ON THE SUBCONVEXITY PROBLEM AND EQUIDISTRIBUTION OF HEEGNER POINTS

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Contents

1. Introduction and main results

1.1. **Statement of the main results.** We are interested in equidistribution of Heegner points in a quite general form. To state our results more precisely, we need to introduce some notations.

Let F be a totally real number field and B be a quaternion algebra over F. Fix an integral ideal $\mathfrak{c}$ of $\mathcal{O}_F$ and $U=\prod_\nu U_\nu \subset \widehat{B}^\times$ an open compact subgroup such that $\widehat{\mathcal{O}}_F^\times \subset U$. Denote by $\mathfrak{S}_D$ the set of quadratic imaginary extension K/F with relative discriminant $D=D_{K/F}$, and $\mathcal{O}_{K,\mathfrak{c}} := \widehat{\mathcal{O}}_K^{\times(\mathfrak{c})} \prod_{\nu|\mathfrak{c}} (1+\mathfrak{c}\mathcal{O}_{K,\nu})$. Define

$$\mathfrak{S}_{D,U,\mathfrak{c}} := \left\{ K\in\mathfrak{S}_D : \text{K admits an embedding } \ \xi_K : K \hookrightarrow B \ \ s.t. \ \ \widetilde{\xi}_K(\mathcal{O}_{K,\mathfrak{c}}) \subseteq U \right\},$$

where $\widetilde{\xi}_K$ is the induced map from ξ_K. Henceforth, we will, for simplicity, identify the notations $\widetilde{\xi}_K$ and ξ_K. Then for any $K\in\mathfrak{S}_{D,U,\mathfrak{c}}$, there exists an embedding ξ_K

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inducing a map

$$\xi_K : K^\times \backslash \widehat{K}^\times / \mathcal{O}_{K,\mathfrak{c}} \to X,$$

where $X = X_U := B^\times \backslash \widehat{B}^\times / U$ is a finite set by definition. For and $G_K \leq Pic\left(\mathcal{O}_{K,\mathfrak{c}}\right)$, define the associated measure as follows:

$$\mu_{G_K}^{(i)} := \frac{1}{|G_K|} \sum_{\sigma \in G_K} \delta_{e_i}\left(\xi_K(\sigma)\right),$$

where δ the Dirac measure.

Assuming the condition:

$$\lim_{|D| \to +\infty} \#\mathfrak{S}_{D,U,\mathfrak{c}} = +\infty. \tag{1}$$

Then we have the following equidistribution law:

Theorem 1. *Let notations be as above. Assuming (1), then there is an absolutely constant $\delta > 0$ such that for any $K \in \mathfrak{S}_{D,U,\mathfrak{c}}$ and any $G_K \leq Pic\left(\mathcal{O}_{K,\mathfrak{c}}\right)$ with index $\leq |D|^\delta$, the weakly-$*$ limits of $\left\{\mu_{G_K}^{(i)}\right\}$ exists and*

$$\lim_{\substack{|D| \to +\infty \\ K \in \mathfrak{S}_{D,U,\mathfrak{c}}}} \mu_{G_K}^{(i)} = \mu_U^{(i)}.$$

Remark. The exponent δ is expected to be at most $\frac{1}{12}$ under Michel-Venkatesh's method. Besides, by definition of weakly-$*$ convergence, the theorem can be restated as follows: if

$$\lim_{|D| \to +\infty} \#\mathfrak{S}_{D,U,\mathfrak{c}} = +\infty,$$

then for any function $V : X \to \mathbb{C}$, we have

$$\lim_{|D| \to +\infty} \frac{1}{|G_K|} \sum_{\sigma \in G_K} V(\xi_K(\sigma)) = \sum_{i=1}^{\#X} \int_X V(x) d\mu_U^{(i)}(x). \tag{2}$$

Hence by Weyl's law (cf. [67]), (2) amounts to saying that:

Corollary 2 (Equidistribution of Heegner points)**.** *Let notations be as in the above theorem. Assuming (1), then as $D \to \infty$, the orbit $\xi_K.G_K$ becomes equidistributed in the set $\{e_1, ..., e_n\}$ relative to the measure μ_U defined in (12). More precisely, there exists an absolute constant $\delta' > 0$ such that*

$$\frac{|\{\sigma \in G_K, \xi_K(\sigma) = e_i\}|}{|G_K|} = \mu_U(\{e_i\}) + O_U\left(|D|^{-\delta'}\right). \tag{3}$$

Remark. We can obtain that (3) is equivalent to Theorem 1 by noting that the space $\mathbb{C}[X]$ is a finitely dimensional reflexive Banach space. Hence we shall only need to show (3). Roughly speaking, this will be done in the Subsection 2.1.

As a straightforward application of Theorem 1, we give a lower bound for the number of Hecke characters χ such that $L(\pi, \chi, \frac{1}{2}) \neq 0$. Precisely, for any $\pi \in \mathcal{A}_0\left(B^\times \setminus \widehat{B}^\times, \omega\right)$, denote by

$$\Omega_{\pi,D,\mathfrak{c}} := \left\{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})} : K \in \mathfrak{S}_{D,U,\mathfrak{c}}, L(\pi, \chi, \frac{1}{2}) \neq 0\right\},$$

we will consider the proportion of $\#\Omega_{\pi,D,\mathfrak{c}}$ in the whole $\widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}$. Mazur conjectured, in a quite special situation, that this proportion is exactly 1, namely, almost all such central values $L(\pi, \chi, \frac{1}{2})$'s are nonzero (cf. [44]). This conjecture predicts that the size of the Mordell-Weil group $E(K_\infty)$ is controlled by the prime factorization of N in K, where E is a (modular) elliptic curve over $\mathbb{Q}$ of conductor N, K is a

CM-field of discriminant D such that $(D,N)=1$, and K_∞ is the anti-cyclotomic $\mathbb{Z}_p$-extension of K. There are various results on the CM case of this conjecture, such as [21] or [53]. The generic case, which occurs either when E has no CM, or when the field of complex multiplications is distinct from K is handled by Vatsal, who shows that under certain conditions on E and K, Mazur's conjecture holds and the group $E(K_\infty)$ is finitely generated (cf. Theorem 1.4 of [62]).

Here we obtain a lower bound of $\#\Omega_{\pi,D,\mathfrak{c}}$ with π a (fixed) general cuspidal automorphic representation for quaternion algebras over a totally real field:

Theorem 3. *Let notations be as above, assuming (1), then there exists an absolute constant δ such that for any small $\varepsilon>0$, we have,*

$$\lim_{D\to\infty}\frac{\#\Omega_{\pi,D,\mathfrak{c}}}{|D|^{\frac{1}{2}+\delta-\varepsilon}}=+\infty,\quad i.e.\ \ \#\Omega_{\pi,D,\mathfrak{c}}\gg_\varepsilon |D|^{\frac{1}{2}+\delta-\varepsilon}. \tag{4}$$

Remark. Clearly, under the Grand Riemann Hypothesis, we can take $\delta=\frac{1}{2}$, giving the exponent $1-\varepsilon$ for any $\varepsilon>0$. While, as for the unconditional case, one can only expect the exponent to be $\frac{7}{12}-\varepsilon$ by a private communication with Han Wu.

Remark. The precise statement of Mazur's Conjecture is as follows:

Let g denote a cuspidal newform of weight 2 on the group $\Gamma_0(N)$, where χ_D is the quadratic Dirichlet character of K, a CM-field over $\mathbb{Q}$ of discriminant D. Suppose that $(N,D)=1$ and $\chi_D(N)=-1$. Then $L(g,\chi,\frac{1}{2})\neq 0$ for all but finitely many χ of conductor p^n.

It is interesting to make the exponent in Theorem 1 explicit. However, it seems hopeless to achieve it by the current methods. Thus we prepare for this goal by considering the subconvexity problem for $GL_2\times GL_1$. Due to (36) and Wu's result (cf. Theorem 1 of [68]), one sees that polynomial-dependence of the analytic conductor is a crucial step for making the exponent δ in (11) explicit, which assures the corresponding exponent in Theorem 1 explicit.

So we shall make Michel-Venkatesh's proof of their subconvex bound for $GL_2\times GL_1$ explicit. As we will see below, both tools from analytic number theory and from representation theory are available. That means that there is essentially no loss working in adelic language.

Let F be a number field and $\mathbb{A}$ be its adele ring. Here we highlight the dependence on π, at a cost of getting an implicit exponent. Precisely, we aim to show that

Theorem 4. *For any cuspidal automorphic representation π of $GL_2(\mathbb{A})$ and arbitrary Hecke character χ, we have*

$$L\left(\pi\otimes\chi,\frac{1}{2}\right)\ll_{F,\varepsilon} C(\pi)^d C(\chi)^{\frac{1}{2}-\delta+\varepsilon}, \tag{5}$$

where d and $\delta>0$ are absolute constants.

Remark. Actually, the subconvex bound also holds when π is Eisenstein series (cf. Theorem 5.1 in [47]). Thus we can take $\pi=1\boxplus 1$ and get the subconvex bound for $L(\chi,\frac{1}{2})$.

Remark. Without considering the explicit dependence on π, Wu gives the subconvexity

$$L\left(\pi\otimes\chi,\frac{1}{2}\right)\ll_{F,\varepsilon,\pi} C(\chi)^{\frac{25}{256}+\varepsilon},$$

bound in [68]. However, one should not expect a good dependence using the method in [68], as was also discussed in Section 5.1 of [69]. In fact, without using relative trace formulae one could not hope the polynomial dependence. In order to obtain such a good dependence, the choice of test vectors must be altered at archimedean places. One may see the local reason from Section 3.2 of [69] and the global one from [24].

So essentially, we give a verification of the claim that the implied constant depends polynomially on the analytic conductor of π in [47], where the constant δ is implicit due to (17), without proof. To achieve that, we use Michel and Venkatesh's method and make every dependence on π effective.

1.2. **Plan of the paper.** This paper is essentially composed of five main parts. The first part contains several main results of this paper. In the second part, we give proofs of these main theorems. As pointed in this section, in order to generalize the equidistrbution laws of Heegner points to totally real field, we need to use representation theory, i.e. we need to correspond a cusp Hecke-eigen modular form to a cuspidal automorphic representation, which is the reason we consider the Dictionary Theorem 12 in the third part with details provided. This section aims to help better connect classical modular forms with modern theory of automorphic representation. Moreover, it serves as a good way to help generalize Michel's result on equidistribution of Heegner points over $\mathbb{Q}$ to that over any totally real number field, because in Michel's case, a group of modular forms can be found as a base of the underlying space, while we use representation theory if the base field in not $\mathbb{Q}$. As we will see, the subconvexity problem is quite widely used in number theory and arithmetic geometry, so we give an introduction to this problem in the forth part, which is an self-contained appendix introducing some basic background on the subconvexity problem (ScP) and also several other important applications are included.

2. Proof of the main results

2.1. **Equidistribution of Heegner points.** We will, in this section, describe the equidistribution of Heegner points. We should caution the reader that the Heegner points considered here are not Heegner points in the classical sense. Particularly, they do not give rise to a family of points on the Jacobian of a modular curve. The points we consider are sometimes referred to as Gross points (cf. [22], or [3]). Also, the equidistribution we consider here is of 'horizontal' type(cf. [33]). To start with, we briefly review some classical results on this subject (cf. [45] and [27]).

2.1.1. *Equidistribution of Heegner points over* $\mathbb{Q}$. Let q be a square-free number and fix a factorization $q = q_+q_-$ with $\mu(q_-) = -1$. Let B_{q_-} be the (unique) quaternion algebra ramified ar the prime divisors of q_- and ∞. Fix $R = R_{q_+,q_-}$ an Eichler order as in [3]. Denote by

$$X_{q_+,q_-} := B^\times \setminus \widehat{B}^\times / \widehat{R}^\times_{q_+,q_-},$$

which is a finite set. Then $Pic(X_{q_+,q_-})$ admits a pair $\langle , \rangle$ such that we can define a probability measure

$$\mu_{q_+,q_-} = \frac{1}{Vol\left(X_{q_+,q_-}\right)} \frac{dxdy}{y^2} \quad \text{with} \quad Vol\left(X_{q_+,q_-}\right) := \int_{X_{q_+,q_-}} \frac{dxdy}{y^2}.$$

Fix an embedding $\varsigma : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $K \subset \bar{\mathbb{Q}}$ be an imaginary quadratic number field of discriminant $-D$ such that $(q, D) = 1$, and denote by $\mathcal{O}_K$ the ring of its integers. Let $Pic(O_K)$ be the Picard group of O_K. In fact, the Heegner point (or, to be precise, the Gross points) are defined over the Hilbert class field H_K of K, and admits a natural free action of $Gal(H_K/K)$ corresponds to that of $Pic(O_K) \cong Gal(H_K/K)$ via the Artin map.

Moreover, to make sure that the set of Heegner points is nonempty, we introduce the *Heegner Condition* (cf. Lemma 2.1 of [3]):

Every prime factor of q_- is innert in K and every prime factor of q^+ splits in K.

Then we have

Theorem 5. *For any continuous function $V : X_{q_+,q_-} \hookrightarrow \mathbb{C}$, then for any imaginary quadratic field K with discriminant $-D$ satisfying the Heegner Condition, for any subgroup $G \leq Pic(O_K)$ such that $[Pic(O_K) : G] \leq |D|^{\frac{1}{5298}}$, and for any Heegner point ξ, one has*

$$\lim_{|D|\to+\infty} \left| \frac{1}{|G|} \sum_{\sigma\in G} V(\xi^\sigma) - \int_{X_{q_+,q_-}} V(x) d\mu_{q_+,q_-}(x) \right| = 0.$$

We, in the following, will aim to generalize this result to obtain the equidistribution laes of general Gross points, i.e. defined over a totally real field. To achieve that, we will use theory of automorphic representation rather than modular forms.

2.1.2. *Auxiliary results.* Let B be a quaternion algebra over a totally real number field F and π a cuspidal automorphic representation on $B_{\mathbb{A}}^\times$ with central character ω. Let K/F be a totally imaginary quadratic field extension with associated quadratic character η on $\mathbb{A}^\times$ and χ be a Hecke character on $K_{\mathbb{A}}^\times$.

Assume that $\omega \cdot \chi \mid_{\mathbb{A}^\times} = 1$, and for any $\nu \in M(F)$,

$$\epsilon(\pi_\nu, \chi_\nu, \frac{1}{2}) = \chi_\nu \eta_\nu(-1)\epsilon(B_\nu).$$

Define the Petersson pairing on $\pi \otimes \widetilde{\pi}$ by

$$\langle f_1, f_2 \rangle_{Pet} := \int_{B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times} f_1(g) f_2(g) dg$$

with the Tamagawa measure such that $\int_{B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times} dg = 2$. Let P_χ denote the period functional on π:

$$P_\chi(f) := \int_{K^\times \mathbb{A}^\times \backslash K_{\mathbb{A}}^\times} f(t)\chi(t) dt$$

for all $f \in \pi$. Then we have the Waldspurger's period formula:

Theorem 6 ([65])**.** *Let notations be as above, then for any pure tensor $f \in \pi$, $f_2 \in \widetilde{\pi}$ with $\langle f_1, f_2 \rangle_{Pet} \neq 0$, we have*

$$\frac{P_\chi(f_1) P_{\chi^{-1}}(f_2)}{\langle f_1, f_2 \rangle_{Pet}} = \frac{L(\pi, \chi, \frac{1}{2})}{2L(1, \pi, Ad) L(2, 1_F)^{-1}} \prod_\nu \beta\left(f_{1,\nu}, f_{2,\nu}\right),$$

where $L(1, \pi, Ad)$ is defined using the Jacquet-Langlands lifting of π, which is determined by Theorem 16.1 in [34]. Here, for any $\nu \in M(F)$, let $\langle , \rangle_\nu : \pi_\nu \times \widetilde{\pi}_\nu \to \mathbb{C}$ be a nontrivial invariant pairing, then

$$\beta\left(f_{1,\nu}, f_{2,\nu}\right) = \frac{L(1, \eta_\nu) L(1, \pi_\nu, Ad)}{L(\pi_\nu, \chi_\nu, \frac{1}{2}) L(2, 1_F)} \int_{K_\nu^\times / F_\nu^\times} \frac{\langle \pi_\nu(t_\nu) f_1, f_2 \rangle_\nu}{\langle f_1, f_2 \rangle} \chi(t_\nu) dt_\nu,$$

where local Haar measure dt_ν are chosen so that $\otimes_\nu dt_\nu = dt$ is the Haar measure on $K_{\mathbb{A}}^\times / \mathbb{A}^\times$ in the definitions of P_χ and $P_{\chi^{-1}}$ and $\int_{K^\times \backslash K_{\mathbb{A}}^\times / \mathbb{A}^\times} dt = 2L(1, \eta)$.

Remark. This remarkable result, which admits significant applications since it connects objects between number theory and arithmetic geometry, was first obtained by Waldspurger in 1985, when $\omega_\pi = 1$ (also cf. [71], Theorem 1.4, in general). An explicit formulation of this connection was given in [8]. If either $\omega \cdot \chi \mid_{\mathbb{A}^\times} = 1$ or $\epsilon(\pi_\nu, \chi_\nu, \frac{1}{2}) = 1$, then $P_\chi(f) = 0$ automatically. Also, the β-factor is equals to 1 for all but finitely many places $\nu \in M(F)$.

Remark. We chose local Haar measures here explicitly as in Section 2 of [8], then

$$Vol\left(K_\nu^\times/F_\nu^\times\right)=\begin{cases}2, & \text{if } F=\mathbb{R} \text{ and } K=\mathbb{C};\\ |\delta|^{\frac{1}{2}}, & \text{if } K_\nu/F_\nu \text{ is unramified;}\\ 2\|D\delta\|_\nu^{\frac{1}{2}}, & \text{if } K_\nu/F_\nu \text{ is ramified,}\end{cases} \tag{6}$$

where $\delta\in F_\nu^\times$ such that $\delta\mathcal{O}_\nu$ is the different of $F_\nu/\mathbb{Q}_\nu$ if $\nu<\infty$ and $\delta=1$ for $\nu\mid\infty$. We will use (6) to bound the β-product later.

Lemma 7. *Let $\pi\in\mathcal{A}_0\left(B^\times\setminus\widehat{B}^\times,\omega\right)$ and $f\in\pi$ be any nonzero pure tensor. Then for $K\in\mathfrak{S}_{D,U,\mathfrak{c}}$ and $\chi\in\widehat{Pic(\mathcal{O}_{K,\mathfrak{c}})}$ such that $\omega\cdot\chi\mid_{\mathbb{A}^\times}=1$, we have,*

$$\prod_{\nu\in M(F)}\beta\left(f_\nu,f_\nu\right)\ll_{\varepsilon,F,\pi}|D|^{-\frac{1}{2}+\varepsilon}. \tag{7}$$

Proof. Note that $\beta\left(f_\nu,f_\nu\right)=1$ if $B_\nu=M_2(F_\nu)$, K_ν/F_ν is an unramified, both π_ν and χ_ν are unramified, dt_ν is normalized such that (6) holds, and f_ν is spherical.

Thus we define $M_{f,K,\chi}(F):=\{\nu\in M(F):\nu<\infty \text{ and } \beta\left(f_\nu,f_\nu\right)\neq1)\}$, and denote by

$$\begin{aligned}M_1(F)&:=\{\nu\in M_{f,K,\chi}(F):\pi_\nu \text{ is ramified or } f_\nu \text{ is not spherical}\}\,;\\ M_2(F)&:=\{\nu\in M_{f,K,\chi}(F)\setminus M_1(F):K_\nu \text{ is non-split}\}\,;\\ M_3(F)&:=\{\nu\in M_{f,K,\chi}(F)\setminus M_1(F):K_\nu \text{ is split}\}\,.\end{aligned}$$

Obviously, we have $\#M_1(F)\ll_{\pi,f}1$ and there are at most $8^{\#M_1(F)}$ local K_ν/F_ν such that $\nu\in M_1(F)$. Hence, the character χ_ν with $\nu\in M_1(F)$ are of at most $O_{\pi,f}(1)$. By taking maximum, one has

$$\begin{aligned}&\left|\prod_{\nu\in M_1(F)}\int_{K_\nu^\times/F_\nu^\times}\frac{\langle\pi_\nu(t_\nu)f,f\rangle_\nu}{\langle f,f\rangle}\chi_\nu(t_\nu)dt_\nu\right|\\ \le&\prod_{\nu\in M_1(F)}\max_{\substack{\nu\in M_1(F)\\ \chi_\nu}}\left|\int_{K_\nu^\times/F_\nu^\times}\frac{\langle\pi_\nu(t_\nu)f,f\rangle_\nu}{\langle f,f\rangle}\chi_\nu(t_\nu)dt_\nu\right|\ll_{\pi,f}1.\end{aligned}$$

For $\nu\in M_2(F)$, $K_\nu^\times$ is a field. We consider the Harish-Chandra function Ξ_ν, which leads to

$$\frac{\langle\pi_\nu(t_\nu)f,f\rangle_\nu}{\langle f,f\rangle}=\Xi_\nu(t_\nu).$$

Since Ξ_ν is bi-$\mathcal{O}_{K,\nu}$-invariant, we can use p-adic Cartan decomposition (cf. Proposition 4.6.2 in [6]) and reduce to compute $\Xi_\nu\left(a(\varpi_\nu^n)\right)$, where $n\ge0$ and ϖ_ν is a uniformazier for K_ν. Then by Macdonald formula (cf. [43]), is that

$$\Xi_\nu\left(a(\varpi_\nu^n)\right)=q_\nu^{-\frac{n}{2}}\left(1+n\frac{1-q_\nu^{-1}}{1+q_\nu^{-1}}\right),\quad \text{for } n\ge0.$$

Therefore one has $\frac{\langle\pi_\nu(t_\nu)f,f\rangle_\nu}{\langle f,f\rangle}\ll1$ for any $\nu\in M_2(F)$, $t_\nu\in K_\nu^\times$, which leads to

$$\int_{K_\nu^\times/F_\nu^\times}\frac{\langle\pi_\nu(t_\nu)f,f\rangle_\nu}{\langle f,f\rangle}\chi(t_\nu)dt_\nu\ll Vol\left(K_\nu^\times/F_\nu^\times\right).$$

For $\nu \in M_3(F)$, $K_\nu^\times/F_\nu^\times \backsimeq F_\nu^\times$ and we fix such an isomorphism $t_\nu \mapsto x_\nu$. Then for $\nu \in M_3(F)$ one has

$$\int_{K_\nu^\times/F_\nu^\times} \frac{\langle \pi_\nu(t_\nu)f, f\rangle_\nu}{\langle f, f\rangle}\chi(t_\nu)dt_\nu = \int_{F_\nu^\times} \Xi_\nu(x_\nu)\chi(x_\nu)dx_\nu$$
$$\ll \sum_{n=0}^{\infty} q_\nu^{-\frac{n}{2}}\left(1+n\frac{1-q_\nu^{-1}}{1+q_\nu^{-1}}\right) \ll 1.$$

On the other hand, for any $n \in \mathbb{N}$, denote by $\omega(n)$ the number of distinct prime factors of n, and $Ker(n)$ the largest square-free factor of n. Then $\omega(n) = \omega(Ker(n))$. Moreover, let $\sigma(n) := \sum_{d|n} 1$, then $\sigma(n) = 2^{\omega(n)}$ if $\mu(n)^2 = 1$. By [49], one has, for $n \geq 3$,

$$\frac{\log \sigma(n)}{\log 2} \leq \frac{1.54 \log n}{\log \log n}, \tag{8}$$

Then we have by (8)

$$\begin{aligned}\# M_{f,D,\chi}(F) &\ll_\pi [F:\mathbb{Q}]\left(\omega(D) + \omega\left(C(\chi)\right)\right)\\ &\leq_{\pi,F} \frac{\log \sigma\left(Ker(D)\right)}{\log 2} + \frac{\log \sigma\left(Ker\left(C(\chi)\right)\right)}{\log 2}\\ &\ll_{\pi,F} \frac{\log Ker(D)}{\log\log Ker(D)} + \frac{\log Ker\left(C(\chi)\right)}{\log\log Ker\left(C(\chi)\right)} \ll_{\pi,F} \frac{\log D}{\log\log D},\end{aligned}$$

since $C(\chi) \ll |D|^{\frac{1}{2}} \log D$, which comes from Class Number Formula (cf. [48]).

Since $\left|L(\pi_\nu, \chi_\nu, \frac{1}{2})\right|^{-1} \leq_{\pi_\nu} 1 + q_\nu^{-\frac{1}{2}}$, there exist constants $C_1 = C_1(\pi, f)$ and $C_2 = C_2(\pi, F)$ such that

$$|\beta\left(f_\nu, f_\nu\right)| \leq C_1 Vol\left(K_\nu^\times/F_\nu^\times\right) \quad \text{for any } \nu \in M_2(F),$$
$$|\beta\left(f_\nu, f_\nu\right)| \leq C_1 \quad \text{for any } \nu \in M_1(F) \cup M_3(F), \quad \text{and} \quad \# M_{f,D,\chi}(F) \leq \frac{C_2 \log D}{\log\log D}.$$

Combining the above results and (6) we obtain that

$$\begin{aligned}\left|\prod_{\nu \in M_{f,K,\chi}(F)} \beta\left(f_\nu, f_\nu\right)\right| &\leq \prod_{\nu \in M_{f,K,\chi}(F)} C_1 \prod_{\nu \in M_2(F)} Vol\left(K_\nu^\times/F_\nu^\times\right)\\ &\ll_F C_1^{\# M_{f,D,\chi}(F)} \prod_{\nu|D} Vol\left(K_\nu^\times/F_\nu^\times\right)\\ &\leq C_1^{\frac{C_2 \log D}{\log\log D}} \|D\|_{\mathbb{A}}^{\frac{1}{2}} = |D|^{-\frac{1}{2} + \frac{C_2 \log C_1}{\log\log D}}.\end{aligned}$$

Also, note that there are only finitely many (relative to the degree of $F/\mathbb{Q}$) types of χ_ν for $\nu \mid \infty$. Hence $\left|\prod_{\nu|\infty} \beta\left(f_\nu, f_\nu\right)\right| \ll_{F,\pi} 1$, proving Lemma 7 when combining with the above estimates. □

Denote by $\mathbb{Z}[X]^0 := \mathrm{Pic}(X)^0$, $\mathbb{C}[X]^0 = \mathbb{Z}[X]^0 \otimes_{\mathbb{Z}} \mathbb{C}$. For any $f \in \mathbb{C}[X]^0$, $\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}$, define

$$P_\chi^0(f) := \sum_{\sigma \in Pic(\mathcal{O}_{K,\mathfrak{c}})} \chi^{-1}(\sigma) f(\xi_K(\sigma)).$$

Note that, we have, by Lemma 2.3 of [8],

$$P_\chi^0(f) \ll_{U,\mathfrak{c}} |D|^{\frac{1}{2}} \|P_\chi(f)\|. \tag{9}$$

Thus by combining (9) with (7) and Theorem 6, one has

Corollary 8. *Let notations be as above, then for any $f \in \pi$ we have*

$$\left\|P_\chi^0(f)\right\|^2 \ll_{\varepsilon,\pi,F} |D|^{\frac{1}{2}+\varepsilon} \left|L(\pi,\chi,\frac{1}{2})\right|.$$

Remark. This is the starting point of Theorem 3.

Let θ be such that no complementary series with parameter larger than θ appear as a component of a cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$, where F is any number field. Then we have

Theorem 9 ([68], Thm 1.1)**.** *For any cuspidal automorphic representation π of $GL_2(\mathbb{A}_F)$ and any Hecke character χ with analytic conductor $C(\chi)$, we have*

$$L(\pi\otimes\chi,\frac{1}{2}) \ll_{F,\varepsilon,\pi} (C(\chi))^{\frac{1}{2}-\delta+\varepsilon}, \quad \text{for any } \ \varepsilon>0, \tag{10}$$

with $\delta = \frac{1-2\theta}{8}$.

Remark. Note that under the Ramanujan-Petersson conjecture, i.e. $\theta = 0$, we have $\delta = \frac{1}{8}$. The best known result on θ is that we can take $\theta = \frac{7}{64}$ (cf. [2]), which gives $\delta = \frac{25}{256}$.

For general automorphic representations of $GL_2(\mathbb{A}_F)$, where F is an arbitrary number field, we have the following subconvex bound:

Theorem 10 ([47], Thm 1.2)**.** *Let F be a number field, then there is an absolute constant $\delta > 0$ such that for π_1, π_2 automorphic representations on $GL_2(\mathbb{A}_F)$ we have:*

$$L(\pi_1\otimes\pi_2) \ll_{F,\pi_2} C(\pi_1)^{\frac{1}{2}-\delta}, \tag{11}$$

where the implied O-constant depends polynomially on Disc(F) (for F varying over fields of some fixed degree) and on $C(\pi_2)$.

Write $e_i = [g_i] \in X$ for the class of the element $g_i \in \hat{B}^\times$. For each g_i, denote by $\Gamma_i = (B^\times \cap g_i U g_i^{-1})/\mathcal{O}^\times$, which is finite and denoted by ω_i its order. Let $\mathbb{Z}[X]$ be the free $\mathbb{Z}$-module (of rank n) of formal sums $\sum_i a_i e_i$. There is a height pairing on $\mathbb{Z}[X]\times\mathbb{Z}[X]$ defined by

$$\langle \sum_i a_i e_i, \sum_j b_j e_j\rangle = \sum_i a_i b_i \omega_i.$$

Thus we have a possibility measure on X:

$$\mu_U(\{e_i\}) = \frac{\omega_i^{-1}}{\sum_{i=1}^n \omega_i^{-1}}. \tag{12}$$

2.1.3. *Proof of Theorem 1.* Now we can prove (3) with the above prep work.

Proof. To begin with, let us define

$$f_0 := \left(\sum_{i=1}^n \omega_i^{-1}\right)^{-\frac{1}{2}} \sum_{i=1}^n \omega_i^{-1} e_i,$$

then we have

$$\mathbb{C}[X] = \mathbb{C}f_0 \oplus \mathbb{C}[X]^0.$$

Note that $\mathbb{C}[X]^0 \subseteq \mathcal{A}_0\left(B^\times\backslash\widehat{B}^\times/U\right) \subseteq \mathcal{A}_0\left(B^\times\backslash\widehat{B}^\times\right)$. By spectral decomposition we have

$$\mathcal{A}_0\left(B^\times\backslash\widehat{B}^\times\right) = \oplus_{\omega\in\widehat{F}^\times}\mathcal{A}_0\left(B^\times\backslash\widehat{B}^\times,\omega\right).$$

Thus by multiplicity one theorem we can write

$$\mathbb{C}[X]^0 = \oplus_j \pi_j^U, \tag{13}$$

where π_j's are cuspidal representations of $B^\times\backslash\widehat{B}^\times$ invariant under the action of U with some central character ω_j.

For $\pi_j' s$ appearing in (13), let $f_{j_k}\in\pi_j$ be pure tensors forming a basis of π_j, thus $\{f_{j_k}\}_{j,k}$ is a basis of $\mathbb{C}[X]^0$. Note that $\#\{f_{j_k}\}_{j,k}<\infty$, we can write

$$e_i=\langle e_i,f_0\rangle f_0+\sum_j\sum_k a_{j_k}f_{j_k},$$

with such $a_i\in\mathbb{C}$ determined by e_i uniquely. Thus we have

$$\begin{aligned}\omega_i\frac{\#\{\sigma\in G,\xi_K(\sigma)=e_i\}}{|G_K|}&=\langle e_i,\frac{1}{|G_K|}\sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\xi_K(\sigma)\rangle\\&=\left(\sum_{i=1}^n\omega_i^{-1}\right)^{-1}+\sum_j\sum_k a_{j_k}\langle f_{j_k},\frac{1}{|G_K|}\sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\xi_K(\sigma)\rangle.\end{aligned}$$

By Fourier inversion, for arbitrary subgroup $G\leq\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})$ we have

$$\begin{aligned}\frac{1}{|G|}\sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\xi(\sigma)&=\frac{1}{|\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})|}\sum_{\chi\in\widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}}\left(\frac{1}{|G|}\sum_{\sigma'\in G}\chi(\sigma')\right)\sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\chi^{-1}(\sigma)\xi(\sigma)\\&=\frac{1}{|\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})|}\sum_{\chi\in\widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}}\left(\frac{1}{|G|}\sum_{\sigma'\in G}\chi(\sigma')\right)P_\chi\\&=\frac{1}{|\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})|}\sum_{\substack{\chi\in\widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\\\chi|_G=1}}P_\chi,\end{aligned}$$

where

$$P_\chi:=\sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\chi^{-1}(\sigma)\xi_K(\sigma).$$

So we have

$$\begin{aligned}\langle f_{j_k},\frac{1}{|G_K|}\sum_{\sigma\in\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\xi_K(\sigma)\rangle&=\frac{1}{|\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})|}\langle f_{j_k},\sum_{\substack{\chi\in\widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})}\\\chi|_{G_K}=1}}P_\chi\rangle\\&\leq\frac{1}{|G_K|}\max_{\substack{\chi\in\widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}\\\chi|_{G_K}=1}}\left|P_\chi^0(f_{j_k})\right|.\end{aligned}$$

For any pair (j,k), suppose the central character of f_{j_k} is ω_j. Then if $\omega_j\cdot\chi\mid_{\mathbb{A}^\times}\neq1$ we must have $\left|P_\chi^0(f_{j_k})\right|=0$ since $Pic(\mathcal{O}_{K,\mathfrak{c}})=K^\times\backslash\widehat{K}^\times/\mathcal{O}_{K,\mathfrak{c}}$ and a character is nontrivial on a group, then the sum of its values over the group must be zero. Also, if $\epsilon(\pi_\nu,\chi_\nu,\frac{1}{2})=-1$, then by a direct computation, $P_\chi^0(f_{j_k})=0$. Henceforth, we may assume that $\omega_j\cdot\chi\mid_{\mathbb{A}^\times}=1$ and $\epsilon(\pi_\nu,\chi_\nu,\frac{1}{2})=1$, when Theorem 6 holds.

If χ factors through the norm $N_{K/F}$ (i.e. χ is of order 2), Θ_χ is an Eisenstein series; in this situation we have

$$L\left(\pi,\chi,\frac{1}{2}\right)=L\left(\pi^{JL}\otimes\Theta_\chi,\frac{1}{2}\right)=L\left(\pi^{JL}\otimes\chi_1,\frac{1}{2}\right)\cdot L\left(\pi^{JL}\otimes\chi_2,\frac{1}{2}\right),$$

where χ_1, χ_2 are Hecke characters such that $\chi_1\chi_2=\chi$. Otherwise, Θ_χ is cuspidal.

Hence, either by (9) in the former case or by (11) in the latter case, together with Theorem 6, (7) and (10) we have, for any $\chi\in\widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})}$,

$$\|P_\chi^0f_{j_k}\|^2\ll_{\varepsilon,\pi_j,f_{j_k}}|D|^{\frac{1}{2}+\varepsilon}\left|L(\pi_j,\chi,\frac{1}{2})\right|\ll_{\pi_j,f_{j_k},\varepsilon}C(\chi)^{\frac{1}{2}-\delta+\varepsilon}\ll_{\varepsilon,U}|D|^{1-\delta+2\varepsilon},$$

which deduces that

$$\max_{\substack{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})} \\ \chi|_{G_K}=1}} \|P_\chi^0(f_{j_k})\| \ll_{\varepsilon,U} |D|^{\frac{1-\delta}{2}+\varepsilon}, \quad \text{for any pair } (j,k) \text{ above}, \tag{14}$$

where δ is an absolute positive constant defined in (10) or in (11) respectively.

By Siegel's theorem (cf. [60]) and Class Number Formula, we have

$$|D|^{\frac{1}{2}-\varepsilon} \ll_\varepsilon |\mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})| \ll |D|^{\frac{1}{2}} \log D.$$

So assuming (14), noting that there are only finitely many such terms, then we have

$$\left| \sum_j \sum_k a_{j_k} \langle f_{j_k}, \frac{1}{|G_K|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{K,\mathfrak{c}})} \xi_K(\sigma) \rangle \right| \ll_{\varepsilon,U} \frac{|D|^{\frac{1-\delta}{2}+\varepsilon}}{|G_K|} \ll_U |D|^{-\delta'},$$

where $\delta' = \frac{\delta}{2}$ with δ the minimal (absolute) exponents between that in (10) and (11). The theorem thus follows. □

Remark. If we take $U = \widehat{R}_{q_+,q_-}^\times$ the standard Eichler order as in [3] and $\mathfrak{c}$ trivial, then Heegner condition affirms that (1) holds. By Theorem 1, the corresponding equidistribution laws hold and they are exactly those in [45] and [27], where a different version of Waldspurger formula is used (from [73]) instead since the test vectors can be chosen explicitly as new forms due to Atkin-Lehner theory (cf. [39]).

Also, we can consider the case where $q_- = 1$, i.e. the space X_{1,q_+} is the usual modular curve $X_0(q_+)$. The proof is quite similar, so we just sketch it here. In fact, the treatment of the discrete spectrum is essentially the same as that given above except that the Jacquet-Langlands correspondence is the identity. However, note that we can take π_2 to be an Eisenstein series, so the contribution from the continuous spectrum can be well bounded via (cf. [23], p.248)

$$\left| \sum_{\sigma \in Pic(\mathcal{O})} \chi^{-1}(\sigma) E\left(\xi(\sigma), \frac{1}{2}+it\right) \right|^2 = \frac{\#\mathcal{O}_F^\times \sqrt{D}}{4} \left| L\left(\Theta_\chi \otimes |.|^{it}, \frac{1}{2}\right) \right|^2.$$

Remark. We should note that (1) is a quite general condition since we do not require the embedding to be optimal. So we shall only consider which kind of CM-extension K/F admits such an embedding defined in $\mathfrak{S}_{D,U,\mathfrak{c}}$. Thus in the last part of this section, we try to give an approach to verify when (1) holds. Let notations be as before Theorem 1. Let R_K (resp. R) be an $\mathcal{O}_F$-order of K (resp. B), we say an embedding $\xi : K \hookrightarrow B$ is a maximal embedding of R_K into R if

$$\xi(K) \cap R = \xi(R_K).$$

Given such an embedding, then for any $\mathfrak{c}$ and open compact subgroup $U \subset \widehat{B}^\times$ such that

$$\mathcal{O}_{K,\mathfrak{c}} \subseteq \widehat{R}_K^\times \quad \text{and} \quad \widehat{R}^\times \subseteq U,$$

$\mathfrak{S}_{D,U,\mathfrak{c}}$ is well defined. If there are infinitely many such K of different relative discriminants and maximal embedding of some R_K into R, a fixed order of B, then we can take $\mathfrak{c}$ and U as above to make 1 hold. By trace formula (cf. Theorem 3.5.11 in [64]), we can reduce to (only finite) local computations, which is much easier to determine.

2.2. Results on Genelized Mazur's Conjecture.

2.2.1. *Proof of Theorem 3.* With the auxiliary results in the last subsection, we are ready to complete the proof of (4).

Proof. We shall assume

$$\lim_{|D|\to+\infty} \#\mathfrak{S}_{D,U,\mathfrak{c}} = +\infty$$

throughout this proof. Then for any $K \in \mathfrak{S}_{D,U,\mathfrak{c}}$, we have, by Corollary 8,

(15)
$$\sum_{\chi\in \widehat{Pic(\mathcal{O}_{K,\mathfrak{c}})}} \left| \sum_{\sigma\in Pic(\mathcal{O}_{K,\mathfrak{c}})} \chi^{-1}(\sigma) f(\xi_K(\sigma)) \right|^2 \ll_{\varepsilon,\pi,F} |D|^{-\frac{1}{2}+\varepsilon} \sum_{\chi\in \widehat{Pic(\mathcal{O}_{K,\mathfrak{c}})}} \left| L\left(\pi,\chi,\frac{1}{2}\right)\right|,$$

where $f \in \pi$ is any nonzero automorphic forms. By orthogonality we see the left sum is equal to $\sum_{\sigma\in Pic(\mathcal{O}_{K,\mathfrak{c}})} |f(\xi_K(\sigma))|^2$; and by (11) the sum in the right hand side is bounded from above by $|D|^{\frac{1}{2}-\delta+\varepsilon}\#\Omega_{\pi,D,\mathfrak{c}}$. Thus we have

$$\frac{1}{|Pic(\mathcal{O}_{K,\mathfrak{c}})|} \sum_{\sigma\in Pic(\mathcal{O}_{K,\mathfrak{c}})} |f(\xi_K(\sigma))|^2 \ll_{\varepsilon} \frac{|D|^{\frac{1}{2}-\delta+\varepsilon}\#\Omega_{\pi,D,\mathfrak{c}}}{|Pic(\mathcal{O}_{K,\mathfrak{c}})|}.$$

By Siegel's theorem and Class Number Formula one sees the term in the right hand side above is $O\left(\frac{\#\Omega_{\pi,D,\mathfrak{c}}}{|D|^{\frac{1}{2}+\delta-\varepsilon}}\right)$. Also, by Theorem 1 we have

$$\lim_{|D|\to+\infty} \left| \frac{1}{|Pic(\mathcal{O}_{K,\mathfrak{c}})|} \sum_{\sigma\in G_K} |f(\xi_K(\sigma))|^2 - \sum_{i=1}^{\#X} \int_X |f(x)|^2 \, d\mu_U^{(i)}(x) \right| = 0.$$

Then (4) follows. □

2.3. **Explicit Subconvex bound for $GL_2 \times GL_1$.** In this subsection, we will introduce how Ergodic principle serves as a tool to deal with ScP above. For the sake of simplicity, we just omit the related precisely definitions and proofs of Soblev norms, which can be find in the second part of [47].

2.3.1. *Notations.* Some important notations used in the following sections are listed as follows:

We shall fix a faithful representation

$$\rho : g \mapsto \begin{pmatrix} g & \\ & (g^T)^{-1} \end{pmatrix} \in SL_4, \quad \text{where } g^T \text{ denoted the transpose.}$$

For any reductive group, let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$, $M(F)$ be the set of places of F, ϖ_v a fixed uniformiser of F_v, q_v the cardinality of the corresponding residue field. Fix a basis for $\mathfrak{g}$, we regard a adjoint embedding as a map $Ad : \mathbb{G} \to GL(dim\mathfrak{g})$.

Denote $X := \mathbb{G}(F) \setminus \mathbb{G}(\mathbb{A})$, and $C^\infty(X)$ the space of smooth functions on X. Note that X can be viewed as an inverse limit of quotients of real Lie groups by some discrete subgroups, and a smooth function means one that can factor through a smooth function on one of these quotients.

Let $\mathcal{S}_d^V$ (resp. $\mathcal{S}_d^X$) be Sobolev norm on the unitary representation (resp. the space X) as defined in [47].

For $v \in M(F)$, define

$$N_v := \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} x \in F_v \right\}, \quad A_v := \left\{ a(x) = \begin{pmatrix} a(x) & \\ 0 & 1 \end{pmatrix} x \in F_v^\times \right\};$$

$\mathcal{A}_0\left(GL_2(F) \setminus GL_2(\mathbb{A}), \omega\right)$ denote the cuspidal automorphic representations, and $\mathcal{A}_0^{gen}\left(GL_2(F) \setminus GL_2(\mathbb{A}), \omega\right)$ the generic subspace.

Define $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2.3.2. *Auxiliary Lemmas.* To start with, we list the bounds for matrix coefficients, which is a consequence of Property (τ) in [9]. Let $\varsigma : \widetilde{GL_2} \longrightarrow GL_2$ be the simply connected covering of GL_2. For $f \in C^\infty(X)$, $x \in X$ set

$$\wp(f) := \int_{g \in \widetilde{GL_2}(F)\backslash \widetilde{GL_2}(\mathbb{A})} f(\varsigma(g)x)dg.$$

Take $g \in GL_2(\mathbb{A})$ and $f_1, f_2 \in C^\infty(X)$, then there exists some constant d only depending on F, such that

$$|\langle g.f_1, f_2\rangle - \langle g.\wp(f_1), g.\wp(f_2)\rangle| \ll \|Ad(g)\|^{\theta-\frac{1}{2}}\mathcal{S}_d^X(f_1)\mathcal{S}_d^X(f_2). \tag{16}$$

Lemma 11 (Quantitative Form of the Ergodic Principle.)**.** *Suppose $H \subset G(\mathbb{A})$ is noncompact, and χ: $H \longrightarrow \mathbb{C}^\times$ a unitary character. Suppose that X has finite measure, i.e. that the center of G is anisotropic. Let ν be a (possibly signed) χ-equivariant measure on X, i.e. $\nu^h = \chi(h)\nu$ for $h \in H$. Let μ be the $G(\mathbb{A})$-invariant (Haar) probability measure, and suppose that, for some $d \geq 0$, we have the majorization:*

$$|\nu|(f) \ll \mu(f) + \epsilon\mathcal{S}_d^X(f), \quad (f \geq 0)$$

Let σ be any probability measure on H. Then, for any f with $\wp(f) = 0$,

$$|\nu(f) - \delta_{\chi=1}\mu(f)| \ll \left(\epsilon\|\sigma\|_{d'}^2 + \|\sigma \star \check{\sigma}\|_{-\beta}\right)\mathcal{S}_{d'}^X(f)^2. \tag{17}$$

Here $\delta_{\chi=1}$ is 1 if χ is trivial and zero otherwise,

$$\|\sigma\|_d := \int_{h\in H} \|Ad(h)\|^d d\sigma(h),$$

and similarly for $\|\sigma \star \check{\sigma}\|$, $\check{\sigma}$ denote the pullback of σ by $g \mapsto g^{-1}$. $\beta = \frac{1}{2} - \theta$, and d' depends only on d.

Proof. For the simplicity, we may assume that $\mu(f) = 0$. Define

$$f *_\chi \sigma := \int_H h.f\chi(h)dh.$$

Thus

$$|\nu f|^2 \leq |\nu|\left(|f *_\chi \sigma|\right)^2 \leq \|f *_\chi \sigma\|^2 + \epsilon\mathcal{S}_d^X(|f *_\chi \sigma|).$$

Recall that we have for $f, h \in C^\infty(X)$, $g \in \mathbb{G}(\mathbb{A})$,

$$\mathcal{S}_d^X(g.f) \ll \|Ad(g)\|^{d'}\mathcal{S}_d^X(f), \quad \mathcal{S}_d^X(fh) \ll_d \mathcal{S}_d^X(f)\mathcal{S}_d^X(h),$$

thus

$$\begin{aligned}
|f *_\chi \sigma|^2 = \langle f *_\chi \sigma, f *_\chi \sigma\rangle &= \int_H\int_H \chi(h_1)\chi(h_2)\langle h_1.f, h_2.f\rangle d\sigma(h_1)d\sigma(h_2) \\
&\ll \int_H d\sigma(h_1)\int_H \left|\langle h_2^{-1}h_1.f, f\rangle\right| d\sigma(h_2) \\
&\ll \int_H d\sigma(h_1)\int_H \left|Ad(h_2^{-1}h_1)\right|^{-\beta} d\sigma(h_2)\mathcal{S}_d^X(f)^2 \\
&= \mathcal{S}_d^X(f)^2\int_H |Ad(h)|^{-\beta} d\sigma\star\check{\sigma}(h) = \mathcal{S}_d^X(f)^2\|\sigma\star\check{\sigma}\|_{-\beta}.
\end{aligned}$$

Then the lemma follows from

$$\mathcal{S}_d^X(|f *_\chi \sigma|) \ll \int_H \|Ad(g)\|^{d'} d\sigma(h)\mathcal{S}_{d'}^X(f).$$

□

Remark. We will, in the following section, for $T=(T_v)\in\mathbb{A}$ take $\sigma=\sigma_T$ to be the average of Dirac measures:

$$\sigma=\frac{1}{M^2}\sum_{v,v'\in I_K}\delta_{a(\varpi_v\varpi_{v'}^{-1})}, \tag{18}$$

where $I_K:=\{v\in M(F):q_v\sim K, T_v=0\}\,,\quad M\ll\frac{K}{\log K}$.

To prove Theorem 4, we need some basic knowledge on Hecke-Langlands functional

$$l^\chi(\phi):=\int_{\mathbb{A}^\times/F^\times}\phi(a(y))\chi(y)d^\times y,$$

where $\phi\in\pi\subseteq\mathcal{A}_0^{gen}(GL_2(F)\setminus GL_2(\mathbb{A}),\omega)$, χ is a Hecke character. Consider the Whittaker intertwiner $\phi\mapsto\left(W_\phi:g\mapsto\int_{\mathbb{A}/F}\phi(n(x)g)\psi(-x)dx\right)$, if ϕ is a pure tensor, i.e. W_ϕ factorizes, then by

$$\phi(g)=\sum_{t\in F^\times}W_\phi\left(a(t)g\right),$$

we can see that for $\Re(s)>1+\theta$,

$$\zeta(s,\phi,\chi):=\int_{\mathbb{A}^\times/F^\times}\phi(a(x))\chi(x)|x|^{s-\frac{1}{2}}d^\times x=\prod_{v\in M(F)}\zeta(s,W_{\phi,v},\chi_v,\psi_v),$$

where

$$\zeta(s,W_{\phi,v},\chi_v,\psi_v):=\int_{F_v^\times}W_{\phi,v}\left(a(x)\right)\chi(x)|x|^{s-\frac{1}{2}}d^\times x.$$

The analysis of local zeta-functions shows that as $W_{\phi,v}$ varies, $\frac{\zeta(s,W_{\phi,v},\chi_v,\psi_v)}{L(\pi_v\otimes\chi_v,s)}$ can be analytically continued into an entire function, and equals 1 for almost all places, where $L(\pi_v\otimes\chi_v,s)$ is the common divisor.

Thus by the functional equation

$$\frac{\zeta(s,W_{\phi,v},\chi_v,\psi_v)}{L(\pi_v\otimes\chi_v,s)}\epsilon(s,\pi_v,\chi_v,\psi_v)=\frac{\zeta(1-s,w.W_{\phi,v},(\omega_v\chi_v)^{-1},\psi_v)}{L(\pi_v\otimes(\omega_v\chi_v)^{-1},1-s)}.$$

Then the analytic continuations and functional equation of $L(\pi\otimes\chi,s)$ follow from that of $\zeta(s,\phi,\chi)$. Thus we have that

$$L(\pi\otimes\chi,\frac{1}{2})=\zeta(\frac{1}{2},\phi,\chi)\prod_{v|\infty}\zeta(\frac{1}{2},W_{\phi,v},\chi_v,\psi_v)^{-1}\prod_{v<\infty}\frac{L(\frac{1}{2},\pi_v\otimes\chi_v)}{\zeta(\frac{1}{2},W_{\phi,v},\chi_v,\psi_v)}. \tag{19}$$

Remark. (19) is the start-point of [68], which we will talk about later.

For $T\in\mathbb{A}$ and h a compactly supported function on $\mathbb{R}_{>0}$, define

$$\mu_\chi(\phi):=\int_{F^\times\backslash\mathbb{A}^{(1)}}\phi(a(y)n(T))\chi(y)d^\times y;$$
$$\mu_\chi^g(\phi):=\mu_\chi(g.\phi);$$
$$\mu_{\chi,h}^{n(T)}(\phi):=\int_0^\infty h(t)\chi(y_t)\mu_\chi^{a(y_t)n(T)}(\phi)d^\times t.$$

2.3.3. *Proof of Theorem 4.* With the prep work above, we are ready to prove (5).

Proof. By Hecke functional (cf. [29]) we have

$$L(\pi\otimes\chi,\frac{1}{2})\ll_\varepsilon(C(\pi)Q)^\varepsilon Q^{\frac{1}{2}}\int_{F^\times\backslash\mathbb{A}^\times}\phi(a(y)n(T))\chi(y)d^\times y,$$

where ϕ is an element in π chosen as in [47], $T_v=\varpi_v^{-C(\pi_v)}$ and $Q=C(\chi)$.

Choose a $y_t \in \mathbb{A}$ such that $|y_t| = t$ for any $t > 0$. To deal with the disadvantage that $\mathbb{A}^\times/F^\times$ is noncompact, we induce a truncation function $h(t)$ as follows. For any nonnegative non-increasing smooth function $h(t)$, which takes value 1 when $t \in (0,1]$ and 0 when $t \geq 2$. For any $\kappa \in (0,1)$, define $h_\kappa(t) := h\left(\frac{t}{A}\right)$, where $A = Q^{-\kappa-1}$.

Let $f(t) := \chi(y_t)\mu_\chi^{a(y_t)n(T)}(\phi)$, then we have

$$\int_0^\infty f(t)d^\times t = \int_0^\infty \chi(y_t)d^\times t \int_{F^\times\backslash\mathbb{A}^{(1)}} a(y_t)n(T).\phi(a(y))\chi(y)d^\times y$$
$$= \int_{F^\times\backslash\mathbb{A}} \phi(a(y)n(T))\chi(y)d^\times y.$$

Thus the Mellin transform of f is

$$F(s) := \int_0^\infty t^{s-1}f(t)dt = \int_{F^\times\backslash\mathbb{A}} \phi(a(y)n(T))\chi(y)|y|^s d^\times y = l^{\chi|\cdot|^s}(n(T).\phi)$$
$$= L^{(S)}(\pi\otimes\chi, \frac{1}{2}+s)\prod_{v\in S} l^{\chi_v|\cdot|_v^s}(n(T).W_{\phi,v})\prod_{v\notin S}\frac{l^{\chi_v|\cdot|_v^s}(n(T).W_{\phi,v})}{L\left(\pi_v\otimes\chi_v, \frac{1}{2}+s\right)},$$

where $S := \{v \in M(F) : v \mid \infty \ \ or \ \ \pi_v \ \ ramified \ \ or \ \ T_v \neq 0 \ \ or \ \ \psi_v \ \ ramified\}$ by (19). Thus the product over $v \notin S$ is 1.

By the convex bound and bounds towards Ramanujan, we have

$$L^{(S)}(\pi\otimes\chi, \frac{1}{2}+s) \ll_\varepsilon (1+|s|)^2 C(\pi)^{\frac{1}{2}+\varepsilon}C(\pi)^{1+2\varepsilon}, (\Re(s) = -\varepsilon);$$

Moreover, for $v \mid \infty$, $l^{\chi_v|\cdot|_v^s}(n(T).W_{\phi,v}) \ll_\varepsilon C(\pi_v)^{\frac{1}{2}+\varepsilon}$ by Lemma 3.3.3 in [47]; By Lemma 4 of [63] we have for $C(\chi_v) > 0$,

$$l^{\chi_v|\cdot|_v^s}(n(T).W_{\phi,v}) \ll (1-q_v^{-1})q_v^{-\frac{C(\chi_v)}{2}};$$

and for $C(\chi_c) = 0$,

$$l^{\chi_v|\cdot|_v^s}(n(T).W_{\phi,v}) = L(\pi_v, \frac{1}{2}+s) \ll C(\pi_v)^{\frac{1}{2}+\varepsilon}(1+|s|)^{\frac{1}{2}+\varepsilon}.$$

Thus we have

$$F(s) \ll_\varepsilon (1+|s|)^3 C(\pi_v)^{1+\varepsilon}Q^{\frac{1}{2}+\varepsilon}.$$

Then by Mellin inversion we have

$$\int_0^\infty h_\kappa(f)f(t)d^\times t = \frac{1}{2\pi i}A^{\frac{1}{2}+\varepsilon}\int_{(-\frac{1}{2}-\varepsilon)} H(-s)F(s)ds \ll_{h,\varepsilon} C(\pi)^{1+\varepsilon}Q^{-\frac{\kappa}{2}+\varepsilon},$$

where $H(s)$ is the Mellin transform of h and hence is rapidly decreasing.

Similarly, we can use $h = h_{Q^{\kappa-1}} - h_{Q^{-\kappa-1}}$ to truncate the range $t \geq Q^{\kappa-1}$.

So we have achieve a weighted (truncated) integral

$$Q^{-\frac{1}{2}-\varepsilon}|L(\pi\otimes\chi, \frac{1}{2})| \ll_{\varepsilon,F} C(\pi)^\varepsilon|\mu_{\chi,h}^{n(T)}| + C(\pi)^{1+\varepsilon}Q^{-\frac{\kappa}{2}},$$

where $y_t \in \mathbb{A}^\times$ with norm t and h is essentially a smoothed characteristic function of interval $[Q^{-1-\kappa}, Q^{-1+\kappa}]$ and κ is a positive (small) constant.

Since $\mu_{\chi,h}^{n(T)}$ is χ-equivariant under the subgroup of elements of $H^{(1)}$ which commute with $n(T)$, we can reduce to the corresponding fact for $\chi = 1$, which is already known by properties of Soblev's norms. Noting that $\mu^{a(y_t)n(T)}$ is orthogonal to all one-dimensional automorphic representation on PGL_2 except the constants, then by Lemma 5.1.5 in [47] we have

$$|\mu_{1,h}^{n(T)}(f)| \leq \int h\int_{X_{PGL_2}} f + \epsilon\mathcal{S}^{X_{PGL_2}}(f),$$

with $\epsilon \ll_{\varepsilon} Q^{\varepsilon}\left(Q^{\frac{\kappa-1}{2}}+Q^{-\delta}\right)$.

Let σ be the average sum of the Dieac measure given by (18). Note that σ is not supported on $H^{(1)}$ but rather on $H^{([\frac{1}{4},4])}:=\{a(y): y\in\mathbb{A}^{\times}, |y|\in[\frac{1}{4},4]\}$.

Since $\|Ad(a(\varpi_v^{-1})a(\varpi_{v'}))\|\asymp M^2$, we have

$$\|\sigma\|_d=\int_H\|Ad(h)\|^d d\sigma(h)\ll M^{2d};$$
$$\|\sigma\star\check{\sigma}\|_{\theta}\ll(\log M)^2\left(M^{-2}+M^{-1-2\theta}+M^{-4\theta}\right).$$

Notice that $supp(\sigma)$ commutes with $n(T)$, one has $\mu_{\chi,h}(\varphi *_{\chi}\sigma)=\mu_{\chi,h*\ \eta}(\sigma)$, where η demote the average of the Dirac measures at $q_v q_{v'}^{-1}$ on $\mathbb{R}_{>0}$ for (v,v') as above. Therefore, we have by Lemma 11:

$$|\mu_{\chi,h*\eta}(\varphi)|^2\ll_{\varepsilon}\mathcal{S}_d^{X_{PGL_2}}(\phi)^2 Q^{\varepsilon}\left(M^{4d}\left(Q^{\frac{\kappa-1}{2}}+Q^{-\delta}\right)+M^{-2}+M^{-1-2\theta}+M^{-4\theta}\right),$$

where d is an absolute constant coming from the Sobolev norm

$$\mathcal{S}_d^{X_{PGL_2}}(\phi)\ll\mathcal{S}_{d'}^{\mathcal{L}^2(\mathcal{A}_0(X,\omega))}(\phi)\ll\mathcal{S}_{d'}^{\pi}(\phi)\ll_{d'}C(\pi)^{2d'},$$

here d' is only determined by d and thus d' is absolute. Take M to be a small power of Q then the subconvex bound comes from the estimate:

$$Q^{-\frac{1}{2}+\varepsilon}L(\frac{1}{2},\pi\otimes\chi)\ll_{\varepsilon,\kappa}C(\pi)^{4d'+2}\left(|\mu_{\chi,h*\eta}(\varphi)|+Q^{-\frac{\kappa}{2}}\right).$$

□

Remark. We see that, by (19) it suffices to show that

$$\zeta(\frac{1}{2},\phi,\chi)\ll_{\varepsilon,\pi}C(\chi)^{-\delta+\varepsilon}\quad\text{and}$$
$$\prod_{v|\infty}\zeta(\frac{1}{2},W_{\phi,v},\chi_v,\psi_v)^{-1}\prod_{v<\infty}\frac{L(\frac{1}{2},\pi_v\otimes\chi_v)}{\zeta(\frac{1}{2},W_{\phi,v},\chi_v,\psi_v)}\ll_{\varepsilon,F}C(\chi)^{\frac{1}{2}+\varepsilon},$$

where ϕ is a test vector chosen as in [68]. The second bound comes from a direct computation if ϕ is assigned. To obtain the first estimate, spectral decomposition is used instead of Lemma 11 to handle $|\mu_{\chi,\sigma*h}^{n(T)}(\phi)|$, which can help make the index δ in (10). Now, we will explain this succinctly as follows. Define

$$\sigma_{\chi}=\frac{1}{M^2}\sum_{v,v'\in I_K}\chi(\varpi_v\varpi_{v'}^{-1})\delta_{a(\varpi_v\varpi_{v'}^{-1})},$$

then a direct computation gives that

$$\mu_{\chi,\sigma*h}^{n(T)}(\phi)=\int_{\mathbb{A}^{\times}/F^{\times}}\sigma*h(|x|)\phi(a(x)n(T))\chi(x)d^{\times}x=\mu_{\chi,h}^{n(T)}(\sigma_{\chi}*\phi).$$

By inequality of Cauchy-Schwarz,

$$\left|\mu_{\chi,h}^{n(T)}(\sigma_{\chi}*\phi)\right|^2\le\int_{\mathbb{A}^{\times}/F^{\times}}h(|x|)d^{\times}x\int_{\mathbb{A}^{\times}/F^{\times}}h(|x|)\left|\sigma_{\chi}*\phi(a(x)n(T))\right|^2 d^{\times}x.$$

Then we spectrally decompose $|\sigma_{\chi}*\phi(a(x))|^2$ in $\mathcal{L}^2(GL_2(F)\setminus GL_2(\mathbb{A}),1)$. The contribution from cuspidal automorphic representations and Eisenstein terms are

$$\sum_{\pi'\quad\text{cuspidal}}\mu_{1,h}(n(T))P_{\pi'}\left(|\sigma_{\chi}*\phi(a(x))|^2\right)\quad\text{and}$$

$$\sum_{\xi\in\widehat{\mathbb{A}^{(1)}/F^{\times}}}\int_{-\infty}^{\infty}4\mu_{1,h}(n(T))\left(P_{\xi,i\tau}\left(|\sigma_{\chi}*\phi(a(x))|^2\right)-P_{\xi,i\tau}\left(|\sigma_{\chi}*\phi(a(x))|^2\right)_N\right)\frac{d\tau}{\pi}$$

respectively (cf. Theorem 2.17 in [68] verifies the interchanging of integrals), where $P_{\pi'}$ (resp. $P_{\xi,i\tau}$) denotes the projection on the corresponding space, i.e. π' (resp. $\pi\left(\xi|.|^{i\tau}, \xi^{-1}|.|^{-i\tau}\right)$).

If we apply the $n(T)$ translation before the projections $P_{\pi'}$ or $P_{\xi,i\tau}$, and use a more general bound concerning the decay of matric coefficients, then we come back to the setting in [47], where all the technique computations are folded in (17) above.

3. View primitive modular forms as automorphic forms

We can view a classical cusp form as an automorphic form, namely, there is a lifting from a cusp form to an automorphic form. Moreover, one can show that this lifting is unique to some extent. The precise statement is the following theorem:

Theorem 12 (The dictionary). *There is a bijection $f \leftrightarrow \pi$ between*

- $f \in \mathcal{S}_k^{prim}(N,\chi)$, *that is f is a primitive holomorphic modular form (or Maass form) of weight k, level N and nebentypus $\chi \in \widehat{\mathbb{Z}/N\mathbb{Z}}$; by primitive we mean it is a Hecke-eigen form for all Hecke operators T_p, a newform which is normalized as $\rho_1(f) = 1$.*
- $\pi \in \mathcal{A}_0\left(GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_{\mathbb{Q}}), \omega\right)$ *is a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ whose representation at infty $\pi_\infty = D_{k-1}$, of conductor N, and central character $\omega = \omega_\pi = \widetilde{\chi}$-the adelization of χ; here D_{k-1} is the discrete series representation of $GL_2(\mathbb{R})$ with lowest non-negative K-type being the character $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{-ik\theta}$, and central character $x \mapsto sgn(x)^k$.*

Moreover, for any finite order character χ of $Q^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times$ we have an equality of L-functions:

$$L(\pi(f)\otimes\chi, s) = L\left(f, \chi, s + \frac{k-1}{2}\right),$$

where the left-hand side is the standard L-function defined as in [34], and the right hand side is defined by a Dirichlet series as in [58].

Remark. This result can be substantially generalized to Hilbert modular forms (cf. [58] or [51]).

3.1. Notations and Preliminaries.

Let

$$\mathcal{H} := \{x + iy \in \mathbb{C} : y > 0\}$$

be the upper half plane and

$$GL_2(\mathbb{R})^+ := \{g \in GL_2(\mathbb{R}) : det(g) > 0\},$$

which can act on $\mathcal{H}$ by fractional linear transformations.

Lemma 13 (Iwasawa Decomposition). *For any $g \in GL_2(\mathbb{R})^+$, we can write g into the form uniquely*

$$g = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ & y^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\lambda > 0$, $y > 0$ and $\theta \in [0, 2\pi)$.

This decomposition gives a topology on $\mathcal{H}$ by the isomorphic

$$\mathcal{H} \xrightarrow{\sim} GL_2(\mathbb{R})^+/SO_2(R),$$

which implies that using (x, y, θ, λ) as coordinates on $GL_2(\mathbb{R})^+$ (subject to the condition above) gives a convenient coordinate system connecting to the standard coordinates on $\mathcal{H}$ in sense that $g.i = x + yi$.

Since the class number of $\mathbb{Q}$ is 1, we have

Lemma 14 (Strong Approximation Theorem)**.**

$$GL_2(\mathbb{A}_{\mathbb{Q}}) = GL_2(\mathbb{Q})GL_2(\mathbb{R})K_0(N), \tag{20}$$

where

$$K_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) : c \equiv 0 (mod \;\; N) \right\}.$$

The connection between $GL_2(\mathbb{R})^+$ and $GL_2(\mathbb{A}_{\mathbb{Q}})$ is the following result:

Proposition 15. *For any positive integer N, there are natural isomorphism:*

$$\Gamma_0(N) \setminus GL_2(\mathbb{A}_{\mathbb{Q}})^+ \tilde{\longrightarrow} GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_{\mathbb{Q}})/K_0(N). \tag{21}$$

Proof. As $GL_2(\mathbb{Q})$ contains elements with negative determinant, we thus have by (20) that $GL_2(\mathbb{A}_{\mathbb{Q}}) = GL_2(\mathbb{Q})GL_2(\mathbb{R})K_0(N)$.

Now consider the map given by including $GL_2(\mathbb{R})^+$ at the archimedean place and passing to the quotient, thus we have

$$GL_2(\mathbb{R})^+ \hookrightarrow GL_2(\mathbb{A}_{\mathbb{Q}}) \twoheadrightarrow GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_{\mathbb{Q}})/K_0(N).$$

Now let's decide the kernel of the homomorphism. Suppose that g_∞ and g'_∞ have the same image, which means that there exist some $\eta \in GL_2(\mathbb{Q})$ and $k_0 \in K_0(N)$ such that $g'_\infty = \eta g_\infty k_0$. We can write $\eta = \eta_\infty \eta_f$ with $\eta_\infty \in GL_2(\mathbb{R})$ and $\eta_f \in GL_2(\mathbb{A}_f)$ as η is embedded diagonally. Thus we have

$$g'_\infty = \eta_\infty g_\infty \quad \text{and} \quad \eta_f = k_0^{-1}.$$

The first means that $det(\eta_\infty) > 0$ and with the other condition it implies that $\eta_f \in GL_2(\mathbb{Q}) \cap K_0(N) = \Gamma_0(N)$. Thus g_∞ and g'_∞ differ by an element from $\Gamma_0(N)$, which proves the proposition. □

3.2. **Constructing the automorphic form associated to a modular form.** The basic idea is to take the identification of spaces in the last section and set up a correspondence between functions on $\mathcal{H}$, $GL_2(\mathbb{R})^+$ and $GL_2(\mathbb{A}_{\mathbb{Q}})$. Instead of actually working with the quotients, we will work will functions satisfying certain transformation laws.

Let $f(z)$ be a classical modular form of level N and weight $k \geq 2$ with nebentypus character χ, i.e.

- f is a holomorphic function on $\mathcal{H} := \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$;
- For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have
$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z);$$
- f is holomorphic at all cusps.

For the sake of convenience, we define the slash operator as

$$\left(f \mid_k \eta\right)(z) = (det(\eta))^{\frac{k}{2}} j(\eta, z)^{-k} f(\eta.z) = (det(\eta))^{\frac{k}{2}} (cz+d)^{-k} f(\eta.z),$$

where $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. So we have $f \mid_k \eta = \chi(d) f$.

The observation that $GL_2(\mathbb{R})^+$ acts on $\mathcal{H}$ with stabilizer $K = SO_2(\mathbb{R})$ suggests how to connect modular forms on $\mathcal{H}$ and automorphic forms on $GL_2(\mathbb{R})^+ = PGL_2(\mathbb{R})$.

Given a cusp form f, we consider the function defined on $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ by

$$\mathcal{F}(g) := \left(f \mid_k g\right)(i) = (ad-bc)^{\frac{k}{2}} (ci+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

This is the automorphic forms for $GL_2(\mathbb{R})$ associated to f.

Proposition 16. *Let $\mathcal{F}$ be defined as above, then we have*

(1) *For* $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ *it satisfies*

$$\mathcal{F}(\eta g) = \chi(d)\mathcal{F}(g).$$

(2) *For* $\kappa = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K = SO_2(\mathbb{R})$, *$\mathcal{F}$ is K-finite.*

(3) *For* $\kappa = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \in Z$, $\mathcal{F}(\eta g) = \omega(\eta)\mathcal{F}(g)$, *where*

$$\omega(\eta) = \begin{cases} 1, & if \lambda > 0; \\ \chi(-1), & otherwise. \end{cases}$$

(4) *$\mathcal{F}$ is bounded.*

(5) *$\mathcal{F}$ is an eigenfunction of Δ with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$.*

(6) $\int_0^1 f(n_t g)dt = 0$ *for any* $n_t := \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$, *any* $g \in GL_2(\mathbb{R})^+$.

Proof. (1) $\mathcal{F}(\eta g) = (f\mid_k \eta g)(i) = (f\mid_k \eta \mid_k g)(i) = \chi(d)(f\mid_k g)(i) = \chi(d)\mathcal{F}(g)$;

(2) $det(g\kappa)^{\frac{k}{2}} j(g\kappa, i)^{-k} = det(g)^{\frac{k}{2}} j(g,i)^{-k} e^{-ik\theta}$ as K is the stabilizer of i. So we have

$$\mathcal{F}(g\kappa) = det(g)^{\frac{k}{2}} j(g\kappa, i)^{-k} f(g.(\kappa.i)) = e^{-ik\theta}\},$$

which implies that $\mathcal{F}$ is K-finite.

(3) $\mathcal{F}(\eta g) = (f\mid_k \eta g)(i) = (\lambda^2)^{\frac{k}{2}} det(g)^{\frac{k}{2}} j(\eta g, i)^{-k} f(g,i) = (\lambda^2)^{\frac{k}{2}}\lambda^{-k}\mathcal{F}(g)$, which gives the conclusion.

(4) Note that $\mathfrak{Im}\left(\frac{ai+b}{ci+d}\right)^{\frac{k}{2}} = \left(\frac{det(g)}{|ci+d|^2}\right)^{\frac{k}{2}}$, so it suffices to show that $|\mathfrak{Im}(z)|^{\frac{k}{2}}|f(z)|$ is bounded. At infinity, using the q-expansion we see that for $|\mathfrak{Im}(z)| \to \infty$,

$$|f(z)| = \left|a_1 q + a_2 q^2 + ...\right| \le C' |q|,$$

where $|q| = e^{-2\pi|\mathfrak{Im}(z)|} \to 0$.

(5) We will prove this in the later section.

(6) $\int_0^1 f(z+t)dt = 0$ since f is a cusp form. By Lemma 13 we have that $g.i = x + yi$. So $\mathcal{F}(g) = (f\mid_k g)(i) = \omega(\eta)e^{ik\theta}y^{\frac{k}{2}}f(z)$, which gives that

$$\mathcal{F}(g) = \omega(\eta)e^{ik\theta}y^{\frac{k}{2}}f(z+t), \quad \text{and the conclusion follows.}$$

□

Thus $\mathcal{F} \in \mathcal{A}(\Gamma_0(N)\setminus GL_2(\mathbb{R})^+, \chi, \omega)$.

We use (21) to see a relation between the spaces and then connect functions on $GL_2(\mathbb{R})$ with functions on $GL_2(\mathbb{A}_\mathbb{Q})$ to satisfy certain transformation properties. The adelic perspective makes it much clearer that modular forms are arithmetic in mature and have the local information ar each prime.

Given a cusp form f, we define a function ϕ_f on $GL_2(\mathbb{A}_\mathbb{Q})$ as follows. Using (20) to write an element of $GL_2(\mathbb{A}_\mathbb{Q})$ as a product where $\eta \in GL_2(\mathbb{Q}), g_\infty \in GL_2(\mathbb{R})^+$ and

$$k_0 \in K_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathbb{Z}}) : c \equiv 0 (mod \ \ N)\right\}.$$

Define $\phi_f(\eta g_\infty k_0) := \mathcal{F}(g_\infty)\lambda(k_0) = (f\mid_k g_\infty)(i)\lambda(k_0)$, where the function λ is an adelization of the Dirichlet character χ. Henceforth, we aim to define a λ such that ϕ_f is well-defined.

Considering the map

$$\prod_p \mathbb{Z}_p^\times \xrightarrow{\widetilde{\pi}} (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi^{-1}} \mathbb{C}^\times,$$

we define the map

$$\omega = \omega_\pi := \chi^{-1} \circ \widetilde{\pi} = \prod_p \omega_p : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times \quad \text{s.t.} \quad \omega\mid_{\mathbb{R}_{>0}^\times} = 1;$$

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \omega(d) = \prod_{p|N} \omega_p(d_p),$$

where d_p denotes the $\mathbb{Q}_p^\times$ component of d.

Let $\pi_p \in \mathbb{A}_\mathbb{Q}^\times$ be the image of p under the inclusion $\mathbb{Q}^\times \hookrightarrow \mathbb{A}_\mathbb{Q}^\times$.

Lemma 17. *If $p \nmid N$, we have $\omega\mid_{\mathbb{Z}_p^\times} = 1$, $\omega(\pi_p) = \chi(p)$. The archimedean component ω_∞ is trivial on $\mathbb{R}_{>0}^\times$. Thus if d is a positive integer prime to N, $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)^{-1}$.*

Proof. By construction, $\omega\mid_{\mathbb{Z}_p^\times} = 1$ for $p \nmid N$ and $\omega_\infty\mid_{\mathbb{R}_{>0}^\times} = 1$. Note that we have the decomposition

$$\pi_p = p \cdot 1 \cdot \gamma, \quad \text{where} \quad \gamma = (\frac{1}{p}, ..., 1, \frac{1}{p}, ...) = \frac{1}{p} \in (\mathbb{Z}/N\mathbb{Z})^\times,$$

which induces that $\omega(\pi_p) = \chi^{-1}(\frac{1}{p}) = \chi(p)$.

ω is trivial on $\mathbb{Q}^\times$, hence

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \omega(d) = \prod_{p|N} \omega_p(d_p) = \prod_{p\nmid N} \omega_p^{-1}(d_p) = \prod_{p|d} \prod_{p\nmid N} \chi^{-1}(p) = \chi^{-1}(d).$$

□

Proposition 18. *The function ϕ_f is well-defined. It is an automorphic form with central character ω, and is a cusp form.*

Proof. Since the strong approximation is not canonical, we consider

$$\eta g_\infty k_0 = \eta' g'_\infty k'_0.$$

So we have $\eta_\infty g_\infty = \eta'_\infty g'_\infty$ and $\eta_f k_0 = \eta'_f k'_0$, which gives that $det(\eta_\infty^{-1}\eta'_\infty) > 0$ and $\eta_f^{-1}\eta'_f \in K_0(N)$.

Hence $\eta^{-1}\eta' \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \cap GL_2(\mathbb{Q})^+ = \Gamma_0(N)$, which gives that

$$\mathcal{F}(g_\infty) = \mathcal{F}(\eta_\infty^{-1}\eta'_\infty g'_\infty) = \mathcal{F}(\begin{pmatrix} a & b \\ c & d \end{pmatrix} g'_\infty) = \chi(d)\mathcal{F}(g'_\infty).$$

By Lemma 17, $\chi^{-1}(d) = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda(\eta^{-1}\eta') = \lambda(\eta_f^{-1}\eta'_f) = \lambda(k_0 k_0'^{-1})$.

Thus $\mathcal{F}(g_\infty)\lambda(k_0) = \mathcal{F}(g'_\infty)\lambda(k'_0)$, namely, ϕ_f is well-defined, and henceforth, we shall show that it is an automorphic form.

ϕ_f is smooth since $\mathcal{F}$ is and λ is locally constant. Based on $\mathcal{F}$'s properties, we have

(1) Left invariance under $GL_2(\mathbb{Q})$ follows from the definition.

(2) Taking $K = K_0(N)SO_2(\mathbb{R})$, for $k = k_0 k_\infty \in K$, we have

$$\begin{aligned}\phi_f(gk) &= \phi_f(g_\infty k_\infty k_0' k_0) = \mathcal{F}(g_\infty k_\infty)\lambda(k_0 k_0') \\ &= e^{-ik\theta}\lambda(k_0)\mathcal{F}(g_\infty)\lambda(k_0') = e^{-ik\theta}\lambda(k_0)\phi_f(g).\end{aligned}$$

In particular, ϕ_f is K-finite.

(3) For $g \in GL_2(\mathbb{A}_{\mathbb{Q}})$ and $z \in \mathbb{A}_{\mathbb{Q}}^\times$, we have

$$\phi_f\left(\begin{pmatrix} z & \\ & z \end{pmatrix} g\right) = \omega(z)\phi_f(g).$$

This is immediate for $z \in \mathbb{Q}^\times$, $z \in \mathbb{R}^\times$ and $z \in \mathbb{Z}_p^\times$, and follows in general by (20).

(4) ϕ_f is bounded because $\mathcal{F}$ is.

(5) Let $\mathcal{Z}$ denote the center of $\mathcal{U}(\mathfrak{gl}_2(\mathbb{R}))$, then ϕ_f is $\mathcal{Z}$-finite because $\mathcal{F}$ is, and the action of $\mathcal{Z}$ is just in the archimedean component.

(6) Note that $\mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times \xrightarrow{\sim} \mathbb{R}_{>0}^\times \widehat{\mathbb{Z}}^\times$, thus we have

$$\begin{aligned}&\int_{\mathbb{Q}^\times\setminus\mathbb{A}_{\mathbb{Q}}^\times} \phi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx \\ &= \int_0^\infty \mathcal{F}\left(\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix} g_\infty\right) dx \int_{\widehat{\mathbb{Z}}^\times} \lambda\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}_f g_f\right) dx = 0.\end{aligned}$$

□

Remark. This construction induces an isomorphism between $\mathcal{S}_k(N,\chi)$ and the space of functions on $GL_2(\mathbb{A}_{\mathbb{Q}})$ satisfying certain properties, i.e. we have

$$\mathcal{S}_k(N,\chi) \hookrightarrow \mathcal{L}_0^2\left(GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_{\mathbb{Q}}), \omega\right).$$

The next thing to do is to analyze the automorphic representation generated by ϕ_f. We thus have to review the local archimedean and non-archimedean representation theory. To start with, we first consider the action of $\mathfrak{gl}_2(\mathbb{R})$ and give a proof of (5) of Proposition 16.

3.2.1. *The action of* $\mathfrak{gl}_2(\mathbb{R})$. For any $X \in \mathfrak{gl}_2(\mathbb{R})$ and $g \in GL_2(\mathbb{R})$, define

$$(X.f)(g) := \frac{d}{dt} f\left(g exp(tX)\right)|_{t=0}.$$

Denote by

$$\widehat{R} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \widehat{L} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \widehat{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widehat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\Delta = -\frac{1}{4}\left(\widehat{H}^2 + 2\widehat{R}\widehat{I} + 2\widehat{L}\widehat{R}\right) \in \mathcal{Z}(\mathfrak{gl}_2(\mathbb{R})) = \mathbb{C}[\Delta, I].$$

It is somewhat difficult to calculate Δ directly, so we turn to consider another expression of it by complexification. Recall Cayley transform, which is conjugation by $C = -\frac{1+i}{2}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$.

By Cayley transform on $\widehat{R}$, $\widehat{L}$ and $\widehat{H}$ we have their complexification:

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\Delta = -\frac{1}{4}\left(H^2 + 2RL + 2LR\right).$$

Proposition 19. *On the space of smooth functions, the action of Δ under the Iwasawa coordinate system is given by*

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-y\frac{\partial^2}{\partial x\partial\theta}. \tag{22}$$

Proof. It is just a direct but complicate computation, so we omit the process here. Specific calculation can be find in Chapter 2, [3]. □

Corollary 20. *Let f be a holomorphic modular form of weight k and $\mathcal{F}$ the associated automorphic form on $GL_2(\mathbb{R})^+$, then*

$$\Delta\mathcal{F}=\frac{k}{2}(1-\frac{k}{2})\mathcal{F}.$$

Proof. Unraveling the definition of $\mathcal{F}$, we compute that

$$\mathcal{F}(g)=(f\mid_k g)(i)=e^{-ik\theta}y^{\frac{k}{2}}f(x+yi).$$

Thus we have

$$\Delta\mathcal{F}=e^{-ik\theta}\left(y^{\frac{k}{2}+2}f_{xx}+y^{\frac{k}{2}+2}f_{yy}+\frac{k}{2}(1-\frac{k}{2})y^{\frac{k}{2}}f-iky^{\frac{k}{2}+1}f_x\right).$$

Since f is holomorphic, we have $f_{xx}+f_{yy}=0$ and $if_x=f_y$, which gives the conclusion. □

3.2.2. *Representations of $GL_2(\mathbb{Q}_p)$ and Hecke algebra.* A representation of $GL_2(\mathbb{Q}_p)$ is admissible if it is smooth and $dim(V^{K'})<+\infty$ for any $K'\subseteq GL_2(\mathbb{Q}_p)$ open compact.

Theorem 21 (Classification of irreducible admissible representations)**.** *Irreducible admissible representations can be divided into the following types:*

1. *All finite dimensional irreducible admissible representations are one-dimensional, and factor through the determinant;*
2. *Irreducible principle series;*
3. *Special series;*
4. *Super-cuspidal representations.*

Proof. The compact proof of this deep result can be found in the section 2-4 in [34]. Thus for the sake of simplicity, we omit the proof here. □

The Hecke algebra $\mathcal{H}(G)$ for $G=GL_2(\mathbb{Q}_p)$ is the convolution algebra of locally constant compactly supported functions. There is a natural action of $\mathcal{H}(G)$ on the representations of $GL_2(\mathbb{Q}_p)$ given by

$$\pi(\phi)v:=\int_G\phi(g)\pi(g)vdg.$$

Remark. Smooth representations of G are the same as $\mathcal{H}(G)$-modules.

The special Hecke algebra $\mathcal{H}(G,K)$ is defined to be $1_K\mathcal{H}(G)1_K$, the space left and right K-invariant of $\mathcal{H}(G)$. A smooth admissible representation (π,V) is called spherical (or unramified) if $V^K\neq 0$. A non-zero element is called a spherical vector.

The spherical vectors are a $\mathcal{H}(G,K)$-module.

Define

$$T_p:=\text{ the characteristic function of } K\begin{pmatrix}p & \\ & 1\end{pmatrix}K;$$

$$R_p:=\text{ the characteristic function of } K\begin{pmatrix}p & \\ & p\end{pmatrix}K.$$

Theorem 22. *$\mathcal{H}(G,K)$ is communicative, actually, $\mathcal{H}(G,K)=\mathbb{C}[T_p,R_p,R_p^{-1}]$.*

Proof. This comes from the p-adic Cartan decomposition (cf. Section 4.6 in [6]). □

Remark. The identification of $\mathcal{H}(G, K)$ with the polynomial algebra is known as the Satake isomorphism.

The other key fact says that the spherical vectors determine the representation.

Theorem 23. *For an irreducible unramified representation* (π, V), $dim(V^K) = 1$. *Moreover, there is an equivalence of categories between irreducible unramified representations and irreducible* $\mathcal{H}(G, K)$*-modules sending* V *to* V^K.

Remark. Thus the way T_p and R_p act completely determines an unramified representation. It is convenient to record their action using Satake parameters: if T_p and R_p act by λ and μ, the Satake parameters are the roots of $x^2 - p^{\frac{k}{2}-1}\lambda x + \mu p^{k-1}$.

Theorem 24 (Classification of irreducible spherical representations)**.** *The only irreducible spherical representations are* $\chi \circ det$ *with* χ *unramified, and irreducible* $\pi(\chi_1, \chi_2)$ *with* χ_1, χ_2 *unramified.*

Proof. Let (π, V) be spherical. Let T_p act on V^K by λ and R_p by μ as above, thus the Satake parameters α_1 and α_2 are the roots of $x^2 - p^{\frac{k}{2}-1}\lambda x + \mu p^{k-1}$. Take χ_1 and χ_2 to be unramified quasi-characters such that

$$\chi_1(p) = \frac{\alpha_1}{p^{k-\frac{1}{2}}} \quad \text{and} \quad \chi_2(p) = \frac{\alpha_2}{p^{k-\frac{1}{2}}}.$$

If $\pi(\chi_1, chi_2)$ is irreducible, its K-fixed vectors have the same Satake parameters. This forces $V \simeq \pi(\chi_1, chi_2)$. So it remains to handle the case where $\pi(\chi_1, \chi_2)$ is reducible, which happens only if $\alpha_1\alpha_2^{-1} = p^{\pm 1}$. Clearly, we may assume that $\alpha_1\alpha_2^{-1} = p$. So $\pi(\chi_1, \chi_2)$ has a one-dimensional sub-representation. However, this must be K-invariant, so there must be $dim(\pi) = 1$. □

Let f be a holomorphic cusp form, eigenfunction associated for Hecke operators T_p for $p \nmid N$; moreover, suppose f is of level N and nebentypus χ. So by the previous results, we have $\phi_f \in \pi(f) \subset \mathcal{A}_0\left(GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_\mathbb{Q}), \omega\right)$.

The representation $\pi(f)$ can be expressed as a restricted tensor product $\otimes'_\nu \pi_{f,\nu}$. We would like to describe $\pi_{f,\nu}$ in terms of the properties of f. After reviewing some facts about the Hecke algebra and connecting it to the classical Hecke operators, we state the strong multiplicity one theorem and then turn to prove the main theorem and discussing the local components.

3.2.3. *The global Hecke algebra and Hecke operators.* The global Hecke algebra $\mathcal{H}_{GL_2(\mathbb{A}_\mathbb{Q})}$ is defined to be the restricted tensor product of the local Hecke algebras for the archimedean and non-archimedean places of $\mathbb{Q}$. For this to be defined, we need to specify a spherical idempotent in all but finitely many of the local Hecke algebras. We use the characteristic function of $GL_2(\mathbb{Z}_p)$ in $\mathcal{H}_{GL_2(\mathbb{A}_\mathbb{Q})}$.

The representation $\pi(f)$ can be viewed as a $\mathcal{H}_{GL_2(\mathbb{A}_\mathbb{Q})}$-module. We have

Theorem 25 (Tensor Product Theorem, [11])**.** *Let* (π, V_π) *be an automorphic representation of* $GL_2(\mathbb{A}_\mathbb{Q})$. *Then* π *is the restricted tensor product of the local representations* π_p, *where* p *runs through all rational primes, and each* π_p *is an irreducible admissible representation of* $GL_2(\mathbb{Q}_p)$.

Then there is a factorization $\pi(f) = \otimes'_\nu \pi_{f,\nu}$ where $\pi_{f,\nu}$ is a $\mathcal{H}_{GL_2(\mathbb{A}_\mathbb{Q})}$-module. This also means that there is a spherical vector is almost all of the $\pi_{f,\nu}$ (a vector invariant under the action of $GL_2(\mathbb{Z}_\nu)$).

The first step is to connect the classical Hecke operators with the adelic ones. We studied the Hecke operators $T_p \in \mathcal{H}_{GL_2(\mathbb{Q}_p)}$ given by convolution with the characteristic function of $H_p = GL_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1 \end{pmatrix}GL_2(\mathbb{Z}_p)$. In the following proposition,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_p$ will denote an element of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with specified matrix at p and the identity elsewhere.

Proposition 26. *Let f be a modular form of level N, and suppose p is a prime such that $p \nmid N$. Then*

$$T_p(\phi_f) = p^{1-\frac{k}{2}} a_p \phi_f = \phi_{p^{1-\frac{k}{2}} T_p f}, \tag{23}$$

where a_p is the T_p-eigenvalue of f.

Proof. Recall that the double set decomposes as

$$\mathcal{H}_p = \cup_{b=0}^{p-1} \begin{pmatrix} p & b \\ & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \cup \begin{pmatrix} 1 & \\ & p \end{pmatrix} GL_2(\mathbb{Z}_p).$$

Thus we have

$$T_p(\phi_f)(g) = \sum_{b=0}^{p-1} \int_{\mathbb{Z}_p} \phi_f \left(g \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p k \right) dk + \int_{\mathbb{Z}_p} \phi_f \left(g \begin{pmatrix} 1 & \\ & p \end{pmatrix}_p k \right) dk.$$

As $p \nmid N$, the p component of $K_0(N)$ is $GL_2(\mathbb{Z}_p)$ and λ is trivial on it. So ϕ_f is right-invariant under $GL_2(\mathbb{Z}_p)$; then we have

$$\int_{\mathbb{Z}_p} \phi_f \left(g \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p k \right) dk = \phi_f \left(g \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right),$$

since $Vol(\mathbb{Z}_p) = 1$.

Thus

$$T_p(\phi_f)(g) = \sum_{b=0}^{p-1} \phi_f \left(g \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right) + \phi_f \left(g \begin{pmatrix} 1 & \\ & p \end{pmatrix}_p \right).$$

We may assume $g = g_\infty$ since we can view $\begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \in \mathbb{A}_f$. Actually, if $g = \eta g_\infty k_0$, then

$$\phi_f \left(g \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right) = \phi_f \left(g_\infty k_0 \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right) = \mathcal{F}(g_\infty)\lambda(k_0)\lambda \left(\begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right)$$
$$= \lambda(k_0)\phi_f \left(g_\infty \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right).$$

We can write

$$g_\infty \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p = \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_{\mathbb{Q}} \left(\begin{pmatrix} p & b \\ & 1 \end{pmatrix}^{-1} g_\infty \right) \gamma,$$

where $\gamma_\nu = \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_\nu^{-1}$ when $\nu \neq p, \infty$; and $\gamma_\nu = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}_\nu^{-1}$ otherwise.

Thus γ reduces to $\begin{pmatrix} p & b \\ & 1 \end{pmatrix}^{-1}$ in $GL_2\left((\mathbb{Z}/N\mathbb{Z})^\times\right)$, deducing that $\lambda(\gamma) = \omega(1) = 1$. So we have now

$$\phi_f \left(g_\infty \begin{pmatrix} p & b \\ & 1 \end{pmatrix}_p \right) = \mathcal{F} \left(\begin{pmatrix} p & b \\ & 1 \end{pmatrix}_\infty^{-1} g_\infty \right) = p^{-\frac{k}{2}} det(g_\infty)^{\frac{k}{2}} j(g_\infty, i)^{-k} f\left(\frac{z-b}{p} \right).$$

Doing the same for the remaining coset gives

$$\phi_f \left(g_\infty \begin{pmatrix} p & b \\ & 1 \end{pmatrix} \right) = p^{\frac{k}{2}} det(g_\infty)^{\frac{k}{2}} j(g_\infty, i)^{-k} f\left(pz\right).$$

Therefore, we deduce that

$$\begin{aligned} p^{\frac{k}{2}-1}T_p(\phi_f)(g) &= det(g_\infty)^{\frac{k}{2}} j(g_\infty, i)^{-k} \left(\sum_{b=0}^{p-1} \frac{1}{p} f\left(\frac{z-b}{p}\right) + p^{k-1} f(pz) \right) \\ &= det(g_\infty)^{\frac{k}{2}} j(g_\infty, i)^{-k} (T_p f)(z) \\ &= a_p det(g_\infty)^{\frac{k}{2}} j(g_\infty, i)^{-k} f(z) = a_p \phi_f(g_\infty). \end{aligned}$$

□

Similarly, we interpret the operator R_p given by convolution with the characteristic function of $\begin{pmatrix} p & \\ & p \end{pmatrix} GL_2(\mathbb{Z}_p)$ to

Proposition 27. *For $p \nmid N$, we have $R_p\phi_f = \chi(p)\phi_f$.*

Proof.

$$(R_p\phi_f)(g_\infty) = \int_{\mathbb{Z}_p} \phi_f\left(g_\infty \begin{pmatrix} p & \\ & p \end{pmatrix}_p k \right) dk = \phi_f\left(g_\infty \begin{pmatrix} p & \\ & p \end{pmatrix}_p \right).$$

Note that

$$g_\infty \begin{pmatrix} p & \\ & p \end{pmatrix}_p = \begin{pmatrix} p & \\ & p \end{pmatrix}_{\mathbb{Q}} \left(\begin{pmatrix} p & \\ & p \end{pmatrix}_\infty^{-1} g_\infty \right) \gamma,$$

where $\gamma_\nu == \begin{pmatrix} p & \\ & p \end{pmatrix}^{-1}$ for $\nu \neq \infty$ or p; and is the identity elsewhere.

Hence

$$\begin{aligned} \phi_f\left(g_\infty \begin{pmatrix} p & \\ & p \end{pmatrix}_p \right) &= \mathcal{F}\left(\begin{pmatrix} p & \\ & p \end{pmatrix}_\infty^{-1} g_\infty \right) = \chi(p)\mathcal{F}\left(\begin{pmatrix} p & \\ & p \end{pmatrix}^{-1} g_\infty \right) \\ &= \chi(p)\phi_f(g_\infty), \end{aligned}$$

since $\mathcal{F}$ is left invariant under $\mathcal{Z}(\mathbb{R})^+$. □

Theorem 28 (Strong Multiplicity One, [11])**.** *Let (π, V) and (π', V') be two irreducible admissible sub-representations of $\mathcal{A}_0\left(GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_\mathbb{Q}), \omega\right)$. Assume that $\pi_\nu = \pi'_\nu$ for all but finitely many places. Then $V \simeq V'$.*

Remark. This form of the theorem is stronger than that in Section 3.5 of [6], where $\pi_\nu = \pi'_\nu$ for all archimedean places is required.

Let f be a primitive form as above, and $f^{GL_2(\mathbb{A}_\mathbb{Q})}$ represents the space spanned by right translations of f under $GL_2(\mathbb{A}_\mathbb{Q})$. Then the resulting representation $\pi(f)$ on this space occurs in the standard regular representation on the cusp forms. We have the following proposition:

Proposition 29. *Let notations be as above, then the representation $\pi(f)$ on the space $f^{GL_2(\mathbb{A}_\mathbb{Q})}$ is irreducible. Furthermore, the local representation π_∞ is the discrete series representation D_{k-1} of lowest weight k.*

To prove this result, we need one more classical theorem from representation theory:

Theorem 30 (Multiplicity One Theorem, [3])**.** *The right regular representation decomposes into the direct sum of irreducible representations, and each of which occurs with multiplicity one.*

Proof of Proposition 29. Let f be a cusp form and Hecke eigen-form. Then Theorem 30 and Theorem 25 ensure that we can write $\pi(f) = \oplus_i \pi_i(f)$, with each irreducible constituent $\pi_i(f)$ being a restricted tensor product of local representations, i.e. $\pi_i(f) = \otimes_p \pi_{i,p}(f)$. Thus, to show that $\pi_(f)$ is irreducible, it suffices to show that $\pi_{i,p}(f) \simeq \pi_{j,p}(f)$ for almost all rational primes p and for all i and j according to Theorem 28.

Let $p \nmid N$ be a prime. We know that ϕ_f is invariant under right translation by $K_0(N)$, so in particular the local component π_p contains an element which is left and right $GL_2(\mathbb{Z}_p)$-translation invariant. Actually, we have for $g = \eta g_\infty k_0 \in GL_2(\mathbb{Q})GL_2(\mathbb{R})^+K_0(N)$ and $k \in K_0(N)$,

$$\begin{aligned}\phi_f(kg) = \phi_f(k\eta g_\infty k_0) = \phi_f(\eta_\infty g_\infty k \eta_f k_0) = \mathcal{F}(\eta_\infty g_\infty)\lambda(k\eta_f k_0)\\ = \mathcal{F}(\eta_\infty g_\infty)\lambda(k)\lambda(\eta_f)\lambda(k_0) = \phi_{(g)}\lambda(k) = \phi_f(g),\end{aligned}$$

since λ is multiplicative.

Thus ϕ_f is an eigenvector for T_p and R_p. We can calculate the eigenvalues as in (23) and Proposition 27, which shows that the eigenvalues are only determined by f, and this action determines the action of $\mathcal{H}\left(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p)\right)$. In other words, the local component π_p is an spherical representation of $\mathcal{H}_p$ that is completely specified in terms of f. By *Strong Multiplicity One Theorem*, this forces ϕ_f to lie in a unique irreducible (π, V), proving the first part of the proposition.

For archimedean places, note that the local representation π_∞ is a $(\mathfrak{gl}(2), O(2))$-module, so it suffices to consider the eigenvalue ρ for the Casimir operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x \partial \theta}$ given in (22). By Proposition 16 we have $\rho = \frac{k}{2}\left(1 - \frac{k}{2}\right)$. Note that an irreducible admissible infinite-dimensional representation of $GL_2(\mathbb{R})$, with infinitesimal character determined by $frack2\left(1 - \frac{k}{2}\right)$ and central trivial on $\mathbb{R}_{>0}$ (by our construction) has to be the discrete series representation D_{k-1}, which amounts to saying that $\pi_\infty = D_{k-1}$, proving the Proposition. ☐

Remark. Infinite-dimensionality of π_∞ is guaranteed by the existence of Whittaker models (cf. Theorem 31 and Theorem 32 in the next subsection).

3.3. **Retrieving a modular form from an automorphic representation.** Let (π, V) be a cuspidal automorphic representation with conductor N, central character ω which is trivial on $\mathbb{R}_{>0}$, and such that the representation ar ∞ is equivalent to D_{k-1}, where D_{k-1} is a discrete series representation of the lowest weight k. Suppose the conductor of π is N. Then for any $p \nmid N$, π_p is equivalent to a spherical representation induced from some unramified character $\chi_{1,p} \otimes \chi_{2,p}$ (cf. Theorem 24). In order to retrieving a modular form from an automorphic representation, it will be convenient to consider a Whittaker model of π.

3.3.1. *Whittaker Models.* To begin with, we recall and fix, once and for all, an additive character ψ of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ as in Tate's thesis, i.e. $\psi(x) = e^{2\pi i \iota(x)}$ with ι as defined in [61]. Particularly, $\iota = \sum_{p \le \infty} \iota_p$; $\iota_\infty(x) = -x$ for any $x \in \mathbb{R}$; for any $x \in \mathbb{Q}_p$, $\iota_p(x)\mathbb{Q}$ is the rational number with only p-power in the denominator such that $x - \iota_p(x) \in \mathbb{Z}_p$. Hence we have $\psi = \psi_\infty \otimes \otimes_p \psi_p$, where $\psi_\infty(x) = e^{-2\pi i x}$ and $\psi_p \mid_{\mathbb{Z}_p} = 1$ and is nontrivial on $p^{-1}\mathbb{Z}_p$ for any rational prime p.

Also, we need to recall some basic background of Whittaker models, which we will work with to retrieve a modular form from a representation. Here we quote some standard results (cf. Chapter 3, 4 of [6]):

Theorem 31 (Local Whittaker Model)**.** *For any prime number p, let π_p be an irreducible admissible infinite-dimensional representation of $GL_2(\mathbb{Q}_p)$. Then there* *exists a unique space $\mathcal{W}(\pi_p,\psi_p)$ of smooth functions invariant under right translations by elements of $GL_2(\mathbb{Q}_p)$ such that for any $W\in\mathcal{W}(\pi_p,\psi_p)$, one has*

$$W\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)=\psi_p(x)W(g),$$

for any $x\in\mathbb{Q}_p$ and $g\in GL_2(\mathbb{Q}_p)$, and the induced representation of $GL_2(\mathbb{Q}_p)$ on $\mathcal{W}(\pi_p,\psi_p)$ is equivalent to π.

Remark. $\mathcal{W}(\pi_p,\psi_p)$ is called a local Whittaker model for π_p.

Theorem 32 (Global Whittaker Models)**.** *For any π being a cuspidal sutomorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$, there exists a unique Whittaker model $\mathcal{W}(\pi,\psi)$ for π with respect to a nontrivial additive character ψ, that is spanned by functions satisfying:*

$$W_\varphi(g):=\int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}}\varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)\overline{\psi(x)}dx,$$

where $\varphi\in\pi$ and $g\in GL_2(\mathbb{A}_{\mathbb{Q}})$.

Remark. We can show by expanding φ into its Fourier expansion that $W_\varphi(g)$ is a product of local integrals. Hence, the space $\mathcal{W}(\pi,\psi)$ decomposes as a restricted tensor product of local Whittaker models.

3.3.2. *Retrieving a cusp Hecke-eigen form from π.* The isomorphism between π and $\mathcal{W}(\pi,\psi)$ allows us to determine a unique holomorphic cusp modular form that corresponding to π by picking up a suitable element from each $\mathcal{W}(\pi_p,\psi_p)$. For almost all p, $W_p\in\mathcal{W}(\pi_p,\psi_p)$ is a spherical vector, and is normalized so that $W_p(x_p)=1$ for all $x_p\in GL_2(BZ_p)$. The choices for the local vectors can be made in the following process.

For the archimedean place, let W_∞ be the lowest weight vector in $\mathcal{W}(\pi_\infty,\psi_\infty)$, namely,

$$W_\infty^0\left(\begin{pmatrix}z & \\ & z\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y & \\ & 1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\right)=\omega_\infty(z)\psi_\infty(x)e^\eta,$$

where $\eta=-2\pi y-ik\theta$. For non-archimedean places, let W_p^{new} be the new vector in $\mathcal{W}(\pi_p,\psi_p)$, i.e. W_p^{new} is an element such that

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}.W_p^{new}=\omega_d(p)W_p^{new}\quad\text{for all}\quad\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(p^f),$$

where $f\in\mathbb{N}$ such that $p^f\parallel N$, and normalized as below.

Denote $\mathcal{K}_p$ the new-vector in the Kirillov model $\mathcal{K}(\pi_p,\psi_p)$ corresponding W_p^{new}, then by definition we have $\mathcal{K}_p(y)=W_p^{new}\begin{pmatrix}y & \\ & 1\end{pmatrix}$, and $\mathcal{K}_p(1)\neq 0$ (cf. Section 2.4 of [55]). Hence $W_p^{new}\begin{pmatrix}1 & \\ & 1\end{pmatrix}\neq 0$ and thus we can normalize this vector as W_p^0 such that $W_p^0\begin{pmatrix}1 & \\ & 1\end{pmatrix}=1$.

Set $W^0:=W_\infty^0\otimes\otimes_{p<\infty}W_p^{new}\in\mathcal{W}(\pi,\psi)$, then by $\pi\simeq\mathcal{W}(\pi,\psi)$ we have a corresponding $f\in\pi$. By Fourier inverse transform, one has

$$f(g)=\sum_{y\in\mathbb{Q}^\times}W^0\left(\begin{pmatrix}y & \\ & 1\end{pmatrix}g\right).$$

Take $g=\begin{pmatrix}y_\infty & \\ & 1\end{pmatrix}\in GL_2^+(\mathbb{A}_{\mathbb{Q}})$ where $y_\infty\in\mathbb{R}_{>0}\hookrightarrow\mathbb{A}_{\mathbb{Q}}$. Then it is clear that $W^0\left(\begin{pmatrix}y_\infty x & \\ & 1\end{pmatrix}\right)=e^{-2\pi y_\infty x}\prod_{p<\infty}W_p^0\left(\begin{pmatrix}x & \\ & 1\end{pmatrix}g\right)$ must vanish unless

$x_\infty > 0$. Hence the Fourier expansion of f can be reduced to

$$f(g) = \sum_{y \in \mathbb{Q}_+^\times} W^0 \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} g \right). \tag{24}$$

In the rest of this subsection, we aim to show that the f above is the desired holomorphic modular form. Precisely, we have:

Theorem 33. *Let $\mathcal{A}_0(k, N, \omega) \subset \mathcal{A}_0(GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A}_\mathbb{Q}), \omega)$ that consists such elements that:*

(1) $\varphi(gk_\theta) = e^{-ik_\theta}\varphi(g)$, *where* $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$;
(2) $\varphi(gk_0) = \omega_f(k_0^\iota)\varphi(g)$, *where ω_f denotes the finite part of ω and* $k_0 \in K_0(N)$, *where* $k_0^\iota = w_0 k_0^t w_0^{-1}$ *and t the transpose and* $w_0 := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$;
(3) $\Delta\varphi = \frac{k}{2}\left(1 - \frac{k}{2}\right)\varphi$.

Then we have $\mathcal{A}_0(k, N, \omega) \simeq \mathcal{S}_k(N, \omega)$.

Proof. It is obvious that $\mathcal{S}_k(N,\omega) \hookrightarrow \mathcal{A}_0(k, N, \omega)$, so it suffices to recover a holomorphic cusp form from any $\varphi \in \mathcal{A}_0(k, N, \omega)$, which can be done by putting, for any $g = \eta g_\infty k_0 \in GL_2(\mathbb{A}_\mathbb{Q})$,

$$f_\varphi(z) := \varphi(g_\infty)\,(det g_\infty)^{-\frac{k}{2}}\, j\,(g_\infty, i)^k,$$

where $g_\infty(i) = z \in \mathcal{H}$. Then by a direct computation we have $\left(\frac{\partial}{\partial \bar{x}}\right) f = 0$ and $f_\varphi = \varphi$ (As mentioned in the proof of Proposition 2.1 in [18], specified details appear in Chapter 1, Section 4 of [19]). ∎

Now we shall show that f is primitive. It suffices to show that f is a common eigenfunction of the Hecke operators T_p for all $p \nmid N$. By definition W_p^0 is an eigenfunction for such T_p's with the same eigenvalue. It follows from (24) that f is also an eigenfunction for T_p with same eigenvalue.

Finally, f is normalized, i.e. $\rho_f(1) = 1$, as shown in [17]. Thus, so far, we have proved the first part of Theorem 12.

3.4. Comparing the associated L-functions.

In this section, we will show the rest part of Theorem 12, namely, the L-functions under the corresponding are actually the same, despite that they are defined in different ways.

3.4.1. *L-functions attached to a cuspidal automorphic representation.*

Let's recall the definition of L-functions attached to a cuspidal automorphic representation π. First review the GL_1-theory. For a Hecke character $\chi = \otimes_p \chi_p$, the local L-functions at the finite places are given by

$$L_p(s, \chi_p) := \left(1 - \chi_p(\varpi_p)p^{-s}\right)^{-1} \quad \text{if } \chi_p \text{ is unramified}$$
$$L_p(s, \chi_p) := 1 \quad \text{if } \chi_p \text{ is ramified.}$$

Define the GL_2 L-functions as follows: if $\pi_p \simeq \chi_{1,p} \boxplus \chi_{2,p}$, then put

$$L_p(s, \pi_p) := L_p(s, \chi_{1,p}) L_p(s, \chi_{2,p}).$$

For other places, define $L_p(s, \pi_p) := 1$ for a supercuspidal representation π_p, and

$$L_p(s, \pi_p) := L_p\left(s + \frac{1}{2}, \chi_p\right)$$

for $\pi_p = St\left(|.|_p^{-\frac{1}{2}}, |.|_p^{\frac{1}{2}}\right) \otimes \chi_p = St\left(|.|_p^{-\frac{1}{2}}\chi_p, |.|_p^{\frac{1}{2}}\chi_p\right)$, the twist of Steinberg representation by χ_p (cf. Theorem 21; or see Section 3 of [37] for more details). And at the archimedean place, we define

$$L_\infty(s, \pi_\infty) := (2\pi)^{-s-\frac{k-1}{2}}\Gamma\left(s + \frac{k-1}{2}\right).$$

The global L-function attached to π is defined as $L(s,\pi) := \otimes_{p<\infty} L_p(s,\pi_p)$ and $\Lambda(s,\pi) := L_\infty(s,\pi_\infty)L(s,\pi)$ the completed L-function. This completed L-function has an analytic continuation to $\mathbb{C}$ and satisfies certain functional equation (cf. [34]).

3.4.2. *L-functions attached to a primitive modular form.* Let f be a primitive holomorphic cusp modular form of weight k, level N and nebentypus character χ. Recall that $\lambda_f(n)$ is defined to be $a_n n^{-\frac{k}{2}}$ for any integral integer n, where $f(z) = \sum_n a_n e^{nz}$. The (finite) L-function attached to f is

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{s-\frac{k}{2}}}.$$

Let $\widetilde{\chi}$ be a character of $\mathbb{Z}/N\mathbb{Z}$ defined by $\widetilde{\chi}(p) := \chi(p)$ for all primes $p \nmid N$; and $\widetilde{\chi}(p) = 0$ if $p \mid N$. Then the L-function has an Euler product:

$$L(s,f) = \prod_{p<\infty}\left(1 - \lambda_f p^{-s+\frac{k}{2}} + \widetilde{\chi}(p)p^{k-1-2s}\right)^{-1}.$$

Define the local factor at ∞ by $L_\infty(s,f) := (2\pi)^{-s}\Gamma(s)$. Also, define the complete L-function as $\Lambda(s,f) := L_\infty(s,f)L(s,f)$, which also has an analytic continuation to $\mathbb{C}$ and satisfies certain functional equation by a standard Hecke theory.

3.4.3. *Relation between the two types of L-functions.* In this section, we shall reveal the relation between $L(s,\pi)$ and $L(s,f)$ defined as above. Our main theorem is :

Theorem 34. *Let $f \in \mathcal{S}_k^{prim}(N,\chi)$ and $\pi(f)$ a cuspidal automorphic representation associated to f. The we have*

$$\Gamma(s,\pi(f)) = L\left(s + \frac{k-1}{2}, f\right).$$

The same relation holds between the finite (and hence the infinite) parts of the two L-functions.

Proof. Let $\Re(s) \gg 0$. For the $W_p^0 \in \mathcal{W}(\pi(f)_p, \psi_p)$ (cf. Section 3.3.2), define a local ζ-integral by:

$$\zeta_p(s, W_p^0) := \int_{\mathbb{Q}_p^\times} W_p^0\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}}d^\times y.$$

A global ζ-integral for $W^0 \in \mathcal{W}(\pi(f),\psi)$ is defined as

$$\zeta^*(s, W^0) := \int_{\mathbb{A}_\mathbb{Q}^\times} W^0\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}}d^\times y.$$

If $\pi(f)$ is a pure tensor, then W^0 factorizes as $W^0 = \otimes_p W_p^0$ into local Whittaker models, then we have $\zeta^*(s,W^0) = \prod_{p\leq\infty}\zeta_p(s,W_p^0)$. By Proposition 6.17 of [18], $L_p(s,\pi(f)_p) = \zeta_p(s,W_p^0)$. Thus we have

$$\Lambda(s,\pi(f)) = \prod_{p\leq\infty} L_p(s,\pi(f)_p) = \prod_{p\leq\infty}\zeta_p(s,W_p^0) = \zeta^*(s,W^0).$$

On the other hand, under the Iwasawa coordinate system, we have

$$\int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}}d^{\times}y = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} \sum_{x\in\mathbb{Q}^{\times}} W^0\left(\begin{pmatrix} xy & \\ & 1 \end{pmatrix}\right)|xy|^{s-\frac{1}{2}}d^{\times}x$$
$$= \int_{\mathbb{A}^{\times}} W^0\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}}d^{\times}y.$$

Note that we have the *Strong Approximation Theorem* : $\mathbb{A}_{\mathbb{Q}}^{\times} \backsimeq \mathbb{Q}^{\times}\mathbb{R}_{>0}^{\times}\widehat{\mathbb{Z}}$, and it follows that for any $y \in \mathbb{A}_{\mathbb{Q}}^{\times}$,

$$f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}} = y_{\infty}^{\frac{k}{2}} f(iy_{\infty})|y_{\infty}|^{s-\frac{1}{2}}.$$

Hence, combining the above we have

$$\Lambda(s,\pi(f)) = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)|y|^{s-\frac{1}{2}}d^{\times}y = \int_0^{+\infty} f(iy)y^{s+\frac{k-3}{2}}dy$$
$$= \sum_{n=1}^{\infty} \frac{a_f(n)n^{-\frac{k}{2}}}{(2\pi n)^{s-\frac{1}{2}}} \int_0^{+\infty} e^{-y}y^{s+\frac{k-1}{2}}\frac{dy}{y}$$
$$= (2\pi n)^{\frac{k-1}{2}-s}\Gamma\left(s+\frac{k-1}{2}\right)\sum_{n=1}^{\infty}\frac{\rho_f(n)}{n^{s-\frac{1}{2}}} = \Lambda\left(s+\frac{k-1}{2}, f\right),$$

for $\Re(s) \gg 0$. Hence we have the inequality for any $s \in \mathbb{C}$ since both sides have analytic continuation to the whole complex plane.

Moreover, by definition, it is clear that $L_{\infty}(s,\pi(f)_{\infty}) = L_{\infty}(s+\frac{k-1}{2}, f)$, and thus we have $L(s,\pi(f)) = \frac{\Lambda(s,\pi(f))}{L_{\infty}(s,\pi(f)_{\infty})} = \frac{\Lambda(s,f)}{L_{\infty}(s,f)} = L(s,f)$. □

Remark. Clearly, Theorem 12 comes from Theorem 33 and Theorem 34.

Appendix A. Background on automorphic L-functions

An L-function is a type of generating function formed out of local data associated with either an arithmetic geometric object (e.g. an Abelian variety defined over a number field) or an automorphic form (or an automorphic representation). The former type is usually called motivic L-functions, while the other type is named as aotomorphic L-functions. According to Shimura-Taniyama for special cases (e.g. elliptic curves) and Langlands in general, it is expected that the latter set contains the former one, which is formulated as Langlands Conjecture (cf. [38]).

Generally speaking, there are four usual ways to define a L-function, namely, via constant terms of Eisenatein series (the Langlands-Shahidi method), via periods of automorphic forms (the theory of integral representations, which begins with the work of Hecke, or actually, the work of Riemann), via a Dirichlet series (which is often taken as their defining propery), and via Rankin-Selberg convolution method.

In this paper, we shall mainly focus on the automorphic L-functions and their subconvexity problem (short for ScP) as defined in the following section. As it will be shown in the last section, ScP has many significant applications in number theory and arithmetic geometry, e.g. by Gross-Zagier formula ([23]) or Waldspurger formula ([65]) we will obtain some equidistribution information on Heegner points supposing some kind of subconvex upperbound and some other conditions such as Heegner condition (cf. Section 6.3 of [27]) is available. Therefore, we begin with introduction to principal L-functions and Rankin-Selberg method.

Fix a number field K with $deg\,(K/\mathbb{Q}) = d$, then for $F \in L^2\left(GL_m(K)\setminus GL_m(\mathbb{A}_K)\right)$. An L-function takes the form of a product of degree $m \geq 1$ over all primes over K:

$$L(s,F) := \prod_{\mathfrak{P}} L_{\mathfrak{P}}(s,F),$$

where the local factors are

$$L_{\mathfrak{P}}(s,F) = \prod_{i=1}^{m}\left(1-\alpha_i(\mathfrak{P})\,(N\mathfrak{P})^{-s}\right)^{-1}$$

for suitable complex numbers $\alpha_i(\mathfrak{P})$ (Satake parameters, which are determined by F) and where $N\mathfrak{P}$ is the norm of $\mathfrak{P}$. A known result is that $L(s,F)$ for K is a product of $L(s,F')$ for $\mathbb{Q}$ with $m' = md$, the degree of the automorphic representation (see [1]). Moreover, given m_i $(i \leq r)$ such that $\sum_{i=1}^{r} m_i = m$, the Langlands theory of Eisenstein series associates to an r-tuple $(\pi_1,...,\pi r)$ of (not necessarily unitary) cuspidal representations a distinguished automorphic representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$, denoted by $\pi = \pi_1 \boxplus ... \boxplus \pi_r$, the *isobaric* sum of the π_i, $i = 1,...,r$. By construction, cuspidal representations are isobaric and it is a result of Shalika that the π_i appearing in the construction of π (i.e. the constituents of π) are unique up to permutation ([10]). Then the L-function of π is given by the product

$$L(\pi_1 \boxplus ... \boxplus \pi_r, s) = \prod_{i=1}^{m} L(\pi_i, s).$$

Langlands also proved that any automorphic representation π is nearly equivalent to an isobaric sum π' (i.e. for almost every place v, $\pi_v \simeq \pi_{v'}$), and as a consequence $L(\pi,s)$ and $L(\pi',s)$ coincide up to finitely many local places.

Therefore, we may mostly concentrate on cuspidal L-functions because they form the building blocks for L-functions of general automorphic representations. We should mention that our main reference is the remarkable lecture notes [46].

A.1. **Principal L-functions.** Let $\pi = \otimes\pi_p$ be an automorphic cuspidal (irreducible) representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with unitary character (denoted by $\mathcal{A}_d^0(\mathbb{Q})$ the set of all such representations). By the general theory (see [10]), π admits an L-function:

$$L(\pi,s) = \prod_{p<\infty} L_p(\pi,s) = \sum_{n\geq 1}\frac{\lambda_\pi(n)}{n^s}.$$

This is an Euler product absolutely convergent for $\operatorname{Re} s$ sufficiently large where for each finite prime p,

$$L_p(\pi,s)^{-1} = L(\pi_p,s)^{-1} = \prod_{i=1}^{m}\left(1-\frac{\alpha_{\pi,i}(p)}{p^s}\right),$$

and $L(\pi,s)$ is completed by a local factor at the infinite place, given by

$$L_\infty(\pi,s) = L(\pi_\infty,s) = \prod_{i=1}^{m}\Gamma_{\mathbb{R}}(s-\mu_{\pi,i}), \quad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2);$$

the coefficients $\gamma_{\pi,i}(p)_{1\leq i\leq d}$ (resp. $\mu_{\pi,i_{1\leq i\leq d}}$) will be called the local parameters of π at p (resp. ∞). The completed L-function $\Lambda(\pi,s) = L_\infty(\pi,s)L(\pi,s)$ has a meromorphic continuation to the complex plane with at most two simple poles, which occurs only if $m = 1$ and $\pi = |.|^{it}$ for some $t \in \mathbb{R}$.

Moreover, $\Lambda(\pi,s)$ satisfies a functional equation of the form

$$q_\pi^{s/2}\Lambda(\pi,s) = \omega(\pi)q_\pi^{(1-s)/2}\Lambda(\tilde{\pi},1-s),$$

where $q_\pi \geq 1$ is an integer (the arithmetic conductor of π) supported at the finite ramified places for π, $\omega(\pi) \in \{z \in \mathbb{C} : |z| = 1\}$ (we call $\omega(\pi)$ the root number), and $\tilde{\pi}$ is the contragredient of π.

It is convenient to encapsulate the main parameters attached to π in a single quantity that occurs in many problems as a normalizing factor. For that purpose, we follow ([32]) to introduce the *analytic conductor* of π:

$$C_\pi := q_\pi \prod_{i=1}^{m} (1 + |\mu_{\pi,i}|),$$

which is actually a special case of the general definition but is sufficient for our purpose. As we will see in the later sections, the quantity C_π helps to give an elegant subconvex upper bound for automorphic L-functions of both $GL_{2,\mathbb{Q}} \times GL_{1,\mathbb{Q}}$ and $GL_{2,\mathbb{Q}} \times GL_{2,\mathbb{Q}}$.

Remark. While in the number field K, p^{-s} shall be replaced by $N(\mathfrak{P})^{-s}$ and

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2}\Gamma(\frac{s}{2}), & \text{if} \quad K_v \sim \mathbb{R} \\ (2\pi)^{-s}\Gamma(s), & \text{if} \quad K_v \sim \mathbb{C}; \end{cases}$$

and

$$C_K(\pi) = N_\pi \prod_{i=1}^{m} \prod_{v|\infty} \left(1 + |\mu_{\pi,i}|^{d(v)}\right),$$

where N_π is the arithmetic conductor; $d(v) = 1$ if $K_v \sim \mathbb{R}$, while $d(v) = 2$ if $K_v \sim \mathbb{C}$.

A.2. **Rankin-Selberg method: L-functions of pairs.** As mentioned in A, another class of L-functions fundamental to the whole theory are the so-called Rankin-Selberg type L-functions $L(\pi \otimes \pi', s)$ associated to pairs of automorphic representations $(\pi, \pi') \in \mathcal{A}_m^0(\mathbb{Q}) \times \mathcal{A}_{m'}^0(\mathbb{Q})$. This theory was initiated by Rankin and Selberg in the case of classical modular forms ([56]). For general automorphic forms, the analytic theory of L-functions of pairs was initiated and developed by series of work by Jacquet, Piatetsky-Shapiro and Shalika and completed in works of Shahidi, Moeglin/ Walspurger and Gelbart. We refer [10] again for a detailed exposition of their construction and derivation of their basic properties.

Given $(\pi, \pi') \in \mathcal{A}_m^0(\mathbb{Q}) \times \mathcal{A}_{m'}^0(\mathbb{Q})$, the Rankin-Selberg type L-function $L(\pi \otimes \pi', s)$ is a Dirichlet series

$$L(\pi \otimes \pi', s) = \prod_p L_p(\pi \otimes \pi', s) = \sum_{n \geq 1} \frac{\lambda_{\pi \otimes \pi'}(s)}{n^s},$$

absolutely convergent for $\mathrm{Re}\, s$ large enough. It is an Euler product of degree mm' with local factors of the form

$$L_p(\pi \otimes \pi', s) = L(\pi_p \otimes \pi'_p, s) = \prod_{i=1}^{mm'} \left(1 - \frac{\gamma_{\pi \otimes \pi', i}(p)}{p^s}\right)^{-1}$$

at the finite places, and at the infinite one, define

$$L_\infty(\pi \otimes \pi', s) = L(\pi_\infty \otimes \pi'_\infty, s) = \prod_{i=1}^{mm'} \Gamma_{\mathbb{R}}\left(s - \mu_{\pi \otimes \pi', i}\right).$$

Moreover, at places v for which π_v is unramified, $L_v(\pi \otimes \pi', s)$ has the explicit expression:

$$L_p(\pi \otimes \pi', s) = \prod_{i=1}^{m} \prod_{j=1}^{m'} \left(1 - \frac{\gamma_{\pi,i}(p)\gamma_{\pi',j}(p)}{p^s}\right)^{-1}$$

at place $v = p > \infty$, and at the infinite place,

$$L_\infty(\pi \otimes \pi', s) = \prod_{i=1}^{m} \prod_{j=1}^{m'} \Gamma_{\mathbb{R}}\left(s - \mu_{\pi,i} - \mu_{\pi',j}\right).$$

The completed L-functions also have similar properties such as meromorphic continuation to the whole plane with at most two simple poles, bounded in vertical strips, and functional equation with analytic conductor $C_{\pi\otimes\pi'}$ satisfying

$$C_{\pi\otimes\pi'} \ll_{m,m'} C_\pi^{m'} C_{\pi'}^{m},$$

which will be used in the later of this paper.

A.3. **Fundamental conjectures.** We now introduce some basic conjectures which lies in the center of this subject. The first is the well known generalization of Riemann's conjecture.

Conjecture 35. *(Grand Riemann Hypothesis)*
The nontrivial zeros of any $\Lambda(\pi, s)$ are all on the critical line, i.e. having real part equal to $\frac{1}{2}$.

Remark. Unfortunately, until now there is no $L(\pi, s)$ for which GRH is known, while the function field version (Weil's conjecture) has been solved. For families of L-functions such as $L(s, \chi)$, the so-called "Density theorems", which asserts that almost all their zeros lies near $\mathrm{Re}\, s = \frac{1}{2}$, is known.

GRH implies the so-called Lindelöff Hypothesis, which, if true, is a very useful bound for L-functions on the critical line. Lindelöff Hypothesis asserts that for any $\epsilon > 0$,

$$L(\frac{1}{2}, \pi) \ll_\epsilon (C_K(\pi))^\epsilon.$$

It follows from the functional equation for $\Lambda(s, \pi)$ and the convex upper bounds of Phragmen-Lindelöff that for any $\epsilon > 0$

$$L(\frac{1}{2}, \pi) \ll_\epsilon (C_K(\pi))^{\frac{1}{4}+\epsilon}.$$

Remark. We will often refer to this bound as the *trivial* bound, however, it is, in this generality, not quite a trivial result as we will give a concise proof later.

Conjecture 36. *(Subconvexity Problem)*
For any automorphic cuspidal representation π of $GL_m(K)$, there is an absolute constant $\delta > 0$ such that

$$L(\frac{1}{2}, \pi) \ll (C(\pi))^{\frac{1}{4}-\delta}.$$

Remark. The estimates of $2k$-th ($k \in \mathbb{N}$) moments of Riemann ζ-function on the critical line $\left\{s \in \mathbb{C} : \Re(s) = \frac{1}{2}\right\}$ lies in the center of classical analytic number theory. Assuming Riemann's Hypothesis (short for "RH"), asymptotic terms of the moments can be achieved(cf. [26]). Conversely, the moments may provide some hints to RH. Note that the value of $\zeta(s)$ on the critical line can be identified with the central value of some twisted L-function. Thus we restrict ourselves to the central value of the corresponding L-functions. One of the most fundamental problems in this direction is the so-called Subconvexity Problem (short for "ScP"). We, in this thesis, aim to discuss this topic via a somewhat higher perspective as we consider more general L-functions, i.e. automorphic L-functions. The central value of automorphic L-functions is of great interest due to its natural appearance in number theory and arithmetic geometry (e.g. BSD conjecture; or latest results in [70]); also, it provides an approach to arithmetic problems via modern analytic methods (e.g. spectral decomposition, Voroni-type summation formula). This interesting problem has a long history. The first subconvexity result is due to Weyl ([66]) for Riemann zeta funation:

$$\zeta(\frac{1}{2}+it) \ll |t|^{\frac{1}{6}+\varepsilon}.$$

However, a lot of methods had been developed to improve Weyl's exponent of $\frac{1}{6}$, such as theory of exponential pairs; the current record (to date), $\frac{32}{205}$, is obtained by M. Huxley (cf. [28]). Since $L(\frac{1}{2}+it,\pi) = L(\frac{1}{2},\pi\otimes|.|^{it})$, it will be sufficient to study the subconvex bound for $L(\frac{1}{2},\pi)$ in practise. Actually, in applications we usually consider some subfamily of such L-functions (e.g. only one of N_π or π_∞ varies) and we seek subconvex bounds uniformly for the subfamily. The ScP has been solved in lots of situations and we will discuss this and some of their applications in the next two sections.

Next we turn to the generalized Ramanujan conjecture, which is the local analogue of GRH and is a spectral concerning the local representations π_v of $GL_m(K_v)$ of the global automorphic cuspidal reporesentation π. It asserts that for a place v such that π_v is unramified, π_v should be tempered (see [54]). Equivalently, this can be formulated in terms of $L(s,\pi_v)$:

Conjecture 37. *Generalized Ramanujan Conjecture (GRC)*
Let $\pi \in \mathcal{A}_0\left(GL_m(\mathbb{A}_K)\setminus GL_m(\mathbb{A}_K),\omega\right)$ which is unramified at a place v. Then for $v<\infty$, $|\gamma_{pi,i}(v)| = 1$ while for $v \mid \infty$, $\Re(\mu_{\pi,i}(v)) = 0$.

Remark. There are nontrivial and quite useful general bounds towards GRC. Firstly, Jacquet and Shalika ([35]) gave the bounds:

$$\left|\log_{N(v)}|\gamma_{\pi,i}(v)|\right| < \frac{1}{2}, \quad \text{for} \quad v<\infty;$$
$$|\operatorname{Re}(\mu_{\pi,i}(v))| < \frac{1}{2}, \quad \text{for} \quad v \mid \infty.$$

This can be viewed as the analogue of the subconvexity bound for L-functions, and a general subconvex bound for $\pi \in \mathcal{A}_0\left(GL_m(\mathbb{A}_K)\setminus GL_m(\mathbb{A}_K),\omega\right)$ is known ([42]):

$$\left|\log_{N(v)}|\gamma_{\pi,i}(v)|\right| < \frac{1}{2}-\frac{1}{m^2+1}, \quad \text{for} v<\infty;$$
$$|\operatorname{Re}(\mu_{\pi,i}(v))| < \frac{1}{2}-\frac{1}{m^2+1}, \quad \text{for} v \mid \infty.$$

Combining this and results on symmetric square lift from $GL(2)$ to $GL(3)$ (see [20]) yields a "subconvix bound":

$$\left|\log_{N(v)}|\gamma_{\pi,i}(v)|\right| < \frac{1}{5}, \quad \text{for} \quad v<\infty; \quad |\operatorname{Re}(\mu_{\pi,i}(v))| < \frac{1}{5}, \quad \text{for} \quad v \mid \infty.$$

We will use this bound combined with some kind of subconvex bound and Kuznetsov's trace formula to give an nontrivial estimate on Kloosterman sum, which is much better than using Weil's bound directly.

A.4. **Effective methods.** In this subsection, we will mainly discuss three methods which are successful in handling ScP. To help better understand their ideas, we will either use them directly to prove some subconvex bound or illustrate their original ideas and developments. Roughly, we will talk about Weyl's shift, which is fundamental in classical approaches to ScP and necessary for any movement methods; then we come to the amplification method, which, basically, takes the advantages of averaging so as to breakthrough barriers from off-diagonal contribution; last, we would like to introduce the latest and powerful tool, the Ergodic method, and briefly show how to use it to prove Theorem 42.

A.4.1. *Around Weyl's shift.* This method provides non-trivial bounds for exponential sums of the form:

$$S_f(X) = \sum_{1\leq n\leq X} e(f(n)),$$

where f is a "good enough" function (e.g. well-approximable by polynomials). It is easy to see that

$$|S_f(X)|^2 = \sum_{1\leq n\leq X}\sum_{1\leq m\leq X} e(f(n)-f(m)) = \sum_{|l|<X}\sum_{\substack{1\leq n\leq X\\ 1\leq n+l\leq X}} e(f(n+l)-f(n)).$$

Now if f is exactly a polynomial (e.g. in Waring problem), $f(l+x)-f(x)$ is a polynomial in x of degree reduced by one. One can continue this process until one reaches a sum for a linear polynomial, where we have some sharp bounds to apply. Generally, this methods applies also to $f(n) = -\frac{it\log n}{2\pi}$, which is the case occuring for ζ. Then we will discuss a variant of the Weyl shifting technique but in a purely arithmetic context. By this method Burgess (cf. [7]) obtained the famous result on classical ScP:

Theorem 38. *For χ a Dirichlet character of character $q = q_\chi$ and order $k = k_\chi$, we have*

$$L(\chi, \frac{1}{2}+it) \ll_\varepsilon (1+|t|)^A q_\chi^{\frac{3}{16}+\varepsilon},$$

for some absolute constant A.

Remark. This bound resisted any improvement for the next 40 years until Conrey/Iwaniec improve the exponent to $\frac{1}{6}$ ([12]). Nowdays, the best result is due to Han Wu using methods from [47], which we will discuss later (see [68]):

$$L(\chi, \frac{1}{2}+it) \ll_\varepsilon (1+|t|)^A q_\chi^{\frac{1}{8}-\delta+\varepsilon},$$

where $\delta = \frac{1-2\theta}{8}$ and θ is such that no complementary series with parameter $>\theta$ appear as a component of a cuspidal automorphic representation of $G(\mathbb{A})$. We see in the remark of Conjecture (37).

A.4.2. *The amplification method.* Fix F a self-dual cusp form on $GL_m(\mathbb{A}_K)$. By contour shifts and the functional equations, we obtain

$$L(\frac{1}{2}, F) = 2\sum_{\mathfrak{a}\neq 0}\frac{c_F(\mathfrak{a})}{\sqrt{N\mathfrak{a}}} W\left(\frac{N\mathfrak{a}}{X}\right),$$

where $W(x)$ is a smooth function decaying rapidly as $x \longrightarrow \infty$ and $X = \sqrt{N\mathfrak{a}}$.

$c_F(\mathfrak{a})$'s are known in some cases to satisfy the GRC so that

$$c_F(\mathfrak{a}) \ll_\varepsilon (N\mathfrak{a})^\varepsilon.$$

In any case for F on $GL_m(\mathbb{A}_K)$ we have the above on average:

$$\sum_{N\mathfrak{a}\leq Y} |c_F(\mathfrak{a})|^2 \ll_\varepsilon Y(C(F))^\varepsilon.$$

So the trivial bound is $L(\frac{1}{2}, F) \ll (C(F))^{\frac{1}{4}+\varepsilon}$ by Cauchy's inequality.

Now set

$$\mathcal{S}(F) := \sum_{F\in\mathcal{F}} |L(\frac{1}{2}, F)|^2,$$

where the analytic conductors $C(F)$'s are all assumed to be the same (or nearly the same) size. In some cases we may take higher moments of L in $\mathcal{S}(F)$ (However, it may be even better to consider

$$\tilde{\mathcal{S}}(F) := \sum_{F\in\mathcal{F}} |L(\frac{1}{2}, sym^n F \otimes sym^n F)|^2).$$

GRH (via Lindelöff Hypothesis) induces $L(\frac{1}{2}, F) \ll_\varepsilon (C(F))^\varepsilon$, so we can expect that:

$$\mathcal{S}(F) \ll_\varepsilon |\mathcal{F}|(C(F))^\varepsilon, \tag{25}$$

which implies that

$$L(\frac{1}{2}, F) \ll_\varepsilon |\mathcal{F}|^{\frac{1}{2}}(C(F))^\varepsilon. \tag{26}$$

So if the family $\mathcal{F}$ is sufficiently small (precisely, $|\mathcal{F}| \ll (C(F))^{1/2-\delta}$) and at the same time rich enough to establish (25), then (26) will yield a subconvex bound.

In practice, when this method succeeds one finds that one can establish (25) with $|\mathcal{F}| \asymp (C(F))^{1/2}$ in a relatively straight-forward analysis involving summing over $\mathcal{F}$ and analyzing only the "diagonal" contribution. This however simply recovers the convexity bound and the heart of the problem is to decrease somewhat the size of $|\mathcal{F}|$.

We next discuss the technique named as "amplification" which can arithmetically sharp $\mathcal{F}$ by introducing weights. The method was introduced to apply on $L(\frac{1}{2}, \chi)$ (see [16]), however its true power can be shown in the more general setting of $L(s, \pi)$. Roughly the idea is as follows:

$$\mathcal{A} := \sum_{F\in\mathcal{F}} \left| L(\frac{1}{2}, F) \sum_{N\mathfrak{b}\le M} a(\mathfrak{b}) c_F(\mathfrak{b}) \right|^2,$$

where $a(\mathfrak{b})$ is arbitrary constants with modulus ≤ 1. This time we expect the bound

$$\mathcal{A} \ll_\varepsilon M|\mathcal{F}|X^\varepsilon. \tag{27}$$

In order to establish (27), one faces off-diagonal terms and if these can be successfully estimated then choosing $a(\mathfrak{b}) = \overline{c_F(\mathfrak{b})}$ i.e. amplifying F_0, we obtain

$$\left| L(\frac{1}{2}, F_0) \sum_{N\mathfrak{b}\le M} |c_F(\mathfrak{b})|^2 \right|^2 \ll_\varepsilon |\mathcal{F}|MX^\varepsilon,$$

which implies a subconvex bound for $L(\frac{1}{2}, F_0)$.

So the key features are the family and dealing with the off-diagonal sums. For example in Theorem 40, the family used is $L(s, F\otimes\chi)$ with χ a Dirichlet character of conductor q. The key off-diagonal sums that used to be treated are of type

$$\sum_{\nu\gamma-\mu\beta=h} c_F(\gamma)c_F(\beta)W\left(\frac{N(\gamma)}{X}\right)W\left(\frac{N(\beta)}{X}\right)G(\gamma,\beta),$$

where ν and μ are fixed small integers in K, $h \neq 0$ and G is a smooth function depending in the arguments of γ and β in the embeddings of K into $\mathbb{R}$.

Over $\mathbb{Q}$ Duke/Friedlander/Iwanoec use Weil's bounds on Kloosterman sums to handle this (see [14] for more details). In general, one uses the full Maass form spectral theory for $GL_2(\mathbb{A}_K)$ and a suitable theory of Poincaŕe series. Crucial ingredients are the GRC bounds:

$$\left|\log_{N(v)} |\gamma_{\pi,i}(v)|\right| < \frac{1}{5}, \quad \text{for} \quad v < \infty; \quad |\mathrm{Re}(\mu_{\pi,i}(v))| < \frac{1}{5}, \quad \text{for} \quad v \mid \infty,$$

and the spectral method.

For the case of $L(s, F \otimes G)$, where G is fixed, F varies over holomorphic newforms of the same weight as G but of level $N \to +\infty$. Then we have

$$L(s, F \otimes G) \ll N^{\frac{1}{2} - \frac{1}{96} + \varepsilon}.$$

The averaging can be handled by means of the Prtersson formula. Let $B_k(N)$ be an orthogonal basis for $S_k(N)$, the space of cusp forms of weight k, level N. Normalize the Fourier coefficients of $\lambda_F(n)$ for $F \in S_k(N)$ by setting

$$\psi_F(n) = \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \right)^{\frac{1}{2}} \frac{\lambda_F(n)}{\|F\|_{Pet}},$$

where $\|\|_{Pet}$ denotes Petersson inner product. Then the formula reads that for m,n ≥ 1,

$$\sum_{F \in B_k(N)} \overline{\psi_F(n)} \psi_F(m) = \delta(m,n) + 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c} \right), \tag{28}$$

where $\delta(m,n)$ is the Kronecker delta function, $J_k(x)$ is the Bessel function and $S(m,n;c)$ is the Kloosterman sum:

$$S(m,n;c) := \sum_{\substack{x(c) \\ x\overline{x} \equiv 1(c)}} e\left(\frac{mx + n\overline{x}}{c} \right).$$

The Petersson formula (28) and its generalization due to Kuznetsov are important tools in the subject. They lie at the bottom of many of the applications of the $GL_2/\mathbb{Q}$ analytic number theory. Particularly, these formulas can give somewhat better estimates of weighted Kloosterman sums in average. For instance, Kuznetsov ([36]) used his trace formula and the fact that $\lambda_1(SL(2,\mathbb{Z}) \setminus \mathbb{H}) \geq \frac{1}{4}$, to show that for m, n fixed

$$\sum_{c \leq X} \frac{S(m,n;c)}{c} \ll_\varepsilon X^{\frac{2}{3} + \varepsilon}.$$

Note that Weil's bound gives the bound $O\left(X^{1+\varepsilon}\right)$.

A.4.3. *The Ergodic method.* In this subsection, we will introduce how Ergodic principle serves as a significant powerful (though always not effective) tool to deal with ScP via sketching the proof of Theorem 42. To start with, we list the so-called quantitative form of the Ergodic principle.

Let $\iota : \widetilde{G} \longrightarrow G$ be the simply connected covering of G. For $f \in C^\infty(X)$, for $x \in X$ set

$$\wp(f) := \int_{g \in \widetilde{G}(K) \setminus \widetilde{G}(\mathbb{A})} f(\iota(g)x) dg.$$

Then for every $g \in G$, there exists $\beta > 0$ and d, such that

$$|\langle g.f_1, f_2 \rangle - \langle g.\wp(f_1), g.\wp(f_2) \rangle| \ll \|Ad(g)\|^{\theta - \frac{1}{2}} \mathcal{S}_d^X(f_1) \mathcal{S}_d^X(f_2), \tag{29}$$

where θ is given in the last section as a standard notation. Now we can take $\theta = \frac{7}{64}$ due to [2].

To start with, let briefly recall the Quantitative Form of the Ergodic Principle (cf. Lemma 2.5.3 in [47]): Suppose $H \subset G(\mathbb{A})$ is noncompact, and χ: $H \longrightarrow \mathbb{C}^\times$ a unitary character. Assume that X has finite measure. Let ν be a (possibly signed) χ-equivariant measure on X. Let μ be the $G(\mathbb{A})$-invariant (Haar) probability measure, and suppose that, for some $d \geq 0$, we have the majorization:

$$|\nu|(f) \ll \mu(f) + \epsilon \mathcal{S}_d^X(f), \quad (f \geq 0).$$

Let σ be any probability measure on H. Then, for any f with $\wp(f) = 0$,

$$|\nu(f) - \delta_{\chi=1}\mu(f)| \ll \left(\epsilon \|\sigma\|_{d'}^2 + \|\sigma \star \check{\sigma}\|_{-\beta} \right) \mathcal{S}_{d'}^X(f)^2.$$

Here $\delta_{\chi=1}$ is 1 if χ is trivial and zero otherwise,

$$\|\sigma\|_d := \int_{h\in H} \|Ad(h)\|^d d\sigma(h),$$

and similarly for $\|\sigma\star\check{\sigma}\|$, $\check{\sigma}$ denote the pullback of σ by $g\mapsto g^{-1}$. β is as in (29), and d' depends only on d.

It should be pointed out that *Ergidic Method* is based on *Property* (τ), which has been established through the work of various mathematicians, and the proof was completed in [9]. This methods should be used together with *Amplification Method* since it just provides a frame where we can pick a good measure σ (cf. 5.2.3 [47]) as an effective amplifier. We explain this idea by sketching the proof of (34). In fact, Theorem 42 can be reduced by period integral and regularization to case there $\pi_1=\chi$, which is exactly Theorem 4. Here the measure σ (as constructed in Section 4.1 of [63]) plays a role similar to that of an amplifier in the amplification method. Precise description will show in the proof of Theorem 42.

A.5. **Results on ScP.** To start with, we consider the following result (cf. [7], with a slight modification):

Theorem 39. *For χ a Dirichlet character of character $q=q_\chi$ and order $k=k_\chi$, we have*

$$L(\chi,\frac{1}{2}+it)\ll_\varepsilon (1+|t|)^\delta q^{\frac{3}{16}+\varepsilon},$$

for some absolute constant δ.

Remark. This bound resisted any improvement for the next 40 years until Conrey/Iwaniec improve the exponent to $\frac{1}{6}$ ([12]).

Proof. Note that, duo to the approximate functional equation for $L(\chi,s)$, it suffices to give a bound for the character sum:

$$S_W(\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^{\frac{1}{2}}}W\left(\frac{n}{q^{1/2}}\right),$$

where $q=q_\chi$ and $W(x)$ is a smooth function decaying rapidly as $x\longrightarrow\infty$. Integrating by part it suffices to bound the partial sum

$$S(\chi,N)=\sum_n \chi(n)$$

non-tivially for $q^{\frac{1}{2}-\delta}\leq N<q$, for some fixed $\delta>0$. For simplicity, we shall handle a smoothed version of $S(\chi,N)$, i.e.

$$S_\chi(N)=\sum_{n\leq N}\chi(n)h(n),$$

where h is a smooth function which is 1 on $[1,N]$ and zero outside $[0,N+1]$. Note that

$$\hat{h}(y)=\int_{\mathbb{R}} h(x)e(-xy)dx\ll \min\{N,y^{-1},y^{-2}\},\quad \int_{\mathbb{R}}|\hat{h}(t)|dt\leq \log 3N.$$

We have

$$S_\chi(N)=\sum_n \chi(n)h(n)=\sum_n \chi(n+ab)h(n+ab),$$

for all a, b ≥ 1. Set $AB = N$, A, B ≥ 1, then

$$\begin{aligned}\varphi_q(\lfloor A \rfloor)\lfloor B \rfloor S_\chi(N) &= \sum_{\substack{a \leq A \\ (a,q)=1}} \sum_{b \leq B} \sum_{n} \chi(n+ab)h(n+ab) \\ &= \sum_{\substack{a \leq A \\ (a,q)=1}} \chi(a) \sum_{|n|\leq N} \sum_{b \leq B} \chi(\bar{a}n+b)h(n+ab) \\ &\leq \sum_{\substack{a \leq A \\ (a,q)=1}} \sum_{|n|\leq N} \left| \sum_{b \leq B} \int_{-\infty}^{+\infty} \frac{1}{a}\hat{h}(\frac{t}{a})e(\frac{nt}{a})\chi(\bar{a}n+b)e(bt)dt \right|,\end{aligned}$$

by Fourier inversion. Hence

$$\varphi_q(\lfloor A \rfloor)\lfloor B \rfloor S_\chi(N) \ll \log N \sum_{\substack{a \leq A \\ (a,q)=1}} \sum_{|n|\leq N} |\sum_{b \leq B} \chi(\bar{a}n+b)e(bt_*)|$$

for some $t_* \in \mathbb{R}$. For $u \in \mathbb{F}_q$, we set

$$\nu(u) = \#\left\{1 \leq a \leq A, (a,q)=1, |n| \leq N, \bar{a}n \equiv u(q)\right\},$$

then

$$\begin{aligned}\varphi_q(\lfloor A \rfloor)\lfloor B \rfloor S_\chi(N) &\ll \log N \sum_{u(q)} \nu(u) \left| \sum_{b \leq B} \chi(u+b)e(bt_*) \right| \\ &\leq \left(\sum_{u(q)} \nu(u) \right)^{1-\frac{1}{r}} \left(\sum_{u(q)} \nu(u)^2 \right)^{\frac{1}{2r}} \left(\sum_{u(q)} \nu(u) \left| \sum_{b \leq B} \chi(u+b)e(bt_*) \right|^{2r} \right)^{\frac{1}{2r}},\end{aligned}$$

where φ_q represents Euler function *mod* q. Note that

$$\sum_{u(q)} \nu(u) = \varphi_q(\lfloor A \rfloor)N,$$

and for $AN < \frac{q}{2}$ we have

$$\sum_{u(q)} \nu(u)^2 = \#\left\{1 \leq a_1, a_2 \leq N, |n_1|, |n_2| \leq N, a_2n_1 - a_1n_2 = 0\right\} \ll AN(\log AN)^3.$$

Hence we conclude that

$$S_\chi(N) \ll (\log q)^{\frac{2r+3}{2r}} A^{\frac{-1}{2r}} N^{\frac{2r-1}{2r}} B^{-1} \left(\sum_{\substack{i \leq r \\ b_i, b_i' \leq B}} \left| \sum_{u(q)} \chi(\prod_{j \leq r} u+b_j)\bar{\chi}(\prod_{k \leq r} u+b_k') \right| \right)^{1/2r}.$$

Now, we will use Heath-Brown's result (cf. Lemma 10 of [25]) for the algebraic exponential sum:

If $\prod_{i=1}^{r} \frac{u+b_i}{u+b_{i'}}$ is not a k-th power, one has

$$\left| \sum_{u(q)} \chi(\prod_{j \leq r} u+b_j)\bar{\chi}(\prod_{k \leq r} u+b_k') \right| \ll_r q^{1/2}.$$

The number of $(b_1, ..., b'_r)$ not satisfying this criterion is $\ll_r B^r$, and in this case we bound the sum trivially by q. Hence

$$\sum_{\substack{i\leq r\\ b_i,b'_i\leq B}} \left|\sum_{u(q)} \chi(\prod_{j\leq r} u+b_j)\bar{\chi}(\prod_{k\leq r} u+b'_k)\right| \ll B^{2r}q^{1/2} + B^r q.$$

Then we just take

$$\delta = Nq^{-1/2r}, \quad B = q^{1/2r} \quad \text{with} \quad r = 4,$$

and Theorem 39 follows. □

The theorem above is a typical result from analytic number theory. While, as we will see, some generalizations in terms of automorphic L-functions are available duo to the effective methods listed in the last section. We will quote the most classical one (as below), and many other results. In the end of this section, we will mention Michel and Venkatesh's general solution to subconvexity problem as give a sketch of their proof.

Theorem 40 ([14])**.** *For a fixed π on $GL(2)/\mathbb{Q}$ a cuspidal eigen-form and χ a primitive Dirichlet character with conductor q, then for any $s \in \mathbb{C}$, $\mathrm{Re}(s) = \frac{1}{2}$, we have*

$$L(s, \pi \otimes \chi) \ll q^{\frac{5}{12}+\varepsilon}.$$

Remark. The subconvex bound is $q^{\frac{1}{2}}$. While Theorem 40 gives nontrivial bounds for the square-free Fourier coefficients of half-integral weight holomorphic cusp forms. Nowdays, the best exponent (can be taken to be $\frac{25}{256}$ so far) is due to Han Wu using methods from [47], which we will discuss later (see [68]). Moreover, Lindelöff Hypothesis in the quadratic twisting χ aspect for $L(\frac{1}{2}, \pi \otimes \chi)$, is equivalent to (or determines) the half-integral weight GRC.

From Theorem 40 we can see an application of ScP, while, we will show more and concrete instances where subconvex bounds play significant roles in various aspects of number theory. Now, we would like to conclude some classical results on ScP by combining main results on ScP by 2006 and give some brief comments on them respectively. Finally, we illustrate concisely the completed results due to Michel and Venkatesh ([47]).

Theorem 41. *Let K be a fixed number field and π_2 be a fixed cuspidal automorphic representation of $GL_2(\mathbb{A}_K)$. Let χ_1, π_1 denote respectively a $GL_1(\mathbb{A}_K)-$automorphic representation (i.e. a Grossencharacter), a $GL_2(\mathbb{A}_K)-$automorphic representation and let $\mathfrak{q}_1$ denote either the conductor of χ_1 or π_1 and $q_1 = N_{K/\mathbb{Q}}(\mathfrak{q}_1)$. There exists an absolute $\delta > 0$ (independent of χ_1, π_1, π_2 and K) such that for $\mathrm{Re}(s) = \frac{1}{2}$ one has:*

$$L(\chi_1, s) \ll_s q_1^{1/4-\delta}, \tag{30}$$

$$L(\chi_1 \times \pi_1, s) \ll_{s,\pi_2,\pi_{1,\infty}} q_1^{1/2-\delta}, \tag{31}$$

$$L(\pi_1, s) \ll_{s,\pi_{1,\infty}} q_1^{1/4-\delta}, \tag{32}$$

$$L(\pi_1 \times \pi_2, s) \ll_{s,\pi_2,\pi_{1,\infty}} q_1^{1/2-\delta}. \tag{33}$$

Remark. For $K = \mathbb{Q}$, (30) is a generalization of Burgess bound and can be further improved when χ is quadratic (see [12]). The initial and somewhat easier case (i.e. the central character of π_1 is trivial) of bound (31) and (32) are mainly due to series of works by Duke/Friedlander/Iwaniec (see [14] and [15]). It was completed for π_1 with arbitrary central character in [5]. The bound for the Rankin-Selberg L-functions (33) for π_1 with arbitrary central character was obtained in [27].

In the case of a number field of higher degree, the first general subconvex result is due to Cogdell/Piatetski-Shapiro/Sarnak [10]: It consists of (31) when F is a totally real field and $\pi_{2,\infty}$ is in the holomorphic discrete series (i.e. corresponds to a holomorphic Hilbert modular form). In [63] ScP (31),(32) and (33) were established for π_1 with trivial central character. Eventually, Michel and Venkatesh combined their respective methods from [45] and [63] to obtain (32) and (33) for π_1 with arbitrary central character.

Now let's further discuss the most general solution to ScP for $GL_{1,\mathbb{Q}}$ and $GL_{2,\mathbb{Q}}$:

Theorem 42 (Theorem 1.2 in [47]). *There is an absolute constant $\delta > 0$ such that: for π_1, π_2 automorphic representations on $GL_2(\mathbb{A}_K)$ (with unitary central character, not necessary of finite order), we have*

$$L(\pi_1 \otimes \pi_2, \frac{1}{2}) \ll_{K,\pi_2} (C(\pi_1 \otimes \pi_2))^{\frac{1}{4}-\delta}. \tag{34}$$

More precisely, the constant implied depends polynomially on the discriminant of K (for K varying over fields of given degree) and on $C(\pi_2)$.

Sketch of the proof. Precisely, we have

$$\begin{aligned}\frac{L(\pi_1 \otimes \pi_2, \frac{1}{2})}{C(\pi_1)^{\frac{1}{2}+\varepsilon}} &\ll_{F,\varepsilon,\pi_2} \left| L^*(\pi_2, Ad, 1)\, L^*(\pi_3, Ad, 1) \int_X \phi_1\phi_2 E(g)dg \right| \\ &\ll_{F,\varepsilon,\pi_2} C(\pi_1)^{\frac{\delta}{9}} \langle \phi_1, \overline{\phi_2 E}\rangle,\end{aligned}$$

where $E \in \pi_3 = 1 \boxplus (\omega_1\omega_2)^{-1}$ and $\phi_i \in \pi_i$ $(i = 1, 2)$ as chosen in Section 5.2.2 in [47]. Set $Q := C(\pi_1)C(\pi_2)$ then it suffices to show that $\langle\ \phi_2 E, \phi_2 E\rangle \ll_{\pi_2} Q^{-\delta}$ by Cauchy-Schwarz's inequality. To achieve that, we shall use amplification method by choosing a compactly supported real signed measure σ on $GL_2(\mathbb{A}_f)$ which satisfies:

1. For every $g \in supp(\sigma)$, we have $||g|| \leq K$, where K is to be taken as a fixed positive (small) power of Q.
2. supp(σ) commutes with $GL_2(F_\nu)$ at all such ν that $\nu \mid \infty$ or $\pi_{i,\nu}$ (i=1,2) is not spherical.
3. $(\phi_2 E) \star \check{\sigma}$ is rapidly decreasing over X.
4. There is an absolute constant B such that $\int X d|\sigma| \leq K^B$. Moreover,
$$\int_X ||Ad(g)||^\beta d|\sigma| \star |\check{\sigma}|(g) \leq K^{-\gamma}$$
holds, where $\beta > \frac{1}{2}$ and $\gamma > 0$ is an constant uniquely determined by β.
5. There is a $\lambda \gg_\varepsilon Q^{-\varepsilon}$ such that $\phi_1 \star \sigma = \lambda_1 \cdot \phi_1$ for any $\varepsilon > 0$.

Actually, we can take σ to be an truncated linear combination of weighted Dirac measure similar to (18). By Cauchy-Schwarz we have

$$\begin{aligned}\left|\lambda\langle \phi_1, \overline{\phi_2 \cdot E}\rangle\right|^2 = \left|\langle \phi_1 \star \sigma, \overline{\phi_2 \cdot E}\rangle\right|^2 &= \left|\langle \phi_1, \overline{(\phi_2 \cdot E) \star \check{\sigma}}\rangle\right|^2 \\ &\leq \langle (\phi_2 \cdot E) \star \check{\sigma}, (\phi_2 \cdot E) \star \check{\sigma}\rangle.\end{aligned}$$

Now we make use of regularized integrals to expand the integral. This is justified by definition. Thus we have

$$\begin{aligned}\left|\lambda\langle \phi_1, \overline{\phi_2 \cdot E}\rangle\right|^2 &\leq \int_X \left|\langle \phi_2^g \cdot E^g, \phi_2 \cdot E\rangle_{reg}\right| d|\sigma| \star |\check{\sigma}|(g) \\ &= \int_X \left|\langle \phi_2^g \cdot \overline{\phi_2}, \overline{E^g} \cdot E\rangle_{reg}\right| d|\sigma| \star |\check{\sigma}|(g).\end{aligned}$$

Then it suffices to show that

$$(35)\qquad \left|\langle \phi_2^g \cdot \overline{\phi_2}, \overline{E^g} \cdot E\rangle_{reg}\right| \ll_{\pi_2,\varepsilon} C(\pi_1)^{\frac{\delta}{9}} ||Ad(g)||^{1-\theta} + ||g||^C Q^{\frac{1-2\theta}{8}} =: \mathfrak{E}_1 + \mathfrak{E}_2,$$

where $\mathfrak{E}_1$ comes from the degenerate terms and $\mathfrak{E}_2$ comes from the generic term of regularized Plancherel formula (cf. [68]) together Wu's estimate (cf. [47]). Here C is a fixed unspecified (absolute) constant.

Remark. Note that the contribution of $\mathfrak{E}_2$ is determined mainly by the subconvex bound for $GL_2 \times \mathrm{GL}_1$, convex bound as well as property (τ). Since for any constant $N > 0$ one has

$$(36)\qquad \int_\pi C(\pi)^N \mathcal{S}_d^\pi \left(\overline{E^g}_s \cdot E_s\right) d\mu_P(\pi) \ll_N \int_\pi \mathcal{S}_{d+O(N)}^\pi \left(\overline{E^g}_s \cdot E_s\right) d\mu_P(\pi) \ll_{\pi_2,N} 1,$$

thus if we have a polynomial dependence on $C(\pi)$ in [68] (or Theorem 4 with an explicit exponent), and the exponent C is specified, then we come to the conclusion that the exponent δ in Theorem 42 can be explicit! That is why we are interested in making the implied constants explicit in (5).

By taking π_2 to be a suitable Eisenstein series, i.e. $1 \boxplus 1$ in the above theorem, we get the following result:

Corollary 43 (Theorem 1.1 in [47])**.** *There exists an absolute constant $\delta > 0$ such that: for π an automorphic representation of $GL_1(\mathbb{A}_K)$ or $GL_2(\mathbb{A}_K)$ (with unitary central character, not necessary of finite order), one has*

$$L(\pi, \frac{1}{2}) \ll_K (C(\pi))^{\frac{1}{4}-\delta}.$$

□

Appendix B. Applications of ScP

B.1. **Equidistribution problems.** In this section, we will describe some applications of the subconvex bound to various equidistribution problems. Among those problems, we pick up the most classical or remarkable ones to discuss, for instance, Linnik problem and equidistribution of Heegner points.

B.1.1. *Equidistribution of lattice points of the sphere.* Given $n \geq 1$, it goes back to Gauss that n is representable as the ternary quadratic form: $X^2 + Y^2 + Z^2$ if and only if n is not the form $4^k(8l-1)$. We denote by

$$R_3(n) := \{\overrightarrow{x} = (x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = n\}$$

and denoted by $r_3(n) := |R_3(n)|$ the number of such representations. By a result of Cornut (see [13]) we know that

$$(37)\qquad r_3(n) \gg_\varepsilon n^{\frac{1}{2}-\varepsilon} \quad \text{if} \quad n \neq 0, 4, 7(8),$$

which means that there are sufficiently many vectors in $R_3(n)$ if $n \neq 0, 4, 7(8)$. So one may then look at the distribution of their projections on the unit sphere S^2 as $n \to \infty$. This question was studied by Linnik ([40]): by usinbg ergodic methods, he proved the equidistribution of the projections under some annoying condition, which was removed by Iwaniec by different methods (see [30]).

Theorem 44. *As n goes to $+\infty$ through integers $n \neq 0, 4, 7(8)$, the set $\frac{R_3(n)}{\sqrt{n}}$ becomes eqiudistributed on the unit sphere S^2 with respect to the standard Lebesgue measure; i.e. for any continuous functions V on S^2,*

$$\frac{1}{r_3(n)} \sum_{\overrightarrow{x} \in R_3(n)} V\left(\frac{R_3(n)}{\sqrt{n}}\right) \to \int_{S^2} V(u) d\mu.$$

Proof. By Weyl's equidistribution criterion, it is sufficient to show that for any harmonic polynomial P on $\mathbb{R}^3$ of degree $k \geq 1$, say, the Weyl sum

$$W(n,P) := \frac{1}{r_3(n)} \sum_{\vec{x} \in R_3(n)} P\left(\frac{R_3(n)}{\sqrt{n}}\right) \to \int_{S^2} P(u)d\mu = 0.$$

Observe that $W(n,P) = 0$ if k is odd. If k is even, one has

$$W(n,P) = \frac{n^{-\frac{k}{4}}}{r_3(n)} \sum_{\vec{x} \in R_3(n)} P\left(R_3(n)\right) = \frac{n^{-\frac{k}{4}}}{r_3(n)} r_P(n),$$

say. In view of (37) it suffices to show that $r_P(n) \ll n^{\frac{k+1}{2}-\delta}$ for some absolute $\delta > 0$. The theta series

$$\Theta_P(z) := \sum_{n \geq 0} r_P(n) e(nz)$$

is in fact a holomorphic modular form of weight $k + \frac{3}{2}$ and level 4 and a cusp form when $k \geq 1$ (see [31]). For $k \leq 1$ we have the Fourier expansion:

$$f(z) = \sum_{n \geq 1} \rho_f(n) n^{\frac{1}{2}} e(nz).$$

By Petersson's formula we have $\rho_f(n) \ll_{f,\varepsilon} n^{-\frac{1}{4}+\varepsilon}$ for any $\varepsilon > 0$. This bound yields $r_P(n) \ll_{P,\varepsilon} n^{\frac{k+1}{2}+\varepsilon}$, which is barely not sufficient for our equidistribution problem. To conquer this, we consider Waldspurger's formula, which relates the fourier coefficient $\rho_f d$ to a central value of a twisted L-function. If D is the discriminant of some quadratic field K, with $|D|^{k+1} > 0$, then Waldspurger's formula has the form

$$|\rho_f(|D|)|^2 = C(f,g,D) L(g \otimes \chi_K, 1/2),$$

where g is a cusp form (see [57]), χ_K is the associated Kronecker symbol and $C(f,g,D)$, the proportionality constant, is bounded independently of D. In particular, for $D = Disc\left(\mathbb{Q}(\sqrt{(-1)^{k+1}d})\right)$ we see by Deligen's bound, that

$$d^{1/2} \rho_f(d) \ll_{\varepsilon,f} d^{\varepsilon} |L(g \otimes \chi_K, 1/2)|^{1/2}.$$

Hence, any improvement over the $1/4$ exponent in $\rho_f(n)$ is equivalent to the solution to the ScP for $L(g \otimes \chi_K, 1/2)$, which can be solved by Theorem 42. $\square$

B.1.2. *Equidistribution of Heegner points (classical case).* We will, in this subsection, describe the equidistribution of Heegner points and their applications in elliptic curves. Let F be a general totally real field. Let B be a quaternion algebra over F and π a cuspidal automorphic representation of $B_{\mathbb{A}}^{\times}$ with central character ω. Let K be a quadratic field extension of F and η the quadratic Hecke character on $F^{\times} \setminus \mathbb{A}^{\times}$ associated to the quadratic extension. Let χ be a Hecke character on $K_{\mathbb{A}}^{\times}$. Write $L(s,\pi,\chi)$ for the Rankin-Selberg L-series $L(s, \pi^{JL} \otimes \pi_{\chi})$, where π^{JL} is the Jacquet-Langlands correspondence of π on $GL_2(\mathbb{A})$ and π_{χ} is the automorphic representation of $GL_2(\mathbb{A})$ corresponding to the theta series of χ, so that $L(s,\pi_{\chi}) = L(s,\chi)$. Assume that $\omega\chi\mid_{\mathbb{A}^{\times}} = 1$.

Let N be the conductor of π^{JL}, D the relative discriminant of K/F, $\mathfrak{c} \subset \mathcal{O}_F$ a fixed ideal.

Further, we assume that for any place $v \mid \infty$ of F, π_v^{JL} is a unitary discrete series of weight 2 and $(\mathfrak{c}, N) = 1$. Let $X = B^{\times} \setminus \hat{B}^{\times}/\hat{R}^{\times}$. Since $B^{\times} \setminus \hat{B}^{\times}$ is compact and $\hat{R}^{\times}$ is open, so X is a finite set. Let $g_1, ..., g_n \in \hat{B}^{\times}$ be a complete set of representations of X. Write $e_i = [g_i] \in X$ for the class of the element $g_i \in \hat{B}^{\times}$. For each g_i, denote by $\Gamma_i = (B^{\times} \cap g_i \hat{B}^{\times} g_i^{-1})/\mathcal{O}^{\times}$, which is finite and denoted by ω_i its order. Let $\mathbb{Z}[X]$ be the free $\mathbb{Z}$-module (of rank n) of formal sums $\sum_i a_i e_i$. There is a height pairing on $\mathbb{Z}[X] \times \mathbb{Z}[X]$ defined by

$$\langle \sum_i a_i e_i, \sum_j b_j e_j \rangle = \sum_i a_i b_i \omega_i.$$

Define $\mathbb{Z}[X]^0 := \mathrm{Pic}(X)^0$, $\mathbb{C}[X]^0 = \mathbb{Z}[X]^0 \otimes_{\mathbb{Z}} \mathbb{C}$.

Denote by $H_N(\mathfrak{c})$ the set of Gross-Heegner points of level $\mathfrak{c}$, which is nonempty under Heegner conditions given in [3]. Consider a Heegner point which is associated to an (optimal) embedding $\xi : K \hookrightarrow B$ which can be lifted as a map $\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}}) \to X$. Hence $\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})$ has an action on $H_N(\mathfrak{c})$. Define an element in $\mathbb{C}[X]$,

$$P_\chi := \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \chi^{-1}(\sigma)\xi(\sigma).$$

For any subgroup $G \leq \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})$ we have, by Fourier inversion, that

$$\begin{aligned}\frac{1}{|G|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \xi(\sigma) &= \frac{1}{|\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})|} \sum_{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})}} \left(\frac{1}{|G|} \sum_{\sigma' \in G} \chi(\sigma') \right) \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \chi^{-1}(\sigma)\xi(\sigma) \\ &= \frac{1}{|\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})|} \sum_{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})}} \left(\frac{1}{|G|} \sum_{\sigma' \in G} \chi(\sigma') \right) P_\chi \\ &= \frac{1}{|\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})|} \sum_{\substack{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \\ \chi|_G = 1}} P_\chi.\end{aligned}$$

Thus, for any $0 \neq f$ we have

$$\begin{aligned}\langle f, \frac{1}{|G|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \xi(\sigma) \rangle &= \frac{1}{|\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})|} \langle f, \sum_{\substack{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \\ \chi|_G = 1}} P_\chi \rangle \\ &\leq \frac{1}{|G|} \max_{\substack{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \\ \chi|_G = 1}} |\langle f, P_\chi \rangle| \\ &= \frac{\langle f, f \rangle^{1/2}}{|G|} \max_{\substack{\chi \in \widehat{\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}})} \\ \chi|_G = 1}} \langle P_\chi^\pi, P_\chi^\pi \rangle^{1/2}.\end{aligned}$$

If such f's can form a basis of $\mathbb{C}[X]^0$, and $\frac{\langle f, f \rangle^{1/2}}{|G|} \max_\chi |P_\chi^\pi| \ll |G|^{-\delta}$, then by a direct computation we will have essentially that

$$\frac{|\{\sigma \in G, \xi(\sigma) = e_i\}|}{|G|} = \mu_N(e_i) + O_N\left(|G|^{-\delta}\right),$$

if G is not too small, where μ_N is a probability measure on X defined by:

$$\mu_N(\{e_i\}) = \frac{\omega_i^{-1}}{\sum_{i=1}^n \omega_i^{-1}}. \tag{38}$$

Precisely, we have the following theorem:

Theorem 45. *Let B, K defined as above satisfying the properties, while $R = R_{\mathfrak{n}_1, \mathfrak{n}_2}$ the classical Eichler order of level N. Then there are absolute constants $\delta, \delta' > 0$ such that for any $\xi \in H_N(\mathfrak{c})$, for any subgroup $G \leq Pic(\mathcal{O}_{\mathfrak{c}})$ of index $[Pic(\mathcal{O}_{\mathfrak{c}}) : G] \leq |D|^\delta$, as $D \to +\infty$, the orbit $G.\xi$ becomes equidistributed in the set $\{e_1, ..., e_n\}$ relative to the measure μ_N defined in (38).*

More precisely, there exists an absolute constant $\delta' > 0$ such that

$$\frac{|\{\sigma \in G, \xi(\sigma) = e_i\}|}{|G|} = \mu_N(\{e_i\}) + O_N\left(|D|^{-\delta'}\right)$$

Proof. Similar to [45], by Jacquet-Langlands correspondence we have finite basis of $\mathbb{C}[X]^0$ consisting of a family of Hilbert modular forms $\{f_i\} \in V(\pi_i, \chi)$ where $i \geq 1$. Define

$$f_0 := \left(\sum_{i=1}^n \omega_i^{-1}\right)^{-\frac{1}{2}} \sum_{i=1}^n \omega_i^{-1} e_i.$$

Write $e_i = \langle e_i, f_0 \rangle f_0 + \sum_{i \geq 1} a_i f_i$. Then

$$\begin{aligned} \omega_i \frac{|\{\sigma \in G, \xi(\sigma) = e_i\}|}{|G|} &= \langle e_i, \frac{1}{|G|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \xi(\sigma) \rangle \\ &= \left(\sum_{i=1}^n \omega_i^{-1}\right)^{-1} + \sum_{i \geq 1} a_i \langle f_i, \frac{1}{|G|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \xi(\sigma) \rangle. \end{aligned}$$

Then by [73] we have

$$|\langle f_i, \frac{1}{|G|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \xi(\sigma) \rangle| \ll \frac{1}{|G|} \max_\chi |D|^{\frac{1}{4}} |L(1/2, \pi, \chi)|^{1/2}.$$

On the other hand, we have, by Siegel's theorem and Class Number Formula,

$$|D|^{-\frac{1}{2} - \varepsilon} \ll_\varepsilon |\mathrm{Pic}(\mathcal{O}_\mathfrak{c})| \ll |D|^{\frac{1}{2}} \log D.$$

Thus by one has

$$\begin{aligned} |\sum_{i \geq 1} a_i \langle f_i, \frac{1}{|G|} \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_\mathfrak{c})} \xi(\sigma) \rangle| &\ll_N \frac{1}{|G|} \max_\chi |D|^{\frac{1}{4}} |L(1/2, \pi, \chi)|^{1/2} \\ &\ll_N \frac{|D|^{1/4 - \delta}}{|G|^{1/2}} \ll_N |D|^{-\delta'}. \end{aligned}$$

The theorem thus follows. □

Remark. δ can be taken to be $\frac{1}{2115}$ if $F = \mathbb{Q}$ as shown in [45]. Furthermore, we quote here the result on elliptic curves corresponding to the theorem above:

Corollary 46 (Theorem 3, [45]). *Let notations be as in Theorem 45. Set $G \subset G_K$ any subgroup of index $\leq |D|^{\frac{1}{2115}}$. For every $e_i \in Ell^{ss}(\mathcal{O}_K)$, we have*

$$\frac{\#\{\sigma \in G : \Phi(E^\sigma) = e_i\}}{|G|} = \mu(e_i) + O_q\left(|D|^{-\delta}\right),$$

where $\Psi = \Psi_q : Ell(\mathcal{O}_K) \to Ell^{ss}(F_{q^2})$ and $\delta > 0$ is an absolute constant.

B.2. **Arithmetic quantum chaos.** Suppose we are giving a compact Riemannian manifold and $\{\varphi_j\}_{j \geq 0}$ an orthogonal basis of $L^2(X)$ composed of Laplace eigenfunctions with eigenvalues ordered in increasing size: $\Delta\varphi_j + \lambda_j \varphi_j = 0$ with $0 = \lambda_0 \leq \lambda_1 \leq \ldots$. Considerations from theoretical physics led to extensive investigations of the distribution properties of φ_j in the limit as $\lambda_j \to +\infty$, and in particular, of the weak-$*$ limits of the sequence of probability measures:

$$d\mu_j := |\varphi_j(x)|^2 dx,$$

where $d\mu_j$ is interpreted in quantum mechanics as the probability density for finding a particle in the state φ_j at the point s and dx is the normalized Riemannian volume. When the geodesic flow is ergodic, a result (see [59] and [72]) shows that at least for a full-density subsequence $\{j_k\}_{k\geq 0}$, $s\mu_j$ weakly-$*$ converges to dx. More precisely, one has for any $V \in C^\infty(X)$,

$$\frac{1}{|\{\lambda_j \leq \lambda\}|} \sum_{\lambda_j \leq \lambda} |\int_X V d\mu_j - \int_X V dx|^2 = o_f(1);$$

this phenomenon is called Quantum Ergodicity.

However, quantum ergodicity does not exhibit an explicit subsequence having dx as its quantum limit, nor does it exclude the possibility of having exceptional (zero density) subsequences $d\mu_{j_k}$ having a quantum limit different from dx. Such exceptional weak limit are called strong scars and indeed have been observed numerically in some related chaotic dynamical systems (such as billiards). Rudnick and Sarnak ([50]) have ruled out the existence of strong scars supported on a finite union of points and closed geodesics. This leads to the conjecture that in many cases dx is the only quantum limit (Quantum Unique Ergodicity).

Conjecture 47. *(QUE) Let X be a negatively curved compact manifold. Then $d\mu_j$ weakly converges to dx as $j \to \infty$.*

So far, the best evidence towards QUE comes from the case of arithmetic surfaces and arithmetic hyperbolic 3-folds. The study of distribution properties of explicit sequences of primitive Hecke eigenforms is sometimes called Arithmetic Quantum Chaos, and one of its conjectures is to prove QUE for such Hecke eigenforms: the Arithmetic QUE Conjecture:

Conjecture 48. *(Arithmetic QUE) For any fixed $q \geq 1$, let f be a primitive weight 0 Maass cusp form (resp. holomorphic cusp form)-with nebentypus trivial or not-for the group $\Gamma_0(q)$ with eigenvalue λ_f (resp. with weight k_f). Then as $\lambda \to +\infty$, the measure*

$$d\mu_f(z) := \frac{|f(z)|^2}{\langle f, f\rangle}\frac{dxdy}{y^2} \quad (resp. \quad d\mu_f(z) := \frac{|f(z)|^2 y^{k_f}}{\langle f, f\rangle}\frac{dxdy}{y^2})$$

weak-$*$ converges on $X_0(q)(\mathbb{C})$ to the normalized Poincaré

$$d\mu_P = \frac{1}{Vol(X_0(q))}\frac{dxdy}{y^2}.$$

Remark. Conjecture 48 has a nice consequence due to Rudnick (see [52]): if f is holomorphic of weight k, then f has $\approx \frac{qk_f}{12}$ zeros on $X_0(q)$; this leads to naturally to the question of the distribution of such zeros. It turns out that the convergence of $s\mu_j$ to $d\mu_P$ implies that the zeros of f are equidistributed with respect to the quantum limit $d\mu_P$.

In the noncompact case, quantum limits of the Eisenstein series can be studied as well. For the full modular curve $X_0(1)$, Luo/ Sarnak ([41]) proved the analog of QUE for E_∞ as follows using subconvex bound.

Theorem 49. *([41]) Set $d\mu_t(z) := |E_\infty(z, 1/2+it)|^2\frac{dxdy}{y^2}$. For V a continuous function compactly supported away from the cusp ∞, one has, as $t \to +\infty$:*

$$\int_{X_0(1)} V(z) d\mu_t(z) = \frac{48}{\pi}\int_{X_0(1)} V(z)\frac{dxdy}{y^2}\log t + o_V(\log t).$$

Proof. It suffices to obtain the above identity for either an incomplete Eisenstein series or a Maass/Hecke-eigenform g. Note that the former case is not trivial, since it requires both a subconvex bound for $\zeta(1/2+it)$ and the Hadamard/de la Vallée-Poussin/Weyl bound (here the savings of the $\log\log t$ is necessary),

$$\frac{\zeta'(1+it)}{\zeta(1+it)} \ll \frac{\log t}{\log\log t},$$

which follows from the zero-free region. For $V = h$ a primitive Maass/Hecke-eigenform, one has to show that

$$\int_{X_0(1)} h(z)d\mu_t(z) = \int_{X_0(1)} |E_\infty(z, 1/2 + it)|^2 h(z)\frac{dxdy}{y^2} = o_h(\log t).$$

By the unfolding method, one has

$$\int_{X_0(1)} h(z)d\mu_t(z) = \frac{2\left|\Gamma(\frac{1+2it}{4})\right|^2\Gamma(\frac{1-2it_h-4it}{4})\Gamma(\frac{1+2it_h-4it}{4})}{\pi^{2it}|\zeta(1+2it)\Gamma(\frac{1+2it}{2})|^2}L(h,\frac{1}{2})L(h,\frac{1}{2}-it).$$

By Stirling's formula and the lower bound $\zeta(1+it) \gg \frac{1}{\log t}$, we thus have

$$\int_{X_0(1)} h(z)d\mu_t(z) \ll_{\varepsilon,h} t^{\varepsilon-1/2}L(h,\frac{1}{2}-it).$$

Here any subconvex bound is sufficient to prove Themrem 49. □

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