

REFINEMENTS OF GÁL'S THEOREM AND APPLICATIONS

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ABSTRACT. We give a simple proof of a well-known theorem of Gál and of the recent related results of Aistleitner, Berkes and Seip [1] regarding the size of GCD sums. In fact, our method obtains the asymptotically sharp constant in Gál's theorem, which is new. Our approach also gives a transparent explanation of the relationship between the maximal size of the Riemann zeta function on vertical lines and bounds on GCD sums; a point which was previously unclear. Furthermore we obtain sharp bounds on the spectral norm of GCD matrices which settles a question raised in [2]. We use bounds for the spectral norm to show that series formed out of dilates of periodic functions of bounded variation converge almost everywhere if the coefficients of the series are in $L^2(\log \log 1/L)^\gamma$, with $\gamma > 2$. This was previously known with $\gamma > 4$, and is known to fail for $\gamma < 2$. We also develop a sharp Carleson-Hunt-type theorem for functions of bounded variations which settles another question raised in [1]. Finally we obtain almost sure bounds for partial sums of dilates of periodic functions of bounded variations improving [1]. This implies almost sure bounds for the discrepancy of $\{n_k x\}$ with n_k an arbitrary growing sequences of integers.

1. INTRODUCTION

Let $1 \leq n_1 < n_2 < \dots < n_k$ be an arbitrary sequence of integers. The problem of bounding GCD sums

$$(1) \quad \frac{1}{k} \sum_{i,j \leq k} \frac{(n_i, n_j)^{2\alpha}}{(n_i n_j)^\alpha}, \quad 0 < \alpha \leq 1$$

arises naturally in metric Diophantine approximation, following Koksma's initial work [19] (see also [10]).

While estimating (1) for many specific sequences is straightforward, the problem of determining the maximal size of (1) among all sequences with k terms is much more subtle and the case $\alpha = 1$ was posed as a prize problem by the Scientific Society at Amsterdam in 1947 on Erdős's suggestion.

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The problem for $\alpha = 1$ was solved by I. S. Gál in 1949 [13]. Gál's proof is a difficult combinatorial argument spanning 20 pages. Gál showed that for $\alpha = 1$ the GCD sum (1) is bounded by $C(\log \log k)^2$ with $C > 0$ an absolute constant and moreover that this bound is optimal up to the value of the constant $C > 0$.

Our first contribution is a simple, two page proof, of Gál's theorem. In addition our proof determines the optimal constant C as $k \rightarrow \infty$, which is new.

Theorem 1. *As $k \rightarrow \infty$,*

$$\sup_{1 < n_1 < \dots < n_k} \frac{1}{k(\log \log k)^2} \sum_{i,j \leq k} \frac{(n_i, n_j)^2}{n_i n_j} \rightarrow \frac{6e^{2\gamma}}{\pi^2}.$$

The extremal sequence in Theorem 1 is supported on very smooth integers. The generalization of (1) to $\frac{1}{2} \leq \alpha < 1$ was studied by Dyer and Harman [10] who were also interested in applications to metric diophantine approximations. Recently, Aistleitner, Berkes and Seip [1] showed that for $1/2 < \alpha < 1$, the GCD sum (1) is bounded by

$$(2) \quad \ll \exp \left(C(\alpha) \cdot \frac{(\log k)^{1-\alpha}}{(\log \log k)^\alpha} \right).$$

This bound is sharp up to the value of the constant $C(\alpha)$. Several authors have remarked on the similarity between this estimate and the conjectured maximal size of the zeta function along a vertical line (see [21, 23]) which states, for $1/2 < \alpha < 1$,

$$(3) \quad \sup_{|t| \leq k} \log |\zeta(\alpha + it)| \asymp \frac{(\log k)^{1-\alpha}}{(\log \log k)^\alpha}.$$

Our method generalizes to the case $\frac{1}{2} < \alpha < 1$ and allows us to rederive in a simple way the results of Aistleitner, Berkes and Seip [1]. In addition, the proof we give shows that the GCD sum (1) is essentially majorized by the square of the supremum of a certain random model of the zeta function, which is in fact used to conjecture (3) (see [21, 23] and Section 9 at the end of this paper). In particular the similarity between (2) and (3) is not a coincidence.

Our method also generalizes to the *spectral norm* case, which is a key ingredient in the applications described later on. In particular, if we let $\mathbf{c} = (c(1), \dots, c(n))$, then one is interested in bounding the quantity

$$\sup_{\|\mathbf{c}\|_2=1} \frac{1}{k} \sum_{i,j \leq k} \frac{(n_i, n_j)^{2\alpha}}{(n_i n_j)^\alpha} \cdot c(n_i) \overline{c(n_j)}.$$

Sharp bounds for the spectral norm have been established for $1/2 < \alpha < 1$ in [1, Theorem 5] but the case $\alpha = 1$ had remained open (see [1], [2]). Our main theorem settles this problem.

Theorem 2. Let $\mathbf{c} = (c(1), \dots, c(n))$. Then, for $\alpha = 1$,

$$(4) \quad \sup_{\|\mathbf{c}\|_2^2=1} \frac{1}{k} \sum_{i,j \leq k} \frac{(n_i, n_j)^{2\alpha}}{(n_i n_j)^\alpha} \cdot c(n_i) \overline{c(n_j)} \leq \left(\frac{6e^{2\gamma}}{\pi^2} + o(1) \right) \cdot (\log \log k)^2.$$

In addition for $0 < \alpha < 1$ we re-derive the bounds obtained for the spectral norm by Aistleitner, Berkes and Seip in [1, Theorem 5]. Precisely, for $\frac{1}{2} < \alpha < 1$ we bound the spectral norm by

$$\exp \left(2C(\alpha) \frac{(\log k)^{1-\alpha}}{(\log \log k)^\alpha} \right)$$

and for $0 < \alpha \leq \frac{1}{2}$ by

$$k^{1-2\alpha} \cdot \exp \left(2C(\alpha) \sqrt{\log k \log \log k} \right)$$

with $C(\alpha)$ an absolute constants depending only on α .

The bound (4) answers a question raised by Aistleitner, Berkes, Seip and Weber [2] regarding the correct power of $\log \log k$ for the spectral norm at $\alpha = 1$, where it is described as a “*a profound problem*” (see remarks after (33) in [2]). We note that inequality (4) also has an asymptotically sharp constant. For example, Hilberdink [17] showed that when $n_i = i$ the bound (4) is attained as $k \rightarrow \infty$ for a certain choice of the coefficients $c(k)$ (this however is not true for $1/2 < \alpha < 1$, see [2]). The bound is also attained when n_k is chosen to be the extremal sequence in Theorem 1 and $c(k) = k^{-1/2}$.

The main idea in the proof of Theorem 2 is that one can write (1) as an integral involving the Riemann zeta-function (or, more precisely, a random model for the Riemann zeta-function) and then appeal to known distributional estimates for this quantity.

We note that while our method recovers completely Theorem 1 from [1] and goes beyond when $\alpha = 1$, the bound for $\alpha = 1/2$ is nonetheless not optimal. It has been conjectured in [2], correcting an older conjecture of Harman [16], that the optimal bound in the limiting case $\alpha = 1/2$ is

$$\exp \left(C \cdot \sqrt{\frac{\log k}{\log \log k}} \right).$$

The best results towards this conjecture are due to Bondarenko and Seip (see [5]), who come within a triple logarithm of the conjecture.

We now focus on applications of Theorem 2, in the spirit of [1]. Let f be a function such that

$$(5) \quad f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0.$$

We are interested in L^2 conditions for the almost everywhere convergence of

$$(6) \quad \sum_{\ell=1}^{\infty} c_{\ell} f(n_{\ell} x)$$

for $1 \leq n_1 < n_2 < \dots$ an arbitrary sequence of integers, and in almost sure bounds for

$$(7) \quad \sum_{i \leq k} f(n_i x).$$

From the point of view of applications to metrical diophantine approximation a natural choice of f is $f(x) = \{x\} - \frac{1}{2}$, a function of bounded variation (which implies that the j -th Fourier coefficient of f is $O(1/j)$). Choosing $f(x) = \chi_I(\{x\}) - |I|$ in (7), with χ_I the indicator function of an interval $I \subset [0, 1]$ relates the problem of bounding (7) to that of obtaining almost sure bounds for the discrepancy of $\{n_k x\}$.

For smooth functions such as $f(x) = \sin 2\pi x$, one can use a deep result of Carleson [6] to show that if c_k is in ℓ^2 then (6) converges almost everywhere. However already for $f(x) = \{x\} - \frac{1}{2}$ the condition $c_k \in \ell^2$ is insufficient. In [1, Section 6] it is shown that for $f(x) = \{x\} - \frac{1}{2}$ and any $\gamma < 2$ there exists an increasing sequence of positive integers n_i and a sequence of real numbers c_{ℓ} , with

$$(8) \quad \sum_{\ell=1}^{\infty} c_{\ell}^2 \cdot (\log \log \ell)^{\gamma} < \infty$$

for which (6) diverges almost everywhere.

Aistleitner, Berkes and Seip [1, Theorem 3] complemented this negative result by showing that if (8) holds for $\gamma > 4$ then (6) converges almost everywhere for any f of bounded variation (and thus also $f(x) = \{x\} - \frac{1}{2}$). In the theorem below we close the remaining gap.

Corollary 3. *Let f be a function of bounded variation satisfying (5). Let c_k be a sequence of real numbers such that*

$$\sum_{\ell \geq 3} c_{\ell}^2 \cdot (\log \log \ell)^{\gamma} < \infty$$

for some $\gamma > 2$. Then for every increasing sequence $(n_{\ell})_{\ell \geq 1}$ the series

$$\sum_{\ell \geq 1} c_{\ell} f(n_{\ell} x)$$

converges almost everywhere.

This result is optimal in the sense that the exponent γ cannot be lowered any further. We note that our condition is roughly equivalent to $c_{\ell} \in L^2 \cdot (\log \log 1/L)^{\gamma}$,

$\gamma > 2$, since the series composed of integers with $|c_\ell| < 1/\ell^2$ converges absolutely. Our method of proof also allows us to recover the recent results in [2] obtained for functions f with Fourier coefficients decaying at a rate of $j^{-\alpha}$ with $1/2 < \alpha < 1$. Corollary 3 also improves a very recent result of Weber [25] where the same conclusion is obtained with a $(\log \log k)^4/(\log \log \log k)^2$ in place of $(\log \log k)^{2+\varepsilon}$ for the special case $n_k = k$.

We also obtain the following improvement of a result in [1, Theorem 2].

Corollary 4. *Let f be a function of bounded variation satisfying (5). Then, for almost every x ,*

$$\sum_{\ell \leq N} f(n_\ell x) \ll \sqrt{N \log N} (\log \log N)^{3/2+\varepsilon}.$$

This improves the exponent $\frac{5}{2}$ obtained in [1, Theorem 2] to $\frac{3}{2}$. The optimal exponent is conjectured to be $\frac{1}{2}$. The problem of obtaining almost everywhere bounds on the quantity $\sum_{\ell \leq N} f(n_\ell x)$ has a long history, partly motivated by the problem of obtaining almost sure bounds for the discrepancy of the sequence $\{n_k x\}$. Indeed weaker estimates on this quantity (or special cases thereof) were obtained by Gál [13] (1949), Erdős and Koksma [11] (1949), Gál and Koksma [14] (1950), Cassels [7] (1950), R. C. Baker [3] (1981), Aistleitner, Mayer and Ziegler (2010), and Aistleitner, Berkes and Seip [1] (2012). See also [2], [4], and [25].

The key estimate in the proofs of Corollary 3 and Corollary 4 is an optimal Carleson-Hunt-type inequality for systems of dilated functions $\{f(n_\ell x)\}$ with f of bounded variation. The theorem below answers a question in [1] regarding the optimal version of the Carleson-Hunt theorem, in this setting (see remarks after Lemma 4 in [1]). It would be fitting to call this a maximal analogue of Gál's theorem.

Theorem 5. *Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be a complex-valued function on the circle with Fourier coefficients, $a(j)$, satisfying the decay condition $|a(j)| = O(|j|^{-1})$. Let n_1, n_2, \dots, n_N be a strictly increasing sequence of positive integers and $c(k)$ a sequence of complex numbers. Then,*

$$\int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c(k) f(n_k x) \right| \right)^2 \ll (\log \log N)^2 \sum_{k=1}^N |c(k)|^2.$$

This inequality with an exponent of 4 instead of 2 was obtained in [1, Lemma 4], it's also shown there that the function $(\log \log N)^2$ cannot be replaced by any slower growing function.

There are several innovations compared to the proof in [1]. First, we perform a splitting according to the largest prime divisor, secondly, we use a majorant principle to handle the tails composed of large primes, and finally we use the ideas that enter

in our proof of Theorem 2 to handle the contribution of the large primes after the application of the majorant principle.

Because of the splitting, which is done according to the largest prime factor, it is tempting to investigate if there are any links with the P-summation method of Fouvry and Tenenbaum, applied to trigonometric series as in de la Bretèche and Tenenbaum's paper [9].

2. NOTATION

We use the usual asymptotic notation. For instance, we write $X \ll Y$ to indicate that there exists a universal constant C such that $|X| \leq C|Y|$. We let \mathbb{T} denote the unit circle, and $e(x) := e^{2\pi ix}$ for $x \in \mathbb{T}$. For $f \in L^1(\mathbb{T})$ we define the j -th Fourier coefficient by the relation

$$c(j) := \int_{\mathbb{T}} f(x)e(-jx)dx.$$

We let $\zeta(s)$ denote the Riemann zeta function.

3. THE RANDOM MODEL

Let $X(p)$ be a sequence of independent random variable, one for each prime p , and equidistributed on the torus \mathbb{T} . For an integer n we let

$$X(n) := \prod_{p^\alpha \parallel n} X(p)^\alpha.$$

The random model of the zeta-function that we will be working with is the following

$$\zeta(\sigma, X) := \prod_p \left(1 - \frac{X(p)}{p^\sigma}\right)^{-1}.$$

Note that the product is convergent almost surely for $\sigma > \frac{1}{2}$ by Kolmogorov's three series theorem. Note also that in an L^p sense (with $p > 0$) for $\frac{1}{2} < \sigma < 1$,

$$\zeta(\sigma, X) = \sum_n \frac{X(n)}{n^\sigma}$$

and that

$$\mathbb{E}[X(n)\overline{X(m)}] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}.$$

Instead of working with the probabilistic model we could also work with the zeta function itself, since for example,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\zeta(\sigma + it)|^2 \cdot n^{-it} dt = \mathbb{E} \left[|\zeta(\sigma, X)|^2 \cdot \overline{X(n)} \right].$$

To re-inforce this point, the distributional estimates that we use in Lemma 7 below are known unconditionally for $\zeta(\sigma + it)$ (see [21, 15, 24]). These results however are often obtained by first passing to the random model, for this reason we did not see the advantage of working with $\zeta(\sigma + it)$ directly which relies on deeper machinery (for example the zeros of $\zeta(s)$ enter the analysis). The random model described above is commonly used in the study of the Riemann zeta-function, we refer the reader to [21, 15, 24, 20, 8]) for more information and examples of its application.

4. THE DISTRIBUTIONAL ESTIMATE

The lemma below is adapted from [21, Lemma 2.1].

Lemma 6. *We have the following bound,*

$$\log \mathbb{E}|\zeta(\alpha, X)|^{2\ell} \leq \begin{cases} 2\ell(\log \log \ell + \gamma + O((\log \ell)^{-1})) & \text{for } \alpha = 1 \\ C(\alpha)\ell^{1/\alpha}(\log \ell)^{-1} & \text{for } \frac{1}{2} < \alpha < 1. \\ C(\frac{1}{2})\ell^2 \log(\alpha - \frac{1}{2})^{-1/2} & \text{for } \alpha \rightarrow 1/2 \end{cases}$$

Proof. Note that

$$\mathbb{E}|\zeta(\alpha, X)|^{2\ell} = \prod_p E_\ell(p) \text{ with } E_\ell(p) = \mathbb{E}\left[\left|1 - \frac{X(p)}{p^\alpha}\right|^{-2\ell}\right].$$

For $p < (2\ell)^{1/\alpha}$ we have the trivial bound $E_\ell(p) \leq (1 - 1/p^\alpha)^{-2\ell}$. For $p > (2\ell)^{1/\alpha}$ we notice that

$$E_\ell(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{e^{i\theta}}{p^\alpha}\right)^{-\ell} \cdot \left(1 - \frac{e^{-i\theta}}{p^\alpha}\right)^{-\ell} d\theta = I_0(2\ell/p^\alpha)(1 + O(\ell/p^{2\alpha}))$$

where $I_0(z)$ is the 0-th modified Bessel function, and $I_0(t) = \sum_n (t/2)^{2n}/(n!)^2$. In particular $\log I_0(t) \ll t^2$ for $0 < t \leq 1$. Combining these bounds we get

$$\log \mathbb{E}|\zeta(\alpha, X)|^{2\ell} \leq 2\ell \sum_{p < (2\ell)^{1/\alpha}} \log\left(1 - \frac{1}{p^\alpha}\right) + C \sum_{p > (2\ell)^{1/\alpha}} \frac{\ell^2}{p^{2\alpha}}.$$

When $\alpha = 1$ the first sum contributes $2\ell(\log \log \ell + \gamma + O(1/\log \ell))$ while the second contributes $O(1)$. When $1/2 < \alpha < 1$ the prime number theorem shows that the above sum is

$$\ll \frac{\ell^{1/\alpha}}{\log \ell} \cdot \left(\frac{\alpha}{1-\alpha} + \frac{\alpha}{2\alpha-1}\right).$$

Finally when α tends to $1/2$ we use the more careful bound

$$\sum_p \frac{\ell^2}{p^{2\alpha}} \ll \ell^2 \cdot \log(\alpha - \frac{1}{2})^{-1}$$

which comes from $\zeta(2\alpha) = 1/(2\alpha - 1) + O(1)$, to conclude. \square

5. PROOF OF THEOREM 2

Let $\mathcal{N} = \{n_1, \dots, n_k\}$. Let

$$D(X) := \sum_{n \in \mathcal{N}} c(n)X(n).$$

Consider the expression

$$\mathbb{E} \left[|\zeta(\alpha, X)|^2 \cdot |D(X)|^2 \right].$$

On the one hand, expanding the square this is equal to

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \sum_{\substack{k|n \\ k \in \mathcal{N}}} c(k) \cdot \frac{k^\alpha}{n^\alpha} \right|^2 &= \sum_{m, n \in \mathcal{N}} c(m) \overline{c(n)} \cdot \frac{(km)^\alpha}{[k, m]^{2\alpha}} \cdot \zeta(2\alpha) \\ (9) \qquad \qquad \qquad &= \zeta(2\alpha) \sum_{i, j \leq k} \frac{(n_i, n_j)^{2\alpha}}{(n_i n_j)^\alpha} \cdot c(n_i) \overline{c(n_j)}. \end{aligned}$$

On the other hand, for any $\ell, V > 0$, and $\frac{1}{2} < \alpha \leq 1$,

$$(10) \qquad |\zeta(\alpha, X)D(X)|^2 \leq e^{2V} \cdot |D(X)|^2 + k \cdot |\zeta(\alpha, X)|^{2(\ell+1)} \cdot e^{-2\ell V}.$$

Indeed to prove this inequality note that if $|\zeta(\alpha, X)| < e^V$ then the left-hand side is less than $e^{2V} \cdot |D(X)|^2$, while if $|\zeta(\alpha, X)| > e^V$, then the left-hand side is less than

$$|D(X)|^2 \cdot |\zeta(\alpha, X)|^{2(\ell+1)} \cdot e^{-2\ell V} \leq k \cdot |\zeta(\alpha, X)|^{2(\ell+1)} \cdot e^{-2\ell V},$$

using the L^∞ bound, $|D(X)|^2 \leq k \cdot \|\mathbf{c}\| \leq k$ coming from Cauchy-Schwarz. Taking the expectation on both sides and using (9) we get

$$(11) \qquad \zeta(2\alpha) \cdot \frac{1}{k} \sum_{i, j \leq k} \frac{(n_i, n_j)^{2\alpha}}{(n_i n_j)^\alpha} \leq e^{2V} \cdot \|\mathbf{c}\|_2^2 + k \cdot \mathbb{E}[|\zeta(\alpha, X)|^{2\ell+2}] \cdot e^{-2\ell V}.$$

In the above equation, if $\alpha = 1$ then we let

$$V = \log \log \log k + \gamma + 2/\psi(k) \text{ and } \ell = \psi(k) \cdot \log k$$

with $\psi(k) \rightarrow \infty$ very slowly as $k \rightarrow \infty$ (say $\psi(k) = \log \log \log k$). Otherwise, we let

$$V = \begin{cases} (C(\alpha) \log k)^{1-\alpha} \cdot (\log \log k)^{-\alpha} & \text{if } \frac{1}{2} < \alpha < 1 \\ (C(\frac{1}{2}) \log k \log \log k)^{1/2}, & \text{if } \alpha = \frac{1}{2} + \frac{1}{\log k} \end{cases} \text{ with } \ell = \frac{\log k}{V}.$$

With this choice of parameters we have $\mathbb{E}[|\zeta(\alpha, X)|^{2\ell+2}] \cdot e^{-2\ell V} \ll k^{-1}$ for a fixed $1/2 < \alpha \leq 1$ and for $\alpha = 1/2 + 1/\log k$. If $1/2 < \alpha \leq 1$ is fixed, then inserting the

choice of ℓ and V made above into (11) gives the claim. In order to prove the claim for $\alpha = \frac{1}{2}$, we use Holder's inequality,

$$\frac{1}{k} \sum_{i,j \leq k} \frac{(n_i, n_j)}{\sqrt{n_i n_j}} \leq \left(\frac{1}{k} \sum_{i,j \leq k} \frac{(n_i, n_j)^{2\alpha}}{(n_i n_j)^\alpha} \right)^{1/(2\alpha)} \cdot k^{1-1/(2\alpha)}$$

with $\alpha = 1/2 + 1/\log k$, and appeal to (11) with the choice of parameters as described above. The result for $0 < \alpha \leq \frac{1}{2}$ follows in the same manner by interpolating with the case $\alpha = \frac{1}{2}$ using Holder's inequality.

6. PROOF OF COROLLARY 1

We have already established the upper bound in Theorem 1. Therefore it suffices to obtain the lower bound. Let $\mathcal{P}(r, \ell) = p_1^{\ell-1} \cdots p_r^{\ell-1}$ where p_1, p_2, \dots are consecutive primes. Gál proves the following identity in [13],

$$\sum_{n_i, n_j | \mathcal{P}(r, \ell)} \frac{(n_i, n_j)^2}{n_i n_j} = \prod_{p | p_1 \dots p_r} \left(\ell + 2 \sum_{v=1}^{\ell-1} \frac{\ell - v}{p^v} \right)$$

where the summation goes over all n_i and n_j dividing $\mathcal{P}(r, \ell)$ ¹. The number of divisors of $\mathcal{P}(r, \ell)$ is ℓ^r . Let r be the largest r such that $(r + \log k)^r < k$. Therefore $r \sim \log k / \log \log k$ and $p_r \sim \log k$ by the prime number theorem. Pick an integer i such that

$$(r + i)^r < k < (r + i + 1)^r.$$

Since $(r + \log k + 1)^{r+1} > k$ but $(r + \log k)^r < k$ it follows that $i \asymp \log k$. In particular $(r + i)^r \sim k$ as $k \rightarrow \infty$. Set $\ell = r + i$ and let $\mathcal{N} = \{n_1, \dots, n_k\}$ be a set containing the $(r + i)^r < k$ divisors of $\mathcal{P}(r, \ell)$ and $k - (r + i)^r$ other integers picked at random. Then according to Gál's identity, highlighted above, the GCD sum

$$\frac{1}{k} \sum_{\substack{i,j \leq k \\ n_i, n_j \in \mathcal{N}}} \frac{(n_i, n_j)^2}{n_i n_j}$$

is at least,

$$\begin{aligned} &\geq (r + i)^r \prod_{p < p_r} \left(1 + 2 \sum_{v=1}^{r+i-1} \frac{1}{p^v} \cdot \left(1 - \frac{v}{r+i} \right) \right) \\ &> (1 + o(1))k \prod_{p < p_r} \left(1 - \frac{1}{p^2} \right)^{-2} \times \prod_{p < p_r} \left(1 + 2 \sum_{v=1}^{r+i-1} \frac{1}{p^v} \cdot \left(1 - \frac{v}{r+i} \right) \right) \left(1 - \frac{1}{p} \right)^2 \end{aligned}$$

¹This identity can be also quickly checked using the fact that $f(m, n) = (m, n)^2 / (mn)$ is a multiplicative function of two variables

By Merten's theorem the first product is asymptotically equal to $(e^\gamma \log p_r)^2 \sim (e^\gamma \log \log k)^2$ as $k \rightarrow \infty$. On the other the second product converges as $k \rightarrow \infty$ to

$$\prod_p \left(1 + 2 \sum_{v=1}^{\infty} \frac{1}{p^v}\right) \left(1 - \frac{1}{p}\right)^2 = \frac{6}{\pi^2}.$$

Combining these two observations the claim follows.

7. CARLESON-HUNT BOUNDED VARIATION

We now turn our attention to the proof of Theorem 5. Our argument will depend on the Carleson-Hunt theorem [18], stated below.

Proposition 7. *There exists an absolute constant $c > 0$ such that*

$$\int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c(k) e(kx) \right| \right)^2 dx \leq c \sum_{k=1}^N |c(k)|^2$$

for any finite sequence $(c(k))$.

We start by writing f as a Fourier series

$$f(x) = \sum_{j \in \mathbb{Z}} a(j) e(jx),$$

where we have the inequality $|a(j)| \ll (1 + |j|)^{-1}$. Next we split the Fourier series of f into two parts based on the factorization of the j . Here A denotes a large real constant to be specified later and $P^+(j)$ corresponds to the largest prime factor of $|j|$. Let

$$r(x) := \sum_{P^+(j) > (\log(N))^{2A+2}} a(j) e(jx), \quad p(x) := \sum_{P^+(j) \leq (\log(N))^{2A+2}} a(j) e(jx)$$

so that $f(x) = p(x) + r(x)$. It suffices to prove, for $g(x) \in \{p(x), r(x)\}$ the inequality:

$$(12) \quad \int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c(k) g(n_k x) \right| \right)^2 dx \ll (\log \log(N))^2 \sum_{k=1}^N |c(k)|^2.$$

For $g(x) = p(x)$ we may write the square root of (12) as

$$\left(\int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M \sum_{P^+(j) \leq (\log(N))^{2A+2}} c(k) a(j) e(jn_k x) \right| \right)^2 dx \right)^{1/2}$$

$$\ll \sum_{P^+(j) \leq (\log(N))^{2A+2}} |a(j)| \left(\int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c(k) e(jn_k x) \right| \right)^2 dx \right)^{1/2}.$$

Applying the classical Carleson-Hunt inequality this is bounded by

$$\ll \left(\sum_{k=1}^N |c(k)|^2 \right)^{1/2} \sum_{P^+(j) \leq (\log(N))^{2A+2}} |a(j)|$$

and it remains to notice that

$$\begin{aligned} \sum_{P^+(j) \leq (\log(N))^{2A+2}} |a(j)| &\ll \sum_{P^+(j) \leq (\log(N))^{2A+2}} j^{-1} \\ &= \prod_{p \leq (\log(N))^{2A+2}} \frac{1}{1-p^{-1}} \ll \log \log N \end{aligned}$$

by Merten's theorem. This completes the analysis of (12).

We now consider the left side of (12) with $g(x) = r(x)$. The key ingredient in the analysis of (12) will be the following almost orthogonality property of the functions $r(n_k x)$, which will be proved shortly.

Lemma 8. *With the notation and conditions stated above, if $I = [M_1, M_2] \subseteq [1, N]$ then*

$$\int_{\mathbb{T}} \left(\sum_{k \in I} c(k) r(n_k x) \right)^2 dx \ll \frac{(\log \log N)^2}{(\log N)^{2A}} \sum_{k \in I} |c(k)|^2.$$

Assuming Lemma 8 for the moment, we may deduce a maximal version of this inequality at the expense of an additional factor of $\log N$ using a Radamacher-Menshov-type argument inequality (see [22] for a systematic discussion of this technique). This lemma will then imply (12) with $g(x) = r(x)$ for fixed $A > 1$.

Lemma 9. *With the notation and conditions stated above,*

$$\int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c(k) r(n_k x) \right| \right)^2 dx \ll \log^2 N \cdot \frac{(\log \log N)^2}{\log^{2A} N} \sum_{k=1}^N |c(k)|^2.$$

Proof. Without loss of generality assume that $N = 2^n$ is a power of 2. We consider the set of dyadic subintervals of $[1, N]$:

$$\mathbb{D} := \{[2^\ell(m-1), 2^{\ell-1}m) : 0 \leq \ell \leq n, 0 \leq m \leq 2^{n-\ell}\}.$$

Let $1 \leq t(x) \leq N$ denote the length of the maximal partial sum at x . We may write the interval $[1, t(x)]$ as a disjoint union of at most $O(\log(N))$ dyadic intervals (elements of \mathbb{D}). It follows, for fixed x , that for any $t(x) \leq N$ there exists a disjoint

decomposition of $[1, t(x)]$ into a union of $O(\log N)$ elements $\{\mathcal{D}_s^{(x)}\}_{s=1}^{\log N}$. Here $\mathcal{D}_s^{(x)}$ are disjoint dyadic intervals depending on x . Hence

$$\left| \sum_{k=1}^t c(k)r(n_k x) \right|^2 \leq \left(\sum_{s=1}^{\log N} \left| \sum_{k \in \mathcal{D}_s^{(x)}} c(k)r(n_k x) \right| \right)^2 \leq \log N \sum_{s=1}^{\log N} \left| \sum_{k \in \mathcal{D}_s} c(k)r(n_k x) \right|^2.$$

Summing over all dyadic intervals the dependence on x may be removed. Indeed we have

$$\begin{aligned} \int_{\mathbb{T}} \left(\max_{1 \leq M \leq N} \left| \sum_{k=1}^M c(k)r(n_k x) \right| \right)^2 &\ll \int_{\mathbb{T}} \log N \sum_{\mathcal{D} \in \mathbb{D}} \left| \sum_{k \in \mathcal{D}} c(k)r(n_k x) \right|^2 \\ &\ll \log N \sum_{\mathcal{D} \in \mathbb{D}} \int_{\mathbb{T}} \left| \sum_{k \in \mathcal{D}} c(k)r(n_k x) \right|^2 dx. \end{aligned}$$

Finally, Lemma 8 combined with the observation that each integer k occurs in $O(\log N)$ diadic intervals, allows us to bound this as

$$\ll \log^2 N \frac{(\log \log N)^2}{\log^{2A}(N)} \sum_{k=1}^N |c(k)|^2.$$

This completes the proof. \square

We need a simple majorant principle, stated below.

Lemma 10. *Let $c_k \geq 0$ be a non-negative sequence. Given a sequence $a_n \geq 0$ and another sequence $b_n \geq 0$ such that $a_n \leq b_n$ we have,*

$$\mathbb{E} \left[\left| \sum_n a_n X(n) \right|^2 \cdot \left| \sum_n c_n X(n) \right|^2 \right] \leq \mathbb{E} \left[\left| \sum_n b_n X(n) \right|^2 \cdot \left| \sum_n c_n X(n) \right|^2 \right].$$

Proof. By non-negativity of c_k, a_n and $a_n \leq b_n$,

$$\begin{aligned} \mathbb{E} \left[\left| \sum_n a_n X(n) \right|^2 \cdot \left| \sum_n c_n X(n) \right|^2 \right] &= \sum_n \left(\sum_{n=k\ell} a_k c_\ell \right)^2 \\ &\leq \sum_n \left(\sum_{n=k\ell} b_k c_\ell \right)^2 = \mathbb{E} \left[\left| \sum_n b_n X(n) \right|^2 \cdot \left| \sum_n c_n X(n) \right|^2 \right] \end{aligned}$$

as claimed. \square

We will also need a simple moment calculation.

Lemma 11. *For any integer $\ell \geq 1$, and any $y \leq z$, and $\sigma > 0$,*

$$\mathbb{E} \left[\left| \sum_{y \leq p \leq z} \frac{X(p)}{p^\sigma} \right|^{2\ell} \right] \ll \ell! \cdot \left(\sum_{y \leq p \leq z} \frac{1}{p^{2\sigma}} \right)^\ell.$$

Proof. See [20, Lemma 3.2]. □

We are now ready to prove Lemma 10.

Proof of Lemma 10. Let $J = (\log N)^{2A+2}$. Expanding r and using orthogonality we get

$$\begin{aligned} \int_0^1 \left(\sum_{k=M_1+1}^{M_2} c(k)r(n_k x) \right)^2 dx &= \frac{1}{2} \sum_{k_1, k_2 \in I} \sum_{\substack{P^+(j_1) > J \\ P^+(j_2) > J \\ j_1 n_{k_1} = j_2 n_{k_2}}} c(k_1)c(k_2)a(j_1)a(j_2) \\ &\ll \sum_{k_1, k_2 \in I} \sum_{\substack{P^+(j_1) > J \\ P^+(j_2) > J \\ j_1 n_{k_1} = j_2 n_{k_2}}} |c(k_1)| \cdot |c(k_2)| \cdot \frac{1}{j_1 j_2} \end{aligned}$$

since $|a(n)| \ll n^{-1}$. The condition $j_1 n_{k_1} = j_2 n_{k_2}$ can be expressed as

$$\mathbb{E}[X(j_1)X(n_{k_1})\overline{X(j_2)X(n_{k_2})}]$$

and therefore we re-write the previous sum as

$$\mathbb{E} \left[\left| \sum_{P^+(n) > J} \frac{X(n)}{n} \right|^2 \cdot \left| \sum_n |c(n)|X(n) \right|^2 \right].$$

If $P^+(n) > J$ then n can be written as $n = pm$ with $p > J$ a prime. Therefore by the majorant principle the above expression is less than or equal to

$$(13) \quad \mathbb{E} \left[\left| \sum_{p > J} \frac{X(p)}{p} \cdot \zeta(1, X) \right|^2 \cdot \left| \sum_n |c(n)|X(n) \right|^2 \right].$$

Proceeding as in our proof of Gál's theorem, for any choice of positive integer $\ell > 0$ we have,

$$(14) \quad \left| \sum_{p > J} \frac{X(p)}{p} \right|^2 < \frac{1}{(\log N)^{2A}} + (\log N)^{2A\ell} \cdot \left| \sum_{p > J} \frac{X(p)}{p} \right|^{2(\ell+1)}.$$

The contribution of $(\log N)^{-2A}$ to (13) is

$$\frac{1}{(\log N)^{2A}} \cdot \mathbb{E} \left[|\zeta(1, X)|^2 \cdot \left| \sum_n |c(n)|X(n) \right|^2 \right] \ll \frac{(\log \log N)^2}{(\log N)^{2A}} \cdot \sum_n |c(n)|^2$$

as is seen by following the same steps as in our proof of Gál's theorem. On the other hand the contribution of $(\log N)^{2A\ell} \cdot \left| \sum_{p>J} X(p)/p \right|^{2(\ell+1)}$ to (13) is bounded by

$$\begin{aligned} &\ll (\log N)^{2A\ell} \cdot \mathbb{E} \left[\left| \sum_{p>J} \frac{X(p)}{p} \right|^{2(\ell+1)} \cdot |\zeta(1, X)|^2 \cdot \left| \sum_n |c(n)|X(n) \right|^2 \right] \\ &\leq (\log N)^{2A\ell} \cdot N \left(\sum_n |c(n)|^2 \right) \cdot \mathbb{E} \left[\left| \sum_{p>J} \frac{X(p)}{p} \right|^{2(\ell+1)} \cdot |\zeta(1, X)|^2 \right]. \end{aligned}$$

By Cauchy-Schwarz and Lemma 13 the expectation is less than

$$\begin{aligned} \mathbb{E} \left[|\zeta(1, X)|^4 \right]^{1/2} \cdot \mathbb{E} \left[\left| \sum_{p>J} \frac{X(p)}{p} \right|^{4(\ell+1)} \right]^{1/2} &\ll \\ &\ll 1 \cdot \left((2(\ell+1))! \cdot \left(\sum_{p>J} \frac{1}{p^2} \right)^{2(\ell+1)} \right)^{1/2} \ll \left(\frac{4\ell}{J} \right)^\ell. \end{aligned}$$

It follows that the total contribution obtained by inserting (14) into (13) is

$$\ll \frac{(\log \log N)^2}{(\log N)^{2A}} \sum_n |c(n)|^2 + \left(\frac{4\ell(\log N)^{2A}}{J} \right)^\ell \cdot N \sum_n |c(n)|^2.$$

Since $J = (\log N)^{2A+2}$ choosing $\ell = \log N$ we see that the second term is negligible compared to the first, and we've obtained the desired bound. \square

8. PROOF OF COROLLARY 3 AND COROLLARY 4

Corollary 3 and Corollary 4 follow immediately from the improved Carleson-Hunt inequality established in Theorem 5. We refer the readers to [1, Proof of Theorem 2 and 3] for the details of this standard deduction.

9. DISCUSSION OF THE CONNECTION WITH THE MAXIMAL SIZE OF $\zeta(s)$

Our bound for Gál's theorem essentially corresponds to $(1/\zeta(2\sigma)) \cdot e^{2V(\sigma)}$ where $V(\sigma)$ is the largest V such that $\mathbb{P}(\log |\zeta(\sigma, X)| > V) < k^{-1-\varepsilon}$. We expect this V to coincide with

$$\sup_{|t|<k} \log |\zeta(\sigma + it)|.$$

The reason for this is the following: if the maximum value M of $|\zeta(\sigma + it)|$, $|t| < k$, is attained at $t = t_0$, then we have

$$|\zeta(\sigma + it_0 + i\varepsilon)| > M/2$$

for $|\varepsilon| \ll 1/\log k$ (see [12]). Thus if the probabilistic model predicts that $|\zeta(\sigma + it)| > e^V$ on a set of measure at most $k^{-\varepsilon}$ for $|t| < k$, then we expect $|\zeta(\sigma + it)| < e^V$ to hold for all $|t| < k$.

We do not expect our approach to deliver sharp estimates for the constants $C(\alpha)$ as defined in Theorem 2. However, for future reference we note that it seems that with more work one can show that for $1/2 < \alpha < 1$ we have

$$C(\alpha) = G_1(\alpha)\alpha^{-2\alpha}(1 - \alpha)^{\alpha-1} + o(1)$$

as $k \rightarrow \infty$, with

$$G_1(\alpha) = \int_0^\infty \log I_0(u)u^{-1-1/\sigma} du$$

and

$$I_0(u) = \sum_{n=0}^{\infty} (u/2)^{2n}/n!^2$$

the modified Bessel function of order 0. This is suggested by Lamzouri's conjecture (see Remark 2 after Theorem 1.5 in [21]) as to the order of

$$\sup_{|t|<k} \log |\zeta(\sigma + it)|.$$

His conjecture is made through an analysis of the random model $\zeta(\sigma, X)$, as described above (see also [12]).

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