SUMMING HECKE EIGENVALUES OVER POLYNOMIALS

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Abstract. In this paper we estimate sums of the form $\sum_{n\leq X}|a_{\mathrm{Sym}^m\pi}(|f(n)|)|$, for symmetric power lifts of automorphic representations π attached to holomorphic forms and polynomials $f(x)\in\mathbb{Z}[x]$ of arbitrary degree. We give new upper bounds for these sums under certain natural assumptions on f. Our results are unconditional when $\deg(f)\leq 4$. Moreover, we study the analogous sum over polynomials in several variables. We obtain an estimate for all cubic polynomials in two variables that define elliptic curves.

1. Introduction

A basic goal of the theory of automorphic forms is to estimate sums of Hecke eigenvalues. In this paper we consider a finer version of this problem, where the sum is taken over values of polynomials $f(x)\in\mathbb{Z}[x]$. Averages over sparse sequences of this type have applications to studying moments of L-functions, and to establishing non-vanishing results. On a related note, sums over values $f(p)$ restricted to primes p, are relevant to some questions arising from the Beyond Endoscopy approach proposed by Langlands.

Notable advances for nonlinear polynomials are rather scarce. The case of quadratic polynomials $f(x)$ has been investigated first by Blomer [Blo08], and later by Templier [Tem11], as well as Templier and Tsimerman [TT13]. By taking absolute values of the summation terms, Kim [Kim07] obtained a conditional result for polynomials of arbitrary degree in the setting of cuspidal automorphic representations π of $\mathrm{GL}(2)$. Assuming the strong Artin conjecture, he showed that $\sum_{n\leq X}|a_\pi(f(n))|^2\ll X$, where $a_\pi(n)$ are the Dirichlet coefficients of the L-function of π. Kim's argument rests on an estimate of Barban and Vehov [BV69] concerning multiplicative functions $g(n)\geq 0$ with the property that there exists a constant c such that $g(p^k)\ll k^c$ for all primes p and positive integers k. It appears to have been overlooked in [Kim07] that, in order to be able to apply [BV69] for $g(n)=|a_\pi(n)|^2$, the Ramanujan conjecture for π must be assumed.

An immediate consequence of Kim's result, under the appropriate assumptions, is the upper bound

$$\sum_{n\leq X}|a_\pi(f(n))|\ll X. \tag{1}$$

Special cases suggest that it should be possible to improve this bound, perhaps by saving a power of $\log X$. For example, if π is generated by a holomorphic cusp form without complex multiplication (CM), the Sato-Tate conjecture implies the asymptotics

$$\sum_{n\leq X}|a_\pi(n)|\sim c\frac{X}{(\log X)^\delta},$$

for some positive constant c and $\delta=1-8/3\pi\approx 0.151$. Furthermore, if ℓ is a fixed nonzero integer and π corresponds to a Maass form, Holowinsky [Hol09] showed that

$$\sum_{n\leq X}|a_\pi(n)a_\pi(n+\ell)|\ll\frac{X}{(\log X)^\delta},$$

for some absolute positive constant $\delta<2(1-8/3\pi)$. Such estimates for shifted convolution sums have played a pivotal role in the resolution of the mass equidistribution conjecture for the surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (cf. [HS10]).

In this first theorem of this article we obtain a logarithm power saving over the bound (1), in a broader context. We maintain assumptions similar to those in [Kim07]. As explained there, one can associate to a monic irreducible polynomial $f(x)\in\mathbb{Z}[x]$ of degree $d\geq 2$ a permutation representation acting on its roots.

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This can be thought of as a Galois representation, which is the sum of the trivial representation and another $(d-1)$-dimensional Artin representation σ_f with the property that the number of solutions of f modulo a prime p is $\rho_f(p) := 1 + \sigma_f(\mathrm{Frob}_p)$. The strong Artin conjecture, or more generally Langlands' principle of functoriality, predicts that there exists an automorphic representation $\pi'(f)$ of $\mathrm{GL}(d-1, \mathbb{A}_{\mathbb{Q}})$ with the same L-function as σ_f. In this case we refer to σ_f as being automorphic (or modular).

A novel feature of our result is that it applies not only to representations π attached to holomorphic forms, but also to every symmetric power $\mathrm{Sym}^m \pi$, provided that σ_f is automorphic. The existence of all symmetric powers lifts has recently been established by Newton and Thorne [NT20] for all cuspidal Hecke eigenforms. Another important aspect is that our estimate holds in short intervals as well.

Theorem 1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ defined by a non-CM holomorphic newform of weight $k \geq 2$. Suppose $f(x) \in \mathbb{Z}[x]$ is a monic irreducible polynomial, with no fixed prime divisor. Let $0 < \varepsilon < 1/10$, $0 < \beta < 1$, and $a, q \in \mathbb{Z}$ with $0 < a \leq q$ and $(q, f(a)) = 1$. If σ_f is automorphic then for every integer $m \geq 1$ we have*

$$\sum_{\substack{X - X_0 < n \leq X \\ n \equiv a \ (mod\ q)}} |a_{\mathrm{Sym}^m \pi}(|f(n)|)| \ll \frac{X_0}{q} \cdot (\log X_0)^{-\frac{m}{2(m+2)^2}}, \tag{2}$$

uniformly for $X^\varepsilon \leq X_0 \leq X$ and $1 \leq q \leq X_0^{1-\beta}$. The implied constant depends on m, π and f.

We remark that f has no fixed prime divisor precisely when $\rho_f(p) < p$ for all primes p. This mild requirement is all we need to apply a very general theorem of Nair and Tenenbaum [NT98] that reduces the problem to bounding sums over primes of the form $\sum_{p \leq X} \rho_f(p) |a_{\mathrm{Sym}^m \pi}(p)|/p$. The modularity of σ_f provides a means of compatibility, in an analytic sense, between the Frobenius traces of σ_f and the Hecke eigenvalues of $\mathrm{Sym}^m \pi$. The rest of the proof combines an insight inspired by Holowinsky's sieve method for shifted convolution sums [Hol09] with certain properties of the adjoint lift; this is detailed in Section 3.

For polynomials of small degree we can explicitly construct the automorphic representation $\pi'(f)$ corresponding to σ_f, so our result is unconditional.

Corollary 1.2. *Let $f(x) \in \mathbb{Z}[x]$ be a monic irreducible polynomial, with no fixed prime divisor. Assume that $\deg f \leq 4$. Then σ_f is automorphic. In particular, the upper bound (2) holds for all such polynomials f.*

Our second main result is concerned with an estimate over cubic polynomials in two variables, which can be viewed as Weierstrass equations defining elliptic curves. This question fits into a more general framework, as developed by de la Brètche and Browning [dlBB06], who investigated the average order of certain multiplicative functions over values taken by general binary forms. We also mention the recent work of Lachand [Lac18] in the case of the special cubic form $X_1^3 + 2X_2^3$.

Theorem 1.3. *Let π be a non-dihedral cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ satisfying the Ramanujan conjecture. Consider an irreducible polynomial*

$$E(x, y) = y^2 - x^3 - ax - b \in \mathbb{Z}[x, y]$$

with discriminant $4a^3 + 27b^2 \neq 0$. Let $\gamma_2 \geq \gamma_1 > 0$, and $X, Y \geq 100$ such that $Y^{\gamma_1} \leq X \leq Y^{\gamma_2}$. Let $\alpha, \beta \in (0, 1)$, and $X^\alpha \leq X_0 \leq X$, $Y^\beta \leq Y_0 \leq Y$. Then

$$\sum_{\substack{X - X_0 < m \leq X \\ Y - Y_0 < n \leq Y}} |a_\pi(|E(m, n)|)| \ll \frac{X_0 Y_0}{(\log X_0 Y_0)^{1/18}}, \tag{3}$$

where the implied constant depends on α, β, γ and π.

Our argument in Theorem 1.3 is robust enough to be applied to any modular variety defined by polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_m]$. In fact, one does not even need explicit polynomials to define the variety; all that is required is the modularity condition. In Section 2 we give all the necessary ingredients for the general case, particularly Proposition 2.2. To simplify notations, we opted to state Theorem 1.3 for elliptic curves.

The principal technical difficulty that we must overcome is adapting the methods of [Nai92] and [NT98] to several variables. In contrast with [Nai92], Brun's sieve does not seem to be amenable to our setting, so we make use of Selberg's upper bound sieve instead. For polynomials in two variables on smooth domains, an approach of similar flavor appears in the work of Khayutin [Kha19], where a conditional proof of the "Mixing Conjecture" of Michel and Venkatesh [MV06] is presented.

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2. Sieve Bounds

We start this section with a brief review of a fundamental result on short sums of certain arithmetic functions in arithmetic progressions due to Nair and Tenenbaum [NT98]. In order to state it, we first need to introduce some notation.

For any $A, B \geq 1$, $\epsilon > 0$ and $k \in \mathbb{Z}^+$, we denote by $\mathcal{M}_k(A, B, \epsilon)$ the class of non-negative arithmetic functions $F(n_1, \cdots, n_k)$ in k variables satisfying

$$F(m_1 n_1, \cdots, m_k n_k) \leq \min\{A^{\Omega(m)}, Bm^{\epsilon}\} \cdot F(n_1, \cdots, n_k), \ m := m_1 \cdots m_k,$$

where $\Omega(m)$ denotes the number of prime factors of m counted with multiplicity; and $(m_j, n_j) = 1$, for $1 \leq j \leq k$.

Let $Q_j \in \mathbb{Z}[x]$, $1 \leq j \leq k$ be polynomials with factorization

$$Q_j(x) = \prod_{i=1}^{r} R_i(x)^{\gamma_{ji}}, \ 1 \leq j \leq k,$$

where the polynomials $R_i(x) \in \mathbb{Z}[x]$ are irreducible over $\mathbb{Q}$. For a polynomial $Q \in \mathbb{Z}[x]$, denote by $\rho_Q(n)$ the number of solutions of the congruence $Q(x) \equiv 0 \pmod{n}$. Write $\rho_i(n) = \rho_{R_i}(n)$ for $1 \leq i \leq r$. Define

$$\vartheta(n; F, \underline{\rho}) := \sideset{}{^\dagger}\sum_{n_1^{\gamma_1} \cdots n_r^{\gamma_r} = n} F\Big(\prod_{i=1}^{r} n_1^{\gamma_{1i}}, \cdots, \prod_{i=1}^{r} n_r^{\gamma_{ki}}\Big) \cdot \frac{\rho_1(n_1) \cdots \rho_r(n_r)}{n_1 \cdots n_r},$$

where the $\dagger$ symbol indicates that the r-fold sum is restricted to pairwise coprime variables. Also, for a polynomial $Q(x) = \sum_i a_i x^i \in \mathbb{Z}[x]$, set $\|Q\| = \max_i |a_i|$.

The Nair-Tenenbaum bound ([NT98], Corollary 1) can now be stated as follows:

Lemma 2.1. *Let $A, B \geq 1$, $g \in \mathbb{Z}^+$, $0 < \epsilon < 1/8g^2$, $0 < \beta < 1$, $0 < \delta \leq 1/2g$, and let r, k be arbitrary positive integers. Let $f \in \mathcal{M}_k(A, B, \epsilon\beta\delta/6)$ and $Q_j \in \mathbb{Z}[X]$ $(1 \leq j \leq k)$ be such that $Q = \prod_{j=1}^{k} Q_j$ has no fixed prime divisor. Let $a, q \in \mathbb{Z}^+$, with $a \leq q$ and $(q, Q(a)) = 1$. Then*

$$\sum_{\substack{x < n < x+y \\ n \equiv a \pmod{q}}} F(|Q_1(n)|, \cdots, |Q_k(n)|) \ll \frac{y}{q} \prod_{\substack{p \leq x \\ p \nmid q}} \left(1 - \frac{\rho_Q(p)}{p}\right) \sum_{\substack{n \leq x \\ (n,q)=1}} \vartheta(n; F, \underline{\rho}) \tag{4}$$

uniformly for $x \geq c_1 \|Q\|^{2\delta}$, $x^{4g^2\epsilon} \leq y \leq x$, $1 \leq q \leq y^{1-\beta}$. The implied constant in $\ll$ sign depends at most on $A, B, \epsilon, \beta, \delta, k, r, g$ and the discriminant of Q. The constant c_1 relies at most on $A, B, \epsilon, \delta, k, r, g$.

Note that the sum on the left hand side is essentially the average of a multiplicative function F evaluated at absolute values of the polynomial $Q_1 Q_2 \cdots Q_k$, which is still a single variable function. To prove Theorem 1.3 we need a similar sieve estimate for polynomials in several variables.

The main result of this section is the following:

Proposition 2.2. *Let $h \geq 0$ be a multiplicative function bounded by the k-th divisor function, for some integer $k \geq 1$. Let $Q \in \mathbb{Z}[x_1, x_2, \cdots, x_n]$ be an irreducible polynomial, and denote by $\rho_Q(p)$ the number of $\mathbb{F}_p$-points on the variety $Q(x_1, x_2, \cdots, x_n) = 0$. Suppose there is a constant $0 < \delta < 1$ such that for all integers $l \geq 1$*

$$|\rho_Q(p^l) - p^{(n-1)l}| \ll p^{(1-\delta)(n-1)l}. \tag{5}$$

For $1 \le j \le n$ consider positive integers $X_j \ge X_j'$ such that

$$\max_{1\le j\le n} \log X_j \ll \min_{1\le j\le n} \log X_j'. \tag{6}$$

Then we have

$$\sum_{X_j - X_j' < m_j \le X_j,\ 1\le j\le n} h(|Q(m_1, m_2, \cdots, m_n)|) \ll \prod_{j=1}^{n} X_j' \cdot \exp\left(\sum_{p \le X_1' X_2' \cdots X_n'} \frac{\omega_Q(p)\cdot(h(p)-1)}{p} \right), \tag{7}$$

where $\omega_Q(p) = \rho_Q(p)/p^{n-1}$, and the implied constant depends only on k, Q and the implied constant in (6).

It is important to emphasize that one cannot apply Brun's sieve, as in [Nai92], to prove Proposition 2.2. The reason is that the crucial condition (R), detailed in Halberstam and Richert [HR74], does not hold in the case of several variables. The remaining part of this section proof is concerned with the proof of Proposition 2.2.

2.1. **Preliminaries.**

First, we obtain an estimate using Selberg's upper bound sieve.

Lemma 2.3. *Suppose $Q(x_1, x_2, \cdots, x_n) \in \mathbb{Z}[x_1, x_2, \cdots, x_n]$ is an irreducible polynomial satisfying (5). For $1 \le j \le n$, let $X_j, X_j', z > 0$ with $z \le X_j' \le X_j$. Then for every positive integer c we have*

$$S := \sum_{\substack{X_1 - X_1' < m_1 \le X \\ P^-(Q(m_1,m_2,\cdots,m_n)) \ge z, \\ (Q(m_1,\cdots,m_n),c)=1}} \cdots \sum_{X_n - X_n' < m_n \le X_n} 1 \ll \prod_{j=1}^{n} X_j' \cdot \prod_{\substack{p<z \\ p\nmid c}} \left(1 - \frac{\rho_Q(p)}{p^n}\right) + z^{2n}(\log z)^3,$$

where $\rho_Q(p)$ is the number of solutions of $Q(x_1, x_2, \cdots, x_n) \equiv 0 \pmod p$ in $\mathbb{F}_p$.

Proof. Set $\mathcal{P}_c = \{p : \ p\nmid c,\ p \text{ prime}\}$ and let $P_c(z) = \prod p$ be the sifting range, where p runs through $p \in \mathcal{P}_c$ with $p < z$. Define

$$S(\mathcal{P}_c, z) := \sum_{\substack{X_1 - X_1' < m_1 \le X \\ (Q(m_1,\cdots,m_n), P_c(z))=1}} \cdots \sum_{X_n - X_n' < m_n \le X_n} 1.$$

Note that $S = S(\mathcal{P}_c, z)$ if $c < z$, and $S \le S(\mathcal{P}_c, z)$ if $c \ge z$. In either case, $S \le S(\mathcal{P}_c, z)$. Let

$$\omega(p) = \begin{cases} \rho_Q(p)/p^{n-1}, & \text{if } p < z \text{ and } p \nmid c, \\ 0, & \text{otherwise.} \end{cases}$$

For $d = \prod p_i^{k_i}$, $k_i \ge 1$, write $\omega(d) = \prod_i \omega(p_i)$. Then by $|\rho_Q(p) - p^{n-1}| \ll p^{(1-\delta)(n-1)}$, one has

$$0 \le \frac{\omega(p)}{p} \le \frac{1}{p} + O(p^{\delta - 1 - \delta n}).$$

Let

$$G(z) = \sum_{\substack{d<z \\ (d,c)=1}} \frac{\mu(d)^2 \omega(d)}{d \prod_{p|d}(1 - \omega(p)p^{-1})} = \sum_{\substack{d<z \\ (d,c)=1}} \mu(d)^2 \prod_{p|d} \frac{\omega(p)p^{-1}}{(1-\omega(p)p^{-1})} = \prod_{\substack{p<z \\ (p,c)=1}} \frac{1}{(1-\omega(p)p^{-1})}.$$

For $1 \le d \le z^2$ define

$$\begin{aligned} A_d &:= \sum_{\substack{X_1 - X_1' < m_1 \le X \\ d | Q(m_1,\cdots,m_n)}} \cdots \sum_{X_n - X_n' < m_n \le X_n} 1 \\ &= \sum_{\substack{r_1 \pmod d \\ d|Q(r_1,\cdots,r_n)}} \cdots \sum_{r_n \pmod d} 1 \sum_{\frac{X_1 - X_1' - r_1}{d} < m_1 \le \frac{X_1 - r_1}{d}} \cdots \sum_{\frac{X_n - X_n' - r_m}{d} < m_n \le \frac{X_n - r_m}{d}} 1. \end{aligned}$$

Let $A = A_1$. Then $A = X_1' X_2' \cdots X_n' + O(1)$, and a straightforward computation shows that

$$R_d := \left| A_d - \frac{\omega(d)}{d} \cdot A \right| \leq \omega(d) d^{n-1}.$$

Denote by $\nu(d)$ is the number of district prime divisors of d. We obtain the following estimate

$$\sum_{\substack{d<z^2 \\ d|P(z)}} 3^{\nu(d)} |R_d| \ll \sum_{\substack{d<z^2 \\ d|P(z)}} 3^{\nu(d)} \omega(d) d^{n-1} \leq z^{2n} \sum_{\substack{d<z^2 \\ d|P(z)}} \frac{3^{\nu(d)} \omega(d)}{d} \leq z^{2n} \prod_{p<z} \left(1 + \frac{\omega(p)}{p}\right)^3 \ll z^{2n} (\log z)^3,$$

where the last inequality follows from Mertens formula in conjunction with the assumption (5). Finally, Selberg's upper bound sieve gives

$$S(\mathcal{P}_c, z) \leq \frac{A}{G(z)} + \sum_{\substack{d<z^2 \\ d|P(z)}} 3^{\nu(d)} |R_d| \ll \prod_{j=1}^{n} X_j' \cdot \prod_{\substack{p<z \\ p\nmid c}} \left(1 - \frac{\rho_Q(p)}{p^n}\right) + z^{4n} (\log z)^3.$$

□

Next, we present a variant of [Nai92, Lemma 2] adapted to our setting.

Lemma 2.4. *Let $\rho_Q(p)$ be the number of $\mathbb{F}_p$-points on the variety $Q(x_1, x_2, \cdots, x_n) = 0$, and set $\omega_Q(p) = \rho_Q(p)/p^{n-1}$. For every $z \geq 2$ the following hold:*

(i). If $1 \leq l \leq \log z$ then

$$\prod_{p<z^{1/l}} \left(1 - \frac{\omega_Q(p)}{p}\right) \ll l \cdot \prod_{p<z} \left(1 - \frac{\omega_Q(p)}{p}\right).$$

(ii). If $c > 0$ is an integer then

$$\prod_{\substack{p<z^{1/2} \\ p\nmid c}} \left(1 - \frac{\omega_Q(p)}{p}\right) \ll \frac{c}{\varphi(c)} \cdot \prod_{p<z} \left(1 - \frac{\omega_Q(p)}{p}\right).$$

(iii). Let H be a multiplicative function bounded by the k-th divisor function, for some $k \geq 2$. Then

$$\sum_{n\leq z} \frac{H(n)\omega_Q(n)}{n} \ll \exp\left(\sum_{p\leq z} \frac{H(p)\omega_Q(p)}{p}\right).$$

2.2. **Proof of Proposition 2.2.** Without loss of generality, we can take

$$z = \min\{X_1'^{\frac{1}{10n}}, X_2'^{\frac{1}{10n}}, \cdots, X_n'^{\frac{1}{10n}}\}$$

to be the sifting level. Consider the following set

$$\mathcal{B} = \{(m_1, m_2, \cdots, m_n) \in \mathbb{Z}^N : \ X_1 - X_1' < m_1 \leq X_1, \ X_2 - X_2' < m_2 \leq X_2, \ \cdots, X_n - X_n' < m_n \leq X_n\}.$$

For $\mathfrak{m} := (m_1, m_2, \cdots, m_n) \in \mathcal{B}$, we define the quantities $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}}$ as follows. First write

$$|Q(m_1, m_2, \cdots, m_n)| = \prod_{i=1}^{l} p_i^{k_i}, \ p_1 < p_2 < \cdots < p_l, \ k_i > 0, \ 1 \leq i \leq l.$$

- If $p_1^{k_1} > z$ set $A_{\mathfrak{m}} = 1$.
- If $p_1^{k_1} \leq z$ take $j \leq l$ to be the largest index such that $p_1^{k_1} \cdots p_j^{k_j} \leq z$, and set $A_{\mathfrak{m}} = p_1^{k_1} \cdots p_j^{k_j}$.
- Let $B_{\mathfrak{m}} = |Q(m_1, m_2, \cdots, m_n)|/A_{\mathfrak{m}}$.

We decompose $\mathcal{B}$ into four disjoint sets, as in [Shi80]:

$$\mathcal{B}_1 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ P^-(B_{\mathfrak{m}}) \geq z^{1/2}\},$$
$$\mathcal{B}_2 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ P^-(B_{\mathfrak{m}}) < z^{1/2}, \ A_{\mathfrak{m}} \leq z^{1/2}\},$$
$$\mathcal{B}_3 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ P^-(B_{\mathfrak{m}}) < \log z \log\log z, \ A_{\mathfrak{m}} > z^{1/2}\},$$
$$\mathcal{B}_4 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ \log z \log\log z < P^-(B_{\mathfrak{m}}) < z^{1/2}, \ A_{\mathfrak{m}} > z^{1/2}\}.$$

Here $P^-(n)$ and $P^+(n)$ denote the least and the greatest prime factor of n, respectively.

In the next four Lemmas (2.5-2.8), we bound the contribution from each set $\mathcal{B}_i$. We maintain the same notations and assumptions as in the statement of Proposition 2.2.

Lemma 2.5. *Let* $\mathcal{B}_1 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ P^-(B_{\mathfrak{m}}) \geq z^{1/2}\}$*. Then*

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1} h(|Q(m_1, m_2, \cdots, m_n)|) \ll_{n,k} \prod_{j=1}^{n} X'_j \exp\left(\sum_{p\leq z} \frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right). \tag{8}$$

Proof. By the definition of $\mathcal{B}_1$ and the multiplicity of h, we have

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1} h(|Q(m_1, m_2, \cdots, m_n)|) \ll \sum_{c\leq z} h(c) \sideset{}{^\dagger}\sum_{(m_1,m_2,\cdots,m_n) in \mathcal{B}_1} h(B_{\mathfrak{m}}), \tag{9}$$

where the symbol $\dagger$ indicates that the inner sum is over the pairs $(m_1, m_2, \cdots, m_n) \in \mathcal{B}_1$ such that $c \mid Q(m_1, m_2, \cdots, m_n)$ and $(c, B_{\mathfrak{m}}) = 1$. Let

$$M_Q(X_j, X'_j : 1 \leq j \leq n) := \max_{X_j - X'_j < m_j \leq X_j, \ 1\leq j\leq n} |Q(m_1, m_2, \cdots, m_n)|.$$

To simplify notation, we shall write M_Q for $M_Q(X_j, X'_j : 1 \leq j \leq n)$; this quantity will also appear in the following Lemmas. Then by (6) one has $\log M_Q \ll \log z$. Hence

$$z^{\Omega(B_{m,n})/2} \leq B_{\mathfrak{m}} \ll \frac{M_Q}{c} \leq M_Q,$$

implying that $\Omega(B_{m,n}) \leq \frac{2\log M_Q}{\log z} + O(1) = O(1)$. So $h(B_{m,n}) \ll A^{\Omega(B_{m,n})} \ll 1$. Substituting this into (9) we obtain

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1} h(|Q(m_1, m_2, \cdots, m_n)|) \ll_k \sum_{c\leq z} h(c) \sideset{}{^\dagger}\sum_{(m,n)\in\mathcal{B}_1} 1. \tag{10}$$

We can express the right-hand side of (10) as

$$\sum_{c\leq z} h(c) \sum_{(r_1,r_2,\cdots,r_m)\in\mathcal{R}_c} \sideset{}{^\dagger}\sum_{\substack{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1\\ m_j\equiv r_j \pmod c, \ 1\leq j\leq n}} 1, \tag{11}$$

where $\mathcal{R}_c \subseteq (\mathbb{Z}/c\mathbb{Z})^{\oplus n}$ denotes the set of solutions of $Q(x_1, x_2, \cdots, x_n) \equiv 0 \pmod c$. Write $m_j = cq_j + r_i$, $1 \leq j \leq n$. Then $Q(m_1, m_2, \cdots, m_n) \equiv Q(r_1, r_2, \cdots, r_n) \pmod c$. Hence, $\frac{Q(m_1,m_2,\cdots,m_n)-Q(r_1,r_2,\cdots,r_n)}{c}$ is a integer. Denote it by $P(q_j, r_j : 1 \leq j \leq n)$. Let $Q(r_1, r_2, \cdots, r_n) = cR(r_1, r_2, \cdots, r_n)$. Then the dagger condition on the inner sum in (11) becomes

$$\begin{cases} \frac{X_j - X'_j - r_j}{c} < q_j \leq \frac{X_j - r_j}{c}, \ 1 \leq j \leq n, \\ P^-(R(r_1, r_2, \cdots, r_n) + P(q_j, r_j : 1 \leq j \leq n)) \geq z^{1/2}, \\ (c, R(r_1, r_2, \cdots, r_n) + P(q_j, r_j : 1 \leq j \leq n)) = 1. \end{cases}$$

Note that by definition of z, one has $X'_j/c \geq z^{1/2}$, $1 \leq j \leq n$. Now we can apply Lemma 2.3 with $\widetilde{Q}(x_1, x_2, \cdots, x_n) = R(r_1, r_2, \cdots, r_n) + P(x_j, r_j : 1 \leq j \leq n)$ to see that

$$\sideset{}{^\dagger}\sum_{\substack{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1\\ m_j\equiv r_i \pmod c, \ 1\leq j\leq n}} 1 \ll \frac{X'_1 X'_2 \cdots X'_n}{c^2} \cdot \prod_{\substack{p<z\\ p\nmid c}} \left(1 - \frac{\rho_{\widetilde{Q}}(p)}{p^n}\right). \tag{12}$$

Since $Q(cx_1+r_1, cx_2+r_2, \cdots, cx_n+r_n) = c\widetilde{Q}(x_1, x_2, \cdots, x_n)$, we conclude that $\rho_Q(p) = \rho_{\widetilde{Q}}(p)$ if $p \nmid c$. So combining (10), (11) with (12) we then obtain

$$\begin{aligned}\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1} h(|Q(m_1,m_2,\cdots,m_n)|) &\ll \prod_{j=1}^n X_j' \sum_{c\le z} \sum_{(r_1,r_2,\cdots,r_n)\in\mathcal{R}_c} \frac{h(c)}{c^2} \cdot \prod_{\substack{p<z\\ p\nmid c}} \left(1-\frac{\rho_Q(p)}{p^n}\right) \\ &= \prod_{j=1}^n X_j' \sum_{c\le z} \frac{h(c)\rho_Q(c)}{c^2} \cdot \prod_{\substack{p<z\\ p\nmid c}} \left(1-\frac{\omega_Q(p)}{p}\right).\end{aligned}$$

By Lemma 2.4 (ii), we have

$$\prod_{\substack{p<z\\ p\nmid c}} \left(1-\frac{\omega_Q(p)}{p}\right) \ll \frac{c}{\varphi(c)} \prod_{p<z} \left(1-\frac{\omega_Q(p)}{p}\right).$$

Using Lemma 2.4 (iii) for the multiplicative function $H(c) = ch(c)/\varphi(c)$ it follows that

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_1} h(|Q(m_1,m_2,\cdots,m_n)|) \ll \prod_{j=1}^n X_j' \prod_{p<z} \left(1-\frac{\omega_Q(p)}{p}\right) \exp\left(\sum_{p<z} \frac{h(p)\omega_Q(p)}{p}\right),$$

which combined with the assumption (5) implies the estimate (8). □

Lemma 2.6. *Let* $\mathcal{B}_2 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_,n) \in \mathcal{B} :\ P^-(B_{\mathfrak{m}}) < z^{1/2},\ A_{\mathfrak{m}} \le z^{1/2}\}$. *Then*

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_2} h(|Q(m_1,m_2,\cdots,m_n)|) \ll_k \prod_{j=1}^n X_j' \cdot \exp\left(\sum_{p\le z} \frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right). \tag{13}$$

Proof. By definition, for each $(m_1, m_2, \cdots, m_n) \in \mathcal{B}_2$ there is a corresponding prime power p^l such that $p^l \| Q(m_1, m_2, \cdots, m_n)$, $p \le z^{1/2}$ and $p^l > z^{1/2}$. For each $p \le z^{1/2}$, denote by l_p the least integer l such that $p^l > z^{1/2}$. Then $l_p \ge 2$ and $\max\{z^{1/2}, p^2\} \le p^{l_p} \le z$. Therefore,

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_2} 1 \le \sum_{p\le z^{1/2}} \sum_{\substack{X_j-X_j'<m_j\le X_j,\ 1\le j\le n\\ p^{l_p}|E(m,n)}} 1 = \sum_{p\le z^{1/2}} \rho_Q(p^{l_p}) \cdot \left(\frac{X_1'X_2'\cdots X_n'}{p^{2l_p}} + O(1)\right).$$

According to (5) we have $\rho_Q(p^{l_p}) \ll p^{l_p-1}$, so

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_2} 1 \ll \sum_{p\le z^{1/2}} \left(\frac{X'}{p^{l_p}} + p^{l_p-1}\right) \le \sum_{p\le z^{1/4}} \frac{X'}{z^{1/2}} + \sum_{z^{1/4}<p\le z^{1/2}} \frac{X'}{p^2} + z^{3/2} \ll \frac{X'}{z^{1/4}} + z^{3/2}, \tag{14}$$

where $X' := X_1'X_2'\cdots X_n'$.

On the other hand, we have

$$h(|Q(m_1,m_2,\cdots,m_n)|) \le d_k(|Q(m_1,m_2,\cdots,m_n)|) \ll_k M_Q^{\frac{A}{\log\log M_Q}} \tag{15}$$

for some absolute constant $A > 0$.

Substituting (15) into (14) gives

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_2} h(|Q(m_1,m_2,\cdots,m_n)|) \ll M_Q^{\frac{A}{\log\log M_Q}} \cdot \left(\frac{X'}{z^{1/4}} + z^{3/2}\right).$$

Note that by assumption (6), $M_Q^{\frac{A}{\log\log M_Q}} \ll z^{1/5}$ and $z = \min\{X_1'^{\frac{1}{10n}}, X_2'^{\frac{1}{10n}}, \cdots, X_n'^{\frac{1}{10n}}\}$. Therefore,

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_2} h(|Q(m_1,m_2,\cdots,m_n)|) \ll \prod_{j=1}^n X_j' \exp\left(\sum_{p\le z} \frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right),$$

proving the estimate (13). □

Lemma 2.7. *Let* $\mathcal{B}_3 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ P^-(B_{\mathfrak{m}}) < \log z \log\log z, \ A_{\mathfrak{m}} > z^{1/2}\}$*. Then*

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_3} h(|Q(m_1, m_2, \cdots, m_n)|) \ll_k \prod_{j=1}^{n} X_j' \exp\left(\sum_{p\le z} \frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right). \tag{16}$$

Proof. By definition, for each $(m_1, m_2, \cdots, m_n) \in \mathcal{B}_3$ there exists some integer $c \mid Q(m_1, m_2, \cdots, m_n)$ such that $z^{1/2} < c \le z$, and $P^+(c) < \log z \log\log z$. Hence

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_3} 1 \le \sum_{\substack{z^{1/2}<c\le z \\ P^+(c)<\log z\log\log z}} \ \sum_{\substack{X_j - X_j' < m_j \le X_j, \ 1\le j\le n \\ c\mid Q(m_1,m_2,\cdots,m_n)}} 1 \ll \prod_{j=1}^{n} X_j' \sum_{\substack{z^{1/2}<c\le z \\ P^+(c)<\log z\log\log z}} \frac{\rho_Q(c)}{c^n}.$$

Appealing to (5) we then deduce that

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_3} 1 \ll X' \sum_{\substack{z^{1/2}<c\le z \\ P^+(c)<\log z\log\log z}} \frac{1}{c} \ll \frac{X'}{z^{1/2}} \sum_{\substack{z^{1/2}<c\le z \\ P^+(c)<\log z\log\log z}} 1 \ll \frac{X'}{z^{1/2}} \cdot \exp\left(\frac{3\log z}{\sqrt{\log\log z}}\right),$$

where $X' = \prod_{j=1}^{n} X_j'$. We remark that the last inequality follows from the standard bound on de Bruijn function. Combining the above with (15) we obtain

$$\begin{aligned}\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_3} h(|Q(m_1, m_2, \cdots, m_n)|) &\ll_k M_Q^{\overline{\frac{A}{\log\log M_Q}}} \cdot \frac{X'}{z^{1/2}} \cdot \exp\left(\frac{3\log z}{\sqrt{\log\log z}}\right) \\ &\ll X' \exp\left(\sum_{p\le z} \frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right).\end{aligned}$$

□

Lemma 2.8. *Let* $\mathcal{B}_4 = \{\mathfrak{m} = (m_1, m_2, \cdots, m_n) \in \mathcal{B} : \ \log z \log\log z < P^-(B_{\mathfrak{m}}) < z^{1/2}, \ A_{\mathfrak{m}} > z^{1/2}\}$*. Then*

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1, m_2, \cdots, m_n)|) \ll_k \prod_{j=1}^{n} X_j' \exp\left(\sum_{p\le z} \frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right). \tag{17}$$

Proof. By the definition of $\mathcal{B}_4$ and the multiplicity of h, we have

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1, m_2, \cdots, m_n)|) \ll \sum_{z^{1/2}<c\le z} h(c) \sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4}^{*} h(B_{\mathfrak{m}}), \tag{18}$$

where the asterisk indicates a sum over $(m_1, m_2, \cdots, m_n) \in \mathcal{B}$ such that $c \mid Q(m_1, m_2, \cdots, m_n)$, $\log z \log\log z < P^-(B_{\mathfrak{m}}) \le z^{1/2}$ and $(c, B_{\mathfrak{m}}) = 1$, with $B_{\mathfrak{m}} = |Q(m_1, m_2, \cdots, m_n)|/c$. Let L be the integral part of $\log z/\log(\log z \log\log z)$. For $2 \le l \le L$, we consider the pairs $(m_1, m_2, \cdots, m_n) \in \mathcal{B}_4$ such that $z^{1/(l+1)} < P^-(B_{\mathfrak{m}}) \le z^{1/l}$. Then by definition, $P^+(A_{\mathfrak{m}}) < z^{1/l}$, and

$$M_Q \gg |Q(m_1, m_2, \cdots, m_n)| \ge B_{\mathfrak{m}} \ge P^-(B_{\mathfrak{m}})^{\Omega(B_{\mathfrak{m}})} \ge z^{\Omega(B_{\mathfrak{m}})/(l+1)}.$$

Hence, $\Omega(B_{\mathfrak{m}}) \ll l$, implying that $h(B_{\mathfrak{m}}) \le C^l$, for some positive constant C depending only on k. Therefore, we have

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1, m_2, \cdots, m_n)|) \ll \sum_{l=2}^{L} C^l \sum_{\substack{z^{1/2}<c\le z \\ P^+(c)\le z^{1/l}}} h(c) \cdot \sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4}^{(l)} 1,$$

where the superscript (l) indicates a sum over $(m_1, m_2, \cdots, m_n) \in \mathcal{B}$ such that $c \mid Q(m_1, m_2, \cdots, m_n)$, $z^{1/(l+1)} < P^-(B_{\mathfrak{m}}) \le z^{1/l}$ and $(c, B_{\mathfrak{m}}) = 1$, with $B_{\mathfrak{m}} = |Q(m_1, m_2, \cdots, m_n)|/c$. Denote by $\mathcal{R}_c \subseteq (\mathbb{Z}/c\mathbb{Z})^{\oplus n}$

the set of solutions of $Q(x_1, x_2, \cdots, x_n) \equiv 0 \pmod c$. Then

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1,m_2,\cdots,m_n)|) \ll \sum_{l=2}^{L} C^l \sum_{\substack{z^{1/2}<c\leq z \\ P^+(x)\leq z^{1/l}}} h(c)\cdot \sum_{(r_1,r_2,\cdots,r_n)\in\mathcal{R}_c} \sum^{(l)}_{\substack{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4 \\ m_j\equiv r_j \pmod c,\ 1\leq j\leq n}} 1. \quad (19)$$

Similarly to (12), we can apply Lemma 2.3 to deduce that

$$\sum^{(l)}_{\substack{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4 \\ m_j\equiv r_j \pmod c,\ 1\leq j\leq n}} 1 \ll \frac{X_1'X_2'\cdots X_n'}{c^2} \prod_{\substack{p<z^{1/(l+1)} \\ p\nmid c}} \left(1-\frac{\rho_Q(p)}{p^2}\right). \quad (20)$$

Inserting (20) into (19) we get

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1,m_2,\cdots,m_n)|) \ll X' \sum_{l=2}^{L} C^l \sum_{\substack{z^{1/2}<c\leq z \\ P^+(c)\leq z^{1/l}}} \frac{h(c)\rho_Q(c)}{c^2} \prod_{\substack{p<z^{1/(l+1)} \\ p\nmid c}} \left(1-\frac{\omega_Q(p)}{p}\right),$$

where $X' = X_1'X_2'\cdots X_n'$. By Lemma 2.4, we obtain

$$\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1,m_2,\cdots,m_n)|) \ll X' \sum_{l=2}^{L} (l+1)C^l \sum_{\substack{z^{1/2}<c\leq z \\ P^+(c)\leq z^{1/l}}} \frac{h(c)\rho_Q(c)}{c\varphi(c)} \prod_{p<z} \left(1-\frac{\omega_Q(p)}{p}\right).$$

Applying [Shi80, Lemma 4] for $H(c) = ch(c)\omega_E(c)/\varphi(c)$ we deduce that

$$\begin{aligned}\sum_{(m_1,m_2,\cdots,m_n)\in\mathcal{B}_4} h(|Q(m_1,m_2,\cdots,m_n)|) &\ll X' \prod_{p<z}\left(1-\frac{\omega_E(p)}{p}\right) \sum_{l=2}^{L}(l+1)C^l \exp\left(\sum_{p\leq z}\frac{h(p)\omega_Q(p)}{p} - \frac{l\log l}{10}\right) \\ &\ll X' \prod_{p<z}\left(1-\frac{\omega_Q(p)}{p}\right) \exp\left(\sum_{p\leq z}\frac{h(p)\omega_Q(p)}{p}\right),\end{aligned}$$

from which (17) is now clear. □

Proof of Proposition 2.2. Since $\mathcal{B}$ is the disjoint union of $\mathcal{B}_1$, $\mathcal{B}_3$, $\mathcal{B}_4$, and $\mathcal{B}_4$, Proposition 2.2 is then obtained from the estimates (8), (13), (16), (17), and the fact that

$$\left|\sum_{p\leq X_1'\cdots X_n'} \frac{\omega_Q(p)\cdot(h(p)-1)}{p} - \sum_{p\leq z}\frac{\omega_Q(p)\cdot(h(p)-1)}{p}\right| \ll \left|\sum_{p\leq X_1'\cdots X_n'}\frac{1}{p} - \sum_{p\leq z}\frac{1}{p}\right| \ll 1.$$

The last step above is a direct consequence of condition (6):

$$\log X_1'X_2'\cdots X_n' \ll \max_{1\leq j\leq n}\log X_j \ll \min_{1\leq j\leq n}\log X_j' \ll \log z.$$

□

3. Proof of Main Theorems

A simple, yet fruitful idea is to bound the Hecke eigenvalues cuspidal representation in terms of its adjoint lifting (assuming it exists). The following elementary inequality led to one of the first estimates of the sum $\sum_{n\leq x}|\tau(n)|$ for the Ramanujan τ-function, without the knowledge of the Sato-Tate conjecture (see [EMS84] for details). As already mentioned in the introduction, similar inequalities were used by Holowinsky [Hol09] in his sieve method for shifted convolution sums.

Lemma 3.1. *Let Π be a cuspidal automorphic representation of $\mathrm{GL}(n,\mathbb{A}_{\mathbb{Q}})$. Suppose that Π admits an adjoint lifting, i.e., $\mathrm{Ad}\,\Pi$ is automorphic. Let p be a prime such that Π_p is unramified and tempered. Then*

$$|a_\Pi(p)| - 1 \leq -\frac{|a_{\mathrm{Ad}\,\Pi}(p)|^2}{2(n+1)^2} + \frac{a_{\mathrm{Ad}\,\Pi}(p)}{2}.$$

Proof. Since $a_{\mathrm{Ad}\,\Pi}(p) = |a_\Pi(p)|^2 - 1$ and $|a_\Pi(p)| \le n$, the inequality follows readily from the fact that

$$|x| \le 1 + \frac{1}{2}(x^2-1) - \frac{1}{2(n+1)^2}(x^2-1)^2$$

is true for all nonnegative real numbers x that are bounded by n in absolute value. □

To ensure the temperedness condition everywhere we shall take π to be a cuspidal representation attached to a holomorphic newform of weight $k \ge 2$. In this case, by a recent breakthrough of Newton and Thorne [NT20] (building on [NT19]) we know the automorphy of the symmetric power lifting $\mathrm{Sym}^m\,\pi$ for all $m \ge 1$. In fact, we are also guaranteed that $\mathrm{Sym}^m\,\pi$ is cuspidal if we insist that the newform be without complex multiplication (cf. [Ram09]).

The following technical result provides a key estimate for the proof of Theorem 1.1.

Proposition 3.2. *Let π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ defined by a non-CM holomorphic newform of weight $k \ge 2$. Suppose $f(x) \in \mathbb{Z}[x]$ is a monic irreducible polynomial. Denote by $\rho_f(p)$ the number of solutions of $f(x) \equiv 0 \pmod p$. If σ_f is automorphic then for every integer $m \ge 1$ we have*

$$\sum_{p\le X} \frac{\rho_f(p)\left(|a_{\mathrm{Sym}^m\,\pi}(p)|-1\right)}{p} \le -\frac{m}{2(m+2)^2}\log\log X + O(1),$$

where the implied constant $O(1)$ depends only on m, π and f.

Proof. Let $\Pi = \mathrm{Sym}^m\,\pi$. As discussed above, we know by [NT20] that Π is a cuspidal automorphic representation of $\mathrm{GL}(m+1)$.

Let N be the arithmetic conductor of π. Since π has trivial central character, for all primes $p \nmid N$ we have

$$a_{\mathrm{Ad}\,\Pi}(p) = \sum_{l=1}^{m} a_{\mathrm{Sym}^{2l}\,\pi}(p). \tag{21}$$

By definition, the quantity $\rho_f(p)$ is nonnegative for all primes p, so Lemma 3.1 gives

$$\sum_{\substack{p\le X\\(p,N)=1}} \frac{\rho_f(p)\left(|a_\Pi(p)|-1\right)}{p} \le -\frac{1}{2(m+2)^2}\sum_{\substack{p\le X\\(p,N)=1}} \frac{\rho_f(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p} + \frac{1}{2}\sum_{\substack{p\le X\\(p,N)=1}} \frac{\rho_f(p)a_{\mathrm{Ad}\,\Pi}(p)}{p}. \tag{22}$$

Note that the contribution from the ramified primes is bounded by a constant. Indeed, since $\rho_f(p) \le \deg f$ and $|a_\Pi(p)| - 1 \le m$ for all primes p, we have

$$\sum_{\substack{p\le X\\p\mid N}} \frac{\rho_f(p)\left(|a_\Pi(p)|-1\right)}{p} \le \sum_{\substack{p\le X\\p\mid N}} \frac{m\deg f}{p} = O(1), \tag{23}$$

where the implied constant depends on m, π and f. Therefore, combining (22) with (23) we obtain

$$\sum_{p\le X} \frac{\rho_f(p)\left(|a_\Pi(p)|-1\right)}{p} \le -\frac{1}{2(m+2)^2}\sum_{p\le X} \frac{\rho_f(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p} + \frac{1}{2}\sum_{p\le X} \frac{\rho_f(p)a_{\mathrm{Ad}\,\Pi}(p)}{p} + O(1). \tag{24}$$

Since σ_f is assumed to be modular, there exists an automorphic representation $\pi'(f)$ of $\mathrm{GL}(\deg f - 1, \mathbb{A}_\mathbb{Q})$ such that $\rho_f(p) = 1 + a_{\pi'(f)}(p)$, as explained in the introduction. For convenience we shall simply refer to $\pi'(f)$ as π'. Therefore, $\rho_f(p)a_{\mathrm{Ad}\,\Pi}(p) = a_{\mathrm{Ad}\,\Pi}(p) + a_{\pi'}(p)a_{\mathrm{Ad}\,\Pi}(p)$.

Let D_f be the discriminant of f. Then π' and π are both unramified at all primes p such that $p \nmid ND_f$. By (21), for $p \nmid ND_f$, one has

$$a_{\pi'}(p)a_{\mathrm{Ad}\,\Pi}(p) = \sum_{l=1}^{m} a_{\pi'}(p)a_{\mathrm{Sym}^{2l}\,\pi}(p) = \sum_{l=1}^{m} a_{\pi'\times\mathrm{Sym}^{2l}\,\pi}(p), \tag{25}$$

where $a_{\pi'\times \mathrm{Sym}^{2l}\pi}(p)$ is the Hecke eigenvalue of the Rankin-Selberg $\pi'_p \boxtimes \mathrm{Sym}^{2l}\pi_p$. When $p \mid ND_f$, we have $|a_{\pi'}(p)a_{\mathrm{Ad}\,\Pi}(p)| \leq m(\deg f-1)(2m+1)$. Note that there are only finitely many such ramified primes, depending only on N and D_f. Hence by (21) and (25) we obtain

$$\sum_{p\leq X}\frac{\rho_f(p)a_{\mathrm{Ad}\,\Pi}(p)}{p}=\sum_{l=1}^{m}\sum_{p\leq X}\frac{a_{\mathrm{Sym}^{2l}\pi}(p)}{p}+\sum_{l=1}^{m}\sum_{p\leq X}\frac{a_{\pi'\times \mathrm{Sym}^{2l}\pi}(p)}{p}+O(1). \tag{26}$$

We claim that the isobaric representation $\pi'(f)$ has no constituents equivalent to $\mathrm{Sym}^{2l}\pi$ for $l\geq 1$. One way to see this is by looking at the corresponding Hodge-Tate weights. Let p be a prime away from ND_f. Fix an isomorphism $\mathbb{C}\simeq\overline{\mathbb{Q}}_p$, where $\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. Under the isomorphism $\mathbb{C}\simeq\overline{\mathbb{Q}}_p$, we can regard σ_f as a $(\deg f-1)$-dimensional p-adic representation. Since σ_f is locally trivial, tensoring with $\mathbb{C}_p$ it becomes $\mathbb{C}_p(0)$, which is a Hodge-Tate representation with Hodge-Tate weights all equal to 0. On the other hand, if $k\geq 2$ is the weight of the newform defining π, we have that the Hodge-Tate weights of $\mathrm{Sym}^{2l}\pi$ are $(0,k-1,2(k-1),\cdots,2l(k-1))$ (e.g., from Proposition 4.3.1 of [CHT08]). This, of course, is not equal to the zero vector since $k\geq 2$.

It follows from the previous paragraph that the Rankin-Selberg L-function $L(s,\pi'\times\mathrm{Sym}^{2l}\pi)$ is regular at $s=1$ for $1\leq l\leq m$. Therefore, both sums on the right-hand side of (26) are bounded, and so

$$\sum_{p\leq X}\frac{\rho_f(p)a_{\mathrm{Ad}\,\pi}(p)}{p}=O(1), \tag{27}$$

where the implied constant depends on m, π and f. Substituting the estimate (27) into (24) we then obtain

$$\sum_{p\leq X}\frac{\rho_f(p)\left(|a_\Pi(p)|-1\right)}{p}\leq -\frac{1}{2(m+2)^2}\sum_{p\leq X}\frac{|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p}-\frac{1}{2(m+2)^2}\sum_{p\leq X}\frac{a_{\pi'}(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p}+O(1). \tag{28}$$

In view of (21) and the fact that Π is self-dual (and tempered) we can write

$$\sum_{\substack{p\leq X\\ p\nmid N}}\frac{|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\substack{p\leq X\\ p\nmid N}}\frac{a_{\mathrm{Sym}^{2i}\pi}(p)a_{\mathrm{Sym}^{2j}\pi}(p)}{p}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\substack{p\leq X\\ p\nmid N}}\frac{a_{\mathrm{Sym}^{2i}\pi\times\mathrm{Sym}^{2j}\pi}(p)}{p}. \tag{29}$$

Moreover

$$\sum_{\substack{p\leq X\\ p\mid N}}\frac{|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p}=O(1)\quad\text{and}\quad\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\substack{p\leq X\\ p\mid N}}\frac{a_{\mathrm{Sym}^{2i}\pi\times\mathrm{Sym}^{2j}\pi}(p)}{p}=O(1). \tag{30}$$

Hence, (29) and (30) combined with Selberg's orthogonality relations (see, for example [LY05]) give

$$\sum_{p\leq X}\frac{|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{p\leq X}\frac{a_{\mathrm{Sym}^{2i}\pi\times\mathrm{Sym}^{2j}\pi}(p)}{p}+O(1)=m\log\log x+O(1). \tag{31}$$

Furthermore, using the decomposition (21) once again, we get

$$\sum_{\substack{p\leq X\\ p\nmid ND_f}}\frac{a_{\pi'}(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\substack{p\leq X\\ p\nmid ND_f}}\frac{a_{\pi'}(p)a_{\mathrm{Sym}^{i}\pi}(p)a_{\mathrm{Sym}^{j}\pi}(p)}{p}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\substack{p\leq X\\ p\nmid ND_f}}\frac{a_{\pi'}(p)a_{\Pi_{i,j}}(p)}{p}, \tag{32}$$

where $\Pi_{i,j}=\mathrm{Sym}^{i}\pi\boxtimes\mathrm{Sym}^{j}\pi$. Note that for $i\geq j$ and $p\nmid N$, we have

$$a_{\Pi_{i,j}}(p)=a_{\mathrm{Sym}^{i}\pi\boxtimes\mathrm{Sym}^{j}\pi}(p)=a_{\mathrm{Sym}^{i+j}\pi}(p)+a_{\mathrm{Sym}^{i+j-2}\pi}(p)+\cdots+a_{\mathrm{Sym}^{i-j}\pi}(p)=\sum_{l=0}^{2j}a_{\mathrm{Sym}^{i-j+2l}\pi}(p), \tag{33}$$

where $\mathrm{Sym}^0 \pi := \mathbf{1}$ the trivial representation and $\mathrm{Sym}^1 \pi = \pi$. Substituting (33) into (32) we then obtain

$$\sum_{\substack{p\le X\\ p\nmid ND_f}} \frac{a_{\pi'}(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p} = 2\sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{l=0}^{2j}\sum_{\substack{p\le X\\ p\nmid ND_f}} \frac{a_{\pi'}(p)a_{\mathrm{Sym}^{i-j+2l}\pi}(p)}{p}. \tag{34}$$

On the other hand, we have

$$\sum_{\substack{p\le X\\ p\mid ND_f}} \frac{a_{\pi'}(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p} = O(1) \quad \text{and} \quad \sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{l=0}^{2j}\sum_{\substack{p\le X\\ p\mid ND_f}} \frac{a_{\pi'}(p)a_{\mathrm{Sym}^{i-j+2l}\pi}(p)}{p} = O(1). \tag{35}$$

It then follows from (34) and (35) that

$$\sum_{p\le X} \frac{a_{\pi'}(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p} = 2\sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{l=0}^{2j}\sum_{p\le X} \frac{a_{\pi'}(p)a_{\mathrm{Sym}^{i-j+2l}\pi}(p)}{p} + O(1). \tag{36}$$

Let S be the set of triples (i,j,l) with $0 \le l \le 2j$ and $1 \le j \le i \le m$ such that the Rankin-Selberg L-function $L(s, \pi' \times \mathrm{Sym}^{i-j+2l}\pi)$ has a pole at $s = 1$. Let $(i,j,l) \in S$. Denote by $m_{i,j,l}$ the multiplicity of the pole of $L(s, \pi' \times \mathrm{Sym}^{i-j+2l}\pi)$ at $s = 1$. Then, using Selberg's orthogonality once more in (36), we get

$$\sum_{p\le X} \frac{a_{\pi'}(p)|a_{\mathrm{Ad}\,\Pi}(p)|^2}{p} = 2\sum_{(i,j,l)\in S} m_{i,j,l}\log\log X + O(1), \tag{37}$$

where the implied constant depends only on π and f.

Substituting (31) and (37) into the inequality (28) we then obtain

$$\sum_{p\le X} \frac{\rho_f(p)\left(|a_{\mathrm{Sym}^n\pi}(p)| - 1\right)}{p} \le -\frac{m\log\log X}{2(m+2)^2}\cdot\Big[1 + \frac{2}{m}\sum_{(i,j,l)\in S} m_{i,j,l}\Big] + O(1) \le -\frac{m}{2(m+2)^2}\log\log X + O(1),$$

which establishes Proposition 3.2. ∎

We are now in position to prove the first main result of this paper.

Proof of Theorem 1.1. Our starting point is Lemma 2.1 applied for the function $F(n) = |a_{\mathrm{Sym}^m\pi}(n)|$ and the polynomial $Q(x) = f(x) \in \mathbb{Z}[x]$. It yields

$$\sum_{\substack{X-X_0<n\le X\\ n\equiv a \pmod q}} |a_{\mathrm{Sym}^m\pi}(|f(n)|)| \ll \frac{X_0}{q}\prod_{\substack{p\le X\\ p\nmid q}}\left(1 - \frac{\rho_f(p)}{p}\right)\sum_{\substack{n\le X\\ (n,q)=1}} \frac{\rho_f(n)|a_{\mathrm{Sym}^m}(n)|}{n}. \tag{38}$$

We turn to the sum on the right-hand side. The multiplicative property of the arithmetic function $|a_{\mathrm{Sym}^m}(n)|$ implies

$$\sum_{\substack{n\le X\\ (n,q)=1}} \frac{\rho_f(n)|a_{\mathrm{Sym}^m}(n)|}{n} \le \prod_{p\le X}\left(1 + \sum_{1\le k\le \frac{\log X}{\log p}} \frac{\rho_f(p^k)|a_{\mathrm{Sym}^m\pi}(p^k)|}{p^k}\right). \tag{39}$$

At the same time, the elementary inequality $\log(1+x) \le x$, true for all $x \ge 0$, gives

$$\sum_{p\le X}\log\left(1 + \sum_{1\le k\le \frac{\log X}{\log p}} \frac{\rho_f(p^k)|a_{\mathrm{Sym}^m\pi}(p^k)|}{p^k}\right) \le \sum_{p\le X}\sum_{1\le k\le \frac{\log X}{\log p}} \frac{\rho_f(p^k)|a_{\mathrm{Sym}^m\pi}(p^k)|}{p^k}. \tag{40}$$

Bounding $\rho_f(p^k)|a_{\mathrm{Sym}^m\pi}(p^k)|$ trivially we see that

$$\sum_{p\le X}\sum_{2\le k\le \frac{\log X}{\log p}} \frac{\rho_f(p^k)|a_{\mathrm{Sym}^m\pi}(p^k)|}{p^k} \le (m+1)\deg f\cdot\sum_{p\le X}\sum_{2\le k\le \frac{\log X}{\log p}} \frac{1}{p^k} \ll \sum_{p\le X}\frac{1}{p^2} \ll 1, \tag{41}$$

where the implied constant depends on m, π and f. Substituting (40) and (41) into (39) we get

$$\sum_{\substack{n\le X\\(n,q)=1}} \frac{\rho_f(n)|a_{\mathrm{Sym}^m}(n)|}{n} \ll \exp\left(\sum_{p\le X}\sum_{k=1}^{\frac{\log X}{\log p}} \frac{\rho_f(p^k)|a_{\mathrm{Sym}^m\pi}(p^k)|}{p^k}\right) \ll \exp\left(\sum_{p\le X} \frac{\rho_f(p)|a_{\mathrm{Sym}^m\pi}(p)|}{p}\right). \quad (42)$$

Using the key estimate proved earlier (Proposition 3.2) together with a well-known result of Erdös [Erd52]:

$$\sum_{p\le X} \frac{\rho_f(p)}{p} = \log\log X + c(f) + o(1), \quad (43)$$

(for some constant $c(f)$ depending of f) we obtain

$$\sum_{p\le X} \frac{\rho_f(p)\cdot|a_{\mathrm{Sym}^m\pi}(p)|}{p} \le \left(1-\frac{m}{2(m+2)^2}\right)\cdot\log\log X + O(1). \quad (44)$$

Hence, combining (42) and (44) we get

$$\sum_{\substack{n\le X\\(n,q)=1}} \frac{\rho_f(n)|a_{\mathrm{Sym}^m}(n)|}{n} \ll (\log X)^{1-\frac{m}{2(m+2)^2}}. \quad (45)$$

Finally, we note that equation (43) also implies that

$$\prod_{\substack{p\le X\\p\nmid q}} \left(1-\frac{\rho_f(p)}{p}\right) \ll \frac{1}{\log X}. \quad (46)$$

Then Theorem 1.1 follows by substituting (45) and (46) into (38).

□

Next, for polynomials f of small degree we explain why the Artin representation σ_f is automorphic.

Proof of Corollary 1.2. Denote by d the degree of f, and by D its discriminant. Recall that the permutation representation acting on the roots of f, viewed as a Galois representation, decomposes as the sum of the trivial representation and another Artin representation $\sigma_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{d-1}(\mathbb{C})$ with the property that $\rho_f(p) = 1 + \sigma_f(\mathrm{Frob}_p)$, where Frob_p is the usual Frobenius automorphism.

For $d \le 4$ we show that σ_f is automorphic, i.e., there exists an automorphic representation $\pi'(f)$ of $\mathrm{GL}(d-1, \mathbb{A}_{\mathbb{Q}})$ with the same L-function as σ_f. We treat each case separately.

- If $d = 2$ then, by quadratic reciprocity, $\rho_f(p) = 1 + \left(\frac{D}{p}\right)$ for all odd primes p (away from D). Since σ_f is simply a character, its automorphy is clear.
- If $d = 3$ then σ_f is induced from a character $\chi : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D})\right) \to \mathbb{C}^\times$. In this case, the existence of $\pi'(f)$ is deduced by automorphic induction from GL(1) to GL(2).
- If $d = 4$ then it can be seen that σ_f is the symmetric square of a 2-dimensional Artin representation $\tau : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ with solvable image (see, e.g., Example 2 and 3 in [Kim07]). By the work of Langlands and Tunnell, the automorphy of τ is known in this case. This, in turn, combined with the adjoint square lift of Gelbart and Jacquet gives the automorphy of σ_f.

We also mention that if $d = 5$ then, by [Kim07] (Example 4), we have that σ_f is the tensor product of two Artin representations of degree 2. Provided that these are automorphic (e.g., if they are odd), the existence of $\pi'(f)$ follows from the functorial product $\mathrm{GL}(2) \times \mathrm{GL}(2)$. Since even icosahedral representations are not yet known to be modular, we cannot deduce the automorphy of σ_f in full generality for $d = 5$.

□

Finally, we give a proof of our second main result.

Proof of Theorem 1.3. In contrast with the proof of Theorem 1.1, neither [Nai92] nor [NT98] can be used here directly, for the sum is over the values of a polynomial in two variables. This is where the sieve methods developed in Section 2 come into play.

Let π_E be the cuspidal automorphic representation associated by modularity to the elliptic curve defined by the polynomial $E(x,y)=y^2-x^3-ax-b$ with $4a^3+27b^2\neq 0$. Then $a_{\pi_E}(p)=p+1-\rho_E(p)$, and the Hasse-Weil bound gives $|\rho_E(p^\ell)-p^\ell-1|\leq 2p^{\ell/2}$ for every integer $\ell\geq 1$

Also, since $Y^{\gamma_1}\leq X\leq Y^{\gamma_2}$ and $X^\alpha\leq X_0\leq X$, $Y^\beta\leq Y_0\leq Y$, it is clear that

$$\max\{\log X,\log Y\}\ll\min\{\log X_0,\log Y_0\}$$

with implied constants depending on α, β, γ_1 and γ_2.

It follows that we can apply Proposition 2.2, which allows us to reduce the problem to showing that

$$\sum_{p\leq X_0Y_0}\frac{\omega_E(p)\cdot(a_\pi(p)-1)}{p}\leq-\frac{1}{18}\log\log X_0Y_0+O(1), \tag{47}$$

where $\omega_E(p)=\rho_E(p)/p$. Appealing to Lemma 3.1 we have

$$\sum_{p\leq X_0Y_0}\frac{\omega_E(p)\cdot(a_\pi(p)-1)}{p}\leq-\frac{1}{18}\sum_{p\leq X_0Y_0}\frac{\omega_E(p)|a_{\mathrm{Ad}\,\pi}(p)|^2}{p}+\frac{1}{2}\sum_{p\leq X_0Y_0}\frac{\omega_E(p)a_{\mathrm{Ad}\,\pi}(p)}{p}. \tag{48}$$

Since π is not dihedral, we note that $\mathrm{Ad}(\pi)$ is a cuspidal automorphic representation of $\mathrm{GL}(3)$, so by Selberg's orthogonality

$$\sum_{p\leq X_0Y_0}\frac{|a_{\mathrm{Ad}\,\pi}(p)|^2}{p}=\log\log X_0Y_0+O(1). \tag{49}$$

Furthermore, the fact that π satisfies the Ramanujan conjecture implies that $|a_{\mathrm{Ad}\,\pi}(p)|\ll_\epsilon p^\epsilon$ for some positive constant (say) $\epsilon<1/100$. Using this upper bound together with the inequality $|a_{\pi_E}(p)|\leq 2p^{1/2}$ we get

$$\sum_{p\leq X_0Y_0}\frac{(1-|a_{\pi_E}(p)|)\cdot|a_{\mathrm{Ad}\,\pi}(p)|^2}{p^2}\ll\sum_p\frac{p^{1/2+2\epsilon}}{p^2}=O(1). \tag{50}$$

On the other hand $L(s,\pi_E\times\mathrm{Ad}\,\pi)$ is entire, since π_E is not isomorphic to $\mathrm{Ad}\,\pi$. As a consequence

$$\sum_{p\leq X_0Y_0}\frac{\omega_E(p)a_{\mathrm{Ad}\,\pi}(p)}{p}=\sum_{p\leq X_0Y_0}\frac{a_{\mathrm{Ad}\,\pi}(p)}{p}+\sum_{p\leq X_0Y_0}\frac{(1-a_{\pi_E}(p))a_{\mathrm{Ad}\,\pi}(p)}{p^2}=O(1). \tag{51}$$

Combining the estimates (48) through (51) establishes (47), and therefore concludes the proof of Theorem 1.3. □

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