Joseph Maher Research Statement

I am interested in geometry and topology, and in low-dimensional manifolds in particular. Much recent work in this area has been influenced by fundamental results and conjectures of Thurston about 3-manifolds. I will describe some of Thurston's conjectures, and how my research relates to them, and then discuss some directions for future work.

The Virtually Haken Conjecture

This conjecture states that every 3-manifold has a finite cover which is Haken. A 3-manifold is Haken if it contains an essential surface, i.e. an embedded surface in the manifold whose fundamental group injects into the fundamental group of the 3-manifold. For example, Haken manifolds include all irreducible 3-manifolds with boundary, and all 3-manifolds with positive Betti numbers. This conjecture is the 3-manifold case of Gromov's more general conjecture that a 1-ended word-hyperbolic group contains a surface group, or else is virtually free. However, even the essential surfaces that arise in finite covers of Haken hyperbolic manifolds are not particularly well understood.

There are many examples of Haken hyperbolic manifolds with a single torus boundary, such as most knot complements. For homology reasons, such 3-manifolds with torus boundary must have at least one embedded essential surface with boundary, and the boundary components of the surface are simple closed curves in the boundary torus, which are usually referred to as slopes. By a result of Hatcher [3] only finitely many such slopes may be realised in a 3-manifold with a single torus boundary. However, its possible that there may be other slopes in a finite cover, which project down to essential immersed surfaces in the original manifold. Its known that a Seifert fibered manifold may have only finitely many slopes realised in its finite covers. I showed in [6] that there are many examples of hyperbolic manifolds in which every boundary slope was realised as the boundary of an essential surface in some finite cover, i.e.

Theorem 1. *Hyperbolic once-punctured torus bundles and hyperbolic two-bridge knots have every boundary slope realised in finite covers.*

The Virtually Fibered Conjecture

Thurston's conjecture that every 3-manifold has a finite cover which is fibered is a much stronger version of the virtually Haken conjecture, as fibered manifolds are automatically Haken, as the fiber is an essential surface. Using minimal surfaces, and links between the geometry of the manifold, and another structure on a 3-manifold, called a Heegaard splitting, I have been able to prove the following theorem [9].

Theorem 2. *Let M be a hyperbolic manifold with a sequence of finite covers with Heegaard splittings of bounded genus. Then M is virtually fibered.*

Every 3-manifold has a Heegaard splitting, which is a decomposition of the manifold into two handlebodies, where a handlebody is a regular neighbourhood of an embedded graph in $\mathbb{R}^3$. If we are interested in hyperbolic manifolds, we can attempt to relate the Heegaard splittings to the geometry of the manifold.

A Heegaard splitting gives rise to a sweepout of the manifold, which we can think of as a one-parameter family of surfaces isotopic to the Heegaard splitting which shrink down to the spines of handlebodies on either side of the Heegaard surface. A "minimax" sweepout, is such a one-parameter family of surfaces, in which the maximum area of the surface during the sweepout is minimized. A result of Pitts and Rubinstein [11] using geometric measure theory shows that the minimax surface is closely related to a minimal surface. In particular, minimal surfaces in hyperbolic manifolds have not just bounded area (from Gauss-Bonnet), but bounded diameter, outside the manifold's Margulis tubes, due to the monotonicity formula for minimal surfaces. Lackenby has used this idea, together with work of Lubotsky [5] on expanders and Property (τ), to investigate covers of hyperbolic manifolds. If the covers have the property that the Heegaard genus of the covers grows more slowly than the degree of the cover, in certain precise ways, and the covers are not too irregular, then there are covers which have positive Betti number, and which are virtually fibered.

I have been able to remove the assumptions needed about the regularity of the covers in the virtually fibered case. In fact I have proved a more general theorem, which only requires the Scharlemann-Thompson width of the splitting to be bounded. I make use of generalized sweepouts, which is a generalization of a Heegaard splitting, developed in my earlier work with Rubinstein [7], and I adapt a construction of Bachman, Cooper and White [1] to straighten the generalized sweepout in the hyperbolic metric, to produce one in which all the leaves are "mostly" negatively curved, and hence have bounded geometry.

The Geometrization Conjecture

Thurston's geometrization conjecture states that any 3-manifold can be divided along essential spheres and tori into pieces which have one of eight 3-dimensional geometries. The eight geometries correspond to Riemannian metrics which are homogeneous, but not necessarily isotropic. This means the metric looks the same at every point, but not necessarily in every direction. Examples include the constant sectional curvature metrics, i.e. spherical, Euclidean and hyperbolic geometry, which are isotropic, and $\mathbb{H}^2 \times \mathbb{R}$, which is not isotropic.

The classification of 3-manifolds with finite fundamental group is a special case of the geometrization conjecture, and these should all have spherical metrics, i.e. constant curvature $+1$. This in turn splits in to two conjectures, known as the Poincaré conjecture, and the spherical spaceform conjecture. The Poincaré conjecture states that the only closed 3-manifold with trivial fundamental group is the 3-sphere, and the spherical spaceform conjecture states that any quotient of 3-sphere is homeomorphic to a quotient of the round three-sphere by isometries. My thesis work was in fact a special case of the spherical spaceform conjecture, and in joint work with Hyam Rubinstein [7] I proved

Theorem 3. *Period three actions on the three-sphere are standard.*

In later work [8], I extended this to all powers of three. An exciting recent development is Perelman's work [10] on Hamilton's program to prove geometrization using Ricci flow. This seems likely to give a complete proof of

Thurston's geometrization conjecture. In particular, this shows that hyperbolic manifolds really are the most important class of 3-manifolds.

Future plans

Another application of these techniques is to show that if a non-Haken hyperbolic manifold has an immersed sweepout of genus g, then it has Heegaard genus $H(g)$, for some function of g. We give a rough sketch of the argument. If the volume of the manifold is large, then there are disjoint immersed surfaces in the sweepout, as the surfaces have bounded geometry. The immersed surfaces can be replaced by embedded surfaces using a result of Gabai [2], and these divide the manifold up into pieces of bounded volume. By the Margulis Lemma, a hyperbolic manifold of bounded volume has bounded Heegaard genus, and by Scharlemann-Thompson thin position for Heegaard splittings, if the manifold is non-Haken this gives an upper bound for the Heegaard genus of the whole manifold.

There is another view point for looking at special surfaces in 3-manifolds. Given a triangulation of the manifold, you can attempt to "straighten" the surface inside each tetrahedron to produce a surface made of triangles and squares called a "normal" surface. There is a combinatorial analogue of area, namely the number of times the surface hits the 1-skeleton, and this can be thought of as putting a metric on the manifold in which all the volume is concentrated along the 1-skeleton of the triangulation.

The advantage of this point of view is that explicit algorithms can be constructed, though many problems are still outstanding. For example, it is not known whether there is an algorithm to tell whether two Heegaard splittings are isotopic. I am working on showing the existence of such an algorithm using the following approach. A strongly irreducible Heegaard splitting may be put in "almost normal" form, i.e. normal everywhere except in at most one tetrahedron, in which it as a specified shape. One way to prove this is to take a sweepout corresponding to the Heegaard splitting and choose a minimax one relative to a suitable notion of complexity. If two such Heegaard splittings are isotopic, then the isotopy between them gives an isotopy between the two sweepouts, so we get a 2-parameter map of $S \times I \times I$ into the manifold. There should be a corresponding minimax argument to show that there is a connecting path of almost normal surfaces across the square between the two sweepouts, and a proof of this would give an algorithm for showing that two strongly irreducible Heegaard splittings were (or were not) isotopic. However, there remain some extensive technicalities to be worked through to check this really works.

References

[1] D. Bachman, D. Cooper, M. White, Large embedded balls and Heegaard genus in negative curvature, Preprint, math.GT/0305290.

[2] D. Gabai, Foliations and the topology of 3-manifolds, *J. Differential Geom.* **18** *(1983), no. 3, 445–503.*

[3] A. E. Hatcher, On the boundary curves of incompressible surfaces, *Pacific J. Math. 99 (1982), no. 2, 373–377.*

[4] M. Lackenby, Heegaard splittings, the virtually Haken conjecture and Property (τ), Preprint, math.GT/0210316.

[5] A. Lubotzky, *Discrete groups, expanding graphs and invarianr measures*, Progress in Mathematics 125, Birkhauser 1994.

[6] J. Maher, Virtually embedded boundary slopes, *Topology And Its Applications (95)1 (1999), 63-74.*

[7] J. Maher, J. H. Rubinstein, Period three actions on the three sphere, *Geometry and Topology,* **Vol. 7** *(2003), 329-397.*

[8] J. Maher, Period three actions on Lens spaces, preprint, math.GT/0311009.

[9] J. Maher, Heegaard gradient and virtual fibers, preprint, math.GT/0411219.

[10] G. Perelman, Ricci flow with surgery on three-manifolds, preprint, math.DG/0303109.

[11] J. Pitts, J. H. Rubinstein, Existence of minimal surfaces of bounded topological type in three-manifolds, *Miniconference on geometry and partial differential equations (Canberra, 1985), 163-176.*