,\Delta_i}(Qx, Qy)| \geqslant \Lambda$. Set $h_i := h_{Q;e_i,\Delta_i}(Qx, Qy)$. We have $\min\{|h_1|, |h_{\ell-1}|, |h_1 - h_2|, \ldots, |h_{\ell-2} - h_{\ell-1}|\} \geqslant \frac{\Lambda}{\ell}$. Arguing exactly as in the proof of Lemma 11 it follows that the total contribution to (5.11) is $\ll_{\ell,B} N^{-B}$ for every $B > 0$.

We infer that the set $\widetilde{\Omega}_{\mathbf{e},\boldsymbol{\Delta},\mathbf{k}}(Q)$ can be replaced by the set $\mathcal{D}_{\mathbf{A},\mathbf{B}}$ defined in (1.3), where we took $A_i = e_i k_i$, $B_i = \Delta_i k_i$, $\mathbf{A} = (A_1, \ldots, A_{\ell-1})$, $\mathbf{B} = (B_1, \ldots, B_{\ell-1})$. Note that $\Psi_{F;\mathbf{e},\boldsymbol{\Delta},\lambda} = \Psi_{F;\mathbf{A},\mathbf{B},\lambda}$, as defined in (4.3). Using Proposition 6 we truncate the series in (5.11) at $\max_i\{|A_i|, |B_i|\} \leqslant \Lambda$ and infer that

$$\mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) = \frac{6Q^2}{\pi^2 N} \sum_{\substack{|A_i|,|B_i| \leqslant \Lambda, \forall i \\ k_i | (A_i, B_i), \forall i}} \mu(\mathbf{k}) \\ \times \iint_{\mathcal{D}_{\mathbf{A},\mathbf{B}}} \Psi_{\widehat{f}_\delta}\Big(\frac{(N/Q^2) B_1}{y(A_1 y - B_1 x)}, \ldots, \frac{(N/Q^2) B_{\ell-1}}{y(A_{\ell-1} y - B_{\ell-1} x)} \Big)\, dxdy + O_{\ell,\varepsilon}(Q^{-\min\{\alpha_\ell, \delta_\ell\} + \varepsilon}).$$

The convention here is that $(A, 0) = A$ if $A \in \mathbb{N}$. Note that if $B_i = 0$ for some i and $\mathcal{D}_{\mathbf{A},\mathbf{B}} \neq \emptyset$, then $A_i \geqslant 1$.

Using again Proposition 6 we can truncate the sum at $\max_i\{|A_i|, |B_i|\} \leqslant \Lambda^{\eta_\ell}$ for some small $\eta = \eta_\ell > 0$ to be fixed, getting

$$\mathfrak{M}_{Q;\delta}^{(\Lambda^\eta)}(\ell) = \mathfrak{M}_{Q;\delta}^{(\Lambda)}(\ell) + O(\Lambda^{-\delta_\ell \eta_\ell}).$$

Using (5.1) we then replace $\Psi_{\widehat{f}_\delta}$ by $\Psi_{\widehat{\chi}_{[0,1]}} = \Psi_{\mathrm{sinc}(\pi \cdot)}$ in the integral at the price of and error term which is $\ll_\varepsilon \Lambda^{(2\ell-2)\eta_\ell + \varepsilon} \leqslant N^{-1+c+(2\ell-2)\eta_\ell d + \varepsilon}$. This is acceptable so long as we choose $\boxed{0 < \eta_\ell < \frac{1-c}{(2\ell-2)d} < 1}$ and we infer

$$\mathfrak{M}^{(\Lambda)}_{Q;\delta}(\ell) = \frac{6Q^2}{\pi^2 N} \sum_{\substack{|A_i|,|B_i| \leqslant \Lambda^\eta, \forall i \\ k_i | (A_i, B_i), \forall i}} \mu(\mathbf{k})$$

$$\times \iint_{\mathcal{D}_{\mathbf{A},\mathbf{B}}} \Psi_{\mathrm{sinc}}\Big(\frac{\pi(N/Q^2)B_1}{y(A_1 y - B_1 x)}, \dots, \frac{\pi(N/Q^2)B_{\ell-1}}{y(A_{\ell-1} y - B_{\ell-1} x)} \Big)\, dxdy + O_\ell(Q^{-\theta_\ell}),$$

with $\boxed{\theta_\ell := \min\{\alpha_\ell, \delta_\ell \eta_\ell\} + \varepsilon < 1}$. Finally, having replaced $\Psi_{\widehat{f_\delta}}$ by $\Psi_{\mathrm{sinc}(\pi\cdot)}$, we now extend the sum to all $A_i \in \mathbb{Z}$ and $B_i \in \mathbb{Z}$ with $\mathcal{D}_{\mathbf{A},\mathbf{B}} \neq \emptyset$ by again using Proposition 6. This contributes an error term of size $O(\Lambda^{-\delta_\ell \eta_\ell})$. After all these manipulations we conclude that

$$\mathfrak{M}^{(\Lambda)}_{Q;\delta}(\ell) = M_\ell\Big(\frac{N}{Q^2}\Big) + O_\ell(Q^{-\theta_\ell}), \tag{5.12}$$

where the quantity $M_\ell(\alpha)$ defined as in (1.4) is absolutely convergent as a result of Proposition 6. Note that in fact the error term is a bit weaker, in the sense that if $0 < \gamma_1 < \gamma_2$ are given, then the error above is $\ll_{\ell,\gamma_1,\gamma_2} Q^{-\theta_\ell}$ whenever it is assumed that $\gamma_1 Q^2 \leqslant N \leqslant \gamma_2 Q^2$. This concludes the proof of part (i) in Theorem 1.

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