HOLOMORPHY OF ADJOINT L-FUNCTIONS FOR $GL(n)$: $n \leq 4$ OUTLINE OF THE PROOF

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ABSTRACT. In this note we give an outline of the proof of the main result in [15] that the adjoint L-functions associated to any cuspidal representations of $GL(3)$ or $GL(4)$ over an arbitrary global field admits a holomorphic continuation to the whole complex plane.

CONTENTS

1. INTRODUCTION

1.1. **Statement of the Main Results.** Let F be a global field, and π be any cuspidal representation of $GL(n, \mathbb{A}_F)$. Then according to Langlands philosophy the adjoint L-function $L(s, \pi, \mathrm{Ad})$ is expected to admit a holomorphic continuation to the whole complex plane.

The first breakthrough was made for classical holomorphic cusp forms by Shimura [11] and independently by Zagier [16]; Shimura's approach was generalized by Gelbart-Jacquet [5] to the adelic setting, while Zagier's method was further developed by Jacquet-Zagier [7] in terms of representation language. Moreover, under the assumption of Dedekind Conjecture and that $\pi \in \mathcal{A}_0(GL(n))$ admits a supercuspidal component, Flicker showed the holomorphy of $L(s, \pi, \mathrm{Ad})$ by a simple

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trace formula (ref. [4]). However, Lemma 4 in [4] is wrong, so Flicker's result is flawed.

In this paper, we will deal with $n \leq 4$ case, leaving the $n \geq 5$ case in the sequel. Our main result is the following.

Theorem 1. *Let F be a global field and $2 \leq n \leq 4$. Let π be a cuspidal representation of $GL(n, \mathbb{A}_F)$ and let τ be a character on $F^\times \backslash \mathbb{A}_F^\times$. Then $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau) = \Lambda(s, \pi \otimes \tau \times \tilde{\pi})/\Lambda(s, \tau)$ is entire, unless $\tau \neq 1$ and $\pi \otimes \tau \simeq \pi$, in which case $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ is meromorphic with only simple poles $s = 0$ and $s = 1$.*

Corollary 2. *Let notation be as before. Then $L(s, \pi, \mathrm{Ad} \otimes \tau) = L(s, \pi \otimes \tau \times \tilde{\pi})/L(s, \tau)$ is entire, unless $\tau \neq 1$ and $\pi \otimes \tau \simeq \pi$, in which case $L(s, \pi, \mathrm{Ad} \otimes \tau)$ is meromorphic with only possible simple poles at $s = 0$ and $s = 1$. In particular, the adjoint L-function $L(s, \pi, \mathrm{Ad}) = L(s, \pi \times \tilde{\pi})/\zeta_F(s)$ is entire.*

Remark. If F is a function field, by using the cohomology of stacks of shtukas and the Arthur-Selberg trace formula, L. Lafforgue showed the Langlands correspondence of cuspidal representations π of $GL_n(\mathbb{A}_F)$ to Galois representations ρ (ref. [9]). Then Theorem 1 follows from the identity $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau) = \Lambda(s, \mathrm{Ad}\, \rho \otimes \tau)$ and analytic properties of $\Lambda(s, \mathrm{Ad}\, \rho \otimes \tau)$, which is known well (ref. [14]). Hence we shall focus on the case that F is a number field, where such a correspondence is not available yet.

Remark. Assuming Piatetski-Shapiro's conjecture on converse theorem (e.g. ref. Chap. 10 in [3]), Theorem 1 would imply that for any cuspidal representation π of $GL(n, \mathbb{A}_F)$, there exists a adjoint lifting $\mathrm{Ad}(\pi)$, which is an representation of $GL(n^2 - 1, \mathbb{A}_F)$, in the sense of [5]. Hence, in principle, Theorem 1 will play a role in Langlands functoriality in this case.

1.2. **The Idea of Proofs.** Our method is similar to [7]. We consider a smooth function $\varphi : G(\mathbb{A}_F) \to \mathbb{C}$ which is left and right K-finite, transforms by a unitary character ω of $Z_G(\mathbb{A}_F)$, and has compact support modulo $Z_G(\mathbb{A}_F)$. Then φ defines an integral operator

$$R(\varphi)f(y) = \int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(x) f(yx) dx,$$

on the space $L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ of functions on $G(F)\backslash G(\mathbb{A}_F)$ which transform under $Z_G(\mathbb{A}_F)$ by ω^{-1} and are square integrable on $G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. This operator can clearly be represented by the kernel function

$$\mathrm{K}(x, y) = \sum_{\gamma \in Z_G(F)\backslash G(F)} \varphi(x^{-1}\gamma y).$$

It is well known that $L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ decomposes into the direct sums of the space $L_0^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ of cusp forms and spaces $L_{\mathrm{Eis}}^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ and $L_{\mathrm{Res}}^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ defined using Eisenstein series and residues of Eisenstein series respectively. Then K splits up as: $\mathrm{K} = \mathrm{K}_0 + \mathrm{K}_{\mathrm{Eis}} + \mathrm{K}_{\mathrm{Res}}$. Selberg trace formula gives an expression for the trace of the operator $R(\varphi)$ restricted to the discrete spectrum, and this is given by

$$\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x, x) dx.$$

We denote by $\mathcal{S}(\mathbb{A}_F^n)$ the space of Schwartz-Bruhat functions on the vector space $\mathbb{A}_F^n$ and by $\mathcal{S}_0(\mathbb{A}_F^n)$ the subspace spanned by products $\Phi = \prod_v \Phi_v$ whose components at real and complex places have the form

$$\Phi_v(x_v) = e^{-\pi \sum_{j=1}^n x_{v,j}^2} \cdot Q(x_{v,1}, x_{v,2}, \cdots, x_{v,n}),\ x_v = (x_{v,1}, x_{v,2}, \cdots, x_{v,n}) \in F_v^n,$$

where $F_v \simeq \mathbb{R}$, and $Q(x_{v,1}, x_{v,2}, \cdots, x_{v,n}) \in \mathbb{C}[x_{v,1}, x_{v,2}, \cdots, x_{v,n}]$; and

$$\Phi_v(x_v) = e^{-2\pi \sum_{j=1}^{n} x_{v,j}\bar{x}_{v,j}} \cdot Q(x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n}),$$

where $F_v \simeq \mathbb{C}$ and $Q(x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n})$ is a polynomial in the ring $\mathbb{C}[x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n}]$.

Denote by Ξ_F the set of characters on $F^\times \backslash \mathbb{A}_F^\times$ which are trivial on $\mathbb{R}_+^\times$. Let $\Phi \in \mathcal{S}_0(\mathbb{A}_F^n)$ and $\tau \in \Xi_F$. Let $\eta = (0, \cdots, 0, 1) \in F^n$. Set

$$f(x, \Phi, \tau; s) = \tau(\det x)|\det x|^s \int_{\mathbb{A}_F^\times} \Phi(\eta t x)\tau(t)^n |t|^{ns} d^\times t,$$

which is a Tate integral (up to holomorphic factors) for $L(ns, x.\Phi, \tau^n)$. It converges absolutely uniformly in compact subsets of $\mathrm{Re}(s) > 1/n$. Since the mirabolic subgroup P_0 is the stabilizer of η. Let $P = P_0 Z_G$ be the full $(n-1, 1)$ parabolic subgroup of G, then $f(x, s) \in \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\delta_P^{s-1/2}\tau^{-n})$, where δ_P is the modulus character for the parabolic P. Then we can define the Eisenstein series

$$E_P(x, \Phi, \tau; s) = \sum_{\gamma \in P(F)\backslash G(F)} f(x, \Phi, \tau; s),$$

which converges absolutely for $\mathrm{Re}(s) > 1$. Also, we define the integral:

$$I^\varphi(s) = \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x, x) E_P(x, \Phi; s) dx.$$

If there is no confusion in the context, we will alway write $I(s)$ (resp. $f(x, s)$) instead of $I^\varphi(s)$ (resp. $f(x, \Phi, \tau; s)$) for simplicity.

According to Proposition 4, Theorem 1 will follow if $I(s) \cdot \Lambda(s, \tau)^{-1}$, $\mathrm{Re}(s) > 1$, admits a holomorphic continuation. To achieve it, we tear $I(s)$ into two parts: geometric side and spectral side. The geometric part is treated in Proposition 5. To deal with the spectral part, which is denoted by $I_\infty(s)$, we develop a mirabolic type of Fourier expansion to further decompose $I_\infty(s)$ as a sum of n distributions: $I_\infty^{(k)}(s)$, $1 \leq k \leq n$ (ref. Prop. 7). Then we continue each $I_\infty^{(k)}(s)$ respectively. There are two major difficulties: the first is showing each $I_\infty^{(k)}(s)$ is well defined when $\mathrm{Re}(s) > 1$, and the other is obtaining continuation of each $I_\infty^{(k)}(s)$. Typically each $I_\infty^{(k)}(s)$ is an infinite sum of meromorphic functions, we need to show its convergence so that it's well defined. Then we have to investigate the analytic property of each $I_\infty^{(k)}(s) \cdot \Lambda(s, \tau)^{-1}$. Furthermore, we also need to get a meromorphic continuation of $I_\infty^{(k)}(s) \cdot \Lambda(s, \tau)^{-1}$. In this process many more infinite sums will show up and after verifying their absolute convergence we get a sum of meromorphic functions, while each individual may have poles. Then the next step is to analyze these possible poles and show that they do cancel with each other. However, by this approach we can only rule out all potential poles of $I(s) \cdot \Lambda(s, \tau)^{-1}$ except for a possible simple pole at $s = 1/2$ when τ is quadratic. This will eventually imply that $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ admits a meromorphic continuation with at most a simple pole at $s = 1/2$. To remedy it, we prove the root number of $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ is always 1 in this case. This would exclude the possibility of existence of a simple pole at $s = 1/2$. Now Theorem 1 follows.

2. Contributions from Geometric Sides

Let $\mathcal{H}(G(\mathbb{A}_F))$ be the Hecke algebra of $\mathcal{H}(G(\mathbb{A}_F))$ and $\varphi \in \mathcal{H}(G(\mathbb{A}_F))$. For any character ω of $\mathbb{A}_F^\times / F^\times$. Let $\varphi \in \mathcal{C}_c^\infty(Z_G(\mathbb{A}_F) \setminus G(\mathbb{A}_F)) \cap \mathcal{H}(G(\mathbb{A}_F))$ be of central character ω. Denote by V_0 the Hilbert space

$$L_0^2\left(G(F) \setminus G(\mathbb{A}_F), \omega^{-1}\right) = \bigoplus_\pi V_\pi,$$

where $\pi \in \mathcal{A}_0\left(G(F)\setminus G(\mathbb{A}_F),\omega^{-1}\right)$, the set of irreducible cuspidal representation of $G(\mathbb{A}_F)$ with central character ω and V_π is the corresponding isotypical component. By multiplicity one, the representation of $G(\mathbb{A}_F)$ on V_π is equivalent to π. For each π, we choose an orthonormal basis $\mathcal{B}_\pi$ of V_π consisting of K-finite vectors. Let $\mathrm{K}_0(x,y)$ be the kernel function for the right regular representation $R(\varphi)$ on V_0. Then we have the decomposition

$$\mathrm{K}_0(x,y)=\sum_{\pi}\mathrm{K}_\pi(x,y),\ \text{where}\ \mathrm{K}_\pi(x,y)=\sum_{\phi\in\mathcal{B}_\pi}\pi(\varphi)\phi(x)\overline{\phi(y)}. \tag{1}$$

All the functions in the summands are of rapid decay in x and y. The sum of $\mathrm{K}_\pi(x,y)$ converges in the space of rapidly decaying functions, by the usual estimates on the growth of cusp forms. The sum over $\mathcal{B}_\pi$ is finitely uniformly in x and y for a given φ because of the K-finiteness of φ.

2.1. **Choice of Test Functions.** Let $\Sigma=\Sigma_\infty\coprod\Sigma_f$ be the set of places of F, where Σ_∞ denotes the subset set of archimedean places of F, and Σ_f denotes the subset of nonarchimedean places of F.

Definition 3. For a place $v\in\Sigma_f$, we say that a test function $\varphi=\otimes_v\varphi_v\in\mathcal{H}\left(G(\mathbb{A}_F)\right)$ is *discrete at* v if φ_v is supported on the intersection of $G(\mathcal{O}_{F_v})$ and the regular elliptic subset of $G(F_v)$.

Let ω be a character of $\mathbb{A}_F^\times/F^\times$. Let $\mathcal{F}^*(\omega)$ be the set of smooth functions $\varphi=\otimes_v'\varphi_v: G(\mathbb{A}_F)\to\mathbb{C}$ which is left and right K-finite, is discrete at some $v\in\Sigma_f$, transforms by the character ω of $Z_G(\mathbb{A}_F)$, and has compact support modulo $Z_G(\mathbb{A}_F)$. Let $\mathcal{F}(\omega)$ be the space spanned linearly by functions in $\mathcal{F}^*(\omega)$.

Then we have an improvement of the Proposition in Section 3.3 of [7]:

Proposition 4. *Let $F(x)$ be a function on $G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ which is K-finite and of polynomial growth in a Siegel domain. Then the following are equivalent:*

- **(a):** $\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)F(x)dx=0$, *for all* $\varphi\in\mathcal{F}(\omega)$;
- **(b):** $\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_\pi(x,x)F(x)dx=0$, *for all* $\varphi\in\mathcal{F}(\omega)$ *and all cuspidal representations* $\pi\in\mathcal{A}_0\left(G(F)\setminus G(\mathbb{A}_F),\omega^{-1}\right)$;
- **(c):** $\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\phi_1(x)\overline{\phi_2(x)}F(x)dx=0$, *for all cuspidal representations* $\pi\in\mathcal{A}_0\left(G(F)\setminus G(\mathbb{A}_F),\omega^{-1}\right)$, *and all K-finite functions* $\phi_1,\phi_2\in\mathrm{V}_\pi$.

2.2. **Contributions from Conjugacy Classes.** Consider the distribution

$$I(s)=\int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)E(x,\Phi;s)dx.$$

Recall that in [7] the calculation of $I(s)$ was based on the decomposition

$$\mathrm{K}(x,x)=\sum_{\mathcal{C}}\mathcal{K}_{\mathcal{C}}(x)+\mathcal{K}_\infty(x),$$

where $\mathcal{C}$ runs through all nontrivial conjugacy classes in $G(F)/Z_G(F)$ and

$$\mathcal{K}_{\mathcal{C}}(x)=\sum_{\substack{\gamma\in\mathcal{C}\\ \gamma\notin P(F)/Z_G(F)}}\varphi(x^{-1}\gamma x),$$

$$\mathcal{K}_\infty(x)=\sum_{\gamma\in P(F)/Z_G(F)}\varphi(x^{-1}\gamma x)-\mathrm{K}_{Eis}(x,x)-\mathrm{K}_{\mathrm{Res}}(x,x).$$

So correspondingly, integrating against the Eisenstein series $E(x,\Phi;s)$ associated to the parabolic subgroup P implies that $I(s)$ can be decomposed as

$$I(s)=\sum_{\mathcal{C}}I_{\mathcal{C}}(s)+I_\infty(s),\ \mathrm{Re}(s)>1.$$

When $G = GL(2)$, Jacquet and Zagier (ref. [7]) computed each $I_{\mathcal{C}}(s)$ and $I_\infty(s)$ for general test function φ. Note that the contribution from non-regular elliptic classes would give Artin L-functions of degree less than n. Therefore, for our particular purpose in this paper, we only use the test functions in $\mathcal{F}_\pi$. This is because for any $\varphi \in \mathcal{F}_\pi$, for any $x \in G(\mathbb{A}_F)$ and any $\gamma \in G(F)$, one has $\varphi(x^{-1}\gamma x) = 0$ unless γ is elliptic regular. Let $\Gamma_{r.e.}(G(F)/Z_G(F))$ be the subset of regular elliptic elements in $G(F)/Z_G(F)$, then $\mathcal{K}_{\mathcal{C}}(s) \equiv 0$ unless $\mathcal{C} \subseteq \Gamma_{r.e.}(G(F)/Z_G(F))$. This helps us simplify the computation of $I(s) = I_{r.e.}(s) + I_\infty(s)$, where

$$I_{r.e.}(s) = \int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \Big\{ \sum_{\gamma \in \Gamma_{r.e.}(G(F)/Z(F))} \varphi(x^{-1}\gamma x) \Big\} \cdot E(x, \Phi; s) dx,$$

$$I_\infty(s) = -\int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \big\{ \mathrm{K}_{Eis}(x,x) + \mathrm{K}_{\mathrm{Res}}(x,x) \big\} \cdot E(x, \Phi; s) dx.$$

We shall deal with $I_{r.e.}(s)$ in this section, and leaving the computation of $I_\infty(s)$ in the next part.

Proposition 5. *Let F be a number field and $I_{reg\ ell}(s)$ be defined as above, then for every field extension E/F of degree n, there is an analytic function $Q_E(s)$ such that*

$$I_{r.e.}(s) = I_{r.e.}^{\varphi}(s) = \frac{1}{n} \sum_{(E:F)=n} Q_E(s) L_E\left(s, \tau \circ N_{E/F}\right), \tag{2}$$

where the summation is taken over only finitely many E's, depending implicitly only on the test function φ.

3. Mirabolic Fourier Expansion of $I_\infty(s)$

Take a test function $\varphi \in \mathcal{F}_\pi$, then by the definition of $E_P(x, \Phi; s)$ we have

$$I_\infty(s) = I_\infty^{\varphi}(s) = -\int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_\infty(x,x) \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma x, s) dx.$$

where $\mathrm{K}_\infty(x,y) = \mathrm{K}_{Eis}(x,y) + \mathrm{K}_{\mathrm{Res}}(x,y)$ is left $N(F)$-invariant. Then

$$I_\infty(s) = -\int_{Z_G(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_\infty(x,x) f(x,s) dx. \tag{3}$$

Now we proceed to compute (3) by considering the Fourier expansion of $\mathrm{K}_\infty(x,y)$.

3.1. Mirabolic Fourier Expansions of Weak Automorphic Forms.

Fourier expansions of automorphic forms of GL_n are well known (ref. [?]). Following the idea of Piatetski-Shapiro in [?], we give a new form of Fourier expansions of weak automorphic forms in terms of generalized mirabolic subgroups, via which a further decomposition of $I_\infty(s)$ is obtained. Here we call a function $f \in \mathcal{C}(G(\mathbb{A}_F))$ a weak automorphic form if it is slowly increasing on $G(\mathbb{A}_F)$, right K-finite and $P_0(F)$-invariant, where P_0 is the mirabolic subgroup of $G = GL_n$.

Fix an integer $n \geq 2$. The maximal unipotent subgroup of $G(\mathbb{A}_F)$, denoted by $N(\mathbb{A}_F)$, is defined to be the set of all $n \times n$ upper triangular matrices in $G(\mathbb{A}_F)$ with ones on the diagonal and arbitrary entries above the diagonal. Let $\psi_{F/\mathbb{Q}}(\cdot) = e^{2\pi i \operatorname{Tr}_{F/\mathbb{Q}}(\cdot)}$ be the standard additive character, then for any $\alpha = (\alpha_1, \cdots, \alpha_{n-1}) \in F^{n-1}$, define a character $\psi_\alpha : N(\mathbb{A}_F) \to \mathbb{C}$ by

$$\psi_\alpha(u) = \prod_{i=1}^{n-1} \psi_{F/\mathbb{Q}}\left(\alpha_i u_{i,i+1}\right), \quad \forall\ u = (u_{i,j})_{n\times n} \in N(\mathbb{A}_F).$$

Write $\psi_k = \psi_{(0,\cdots,0,1,\cdots,1)}$ (where the first $n-k$ components are 0 and the remaining k components are 1) and $\theta = \psi_{(1,\cdots,1)}$, the standard generic character used to define Whittaker functions.

For $1 \leq k \leq n-1$, let B_{n-k} be the standard Borel subgroup (i.e. the subgroup consisting of nonsingular upper triangular matrices) of GL_{n-k}; let N_{n-k} be the unipotent radical of B_{n-k}. For any $i, j \in \mathbb{N}$, let $M_{i\times j}$ be the additive group scheme of $i \times j$-matrices. Define the unipotent radicals

$$N_{(k,1,\cdots,1)} = \left\{ \begin{pmatrix} I_k & B \\ & D \end{pmatrix} : \ B \in M_{k\times(n-k)}, \ D \in N_{n-k} \right\}, \ 1 \leq k \leq n-1 .$$

For $1 \leq k \leq n-1$, set the generalized mirabolic subgroups

$$R_k = \left\{ \left(\begin{array}{cc} A & C \\ 0 & B \end{array} \right) : \ A \in GL_k, \ C \in M_{k\times(n-k)}, \ B \in B_{n-k} \right\}.$$

For $2 \leq k \leq n-1$, define subgroups of R_k by

$$R_k^0 = \left\{ \left(\begin{array}{ccc} A & B' & C \\ 0 & a & D \\ 0 & 0 & B \end{array} \right) : \ \begin{pmatrix} A & B' \\ & a \end{pmatrix} \in GL_k, \ \begin{pmatrix} C \\ D \end{pmatrix} \in M_{k\times(n-k)}, \ B \in B_{n-k} \right\}.$$

Also we define $R_0 = R_1^0 = N_{(0,1,\cdots,1)} := N_{(1,1,\cdots,1)}$ to be the unipotent radical of the standard Borel subgroup of GL_n.

Proposition 6 (Mirabolic Fourier Expansion)**.** *Let h be a continuous function on $P_0(F) \setminus G(\mathbb{A}_F)$. Then we have*

$$h(x) = \sum_{k=1}^{n} \sum_{\delta_k \in R_{k-1}\backslash R_{n-1}} \int_{N_{(k-1,1,\cdots,1)}(F)\backslash N_{(k-1,1,\cdots,1)}(\mathbb{A}_F)} h(n\delta_k x)\psi_{n-k}(n)dn \tag{4}$$

if the right hand side converges absolutely and locally uniformly.

3.2. **Decomposition of $I_\infty(s)$.** Applying Proposition 6 to the kernel function $\mathrm{K}(x,y)$ viewed as a function of x, we thus obtain a formal decomposition of the distribution $I_\infty(s)$ when $\mathrm{Re}(s) > 1$. Convergence problems of this expansion will be settled in the following several sections.

Now for $1 \leq k \leq n$ we write $I_\infty^{(k)}(s)$ for the above (formal) integral, namely,

$$I_\infty^{(k)}(s) = \int_{Z_G(\mathbb{A}_F)R_{k-1}(F)\backslash G(\mathbb{A}_F)} \int_{[N_k^*]} \int_{[N_k']} \mathrm{K}_\infty(n^* n_1 x, x)\theta(n_1)dn_1 dn^* f(x,s)dx.$$

Proposition 7. *Let notation be as before. Then one has, when* $\mathrm{Re}(s) > 1$, *that*

$$I_\infty(s) = \sum_{k=1}^{n} I_\infty^{(k)}(s), \tag{5}$$

where $N_k' = N_{(k,1,\cdots,1)}$ *and*

$$N_k^* = \left\{ \begin{pmatrix} I_{k-1} & C & \\ & 1 & \\ & & I_{n-k} \end{pmatrix} : \ C \in \mathbb{G}_a^{k-1} \right\}.$$

Both sides of (5) *converge absolutely when* $\mathrm{Re}(s) > 1$.

In the following sections these $I_\infty^{(k)}(s)$ will be treated separately because of their different characters. As we will see, $I_\infty^{(1)}(s)$ can be reduced to an infinite sum of Rankin-Selberg convolutions of irreducible generic non-cuspidal representations of $GL(n, \mathbb{A}_F)$ (ref. Section 6), and $I_\infty^{(n)}(s)$ will be handled by Langlands-Shahidi's method after applying some geometric auxiliary results (ref. Section 4); while the remaining terms will be treated by invoking intertwining operators and spectral analysis of tori (ref. Section 5). In particular, according to results in the following sections (Section 4 for $k = n$ and Section 5 for $1 < k < n$), $I_{\infty}^{(k)}(s)$ converges absolutely when $\mathrm{Re}(s) > 1$, and admits a meromorphic continuation to the whole complex plane.When $n \leq 4$, we also obtain a meromorphic continuation of $I_{\infty}^{(1)}(s)$ in Section 7. Hence the expansion (5) is well defined on both sides for $\mathrm{Re}(s) > 1$, and can be regarded as an identity between their continuations when $s \in \mathbb{C}$ is arbitrary and $n \leq 4$.

4. Contributions from $I_{\infty}^{(n)}(s)$

Now we start with handling the last term $I_{\infty}^{(n)}(s)$, since the approach here applies to part of the computation of $I_{\infty}^{(k)}(s)$, $2 \leq k \leq n-1$, as well.

Proposition 8. *Let $\mathcal{C}$ be a regular $G(F)$-conjugacy classes in $G(F)$. Then there exists a $P(F)$-conjugacy class $\mathcal{C}_0$ such that*

$$\mathcal{C} = \mathcal{C}_0 \coprod \bigcup_{k=1}^{n-1} \mathcal{C} \cap Q_k(F)^{P(F)}, \tag{6}$$

where $Q_k(F)^{P(F)} = \{p\gamma p^{-1} : \ \gamma \in Q_k(F), \ p \in P(F)\}$.

Let $\mathfrak{C}_{r.e.}^{P(F)}$ be the union of regular elliptic components of all $G(F)$-conjugacy classes in $G(F)$. Then $\mathfrak{C}_{r.e.}^{P(F)}$ is a disjoint union of $P(F)$-conjugacy classes in $G(F)$ by Proposition 8.

Corollary 9. *Let notation be as before. Set $(F^{\times})^n = \{t^n : \ t \in F^{\times}\}$, and let*

$$\widetilde{\mathcal{R}}_P^* = \left\{ w_1 w_2 \cdots w_{n-1} \begin{pmatrix} I_{n-3} & & \\ & t & \\ & & I_2 \end{pmatrix} \mathfrak{u} : \ t \in F^{\times} / \left(F^{\times}\right)^n, \ \mathfrak{u} \in N_P(F) \right\}. \tag{7}$$

Then $\widetilde{\mathcal{R}}_P^*$ forms a family of representatives of $\left(Z_G(F) \cap \mathfrak{C}_{r.e.}^{P(F)}\right) \backslash \mathfrak{C}_{r.e.}^{P(F)}$.

4.1. Holomorphic Continuation.

Let $P_0(F)$ be the mirabolic subgroup of $G(F)$. For any $\gamma \in G(F)$, write $\gamma^{P_0(F)}$ for the $P_0(F)$-conjugacy class of γ, which is the same as $P(F)$-conjugacy class of γ. Then by Corollary 9 one can decompose $Z_G(F) \backslash G(F)$ as

$$Z_G(F) \backslash G(F) = \coprod_{\gamma \in \widetilde{\mathcal{R}}_P^*} \gamma^{P_0(F)} \coprod \bigcup_{k=1}^{n-1} \left(Z_G(F) \backslash Q_k(F)\right)^{P_0(F)}, \tag{8}$$

where Q_k is maximal parabolic subgroup of type $(k, n-k)$. By the decomposition (8), one can write $I_{\infty}^{(n)}(s) = I_{\infty}^{r.e.}(s) + I_{\infty}^{p.c.}(s)$, where

$$I_{\infty}^{r.e.}(s) = \int_{Z_G(\mathbb{A}_F) R_{n-1}(F) \backslash G(\mathbb{A}_F)} \int_{[N_P]} \sum_{\gamma \in \widetilde{\mathcal{R}}_P^*} \sum_{p \in P_0(F)} \varphi(x^{-1} n^{-1} p^{-1} \gamma p x) dn f(x, s) dx,$$

$$I_{\infty}^{p.c.}(s) = \int_{Y_n} \int_{[N_P]} \sum_{\gamma^{P_0(F)} \in \mathcal{P}} \sum_{p \in P_0(F)} \varphi(x^{-1} n^{-1} p^{-1} \gamma p x) dn f(x, s) dx,$$

where $Y_n = Z_G(\mathbb{A}_F) R_{n-1}(F) \backslash G(\mathbb{A}_F) = Z_G(\mathbb{A}_F) P_0(F) \backslash G(\mathbb{A}_F)$ and

$$\mathcal{P} = \left\{\gamma^{P(F)} : \ \gamma \in Z_G(F) \backslash Q_k(F) \text{ for some } 1 \leq k \leq n-1\right\}.$$

An analysis on the support of φ leads to that $I_{\infty}^{p.c.}(s) = 0$. Now our computation reduces to $I_{\infty}^{(n)}(s) = I_{\infty}^{r.e.}(s)$.

Depending on the purity of n, we can further simplify $I_{\infty}^{r.e.}(s)$. Recall the test function φ has the central character ω, Ξ is the set of idele class characters on $\mathbb{A}_F$, which is trivial on the archimedean places. Denote by $\Xi_{\omega,n}$ the subset $\{\chi\in\Xi:\ \chi^n=\omega\}\subset\Xi$. Also, let $\Xi_{\tau,2}^n=\{\xi\in\Xi:\ \xi^2=\tau\}$ if n is even, and set $\Xi_{\tau,2}^n$ to be the empty set if n is odd. Then when n is odd, we have

$$I_{\infty}^{r.e.}(s)=\frac{1}{c_P}\int_K f(k,s)dk\int_{N(\mathbb{A}_F)}d\mathfrak{u}\int_{[N^P]}d\mathfrak{u}'\int_{N_P(\mathbb{A}_F)}dn\sum_{\chi\in\Xi_{\omega,n}}\int_{\mathbb{A}_F^\times}\Delta_{s,\tau,\chi}^{od}(\mathfrak{t})d^\times t_1$$
$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}\mathfrak{u}\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\widetilde{w}n\mathfrak{u}'k\right)d^\times t_2\cdots d^\times t_{n-1},$$

where we use the fact that $\left(\mathbb{A}_F^\times\right)^n\cdot F^\times/\left(F^\times\right)^n=F^\times\cdot\left(F^\times\backslash\mathbb{A}_F^\times\right)^n$, and $\tau|\cdot|_{\mathbb{A}_F}$ is $F^\times$-invariant, and

$$\Delta_{s,\tau,\chi}^{od}(\mathfrak{t})=\bar{\chi}(t_1)\tau(t_1)^{\frac{n-1}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\tau(t_i^{\frac{n+1}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

When n is even, we also have a similar expansion.

Let $T_*(\mathbb{A}_F^\times)=\{\mathrm{diag}(1,t_1,t_2,\cdots,t_{n-1})\in T(\mathbb{A}_F):\ t_i\in\mathbb{A}_F^\times,\ 1\leq i\leq n-1\}$. Set

$$\iota:\ T^*(\mathbb{A}_F^\times)\longrightarrow T_*(\mathbb{A}_F^\times),\ \mathfrak{t}\mapsto\mathfrak{t}^\iota=\mathrm{diag}(1,t_2^{-1},t_3^{-1},\cdots,t_{n-1}^{-1},t_1).$$

For any $n\in\mathbb{N}_{\geq2}$, define

$$\mathfrak{F}_{\chi,\xi}(x;k,s)=\int_{N(\mathbb{A}_F)}d\mathfrak{u}\int_{[N^P]}d\mathfrak{u}'\int_{T^*(\mathbb{A}_F^\times)}\varphi\left(k^{-1}\mathfrak{u}\mathfrak{t}^\iota x\mathfrak{u}'k\right)\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})d^\times\mathfrak{t},$$

where we write $\delta_n=-\frac{1+(-1)^n}{2}$ and denote by $\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})$ the following character

$$\bar{\chi}(t_1)\xi(t_1)^{-\delta_n}\tau(t_1)^{\frac{n-1-\delta_n}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

Since $[N^P]=N^P(F)\backslash N^P(\mathbb{A}_F)$ is compact and φ is compactly supported, the function $\mathfrak{F}_{\chi,\xi}(x;k,s)$ is well defined for any χ, ξ and $\mathrm{Re}(s)>1$.

Let $b=\mathfrak{u}\mathfrak{t}\in B(\mathbb{A}_F)$, where $\mathfrak{u}\in N(\mathbb{A}_F)$, $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\cdots,t_n)\in T(\mathbb{A}_F)$. Then

$$\mathfrak{F}_{\chi,\xi}(bx;k,s)=\prod_{i=1}^{n}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}\cdot\mathfrak{F}_{\chi,\xi}(x;k,s).$$

Since the modular character of $T(\mathbb{A}_F)$ is $\delta_{T(\mathbb{A}_F)}(\mathfrak{t})=\prod_{i=1}^n t_i^{n+1-2i}$, so one has

$$\mathfrak{F}_{\chi,\xi}(x;k,s)\in\mathrm{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\left(\chi\xi^{\delta_n}\tau^{\lambda_1}|\cdot|_{\mathbb{A}_F}^{\lambda_1 s},\cdots,\chi\xi^{\delta_n}\tau^{\lambda_{n-1}}|\cdot|_{\mathbb{A}_F}^{\lambda_{n-1}s},\chi\xi^{\delta_n}\tau^{\lambda_n}|\cdot|_{\mathbb{A}_F}^{\lambda_n s}\right),$$

where for $1\leq i\leq n$, $\lambda_i=\frac{n+1-\delta_n}{2}-i$. Denote by

$$G_{\chi,\xi}(x;s)=\frac{1}{c_P}\int_K f(k,s)\mathfrak{F}_{\chi,\xi}(x;k,s)dk.$$

Then we have (at least formally) that

$$I_{\infty}^{(n)}(s)=I_{\infty}^{r.e.}(s)=\sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi_{\tau,2}^n}\int_{N_P(\mathbb{A}_F)}G_{\chi,\xi}(\widetilde{w}n;s)dn.$$

Thus by the Langlands-Shahidi method and Tate's thesis one concludes the following.

Theorem 10. *Let notation be as before, then $I_{\infty}^{(n)}(s)$ converges absolutely and locally normally in the domain* $\mathrm{Re}(s) > 1$. *Moreover, $I_{\infty}^{(n)}(s)$ admits a meromorphic continuation. Precisely, one has*

$$I_{\infty}^{(n)}(s) \sim \frac{L(s,\tau)L(2s,\tau^2)\cdots L((n-1)s,\tau^{n-1})L(ns,\tau^n)}{L(s+1,\tau)L(2s+1,\tau^2)\cdots L((n-1)s+1,\tau^{n-1})}.$$

Remark. To make the above formally computation rigorous, one can apply the proceeding computation to $I_{\infty,+}^{(n)}(s) = I_{\infty}^{(n)}(s;|\varphi|,|\Phi|,1)$. Then everything is non-negative, so we can interchange orders of integrals. The convergence of $I_{\infty,+}^{(n)}(s)$ follows from Langlands' theory on intertwining operators. Therefore, the proceeding formal computation is justified by dominant control theorem.

5. The Contributions from $I_{\infty}^{(k)}(s)$ $(2 \leq k \leq n-1)$

In this section we claim that $I_{\infty}^{(k)}(s)$ admits meromorphic continuation to $\mathbb{C}$. The proof is much more subtle than the continuation of $I_{\infty}^{(n)}(s)$. Recall that

$$I_{\infty}^{(k)}(s) = \int_{Z_G(\mathbb{A}_F)R_{k-1}(F)\backslash G(\mathbb{A}_F)}\int_{[N_k^*]}\int_{[N_k']} \mathrm{K}_{\infty}(n^*n_1x,x)\theta(n_1)dn_1dn^*f(x,s)dx.$$

Since our test function is supported in the subset of elliptic regular elements, we can show that the corresponding inner integral of $I_{\infty}^{(k)}(s)$ is supported in a particular Bruhat cell, i.e., the cell corresponding to the longest element. Finding an explicit form of representatives of this cell we then see that $I_{\infty}^{(k)}(s)$ is equal to

$$\int_{Y_k'} f(x,s)dx\int_{\mathcal{N}_l}du_l\int_{[N_k^*]}du\int_{[N_0^{(k)}]}dn\int_{\mathcal{N}_r}\sum_{\delta}\varphi(x^{-1}u_l^{-1}u\delta\widetilde{w}_ku_rnx)\bar{\theta}(u_ru_l^{-1})du_r,$$

where $\delta \in \mathrm{diag}(GL_k,\mathbb{G}_m^{n-k})(F)$, $Y_k' = Z_G(\mathbb{A}_F)R_{k-1}^*(F)N_k'(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ with

$$R_{k-1}^*(F) = \left\{\begin{pmatrix} A & B' & \mathbf{0}\\ 0 & a & \mathbf{0}\\ 0 & 0 & B\end{pmatrix} : \begin{pmatrix} A & B'\\ & a\end{pmatrix} \in GL_k,\ B \in \mathbb{G}_m^{n-k}\right\};$$

$\widetilde{w}_k$ is the longest element, $\mathcal{N}_l' = N_{(k,1,\cdots,1)}(\mathbb{A}_F)$, $\mathcal{N}_r = \mathrm{diag}(I_{k-1},N_{n+1-k})(\mathbb{A}_F)$,

$$N_0^{(k)} = \begin{pmatrix} I_{k-1} & \mathbf{0} & M_{(k-1)\times(n-k)}\\ & 1 & \mathbf{0}\\ & & I_{n-k}\end{pmatrix}.$$

One sees clearly that the integral over Y_k' will decompose into products I_1I_2, where I_1 is an integral over $P_k(F)\backslash GL_k(\mathbb{A}_F)$, where P_k is the mirabolic subgroup of GL_k; and I_2 is an integral over the torus $\mathbb{G}_m^{n-1-k}$. The absolute convergence is handled in Section 5.2 of [15]. So we can switch orders of some integrals and just discuss formal computation here. To deal with I_1, we consider representatives of Bruhat normal form of $P_k(F)\backslash GL_k(\mathbb{A}_F)$ just as Section 4. Then apply the same idea as continuing $I_{\infty}^{(n)}(s)$ in last section, but here we need a more complicated computation since I_1 is not really independent of I_2 due to the action of $\widetilde{w}_k$. For I_2 we shall just apply Poisson summation along torus and the continuation from this part comes from shifting contour. Putting things together, we obtain the following, which appears to be new.

Theorem 11. *Let $2 \leq k \leq n-1$. Then we have*

$$I_{\infty}^{(k)}(s) \sim \frac{L(s,\tau)L(2s,\tau^2)\cdots L\left((k-1)s,\tau^{k-1}\right)L(ns,\tau^n)}{L(s+1,\tau)L(2s+1,\tau^2)\cdots L\left((k-1)s+1,\tau^{k-1}\right)}.$$

Remark. The convergence here is more delicate than that of Theorem 10. We will use intertwining operator theory iteratively and introduce an extra family of dominant integrals with parameters in some cone. Then invoking Poisson summation and absolute convergence we can reduce $I_{\infty}^{(k)}(s)$ to roughly a finite sum of intertwining operators. The holomorphic continuation comes from that of intertwining theory and shifting contour of Mellin transform. See Section 5 of [15] for details.

6. Contributions from $I_{\infty}^{(1)}(s)$

In this section, we shall outline the proof of absolute and locally uniform convergence of $I_{\infty}^{(1)}(s)$. The whole process is lengthy and makes use of a variant of Arthur's truncation technique. See Section 6 of [15] for details.

For any functions $G(x,y)$ on $Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)\times Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$, let $\mathcal{F}_1 G(x,y)$ be the Fourier transform along the x-variable.

Proposition 12. *Let notation be as above. Let $R(x)$ be a slowly increasing function on $\mathcal{S}_0$. Then we have*

$$\int_{Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\sum_{\chi}\left|\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_{\chi}(x,x)\cdot R(x)\right|dx<\infty, \tag{9}$$

where χ runs over all the equivalent classes of cuspidal datum; and Λ_2^T is Arthur's truncation operator with respect to the second variable (ref. [1]).

Proposition 13. *Let notation be as before. Let $\chi\in\mathfrak{X}$ be a cuspidal datum. Then there exists some $T_0\in\mathfrak{a}_0$ depending only on the support of φ, such that for any $T\in\mathfrak{a}_0$ with $T-T_0\in\mathfrak{a}_0^+$, one has*

$$\int_{Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_{\chi}(x,x)\cdot R(x)dx$$

converges absolutely, and it is of the form

$$\sum_{w\in W_n}\sum_{Q}C_1^Q(T_0;w,\chi,R)e^{-\lambda_w(T)}+\sum_{w\in W_n}\sum_{Q}C_2^Q(T_0;w,\chi,R)P_{w,Q}(T;T_0), \tag{10}$$

where $C_1^Q(T_0;w,\chi,R)$ and $C_2^Q(T_0;w,\chi,R)$ are constants depending on w, χ, R and T_0; λ_w is a point $(\mathfrak{a}_0^*)^+$, decided by $w\in W_n$; and $P_{w,Q}(T;T_0)$ is a polynomial depending on w and Q, with $\deg P_{w,Q}(T;T_0)\leq\dim\mathfrak{a}_Q^G$.

Proposition 14. *Let notation be as above. Let $\chi\in\mathfrak{X}$ be a cuspidal datum. Let $R(x)$ be a slowly increasing function on a Siegel domain $\mathcal{S}_0$. Then we have*

$$\int_{Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\left|\mathcal{F}_1\,\mathrm{K}_{\chi}(x,x)\cdot R(x)\right|dx<\infty. \tag{11}$$

Remark. Inequality (11) comes from estimate on gauges considered in [6].

Let R be a slowly increasing function on X_G. Define, at least formally, that

$$J_R=\int_{X_G}\sum_{\chi}\mathcal{F}_1\,\mathrm{K}_{\chi}(x,x)\cdot R(x)dx.$$

Noting the fact that $\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_{\chi}(x,x)=\mathcal{F}_1\,\mathrm{K}_{\chi}(x,x)$ when T is sufficiently regular, one then concludes from Proposition 13 and Proposition 14 that:

Corollary 15. *Let notation be as above. Then for any slowing increasing left $Z_G(\mathbb{A}_F)N(F)$-invariant function R, J_R is well defined. Moreover, we have*

$$J_R=\int_{Y_G}\sum_{\chi\in\mathfrak{X}}\widehat{\mathrm{K}}_{\chi}(x,x)\cdot R(x)dx=\sum_{\chi\in\mathfrak{X}}\int_{Y_G}\widehat{\mathrm{K}}_{\chi}(x,x)\cdot R(x)dx, \tag{12}$$

where $Y_G := Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, *and for any* $\chi \in \mathfrak{X}$,

$$\widehat{\mathrm{K}}_\chi(x,y) = \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{N(F)\backslash N(\mathbb{A}_F)} \mathrm{K}_\chi(n_1x, n_2y)\theta(n_1)\bar{\theta}(n_2)dn_1dn_2.$$

One then further deduces that

Theorem 16. *Let notation be as before. Let* $s \in \mathbb{C}$ *be such that* $\mathrm{Re}(s) > 1$. *Let* $Y_G = Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. *Then the following integral*

$$\sum_{\chi\in\mathfrak{X}} \sum_{P\in\mathcal{P}} \sum_{\phi_1\in\mathfrak{B}_{P,\chi}} \sum_{\phi_2\in\mathfrak{B}_{P,\chi}} \int_{\Lambda^*} \int_{Y_G} \Big|\langle \mathcal{I}_P(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)\Big| dxd\lambda$$

is finite, and is uniformly bounded if s *lies in some compact subset of the right half plane* $\{z : \ \mathrm{Re}(z) > 1\}$. *In particular,* $I_\infty^{(1)}(s)$ *converges absolutely for* $\mathrm{Re}(s) > 1$. *Moreover, when* $\mathrm{Re}(s) > 1$, $I_\infty^{(1)}(s)$ *is equal to*

$$\sum_{\chi} \sum_{P\in\mathcal{P}} \frac{1}{c_P} \sum_{\phi_1\in\mathfrak{B}_{P,\chi}} \sum_{\phi_2\in\mathfrak{B}_{P,\chi}} \int_{\Lambda^*} \langle \mathcal{I}_P(\lambda,\varphi)\phi_2,\phi_1\rangle \int_{Y_G} W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)dxd\lambda,$$

where χ *runs over proper cuspidal data, i.e.,* χ *is not of the form* $\{(G,\pi)\}$. *Particularly, as a function of* s, $I_\infty^{(1)}(s)$ *is analytic in the right half plane* $\{z : \ \mathrm{Re}(z) > 1\}$.

6.1. Absolute Convergence in the Critical Strip $S_{[0,1]}$.

Theorem 17. *Let* $s \in \mathbb{C}$ *be such that* $0 < \mathrm{Re}(s) < 1$, *then*

$$\sum_{\chi} \sum_{P\in\mathcal{P}} \frac{1}{c_P} \sum_{\phi\in\mathfrak{B}_{P,\chi}} \int_{\Lambda^*} R_\varphi(s,\lambda;\phi)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})d\lambda, \tag{13}$$

converges absolutely, normally with respect to s, *where* $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$ *is the complete* L*-function, and*

$$R_\varphi(s,\lambda;\phi_2) = \sum_{\phi_1\in\mathfrak{B}_{P,\chi}} \langle \mathcal{I}_P(\lambda,\varphi)\phi_1,\phi_2\rangle \cdot \frac{\Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})},\ \mathrm{Re}(s) > 1,$$

with $\Psi(s,W_1,W_2;\lambda)$ *the standard Rankin-Selberg integral.*

7. Holomorphic Continuation via Multidimensional Residues

From preceding estimates, we see that when $\mathrm{Re}(s) > 1$, $I_\infty^{(1)}(s)$ is a combination of Rankin-Selberg convolutions for automorphic functions which are not of rapid decay. Zagier [17] computed the Rankin-Selberg transform of some type of automorphic functions and derived the desired holomorphic continuation for $n = 2$ and $F = \mathbb{Q}$ case. However, general Eisenstein series for $GL(n)$ do not have the asymptotic properties as Zagier considered, since there are mixed terms in the Fourier expansion (ref. Proposition 6). Thus one needs to develop a different approach to obtain the continuation. We will make essentially use of zero-free region to obtain a continuation.

7.1. Continuation via a Zero-free Region.

Recall that we fix the unitary character τ. Let $\mathcal{D}_\tau$ be a standard (open) zero-free region of $L_F(s,\tau)$ (e.g. ref. [2]). We fix such a $\mathcal{D}_\tau$ once for all. We thus can form a domain

$$\mathcal{R}(1/2;\tau)^- := \{s\in\mathbb{C} : \ 2s\in\mathcal{D}_\tau\} \supsetneq \{s\in\mathbb{C} : \ \mathrm{Re}(s)\geq 1/2\}. \tag{14}$$

In Section 7.2, we will continue $I_\infty^{(1)}(s)$ to the open set $\mathcal{R}(1/2;\tau)^-$. Invoking (14) with functional equation we then obtain a meromorphic continuation of $I_\infty^{(1)}(s)$ to the whole complex plane.

7.1.1. *Meromorphic continuation of* $J_{P,\chi}(s;\phi,\mathcal{C}_\chi(\boldsymbol{\epsilon}))$ *across the critical line* $\mathrm{Re}(s)=1$. Let $1\leq m,m'\leq n$ be two integers. Let $\sigma\in\mathcal{A}_0(GL_m(F)\backslash GL_m(\mathbb{A}_F))$ and $\sigma'\in\mathcal{A}_0(GL_{m'}(F)\backslash GL_{m'}(\mathbb{A}_F))$. Fix $\epsilon_0>0$. For any $c'>0$, let $\mathcal{D}_{c'}(\sigma,\sigma')$ be

$$\left\{\kappa=\beta+i\gamma:\ \beta\geq 1-c'\cdot\Big[\frac{(C(\sigma)C(\sigma'))^{-2(m+m')}}{(|\gamma|+3)^{2mm'[F:\mathbb{Q}]}}\Big]^{\frac{1}{2}+\frac{1}{2(m+m')}-\epsilon_0}\right\},$$

if $\sigma'\not\cong\widetilde{\sigma}$; and let $\mathcal{D}_{c'}(\sigma,\sigma')$ denote by the region

$$\left\{\kappa=\beta+i\gamma:\ \beta\geq 1-c'\cdot\Big[\frac{(C(\sigma))^{-8m}}{(|\gamma|+3)^{2mm^2[F:\mathbb{Q}]}}\Big]^{-\frac{7}{8}+\frac{5}{8m}-\epsilon_0}\right\},$$

if $\sigma'\simeq\widetilde{\sigma}$. According to [2] and the Appendix of [8], there exists a constant $c_{m,m'}>0$ depending only on m and m', such that $L(\boldsymbol{\kappa},\sigma\times\sigma')$ does not vanish in $\boldsymbol{\kappa}=(\kappa_1,\cdots,\kappa_r)\in\mathcal{D}_{c_{m,m'}}(\sigma,\sigma')\times\cdots\times\mathcal{D}_{c_{m,m'}}(\sigma,\sigma')$. Let $c=\min_{1\leq m,m'\leq n}c_{m,m'}$ and $\mathcal{C}(\sigma,\sigma')$ be the boundary of $\mathcal{D}_c(\sigma,\sigma')$. We may assume that c is small such that the curve $\mathcal{C}(\sigma,\sigma')$ lies in the strip $1-1/(n+4)<\mathrm{Re}(\kappa_j)<1$, $1\leq j\leq r$. Fix such a c henceforth.

Let $\chi\in\mathfrak{X}_P$ be a cuspidal datum with respect to a standard parabolic P, and $\pi=\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\sigma_1,\sigma_2,\cdots,\sigma_r)\in\chi$. For any $\epsilon\in(0,1]$ we set

$$\mathcal{D}_\chi(\epsilon)=\bigcap_{1\leq i\leq r}\bigcap_{i<j\leq r}\Big\{\kappa\in\mathbb{C}:\ \mathrm{Re}(\kappa)\geq 0,\ 1-\kappa\in\mathcal{D}_{c\epsilon}(\sigma_i,\sigma_j)\Big\}.$$

Also, for $\epsilon=0$, we set $\mathcal{D}_\chi(\epsilon)=\big\{\kappa\in\mathbb{C}:\ \mathrm{Re}(\kappa)\geq 0\big\}$. Then by the above discussion, as a function of $\boldsymbol{\kappa}$, $L_S(\boldsymbol{\kappa},\pi,\widetilde{\pi})$ is nonzero in the region $\mathcal{D}_\chi(\boldsymbol{\epsilon})=\big\{\boldsymbol{\kappa}=(\kappa_1,\cdots,\kappa_r)\in\mathbb{C}^r:\ \kappa_l\in\mathcal{D}_\chi(\epsilon_l)\big\}$, where $\boldsymbol{\epsilon}=(\epsilon_1,\cdots,\epsilon_r)\in[0,1]^r$. We can write $\mathcal{D}_\chi(\boldsymbol{\epsilon})$ as a product space $\mathcal{D}_\chi(\boldsymbol{\epsilon})=\prod_{l=1}^r\mathcal{D}_\chi(\epsilon_l)$, and let $\partial\mathcal{D}_\chi(\epsilon_l)$ be the boundary of $\mathcal{D}_\chi(\epsilon_l)$. Then when $\epsilon_l>0$, $\partial\mathcal{D}_\chi(\epsilon_l)$ has two connected components and one of which is exactly the imaginary axis. Let $\mathcal{C}_\chi(\epsilon_l)$ be the other component, which is a continuous curve, where $0\leq\epsilon_l\leq 1$. When $\epsilon_l=0$, let $\mathcal{C}_\chi(\epsilon_l)$ be the maginary axis. Set $\mathcal{C}_\chi(\boldsymbol{\epsilon})=\mathcal{C}_\chi(\epsilon_1)\times\cdots\times\mathcal{C}_\chi(\epsilon_{r-1})$, $0\leq\epsilon_l\leq 1$, $1\leq l\leq r-1$.

Let $\boldsymbol{\epsilon}=(\epsilon_1,\cdots,\epsilon_{r-1})\in[0,1]^{r-1}$. For any $\beta\geq 1/2$, we denote by

$$\mathcal{R}(\beta;\chi,\boldsymbol{\epsilon})=\Big\{s\in 1+\mathcal{D}_\chi(\boldsymbol{\epsilon})\Big\}\bigcup\Big\{s\in 1-\mathcal{D}_\chi(\boldsymbol{\epsilon})\Big\}. \tag{15}$$

Let $\boldsymbol{\epsilon}=(1/n,1/n,\cdots,1/n)\in\mathbb{R}^{n-1}$ and $s\in 1+\mathcal{D}_\chi(\boldsymbol{\epsilon})$ and $\mathrm{Re}(s)>1$. Then $R(s,W_1,W_2;\boldsymbol{\kappa},\phi)\Lambda(s,\pi_{\boldsymbol{\kappa}}\otimes\tau\times\widetilde{\pi}_{-\boldsymbol{\kappa}})$ is equal to a holomorphic function multiplying

$$\prod_{k=1}^r\Lambda(s,\sigma_k\otimes\tau\times\widetilde{\sigma}_k)\prod_{j=1}^{r-1}\prod_{i=1}^j\frac{\Lambda(s+\kappa_{i,j},\sigma_i\otimes\tau\times\widetilde{\sigma}_{j+1})\Lambda(s-\kappa_{i,j},\sigma_{j+1}\otimes\tau\times\widetilde{\sigma}_i)}{\Lambda(1+\kappa_{i,j},\sigma_i\times\widetilde{\sigma}_{j+1})\Lambda(1-\kappa_{i,j},\sigma_{j+1}\times\widetilde{\sigma}_i)}.$$

Let $\mathcal{G}(\boldsymbol{\kappa};s)=\mathcal{G}(\boldsymbol{\kappa};s,P,\chi)$ denotes the above product. Also, for simplicity, we denote by $\mathcal{F}(\boldsymbol{\kappa};s)=\mathcal{F}(\boldsymbol{\kappa};s,P,\chi)$ the function $R_\varphi(s,\boldsymbol{\kappa};\phi)\Lambda(s,\pi_{\boldsymbol{\kappa}}\otimes\tau\times\widetilde{\pi}_{-\boldsymbol{\kappa}})$. Then the Rankin-Selberg theory implies that $\mathcal{F}(\boldsymbol{\kappa};s)/\mathcal{G}(\boldsymbol{\kappa};s)$ can be continued to an entire function. We will write $\mathcal{C}$ for the boundary $C_\chi(1)$, and (0) for the imaginary axis. Then an analysis on the potential poles of $\mathcal{G}(\boldsymbol{\kappa};s)$ leads to an expression for the integral $J_{P,\chi}(s;\phi,\mathcal{C}_\chi(\mathbf{0}))=J_{P,\chi}(s;\phi,\mathcal{C})-\mathcal{J}_1(s)$, where

$$\mathcal{J}_1(s)=\sum_{j=1}^{r-1}\sum_{i=1}^j\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j+1},$$

where $\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)$ is not identically vanishing unless $\sigma_i\otimes\tau\simeq\sigma_{j+1}$, in which case one must have $n_i=n_{j+1}$. To obtain meromorphic continuation of $J_{P,\chi}(s;\phi,\mathcal{C}_\chi(\boldsymbol{\epsilon}))$ inside the critical strip $0<\mathrm{Re}(s)<1$, we start with the following initial step:

Proposition 18. *Let notation be as before. Let* $\boldsymbol{\epsilon} = (1/n, 1/n, \cdots, 1/n)$ *and* $s \in 1 + \mathcal{D}_\chi(\boldsymbol{\epsilon})$ *and* $\operatorname{Re}(s) > 1$. *Then* $\sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} J_{P,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0}))$ *is equal to*

$$\sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} J_{P,\chi}(s; \phi, \mathcal{C}) - \sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{J}(s), \tag{16}$$

where the summand $\mathcal{J}(s)$ *in the last double sum is equal to*

$$\sum_{m=1}^{r-1} \sum_{\substack{j_m, j_{m-1}, \cdots, j_1 \\ 1 \leq j_m < \cdots < j_1 \leq r-1}} \cdots \sum c_{j_1, \cdots, j_m} \int_{\mathcal{C}} \cdots \cdots \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_m} = s-1} \cdots \operatorname*{Res}_{\kappa_{j_1} = s-1} \mathcal{F}(\boldsymbol{\kappa}; s) \frac{d\kappa_{r-1} \cdots d\kappa_1}{d\kappa_{j_m} \cdots d\kappa_{j_1}},$$

where $c_{j_1, \cdots, j_m}$*'s are some explicit integers, and* $d\kappa_{r-1} \cdots d\kappa_1 / (d\kappa_{j_m} \cdots d\kappa_{j_1})$ *means* $d\kappa_{r-1} \cdots \widehat{d\kappa_{j_m}} \cdots \widehat{d\kappa_{j_1}} \cdots d\kappa_1$. *Moreover, the terms in* (16) *converges absolutely and normally inside* $\mathcal{R}(1; \chi, \boldsymbol{\epsilon}) \setminus \{1\}$, *where* $\mathcal{R}(1; \chi, \boldsymbol{\epsilon})$ *is defined in* (15). *Hence* (16) *gives a meromorphic continuation of* $\sum_{\phi \in \mathfrak{B}_{P,\chi}} J_{P,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0}))$ *to* $\mathcal{R}(1; \chi, \boldsymbol{\epsilon})$, *with a potential pole at* $s = 1$.

Denote by $\mathcal{I}_0(s)$ the first term of the right hand side of (16), i.e.,

$$\mathcal{I}_0(s) := \sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} J_{P,\chi}(s; \phi, \mathcal{C}), \quad s \in 1 + \mathcal{D}_\chi(\boldsymbol{\epsilon}), \ \operatorname{Re}(s) > 1.$$

Proposition 19. *Let notation be as before. Let* $s \in 1 + \mathcal{D}_\chi(\boldsymbol{\epsilon})$ *and* $\operatorname{Re}(s) > 1$. *Then*

$$\mathcal{I}_0(s) = \sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} J_{P,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0})) + \sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{J}_m^0(s), \tag{17}$$

where the summand $\mathcal{J}^0(s)$ *in the last double sum is equal to*

$$\sum_{m=1}^{r-1} \sum_{\substack{j_m, j_{m-1}, \cdots, j_1 \\ 1 \leq j_m < \cdots < j_1 \leq r-1}} \cdots \sum \tilde{c}_{j_1, \cdots, j_m} \int_{(0)} \cdots \cdots \int_{(0)} \operatorname*{Res}_{\kappa_{j_m} = 1-s} \cdots \operatorname*{Res}_{\kappa_{j_1} = 1-s} \mathcal{F}(\boldsymbol{\kappa}; s) \frac{d\kappa_{r-1} \cdots d\kappa_1}{d\kappa_{j_m} \cdots d\kappa_{j_1}},$$

where $\tilde{c}_{j_1, \cdots, j_m}$*'s are some explicit integers, and* $d\kappa_{r-1} \cdots d\kappa_1 / (d\kappa_{j_m} \cdots d\kappa_{j_1})$ *means* $d\kappa_{r-1} \cdots \widehat{d\kappa_{j_m}} \cdots \widehat{d\kappa_{j_1}} \cdots d\kappa_1$. *Moreover, the terms in* (17) *converges absolutely and normally inside any bounded strip.*

Let notation be as in Proposition 19. For $\chi \in \mathfrak{X}_P$, denote by $\mathcal{I}_{0,\chi}(s)$ the summand of the first term of the right hand side of (17), i.e.,

$$\mathcal{I}_{0,\chi}(s) = \sum_{\phi \in \mathfrak{B}_{P,\chi}} J_{P,\chi}(s; \phi, \mathcal{C}_\chi(\mathbf{0})) + \sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{J}_\chi^0(s).$$

To prove Theorem 1, we expect a meromorphic continuation of $\sum_{\chi \in \mathfrak{X}_P} \mathcal{I}_{0,\chi}(s)$ to some open region containing the half plane $\operatorname{Re}(s) \geq 1/2$ (then one can apply functional equation to deal with the remaining area). An initial step is to get a continuation of $\sum_{\chi \in \mathfrak{X}_P} \mathcal{I}_{0,\chi}(s)$ to some open region containing the half plane $\operatorname{Re}(s) \geq 1$. Due to the zero-free region it is clear that $\mathcal{I}_{0,\chi}(s)$ is naturally meromorphic (with a possible pole at $s = 1$) when $s \in \mathcal{R}(1; \chi, \boldsymbol{\epsilon})$ (defined in (15)), which does include the half plane $\operatorname{Re}(s) \geq 1$. However, as χ varies over $\mathfrak{X}_P$, the intersection of all these domains $\mathcal{R}(1; \chi, \boldsymbol{\epsilon})$ is exactly the line $\operatorname{Re}(s) = 1$. Hence one cannot expect a continuation of $\sum_{\chi \in \mathfrak{X}_P} \mathcal{I}_{0,\chi}(s)$ to a domain we want in this way. Nevertheless, we can remedy this by considering continuation of each $\mathcal{I}_{0,\chi}(s)$ first, then showing the sum $\sum_{\chi \in \mathfrak{X}_P} \mathcal{I}_{0,\chi}(s)$, viewed as a sum of continuations of each $\mathcal{I}_{0,\chi}(s)$, does converge absolutely and locally normally out of finitely many explicit poles, giving a desired meromorphic continuation of $\sum_{\chi \in \mathfrak{X}_P} \mathcal{I}_{0,\chi}(s)$. This will be carried out in the following parts.

7.2. **Meromorphic Continuation Inside the Critical Strip.** Let $s \in \mathcal{R}(1;\chi,\boldsymbol{\epsilon})$ and $1 \leq m \leq r-1$. Let $j_m, j_{m-1}, \cdots, j_1$ be m integers such that $1 \leq j_m < \cdots < j_1 \leq r-1$. Consider the summand in the second term of (16):

$$\mathcal{I}_{m,\chi}(s) := \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{\mathcal{C}} \cdots \cdots \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_{j_m}=s-1} \cdots \operatorname*{Res}_{\kappa_{j_1}=s-1} \mathcal{F}(\boldsymbol{\kappa}; s) \frac{d\kappa_{r-1} \cdots d\kappa_1}{d\kappa_{j_m} \cdots d\kappa_{j_1}}.$$

Then each $\mathcal{I}_{m,\chi}(s)$ is naturally meromorphic in $\mathcal{R}(1;\chi,\boldsymbol{\epsilon})$ with a possible at $s=1$.

Theorem 20. *Let notation be as before. Let $n \leq 4$. Let $\chi \in \mathfrak{X}_P$. Assume that the adjoint L-function $L(s,\sigma,\mathrm{Ad}\otimes\tau)$ is holomorphic inside the strip $\mathcal{S}_{(0,1)}$ for any cuspidal representation $\sigma \in \mathcal{A}_0(GL(k,\mathbb{A}_F))$, and any $k \leq n-1$. Then for any $0 \leq m \leq r-1$, the function*

$$\sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{I}_{m,\chi}(s), \qquad s \in \mathcal{R}(1;\chi,\boldsymbol{\epsilon}),$$

admits a meromorphic continuation to the area $\mathcal{R}(1/2;\tau)^-$, with possible simple poles at $s \in \{1/2, 2/3, \cdots, (n-1)/n, 1\}$, where $\mathcal{R}(1/2;\tau)^-$ is defined in (14). Moreover, for any $3 \leq k \leq n$, if $L_F((k-1)/k,\tau) = 0$, then $s = (k-1)/k$ is not a pole.

Remark. We restrict ourselves to the case $n \leq 4$ for the following two reasons. On the one hand, we actually need to assume Dedekind Conjecture of degree n to handle the contribution from geometric side. This conjecture has been confirmed when $n \leq 4$, so we will get unconditional results if $n \leq 4$. On the other hand, when $n \geq 5$, the procedure of meromorphic continuation is even more complicated, since we are lack of a symmetrical description of this process. Thus, we will focus on $n \leq 4$ case in this paper.

Remark. In can be seen from the proof that when $n \leq 3$, we can continue the functions $\sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{I}_{m,\chi}(s)$ to $\mathrm{Re}(s) > 1/3$. When $n = 4$, we can only continue $\sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{I}_{m,\chi}(s)$ to $\mathcal{R}(1/2;\tau)^-$, an open set just containing the right half plane $\mathrm{Re}(s) \geq 1/2$. This is because some of its components involve $\Lambda(2s,\tau^2)^{-1}$ as a factor (e.g. ref. (318) of [15]). The key ingredient is that $\mathcal{R}(1/2;\tau)^-$ is uniform with respect to $\chi \in \mathfrak{X}_P$. In fact this is sufficient to give $\sum_{\chi}\sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{I}_{m,\chi}(s)$ a continuation to $\mathcal{R}(1/2;\tau)^-$ (ref. Theorem 21), hence to the whole complex plane when combining with functional equation.

Proof. The proof of Theorem 20 roughly follows from a repeated application of Cauchy's integral formula to shift contour. The complete proof is rather tedious although we take advantage of symmetry to simplify it a little bit. To illustrate the idea more directly, we shall follow Jacquet-Zagier's original approach, for the $n = 3$ case, to give meromorphic continuation of $J(s) = \int_{(0)}\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2$ as follows, which involves 56 terms in total (also some of them are same but locate in different regions). When $s \in \mathcal{R}(1)^+$, we have, by Cauchy integral formula, that

$$\begin{aligned} J(s) = &\int_{\mathcal{C}}\int_{\mathcal{C}} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2 - \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_1 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 - \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2 = s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 \\ &- \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \operatorname*{Res}_{\kappa_1 = s-1} \operatorname*{Res}_{\kappa_2 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1 = 2s-2} \operatorname*{Res}_{\kappa_2 = s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s). \end{aligned}$$

Since the right hand side is meromorphic in $\mathcal{R}(1)$, we get meromorphic continuation of $J(s)$ in $\mathcal{R}(1)^-$. Denote by $J_1(s)$ this continuation. Let $s \in \mathcal{R}(1)^-$. Then

$$\begin{aligned} J_1(s) = &\int_{(0)}\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 \\ &+ \int_{(0)} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \int_{(0)} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 - \\ &\int_{(0)} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \\ &\operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) \\ &+ \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s), \end{aligned}$$

where the right hand side is meromorphic in $1/2 < \mathrm{Re}(s) < 1$. Hence we obtain a meromorphic of $J_1(s)$ to the domain $\mathcal{S}_{(1/2,1)}$. Denote by $J_2(s)$ this continuation. Let $s \in \mathcal{R}(1/2)^+$. Then we have, again, by Cauchy integral formula, that

$$\begin{aligned} J_2(s) = &\int_{(0)}\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2 + \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 + \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 \\ &+ \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 - \\ &\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \\ &\operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2s-1} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \\ &\operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \\ &+ \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_2=s-1-\kappa-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s), \end{aligned}$$

where the right hand side is meromorphic in $\mathcal{R}(1/2)$. Hence we obtain a meromorphic continuation of $J_2(s)$ in $s \in \mathcal{R}(1/2)$. Let $s \in \mathcal{R}(1/2)^-$. Then we have, again, by Cauchy integral formula, that

$$\begin{aligned} J_2(s) = &\int_{(0)}\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2 + \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 + \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 \\ &+ \int_{\mathcal{C}} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \int_{(0)} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 - \\ &\int_{(0)} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \\ &\operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2s-1} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \\ &\operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \\ &+ \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_2=s-1-\kappa-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) + \\ &\operatorname*{Res}_{\kappa_1=1-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_2=1-2s} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=1-2s} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s), \end{aligned}$$

where the right hand side is meromorphic in $1/3 < \mathrm{Re}(s) < 1/2$. Hence we obtain a meromorphic continuation of $J_2(s)$ in $s \in \mathcal{S}(1/3, 1/2)$. Therefore, putting the above computation together, we get a meromorphic continuation of $J(s)$ to the domain $s \in \mathcal{S}_{(1/3,1)}$.

Then one needs to investigate these terms individually. What is worse, the situation would be much more complicated in GL_4 case. $\square$

Recall that we need to investigate the analytic behavior of the function

$$\mathcal{Z}_{m,*}(s)=\sum_{P}\frac{1}{c_P}\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{I}_{m,\chi}(s)\cdot\Lambda(s,\tau)^{-1},$$

where the sum over standard parabolic subgroups P is finite while the sum over cuspidal data χ is infinite. According to Theorem 16 $\mathcal{Z}_*(s)$ converges absolutely and locally normally in the region $\mathrm{Re}(s)>1$. Moreover, by Theorem 20 we see that each summand $\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{I}_{m,\chi}(s)\cdot\Lambda(s,\tau)^{-1}$ admits a meromorphic continuation to the region $\mathcal{R}(1/2;\tau)^{-}$, with possible simple poles at $s\in\{1/2,2/3,3/4\}$ and a pole of order at most 4 at $s=1$. Denote formally by

$$\mathcal{Z}_m(s)=\sum_{P}\frac{1}{c_P}\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\widetilde{\mathcal{I}}_{m,\chi}(s)\cdot\Lambda(s,\tau)^{-1}, \tag{18}$$

where $\widetilde{\mathcal{I}}_{m,\chi}(s)$ is the continuation of $\mathcal{I}_{m,\chi}(s)$. Clearly we only need to show that $(s-1/2)(s-2/3)(s-3/4)(s-1)^4\mathcal{Z}(s)$ converges absolutely and locally normally inside the domain $\mathcal{R}(1/2;\tau)^{-}$. Invoking this with the second part of Theorem 20 will lead to a meromorphic continuation of $Z(s)$ to the region $\mathcal{R}(1/2;\tau)^{-}$ with a possible simple pole at $s=1/2$ and a pole of order 4 at $s=1$.

Theorem 21. *Let notation be as before. Let $0\leq m\leq r-1$. Then $\mathcal{Z}(s)$ admits a meromorphic continuation to the domain $\mathcal{R}(1/2;\tau)^{-}$, where it has possible poles at $s=1/2$ and $s=1$. Moreover, if $s=1/2$ is a pole, then it must be simple.*

Remark. One can formally verify this conclusion by a direct computation without considering convergence problem. The convergence of $m=0$ case follows from Theorem 17, while for $m\geq 1$, the situation is more subtle. We shall prove an estimate of the form

$$\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\mathcal{C}_1}\cdots\cdots\int_{\mathcal{C}_{r-1}}\left|\operatorname*{Res}_{\kappa_{j_m}=\delta_m(s)}\cdots\operatorname*{Res}_{\kappa_{j_1}=\delta_1(s)}\mathcal{F}(\boldsymbol{\kappa};s)\right|\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}}<\infty,$$

where the above inequality holds locally uniformly in the set of regular points of $\operatorname*{Res}_{\kappa_{j_m}=\delta_m(s)}\cdots\operatorname*{Res}_{\kappa_{j_1}=\delta_1(s)}\mathcal{F}(\boldsymbol{\kappa};s)$ in $\mathcal{R}(1/2;\tau)^{-}$. See Theorem 81 of [15] for details.

8. Proof of Main Theorems

Proposition 22. *Let $n\geq 1$ be an integer. Let π be an cuspidal representation of $GL(n,\mathbb{A}_F)$ and τ be a quadratic primitive Hecke character on $F^\times\backslash\mathbb{A}_F^\times$, where F is a number field. Then the root number of $\Lambda_F(s,\pi,Ad\otimes\tau)$ is 1.*

Proof of Theorem 1. Recall that we have shown, for any test function $\varphi\in\mathcal{F}(\omega)$,

$$I(s)=\int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)E(x,\Phi,\tau;s)dx=I_{r.e.}(s)+I_\infty(s),$$

where $I_{r.e.}(s)$ is defined via (2), namely,

$$I_{r.e.}(s)=I_{r.e.}^{\varphi}(s)=\frac{1}{n}\sum_{(E:F)=n}Q_E(s)\Lambda_E\left(s,\tau\circ N_{E/F}\right);$$

and $I_\infty(s)=\sum_{k=1}^{n}I_\infty^{(k)}(s)$. Since $n\leq 4$, then according to Uchida-Van der Waal Theorem (ref. [12] and [13]) and its generalization to twist form (ref. [10]), each $\Lambda_E\left(s,\tau\circ N_{E/F}\right)\cdot\Lambda_F(s,\tau)^{-1}$ admits a holomorphic continuation to the whole complex plane. Since the sum over extensions E/F is finite, the function $I_{r.e.}(s)\cdot\Lambda_F(s,\tau)^{-1}$ admits an entire continuation.

Also, by Theorem 10, Corollary 11 and Theorem 20, the function $I_\infty(s)/\Lambda_F(s,\tau)$ admits a meromorphic continuation to $Re(s)>1/3$, with possible simple poles at

$s \in \{1/2, 2/3, 3/4\}$. Moreover, if $L_F(2/3, \tau) = 0$, then $I_\infty(s) \cdot \Lambda_F(s, \tau)^{-1}$ is regular at $s = 2/3$; if $L_F(3/4, \tau) = 0$, then $I_\infty(s) \cdot \Lambda_F(s, \tau)^{-1}$ is regular at $s = 3/4$.

Let ρ be a zero of $\Lambda(s, \tau)$ of order $r_\rho \geq 1$ such that $\mathrm{Re}(\rho) > 1/3$. Denote by

$$J(\rho; j) = \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x, x) \frac{\partial^j}{\partial s^j} E(x, \Phi, \tau; s) \mid_{s=\rho} dx, \ \ 0 \leq j \leq r_\rho - 1.$$

If $\rho \neq 1/2$, we then see that $J(\rho; j) = 0$ for any $0 \leq j \leq r_\rho - 1$ and $\varphi \in \mathcal{F}(\omega)$. According to Proposition 4, one has, for all cuspidal representations $\pi \in \mathcal{A}_0\left(G(F) \setminus G(\mathbb{A}_F), \omega^{-1}\right)$, and all K-finite functions $\phi_1, \phi_2 \in \mathrm{V}_\pi$, that

$$\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \phi_1(x)\overline{\phi_2(x)} \frac{\partial^j}{\partial s^j} E(x, \Phi, \tau; s) \mid_{s=\rho} dx = 0.$$

Then by Rankin-Selberg theory, we have, for all cuspidal representations $\pi \in \mathcal{A}_0\left(G(F) \setminus G(\mathbb{A}_F), \omega^{-1}\right)$, that $\frac{\partial^j}{\partial s^j}\Lambda(s, \pi \otimes \tau \times \widetilde{\pi}) \mid_{s=\rho}= 0$, $1 \leq j < r_\rho$, implying that the adjoint L-function $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ is regular at $s = \rho$.

Now assume that $\rho = 1/2$, namely, $L_F(1/2, \tau) = 0$. If τ is not quadratic, then by Theorem 21, $I_\infty^{(1)}(s) \cdot \Lambda_F(s, \tau)^{-1}$ is regular at $s = 1/2$. Therefore, we have $J(1/2; j) = 0$, for $1 \leq j \leq r_{1/2} - 1$. Hence, by Proposition 4 and similar analysis as above we see that $\frac{\partial^j}{\partial s^j}\Lambda(s, \pi \otimes \tau \times \widetilde{\pi}) \mid_{s=1/2}= 0$, $1 \leq j \leq r_{1/2} - 1$, implying that the adjoint L-function $\Lambda(s, \pi, \mathrm{Ad})$ is regular at $s = 1/2$. Now we assume that $\tau^2 = 1$. If $r_{1/2} \geq 2$, then by Theorem 10, Theorem 11 and Theorem 21 we see that $J(1/2; j) = 0$, for $1 \leq j \leq r_{1/2} - 2$. Hence, by Proposition 4 and similar analysis as above we see that $\frac{\partial^j}{\partial s^j}\Lambda(s, \pi \otimes \tau \times \widetilde{\pi}) \mid_{s=1/2}= 0$, $1 \leq j < r_{1/2} - 1$, implying that the adjoint L-function $\Lambda(s, \pi, \mathrm{Ad})$ has at most a simple pole at $s = 1/2$. Now we apply Proposition 22 to exclude this possible simple pole at $1/2$. Suppose that $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ has a pole at $s = 1/2$. Since the root number of $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ is trivial, then the order of the pole $s = 1/2$ must be even. So $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ cannot have a simple pole at $s = 1/2$. A contradiction. If $r_{1/2} = 1$, then clearly, the adjoint L-function $\Lambda(s, \pi, \mathrm{Ad})$ has at most a simple pole at $s = 1/2$. The same argument on root number excludes the possibility of pole at $s = 1/2$.

In all, we have shown that $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$ is holomorphic in $\mathcal{R}(1/2; \tau)^- \cup \mathcal{S}_{(1/2,\infty)}$. Now Theorem 1 follows from global functional equation of $\Lambda(s, \pi, \mathrm{Ad} \otimes \tau)$. □

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