ARITHMETIC DISTRIBUTION OF TEMPERED COMPONENTS OF CUSPIDAL REPRESENTATIONS OF $GL(3)$

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ABSTRACT. Let $\pi = \otimes_v' \pi_v$ be an arbitrary unitary cuspidal representation of $GL(3)$ over a number field. We consider an arithmetic aspect of Sato-Tate problem towards π. Specifically, we show that there are infinitely many unramified places v in certain ray classes such that π_v's are tempered with traces of corresponding Hecke operators lying inside the unit open disk. Some quantitative results of Linnik type are provided as well.

CONTENTS

1. INTRODUCTION

1.1. **Infinity of Tempered Components in Ray Classes.** Let π be a unitary cuspidal representation of $GL(n, \mathbb{A}_F)$, where $n \in \mathbb{N}_{\geq 2}$ and F is a number field. Let $\mathcal{R}_n(\pi; F)$ be the set of unramified finite places of F such that the local components of π are tempered at these places. When $\mathcal{R}_n(\pi; F)$ is nonempty, we will call its elements *Ramanujan places* associated to π. Let $S^{ur}(\pi)$ be the set of unramified finite places of F. The generalized Ramanujan conjecture asserts that $\mathcal{R}_n(\pi; F) = S^{ur}(\pi)$, for any $n \geq 2$, any base field F and any such π.

As a first step to attack this conjecture, it is natural to ask whether $\mathcal{R}_n(\pi; F)$ is nonempty. Also, due to the work of Kim and Shahidi, we see the importance of knowing a positive answer to such a question. Moreover, according to [3], if $\mathcal{R}_n(\pi; F)$ has Dirichlet density 100%, then one gets a standard zero-free region for $L(s, \pi \times \widetilde{\pi})$.

For $n = 2$, it is known that for general F and π, the set $\mathcal{R}_2(\pi; F)$ has a lower Dirichlet density at least $34/35$ due to Ramakrishnan (ref. [16]) and Kim and Shahidi (ref. [10]). For $n = 3$ and $F = \mathbb{Q}$, it is showed in [17] that for any unitary cuspidal representations π, the set $\mathcal{R}_3(\pi; F)$ is infinite. For $n = 4$ and general F, Walji (ref. [21]) showed that for unitary, self-dual cuspidal representations π, the set $\mathcal{R}_4(\pi; F)$ has a lower Dirichlet density at least $1/4$. For $n \geq 4$ and F a totally real field or a CM field, and π a cuspidal representation of $GL(n, \mathbb{A}_F)$ which is algebraic, regular, self-dual, with a discrete series component at a finite place, Clozel (ref. [4]) proved that every finite place where π is unramified lies in $\mathcal{R}_n(\pi;F)$. However, no results are known for general π.

It is an interesting observation that if $|a_\pi(\mathfrak{p})|<1$, then $\pi_\mathfrak{p}$ is tempered (e.g. ref. Section 3), where $a_\pi(\mathfrak{p})$ is the Dirichlet coefficient at the place $\mathfrak{p}$ of the standard L-function associated to π. Let $\mathcal{R}_3^-(\pi;F):=\{\mathfrak{p}\in S^{ur}(\pi):\ |a_\pi(\mathfrak{p})|<1\}$. Then $\mathcal{R}_3^-(\pi;F)\subseteq\mathcal{R}_3(\pi;F)$. Although Sato-Tate conjecture predicts that $\mathcal{R}_3^-(\pi;F)$ has positive density for unitary cuspidal representation π not a functorial lifting from other groups, it was unclear if $\mathcal{R}_3^-(\pi;F)$ is empty or not for general π and F. On the other hand, Ramakrishnan [16], using Rankin-Selberg method, obtained a nontrivial lower bound for Dirichlet density of places $v\in S^{ur}(\pi)$ such that $|a_v(\pi)|<1+\epsilon$, for any $\epsilon>0$. However, when $\epsilon=0$, the lower bound becomes zero. It seems likely that the radius 1 here is a threshold for Rankin-Selberg method.

In this paper, we refine Ramakrishnan's approach in [17] to show infinity of $\mathcal{R}_3^-(\pi;F)$ over certain ray classes of a number field, with our previous work on holomorphy of twisted adjoint L-functions for $GL(3)$ (ref. [24]) being one of the new inputs.

Let $\mathfrak{M}$ be a modulus on F. Let $Cl_\mathfrak{M}(F)$ be the associated ray class group with class number $h(\mathfrak{M})$. Let $\mathcal{C}$ be the trivial ray class. Then we have

Theorem A. *Let notation be as before. Assume that $3\nmid h(\mathfrak{M})$. Then one has*

$$\sum_{\substack{\mathfrak{p}^2\in\mathcal{C}\\ |a_\pi(\mathfrak{p})|<1}}\frac{\log N_{F_\mathfrak{p}/\mathbb{Q}_p}(\mathfrak{p})}{N_{F_\mathfrak{p}/\mathbb{Q}_p}(\mathfrak{p})^{1/2}}=+\infty. \tag{1}$$

In particular, there are infinitely many primes $\mathfrak{p}$ such that $\mathfrak{p}^2\in\mathcal{C}$ such that $\pi_\mathfrak{p}$ is tempered and $|a_\pi(\mathfrak{p})|<1$. Moreover, if the arithmetic conductor of π is cube-free, then (1) holds for all $h(\mathfrak{M})$.

Remark. Let σ be a cuspidal representation on $GL(2)$. Let $S_1=\{\mathfrak{p}:\ |a_\sigma(\mathfrak{p})|<\sqrt{2}\}$, and $S_2=\{\mathfrak{p}:\ a_\sigma(\mathfrak{p})\neq 0\}$. Then [16] leads to S_1 has at least 50% Dirichlet density. On the other hand, one can take σ to be some CM modular form to make S_2 has at most 50% Dirichlet density. Hence, in prior, one cannot tell if $S_1\cap S_2$ is non-empty or not. However, we can take $\pi=\mathrm{Ad}\,\sigma$, the adjoint lift of σ (ref. [5]), into Theorem A to conclude that $S_1\cap S_2$ is infinite even if we require certain congruence conditions on the primes in S_1 and S_2.

We restate Theorem A in classical language below, and consider a Linnik type problem related to it in Section 1.2. Let f be a Maass form on $SL(3)/\mathbb{Q}$, with normalized p-th Fourier coefficients $A_f(p,1)=\alpha_{p,1}+\alpha_{p,2}+\alpha_{p,3}$, where $\alpha_{p,i}$'s are Hecke eigenvalues, $1\leq i\leq 3$. Then we have

Corollary 2. *Let notation be as before. Let q be an integer. Then there exists infinitely many primes $p\equiv\pm1 \mod q$, such that $|A_f(p,1)|<1$.*

Remark. Note that $|A_f(p,1)|<1$ implies $|\alpha_{p,1}|=|\alpha_{p,2}|=|\alpha_{p,3}|=1$, i.e., f satisfies Ramanujan conjecture at p.

1.2. **Least Ramanujan Prime in an Arithmetic Progression.** Dirichlet's theorem asserts infinitely many primes in any fixed arithmetic progression $a \mod q$, with $(a,q)=1$. Then a natural question considered by Linnik is to see how large can the least prime in this arithmetic progression be. Note that, by Siegel–Walfisz theorem, the size e^{q^ϵ} is admissible, for any $\epsilon>0$. Moreover, assuming nonexistence of exceptional character modulo q, then the Prime Number Theorem implies that the bound $q^{c\log q}$ is admissible for some positive constant c.

Linnik (ref. [13] and [14]) proved that the least prime can be bounded by q^L for some constant $L > 0$, which is called Linnik's constant. It is known that $L = 2 + \epsilon$ under the Grand Riemann Hypothesis. Unconditionally, the best record is $L = 5$, by Xylouris [23], which is based on the work of Heath-Brown [7]. Attenuating the admissible Linnik constant is one of the core problems in analytic number theory, as it is closely related to three important principles about the zeros of Dirichlet L-functions: the zero-free region, the log-free zero-density estimate, and the exceptional zero repulsion. Since we show, in Theorem 40, that there are infinitely many Ramanujan primes in $a \mod q$, $(a, q) = 1$, it is natural to consider a similar problem of Linnik type; namely, for a given cuapidal representation π of $GL(3, \mathbb{A})$, one can ask if the least prime p such that $|a_\pi(p)| < 1$ is bounded by a power of qC_π, where C_π is the analytic conductor of π.

This problem seems likely to be untouched so far. The answer to such a problem should be a consequence of horizontal "Arithmetic Sato-Tate Conjecture". Since the statement of Sato-Tate is purely analytic, there is no reason in prior that the Hecke eigenvalues distribute nicely in arithmetic progressions. In this section, we study several variants of the above proposed problem, and give explicit upper bounds in terms of powers of the modulus and analytic conductor. We start with a prototype:

Theorem B. *Let notation be as before. Let q_2 be a positive integer. Let $0 < \epsilon < 1/4$. Then there exist an integer n such that $(n, q_2) = 1$, and $n \ll_\epsilon C_\pi^{4+\epsilon} q_2^{30/7-\epsilon}$, and $|a_\pi(n)| < 1$. Moreover, the implied constant is effective.*

In fact, the proof of Theorem B can be further refined to prove a more restrictive version (see Section 5.2):

Theorem C. *Let notation be as before. Let q_1, q_2 be positive integers with $3 \nmid q_1$ and q_2 square-free. Then there exist a computable constant $L > 0$ and an integer n such that $n \equiv 1 \mod q_1$, $(n, q_2) = 1$, and $n \ll_\epsilon (C_\pi q_1^4)^{2+\epsilon} q_2^{8+\epsilon} + q_1^L$, and $|a_\pi(n)| < 1$. Moreover, the implied constant is effective.*

Making use of multiplicativity of $a_\pi(n)$, we can get an upper bound on prime power p^k such that $|a_\pi(p^k)| < 1$. In particular, this would imply an upper bound for the first Ramanujan prime:

Corollary 5. *Let notation be as above. Let q be a positive integer. Then there exist a prime p and an integer k, such that $p \nmid q$, and $p \ll_\epsilon (C_\pi^2 q^3)^{\frac{1+\epsilon}{k}}$, and $|a_\pi(p^k)| < 1$. In particular, π_p is tempered.*

Remark. Theorem B and Theorem C can be generalized to $GL(4)$, although the upper bound is worse. See the remark after Proposition 15.

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2. Nontrivial Zeros of Adjoint L-functions

Let $L(s, \pi, \mathrm{Ad} \otimes \chi) = L(s, \pi \otimes \chi \times \widetilde{\pi})/L(s, \chi)$ be the twisted adjoint L-function of π by χ, where χ is a Hecke character on $\mathbb{A}_F^\times$. It is known that $L(s, \pi, \mathrm{Ad} \otimes \chi)$ can be represented by a Dirichlet series:

$$L(s, \pi, \mathrm{Ad} \otimes \chi) = \sum_{\mathfrak{b} \subseteq \mathcal{O}_F} \frac{a_{\mathfrak{b}}(\pi, \mathrm{Ad} \otimes \chi)}{N_{F/Q}(\mathfrak{b})^s}, \quad \mathrm{Re}(s) > 2,$$

where $\mathfrak{b}$ runs over nonzero integral ideals of $\mathcal{O}_F$. When χ is trivial, we shall denote by $L(s,\pi,\mathrm{Ad})=L(s,\pi,\mathrm{Ad}\otimes\chi)$ and $a_{\mathfrak{b}}(\pi,\mathrm{Ad})=a_{\mathfrak{b}}(\pi,\mathrm{Ad}\otimes\chi)$ for simplicity.

In this section, we will show that for any cuspidal representation π of $GL(3,\mathbb{A}_F)$, and any Hecke character χ on $\mathbb{A}_F^\times$, where F is a number field, the L-function $L(s,\pi,\mathrm{Ad}\otimes\chi)$ has infinitely many zeros in the strip $1/2\leq\mathrm{Re}(s)<1$. To achieve it, we need to investigate analytic properties of $L(s,\pi,\mathrm{Ad}\otimes\chi)$. The following theorem is one of the main results in [24].

Theorem 6. *Let F be a global field. Let π be a cuspidal representation of $GL(3,\mathbb{A}_F)$ and let χ be a character on $F^\times\backslash\mathbb{A}_F^\times$. Then $L(s,\pi,\mathrm{Ad}\otimes\chi)$ is entire, unless $\chi\neq 1$ and $\pi\otimes\chi\simeq\pi$, in which case $L(s,\pi,\mathrm{Ad}\otimes\tau)$ is meromorphic with only possible simple poles at $s=0$ and $s=1$. In particular, the adjoint L-function $L(s,\pi,\mathrm{Ad})=L(s,\pi\times\widetilde{\pi})/\zeta_F(s)$ is entire.*

Let notation be as before. Let $L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)$ be the infinite part of the twisted adjoint L-function $L(s,\pi,\mathrm{Ad}\otimes\chi)$. Then by local Langlands correspondence, one can write $L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)$ as $E_\infty(s)\prod_{v|\infty}\prod_{\mu_v}\Gamma_{F_v}(s+\mu_v)$, where $E_\infty(s)$ is some exponential function and μ_v's are some complex numbers. Moreover, we have the functional equation

$$\Lambda(s,\pi,\mathrm{Ad}\otimes\chi)=\epsilon(\mathrm{Ad}\otimes\chi)Q_{\pi,\mathrm{Ad}\otimes\chi}^{1/2-s}\Lambda(1-s,\pi,\mathrm{Ad}\otimes\overline{\chi}),\tag{2}$$

where $\Lambda(s,\pi,\mathrm{Ad}\otimes\chi)=L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)L_(s,\pi,\mathrm{Ad}\otimes\chi)$ is the complete L-function; $\epsilon(\mathrm{Ad}\otimes\chi)$ is a complex number with modulus one and $Q_{\pi,\mathrm{Ad}\otimes\chi}$ is an integer.

To control the magnitude of $L(s,\pi,\mathrm{Ad}\otimes\chi)$, we shall use the following definition.

Definition 7 (Analytic Conductor)**.** Let $C(\pi_v,\mathrm{Ad}\otimes\chi_v)=\prod_{\mu_v}(2+|\mu_v|)$, and set $C(\pi,\mathrm{Ad}\otimes\chi)=Q(\pi,\mathrm{Ad}\otimes\chi)\prod_{v|\infty}C(\pi_v,\mathrm{Ad}\otimes\chi_v)$, where $Q(\pi,\mathrm{Ad}\otimes\chi)$ is the arithmetic conductor showing up in the functional equation. We then call $C(\pi,\mathrm{Ad}\otimes\chi)$ the analytic conductor function associated to $L(s,\pi,\mathrm{Ad}\otimes\chi)$.

With the above definition, we can bound $L(s,\pi,\mathrm{Ad}\otimes\chi)$ in the critical strip:

Lemma 8 (Preconvex bound)**.** *Let notation be as before. Then for $s\in\mathbb{C}$ such that $0<\mathrm{Re}(s)<1$, we have*

$$L(s,\pi,\mathrm{Ad}\otimes\chi)\ll_{F,\epsilon}\left(1+|s(s-1)|^{-1}\right)C(\pi,\mathrm{Ad}\otimes\chi)^{9/10-\mathrm{Re}(s/2)+\epsilon},\tag{3}$$

where the implies constant is absolute, depending only on ϵ and the base field F.

Proof. By definition, χ extends to a character on $GL(3,\mathbb{A}_F)$ via composing with the determinant map, i.e., by setting $\chi(x)=\chi(|\det x|_{\mathbb{A}_F})$, for any $x\in GL(3,\mathbb{A}_F)$. Thus χ is automorphic and invariant on $N(\mathbb{A}_F)$. Hence $\pi\otimes\chi$ is also cuspidal. We may write the cuspidal representations as $\pi\otimes\chi=\otimes_v(\pi_v\otimes\chi_v)$ and $\widetilde{\pi}=\otimes_v'\widetilde{\pi}_v$. For prime ideals $\mathfrak{p}$ at which neither $\pi_{\mathfrak{p}}$ or $\widetilde{\pi}_{\mathfrak{p}}$ is ramified, let $\{St_{\pi\otimes\chi,i}(\mathfrak{p})\}_{i=1}^3$ and $\{St_{\widetilde{\pi},j}(\mathfrak{p})\}_{j=1}^3$ be the respective Satake parameters of $\pi\otimes\chi$ and $\widetilde{\pi}$. The Rankin-Selberg L-function at such a $\mathfrak{p}$ (there are all but finitely many such primes) is defined to be

$$L_{\mathfrak{p}}(s,\pi_{\mathfrak{p}}\otimes\chi_{\mathfrak{p}}\times\widetilde{\pi}_{\mathfrak{p}})=\prod_{i=1}^3\prod_{j=1}^3\left(1-St_{\pi\otimes\chi,i}(\mathfrak{p})St_{\widetilde{\pi},j}(\mathfrak{p})N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\right)^{-1}.$$

Since χ is unitary, by [11] or [2], we have $\big|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})}|St_{\pi\otimes\tau,i}(\mathfrak{p})|\leq 1/2-1/(3^2+1)=2/5$, and $\big|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})}|St_{\widetilde{\pi},j}(\mathfrak{p})|\leq 2/5$. For the remaining finite places $\mathfrak{p}$, The Rankin-Selberg L-function at such a $\mathfrak{p}$ can be written as

$$L_{\mathfrak{p}}(s,\pi_{\mathfrak{p}}\otimes\chi_{\mathfrak{p}}\times\widetilde{\pi}_{\mathfrak{p}})=\prod_{i=1}^3\prod_{j=1}^3\left(1-St_{\pi\otimes\chi\times\widetilde{\pi},i,j}(\mathfrak{p})N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\right)^{-1},$$

with $\left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})}|St_{\pi\otimes\chi\times\widetilde{\pi},i,j}(\mathfrak{p})|\right| \leq \left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})}|St_{\pi\otimes\chi,i}(\mathfrak{p})|+\right|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})}|St_{\widetilde{\pi},j}(\mathfrak{p})|$, which is bounded by $4/5$. Then an easy estimate implies that for any s such that $\beta = \mathrm{Re}(s) > 1 + 4/5 = 9/5$, we have

$$\begin{aligned}\left|L(s,\pi\otimes\chi\times\widetilde{\pi})\right| = \prod_{\mathfrak{p}}\left|L_{\mathfrak{p}}(s,\pi_{\mathfrak{p}}\otimes\chi_{\mathfrak{p}}\times\widetilde{\pi}_{\mathfrak{p}})\right| &\leq \prod_{\mathfrak{p}}\prod_{i=1}^{3}\prod_{j=1}^{3}\left|1-N_{F/\mathbb{Q}}(\mathfrak{p})^{4/5-\beta}\right|^{-1}\\ &= \prod_{\mathfrak{p}}\left|1-N_{F/\mathbb{Q}}(\mathfrak{p})^{4/5-\beta}\right|^{-9} = \zeta_F(\beta-4/5)^9,\end{aligned}$$

where $\zeta_F(s)$ is the Dedekind zeta function associated to $F/\mathbb{Q}$. In particular,

$$\left|L(s,\pi\otimes\chi\times\widetilde{\pi})\right| \leq \zeta_F(\mathrm{Re}(s)-4/5)^9 \leq \zeta_F(\beta_0-4/5)^9,\ \mathrm{Re}(s)\geq\beta_0>9/5.$$

On the other hand, we have the trivial bound

$$|L(s,\chi)|^{-1} \leq \prod_{\mathfrak{p}}(1+N_{F/\mathbb{Q}}(\mathfrak{p})^{-\mathrm{Re}(s)}) \leq \prod_{\mathfrak{p}}(1+N_{F/\mathbb{Q}}(\mathfrak{p})^{-\beta_0}) = \frac{\zeta_F(\beta_0)}{\zeta_F(2\beta_0)}.$$

Therefore, combining the above estimates we conclude that

$$\left|L(s,\pi,\mathrm{Ad}\otimes\chi)\right| \leq \zeta_F(\beta_0-4/5)^9\zeta_F(\beta_0)\zeta_F(2\beta_0)^{-1},\ \mathrm{Re}(s)\geq\beta_0>9/5. \tag{4}$$

Also, at each infinite place $v \mid \infty$, there exists a set of 9 complex parameters $\{\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i,j}:\ 1\leq i\leq 3,\ 1\leq j\leq 3\}$ such that each local L-factor at v is

$$L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v) = \frac{Q_v(s)}{\Gamma_{F_v(s+\mu_{\chi;v})}}\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma_{F_v}\left(s+\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i,j}\right),$$

where $Q_v(s)$ is of the form $c_1\pi^{c_2 s}$, with c_1, $c_2\in\mathbb{C}$; and according to local Langlands correspondence, there exists some i_0, j_0, such that $\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i_0,j_0} = \mu_{\chi;v}$.

Likewise, we have $\left|\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i,j}\right| \leq 4/5$, $1\leq i,j\leq 3$, according to loc. cit. Moreover, since $\widetilde{\pi_v\otimes\chi_v} = \widetilde{\pi}_v\otimes\widetilde{\chi}_v$, the finite set $\{\overline{\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i,j}}:\ 1\leq i,j\leq 3\}$ is equal to $\{\mu_{\widetilde{\pi}\otimes\widetilde{\chi}\times\pi;v,i,j}:\ 1\leq i,j\leq 3\}$ for any $v\mid\infty$. Note that by Stirling's formula one has, for $s\in\mathbb{C}$ such that $-2<\mathrm{Re}(s)<1/5$, that

$$\Gamma\left(1-s+\bar{\mu}/2\right)\cdot\Gamma\left(s+\mu/2\right)^{-1} \ll \left(1+|s+\mu|\right)^{1/2-\mathrm{Re}(s)},$$

for any $\mu\in\mathbb{C}$ such that $\mathrm{Re}(\mu)>-1+4/5=-1/5$. Then combining these with the duplication formula $\Gamma_{\mathbb{C}}(s)=\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ we have that

$$\prod_{v\mid\infty}\frac{L_v(1-s,\widetilde{\pi}_v,\mathrm{Ad}\otimes\widetilde{\chi}_v)}{L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)} \ll \prod_{v\mid\infty}C(\pi_v,\mathrm{Ad}\otimes\chi_v)^{1/2-\mathrm{Re}(s)},$$

where $-2<\mathrm{Re}(s)<1/5$. Together with the functional equation we have

$$L(s,\pi,\mathrm{Ad}\otimes\chi) = O\left(C(\pi,\mathrm{Ad}\otimes\chi)^{1/2-\mathrm{Re}(s)}\right),\ \mathrm{Re}(s)\leq\beta_0<-4/5. \tag{5}$$

If $\pi\otimes\chi\simeq\widetilde{\pi}$, then according to Theorem 6, $L(s,\pi,\mathrm{Ad}\otimes\chi)$ has simple poles precisely at $s=1$ and possibly at $s=0$. Let $c=\max_{i,j}\{5-\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i,j}\}$. Consider the function $f(s)$ which is defined by the product of $s(s-1)$ and

$$L(s,\pi,\mathrm{Ad}\otimes\chi)\left(s+\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i_0,j_0}\right)^{-\beta+5/2}\prod_i\prod_j\left(s+c+\mu_{\pi\otimes\chi\times\widetilde{\pi};v,i,j}\right)^{\beta-5/2},$$

where the parameter $\beta<-4/5$. Then by (5) we see that $f(s)$ is holomorphic and is bounded in the lines $\mathrm{Re}(s)=-4/5-\epsilon$ and $\mathrm{Re}(s)=9/5+\epsilon$. Since $f(s)$ is holomorphic in $s\in(-4/5-\epsilon-2i,9/5+\epsilon-2i)$, it is bounded in this in this interval. Hence by Phragmén-Lindelöf principle and the fact that $f(s)=\overline{f(\bar{s})}$ we have that $f(s)$

is bounded by $O_\epsilon\left(C(\pi, \mathrm{Ad}\otimes\chi)^{(9/5-\mathrm{Re}(s))/2+\epsilon}\right)$ in the strip $-4/5 \leq \mathrm{Re}(s) \leq 9/5$, leading to the estimate

$$L(s,\pi,\mathrm{Ad}\otimes\chi) \ll_\epsilon |s(1-s)|^{-1} C(\pi,\mathrm{Ad}\otimes\chi)^{\frac{9/5-\mathrm{Re}(s)}{2}+\epsilon}. \tag{6}$$

If $\pi\otimes\chi \not\cong \widetilde{\pi}$, then according to Theorem 6, $L(s,\pi,\mathrm{Ad}\otimes\chi)$ is entire. Then similarly combining Phragmén-Lindelöf principle with (4) and (5) we obtain the following preconvex bound in the strip $-4/5 \leq \mathrm{Re}(s) \leq 9/5$:

$$L(s,\pi,\mathrm{Ad}\otimes\chi) \ll_\epsilon C(\pi,\mathrm{Ad}\otimes\chi)^{\frac{9/5-\mathrm{Re}(s)}{2}+\epsilon}. \tag{7}$$

Now (3) follows from (7) and (6). ☐

With Lemma 8, we can then apply a result in [12] to further prove a convex bound for adjoint L-functions, which plays a role in the proof of Theorem B.

Proposition 9 (Convex bound). *Let notation be as before. Then for $s \in \mathbb{C}$ such that $0 < \mathrm{Re}(s) < 1$, we have*

$$L(s,\pi,\mathrm{Ad}\otimes\chi) \ll_{F,\epsilon} \left(1+|s(s-1)|^{-1}\right) C(\pi,\mathrm{Ad}\otimes\chi)^{\frac{1-\mathrm{Re}(s)}{2}+\epsilon}, \tag{8}$$

where the implies constant is absolute, depending only on ϵ and the base field F.

Proof. Let $g(s) = s(1-s)L(s,\pi,\mathrm{Ad}\otimes\chi)$. Then $g(s)$ is entire by Theorem 6. Moreover, by the above preconvex bound (3), $g(s)$ satisfies the hypothesis of Theorem 2 of [12]. It is then easy to see that (8) follows from functional equation (2), Theorem 2 of [12] and Phragmén-Lindelöf principle. ☐

Now we make use of theory of entire functions of finite orders to investigate zeros of $L(s,\pi,\mathrm{Ad}\otimes\chi)$ is the critical strip $0 < \mathrm{Re}(s) < 1$.

Proposition 10. *There are infinitely many zeros of $L(s,\pi,\mathrm{Ad}\otimes\chi)$ in the strip $1/2 \leq \mathrm{Re}(s) < 1$.*

Proof. It follows from Theorem 1 of [24] that the function

$$G(s) := s(s-1)\Lambda(s,\pi,\mathrm{Ad}\otimes\chi) = s(s-1)L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)L(s,\pi,\mathrm{Ad}\otimes\chi)$$

is entire. According to Theorem 6 and Proposition 9, $s(s-1)L(s,\pi,\mathrm{Ad}\otimes\chi)$ is entire and is order at most 1. On the other hand, by Stirling's formula one sees that $L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)$ is of order exactly 1. Hence $G(s)$ is an entire function of order 1. Then it must has infinitely many zeros. By a explicit construction of zero-free region of Rankin-Selberg L-functions (ref. [1]) and estimates on archimedean Langlands parameters (ref. [2]) one concludes that $G(s)$ has at most a simple zero in the right half plane $\mathrm{Re}(s) \geq 1$. Then by functional equation of $\Lambda(s,\pi,\mathrm{Ad}\otimes\chi)$ one concludes that $G(s)$ has at most a simple zero in the left half plane $\mathrm{Re}(s) \leq 0$. Therefore, there are infinitely many zeros of $G(s)$ in the critical strip, namely, $\Lambda(s,\pi,\mathrm{Ad}\otimes\chi)$ has infinitely many zeros in the critical strip $0 < \mathrm{Re}(s) < 1$. Then by functional equation again, $\Lambda(s,\pi,\mathrm{Ad}\otimes\chi)$ has infinitely many zeros in the strip $1/2 \leq \mathrm{Re}(s) < 1$.

Since $L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)$ is of the form $E_\infty(s)\prod_{v|\infty}\prod_{\mu_v}\Gamma_{F_v}(s+\mu_v)$, where $E_\infty(s)$ is some exponential function and μ_v's are some complex numbers, and the product is finite. Hence there are only possibly many poles of $L_\infty(s,\pi,\mathrm{Ad}\otimes\chi)$ in the strip $1/2 \leq \mathrm{Re}(s) < 1$. Consequently, $L(s,\pi,\mathrm{Ad}\otimes\chi)$ has infinitely many zeros in the strip $1/2 \leq \mathrm{Re}(s) < 1$. ☐

3. Stake Parameters at Unramified Places

Let notation be as before. Write $\pi = \otimes_v' \pi_v$. Recall that $S^{ur}(\pi)$ is the set of finite places where π is unramified. For any $v \in S^{ur}(\pi)$, we will write $\mathfrak{p}_v$ the prime ideal of $\mathcal{O}_F$ corresponding to v. Let ϖ_v be a uniformizer of the local field F_v. Set $q_v = |\varpi_v|_v^{-1}$. Let $S^{ra}(\pi) := \Sigma_{F,fin} \setminus S^{ur}(\pi)$ be the set of finite places where π is ramified. Then $S^{ra}(\pi)$ is a finite set.

Let $v \in S^{ur}(\pi)$. Let $St_v(\pi) = \{\alpha_{v,1}, \alpha_{v,2}, \alpha_{v,3}\}$ be the set of Satake parameters of π_v. On the one hand, suppose that $v \in \mathcal{R}_3(\pi; F)$. Set the Langlands class $St_v(\pi) = \{\alpha_{v,1}, \alpha_{v,2}, \alpha_{v,3}\}$. Then $|\alpha_{v,1}| = |\alpha_{v,2}| = |\alpha_{v,3}| = 1$. Hence $a_{\mathfrak{p}_v^m}(\pi, \mathrm{Ad}) = |a_\pi(\mathfrak{p}_v^m)|^2 - 1 \in [-1, 8]$.

On the other hand, if $v \notin \mathcal{R}_3(\pi; F)$, then there exists some $\alpha_v \in St_v(\pi)$ such that $|\alpha_v| \neq 1$. We may assume $\alpha_v = \alpha_{v,1}$ and write it as uq_v^t, for some $u \in \mathbb{C}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $t \in \mathbb{R} \setminus \{0\}$. On the other hand, since π_v is unitary, we have $\{\overline{\alpha}_{v,1}, \overline{\alpha}_{v,2}, \overline{\alpha}_{v,3}\} = \{\alpha_{v,1}^{-1}, \alpha_{v,2}^{-1}, \alpha_{v,3}^{-1}\}$, which leads to that $St_v(\pi) = \{uq_v^t, uq_v^{-t}, w\}$, for some $w \in \mathbb{C}^1$. We may assume that $t > 0$ and $w = ue^{i\theta}$, where $\theta \in [0, 2\pi)$.

Then the trace of Hecke operator at v is $a_{\mathfrak{p}_v}(\pi) = uq_v^t + uq_v^{-t} + w$. Since π_v is unramified, we then have that

$$|a_\pi(\mathfrak{p}_v)|^2 - 1 = 2 + q_v^{2t} + q_v^{-2t} + 2(q_v^t + q_v^{-t}) \cos\theta > 0.$$

Likewise, for any $m \in \mathbb{N}$, we have $a_\pi(\mathfrak{p}_v^m) = u^m q_v^{mt} + u^m q_v^{-mt} + w^m$, and thus

$$|a_\pi(\mathfrak{p}_v^m)|^2 - 1 = 2 + q_v^{2mt} + q_v^{-2mt} + 2(q_v^{mt} + q_v^{-mt}) \cos m\theta > 0. \tag{9}$$

Therefore, if $a_{\mathfrak{p}_v^m}(\pi, \mathrm{Ad}) \leq 0$ for some m, then v must lie in the set $\mathcal{R}_3(\pi; F)$.

Lemma 11. *Let π be a cuspidal representation on $GL(3)/F$. Suppose that $\pi_\mathfrak{p}$ is unramified and non-tempered at prime $\mathfrak{p}$. Then one has*

$$a_\mathfrak{p}(\pi, Ad) + a_{\mathfrak{p}^2}(\pi, Ad) \geq 16/3. \tag{10}$$

Proof. Since $\pi_\mathfrak{p}$ is unramified and non-tempered at prime $\mathfrak{p}$, we have

$$a_{\mathfrak{p}^m}(\pi, Ad) = |a_\pi(\mathfrak{p}_v^m)|^2 - 1 = (q_v^{mt} + \cos m\theta)^2 + (q_v^{-mt} + \cos m\theta)^2 + 2\sin^2 m\theta.$$

Set $f(x, \gamma) = (x + \cos\gamma)^2 + (x^{-1} + \cos\gamma)^2 + 2\sin^2\gamma$. Now the problem is to find the minimum of $g(x, \gamma) = f(x, \gamma) + f(x^2, 2\gamma)$, under the restriction that $x \geq 1$. Using Lagrange multiplier or numerical computation one finds

$$\min_{\substack{x \geq 1 \\ -\pi < \gamma \leq \pi}} g(x, \gamma) \geq 16/3.$$

Hence (10) follows. □

Lemma 12. *Let π be a cuspidal representation on $GL(3)/F$. Suppose that $\pi_\mathfrak{p}$ is unramified and tempered at prime $\mathfrak{p}$. Assume further that $a_\mathfrak{p}(\pi, Ad) \geq 0$. Then*

$$a_\mathfrak{p}(\pi, \mathrm{Ad})^2 + a_{\mathfrak{p}^2}(\pi, \mathrm{Ad}) \geq 0. \tag{11}$$

Proof. Since $\pi_\mathfrak{p}$ is unramified and tempered at prime $\mathfrak{p}$, we can assume $\pi_\mathfrak{p}$ has Satake parameters $\{e^{i\alpha}, e^{i\beta}, e^{i\gamma}\}$.

$$a_{\mathfrak{p}^m}(\pi, Ad) = |a_\pi(\mathfrak{p}_v^m)|^2 - 1 = 8\cos\frac{\alpha - \gamma}{2}\cos\frac{\beta - \gamma}{2}\cos\frac{\alpha - \beta}{2}. \tag{12}$$

Set $h(x, y) = \cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{x-y}{2}$. One can verify that $h(x, y) + h(2x, 2y) \geq 0$, under the condition that $h(x, y) \geq 0$. Then (11) follows. □

Lemma 13. *Let π be a cuspidal representation on $GL(3)/F$. Suppose that $\pi_\mathfrak{p}$ is unramified and tempered at prime $\mathfrak{p}$. Assume further that $a_{\mathfrak{p}^k}(\pi, Ad) \geq 0$, $1 \leq k \leq 4$. Then*

$$a_\mathfrak{p}(\pi, Ad)^2 + a_{\mathfrak{p}^2}(\pi, Ad) + a_{\mathfrak{p}^4}(\pi, Ad) \geq 2. \tag{13}$$

Proof. The proof makes use of the expression (12) and the auxiliary function $h(x, y)$. Using Lagrange multiplier or numerical computation we can show that

$$h(x, y) + h(2x, 2y) + h(4x, 4y) \geq 2,$$

under the condition that $h(kx, ky) \geq 0$, $1 \leq k \leq 4$. Since this can be done by calculus or by Mathematica, we shall omit the computation as before. □

In [22], Yan proved that there exists some $1 \leq k \leq n$ such that $|a_\pi(\mathfrak{p}^k)| \geq 1$, which in our particular situation, follows readily from (9). But this lower bound is clearly insufficient for proving Theorem B. In fact, any lower bound as $|a_\pi(\mathfrak{p}^k)| \geq 1 + \epsilon$ with $\epsilon > 0$ would lead to results of Theorem B type. In the below, we shall prove that one can essentially take $\epsilon = \sqrt{2} - 1 > 0$. The main idea behind is combining a variant of Brumley's result (ref. Lemma 1 in [1]) with the pigeonhole principle via Hecke relations. We start with

Lemma 14. *Let $\alpha_i \in \mathbb{C}$, $1 \leq i \leq n$. Let b_m be defined via*

$$\sum_{m=1}^{\infty} b_m X^m = \frac{\prod_{s=1}^n \prod_{t=1}^n (1 - \alpha_s \alpha_t^{-1} X) \prod_{u=1}^n \prod_{v=1}^n (1 - \overline{\alpha}_u \overline{\alpha}_v^{-1} X)}{(1 - X) \prod_{i=1}^n \prod_{j=1}^n \prod_{k=1}^n \prod_{l=1}^n (1 - \alpha_i \overline{\alpha}_k \alpha_j^{-1} \overline{\alpha}_l^{-1} X)}, \tag{14}$$

viewed as a formal series. Then $b_{n^2-1} \geq 1$.

Note that an elementary computation shows that the right hand side of (14) can be written into the form $\prod_i \prod_j (1 - \beta_i \overline{\beta}_j X)^{-1}$, with $\prod \beta_i = 1$. Hence Brumley's Lemma (ref. [1]) applies immediately here. Using Lemma 14 one can prove:

Proposition 15. *Let π be a unitary cuspidal representation on $GL(n)/F$, where $n \in \mathbb{N}$ is arbitrary. Suppose π is unramified at $\mathfrak{p}$. Then there exists some $1 \leq k \leq n^2 - 1$ such that either $a_\pi(\mathfrak{p}^k) = 0$ or $|a_\pi(\mathfrak{p}^k)| \geq \sqrt{2}$.*

Proof. Comparing logarithm expansions of both sides of (14) one has,

$$m b_m = \sum_{j=1}^{m} \left\{ \Big| \sum_{i=1}^{n} \alpha_i^k \Big|^2 - 1 \right\} \cdot b_{m-j}. \tag{15}$$

Since π is unramified at $\mathfrak{p}$, we may assume the Satake parameter for $\pi_\mathfrak{p}$ is given by $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then for any integer $k \geq 1$, $a_\pi(\mathfrak{p}^k) = \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k$.

Assume, on the contrary, that $0 < |a_\pi(\mathfrak{p}^k)| < \sqrt{2}$ for all $1 \leq k \leq n^2 - 1$. Then $b_1 = |\alpha_1 + \alpha_2 + \cdots + \alpha_n|^2 - 1 = |a_\pi(\mathfrak{p})|^2 - 1 < 1$. Apply then $m = 2$ to see that $2b_2 = (|a_\pi(\mathfrak{p})|^2 - 1)^2 + (|a_\pi(\mathfrak{p}^2)|^2 - 1) < 1 + 1 = 2$. So $b_2 < 1$. One can continue this procedure, one eventually gets (by induction) that $b_{n^2-1} < 1$, which is a contradiction to Lemma 14. Thus there must be some $1 \leq k \leq n^2 - 1$, such that either $|a_\pi(\mathfrak{p}^k)| \geq \sqrt{2}$ or $a_\pi(\mathfrak{p}^k) = 0$. □

Remark. Although Proposition (15) is not used in this paper, we keep it here as it is of independent interest. For example, one can prove similar results (but with weaker exponents) as Theorem B and Theorem C for $GL(4)$, with Lemma 11, Lemma 12 and Lemma 13 replaced with Proposition 15.

4. Proof of Theorem A

Let $\mathfrak{M}$ be a modulus over F. Let $Cl_\mathfrak{M}(F)$ be the ray class group associated to modulus $\mathfrak{M}$. Then $Cl_\mathfrak{M}(F)$ is a finite abelian group. Let $h_\mathfrak{M}$ be the ray class number. Denote by $\widehat{Cl}_\mathfrak{M}(F)$ the Pontryagin dual group of $Cl_\mathfrak{M}(F)$. Then the cardinality of $\widehat{Cl}_{\mathfrak{M}}(F)$ is equal to $h_{\mathfrak{M}}$, and characters in $\widehat{Cl}_{\mathfrak{M}}(F)$ lifts to Hecke characters naturally. Since $\widehat{Cl}_{\mathfrak{M}}(F)$ is finite and abelian, we have

$$\sum_{\chi\in\widehat{Cl}_{\mathfrak{M}}(F)}\chi(\mathcal{C})=\begin{cases}h_{\mathfrak{M}}, & \text{if } \mathcal{C} \text{ is nontrivial;}\\ 0, & \text{otherwise.}\end{cases} \tag{16}$$

Thus for any L^1 function $f:\Sigma_{F,fin}\longrightarrow\mathbb{C}$ one has by (16) that

$$\sum_{\substack{v\in\Sigma_{F,fin}\\ \mathfrak{p}_v\in\mathcal{C}}}f(\mathfrak{p}_v)=\frac{1}{h_{\mathfrak{M}}}\sum_{\chi\in\widehat{Cl}_{\mathfrak{M}}(F)}\overline{\chi}(\mathcal{C})\sum_{v\in\Sigma_{F,fin}}f(\mathfrak{p}_v)\chi(\mathfrak{p}_v). \tag{17}$$

Let $\mathrm{Re}(s)>2$. Then $L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)$ is nonvanishing. Then $\log L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)$ is well defined. Also, for $v\in S^{ur}(\pi)$, a straightforward computation implies that

$$\log L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)=\sum_{m=1}^{\infty}\frac{a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})\chi_v(\mathfrak{p}_v)^m}{mq_v^{ms}},\ \ \mathrm{Re}(s)>2. \tag{18}$$

Take $f(\mathfrak{p}_v)$ to be $\log L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)$. Putting together (17) and (18) we thus have

$$\sum_{m=1}^{\infty}\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^m\in\mathcal{C}}}\frac{a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})}{mq_v^{ms}}=\frac{1}{h_{\mathfrak{M}}}\sum_{\chi\in\widehat{Cl}_{\mathfrak{M}}(F)}\overline{\chi}(\mathcal{C})\log\frac{L(s,\pi,\mathrm{Ad}\otimes\chi)}{L_{S^{ra}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)},$$

where $\mathrm{Re}(s)>2$ and $L_{S^{ra}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)=\prod_{v\in S^{ra}(\pi)}L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)$.

Let $v\in S^{ra}(\pi)$. Let σ_v be an n-dimensional representation of $W_{F_v}\times GL(2,\mathbb{C})$ for v nonarchimedean associated to π_v via local Langlands correspondence (ref. [8] and [6]). Let $\mathrm{Ad}\,\sigma_v$ be the adjoint representation of σ_v. Then $L_v(s,\pi_v,\mathrm{Ad}\otimes\chi_v)=L_v(s,\chi_v\otimes\mathrm{Ad}\,\sigma_v)$. Since $L_v(s,\chi_v\otimes\mathrm{Ad}\,\sigma_v)$ is nonvanishing everywhere, the partial L-function $L_{S^{ra}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ is nonvanishing everywhere. Hence the partial L-function $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)=L(s,\pi,\mathrm{Ad}\otimes\chi)\cdot L_{S^{ra}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)^{-1}$ is holomorphic at $s_0\in\mathbb{C}$ if $L(s,\pi,\mathrm{Ad}\otimes\chi)$ is regular at s_0.

Note that we assume $h_{\mathfrak{M}}$ is coprime to 3. Then for any nontrivial $\chi\in\widehat{Cl}_{\mathfrak{M}}(F)$, $\chi^3\neq 1$. Suppose $\pi\otimes\chi\simeq\pi$ for some nontrivial $\chi\in\widehat{Cl}_{\mathfrak{M}}(F)$. Let ω_π be the central character of π. Then one has $\omega_\pi=\omega_\pi\cdot\chi\circ\det=\omega_\pi\cdot\chi^3$. Hence $\chi^3=1$, which contradicts our assumption on $h_{\mathfrak{M}}$. Hence π is not self-twist for any $\chi\in\widehat{Cl}_{\mathfrak{M}}(F)\setminus\{1\}$. Then by Theorem 6 and the above analysis we conclude that $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ is entire for any $\chi\in\widehat{Cl}_{\mathfrak{M}}(F)$. By Proposition 10, $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ has infinitely many zeros in the strip $1/2\leq\mathrm{Re}(s)<1$. Now we have, for $\mathrm{Re}(s)>2$, that

$$\sum_{m=1}^{\infty}\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^m\in\mathcal{C}}}\frac{a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})\log q_v}{q_v^{ms}}=-\sum_{\chi\in\widehat{Cl}_{\mathfrak{M}}(F)}\frac{\overline{\chi}(\mathcal{C})\cdot L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)'}{h_{\mathfrak{M}}\cdot L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)}. \tag{19}$$

Proof of Theorem A. Denote by $\mathcal{R}^+(\pi;F)=\{v\in S^{ur}(\pi)\cap\mathcal{R}_3(\pi;F):\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})>0\}$; and $\mathcal{R}^-(\pi;F)=\{v\in S^{ur}(\pi)\cap\mathcal{R}_3(\pi;F):\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0\}$. Let $\mathcal{R}_*^+(\pi;F)=\{v\in S^{ur}(\pi)\setminus\mathcal{R}_3(\pi;F):\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})>0\}$. Set $\mathcal{R}_1^-(\pi;F)=\{v\in S^{ur}(\pi)\cap\mathcal{R}_3(\pi;F):\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})>0,\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0\}\subseteq\mathcal{R}^+(\pi;F)$; and $\mathcal{R}_2^-(\pi;F)=\{v\in S^{ur}(\pi)\cap\mathcal{R}_3(\pi;F):\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0,\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0\}\subseteq\mathcal{R}^-(\pi;F)$.

Let $\mathcal{R}^+(\pi;F,\mathcal{C})=\mathcal{R}^+(\pi;F)\cap\mathcal{C}$, $\mathcal{R}^-(\pi;F,\mathcal{C})=\mathcal{R}^-(\pi;F)\cap\mathcal{C}$, $\mathcal{R}_1^-(\pi;F,\mathcal{C})=\mathcal{R}_1^-(\pi;F)\cap\mathcal{C}$, and $\mathcal{R}_2^-(\pi;F,\mathcal{C})=\mathcal{R}_2^-(\pi;F)\cap\mathcal{C}$. Denote by

$$I_m^+(s)=\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^m\in\mathcal{C},\ a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})>0}}\frac{a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})\log q_v}{q_v^{ms}},\quad \mathrm{Re}(s)>2;$$

$$I_m^-(s)=\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^m\in\mathcal{C},\ a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})<0}}\frac{a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})\log q_v}{q_v^{ms}},\quad \mathrm{Re}(s)>2.$$

Let $I_{\leq 2}^-(s)=I_1^-(s)+I_2^-(s)$; and $I_{\geq 3}^-(s)=\sum_{m\geq 3}I_m^-(s)$. Set $I^+(s)=\sum_{m\geq 1}I_m^+(s)$. Then from the last paragraphs of Section 3 we see that for any $a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})<0$ one must have $a_{\mathfrak{p}_v^m}(\pi,\mathrm{Ad})\geq -1$. Thus $I_{\geq 3}^-(s)$ is dominated by $\sum_{m\geq 3}q_v^{-m\,\mathrm{Re}(s)}\log q_v$, which converges absolutely for $\mathrm{Re}(s)\geq 1/2-\epsilon$, where $0<\epsilon<1/6$. Hence $I_{\geq 3}^-(s)$ is holomorphic in the domain $1/2\leq\mathrm{Re}(s)<1$. Then we have, from proceeding computations, that

$$I_{\leq 2}^-(s)+I^+(s)=-\frac{1}{h_{\mathfrak{M}}}\sum_{\chi\in\widehat{Cl}_{\mathfrak{M}}(F)}\frac{\overline{\chi}(\mathcal{C})L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)'}{L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)}-I_{\geq 3}^-(s). \tag{20}$$

For any $\chi\in\widehat{Cl}_{\mathfrak{M}}(F)$, denote by $Z(\pi;\chi)$ the set of zeros of $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ in the strip $1/2\leq\mathrm{Re}(s)<1$. Let

$$\beta=\max_{\chi\in\widehat{Cl}_{\mathfrak{M}}(F)}\sup_{\rho\in Z(\pi;\chi)}\mathrm{Re}(\rho)\in[1/2,1].$$

Then the right hand side of (20) is meromorphic in the strip $1/2-\epsilon<\mathrm{Re}(s)<2$ and has at least a pole in the strip $\beta-\epsilon<\mathrm{Re}(s)<\beta+\epsilon$, for any $\epsilon>0$.

Then according to (20), $I_{\leq 2}^-(s)+I^+(s)$ admits a meromorphic continuation to the domain $1/2-\epsilon<\mathrm{Re}(s)<2$. Note that $I_{\leq 2}^-(s)$ is a Dirichlet series with negative coefficient, we may denote by $\beta^-\in\mathbb{R}$ its abscissa of absolute convergence. Also, note that $I^+(s)$ is a Dirichlet series with positive coefficients. Let $\beta^+\in\mathbb{R}$ be its abscissa of absolute convergence. If $\beta^-<\beta^+$, then $I_{\leq 2}^-(s)$ converges absolutely in the region $\mathrm{Re}(s)\geq(\beta^++\beta^-)/2$. Note that $s=\beta^+$ is a pole of $I^+(s)$. Denote by $H(s)$ the right hand side of (19).

Since $\mathcal{C}$ is taken to be trivial, $\overline{\chi}(\mathcal{C})=1$. So $H(s)$ must have a pole in the strip $1/2\leq\mathrm{Re}(s)<1$. Then $s=\beta^+$ has to be a pole of one of the meromorphic functions $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)'\cdot L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)^{-1}$, which asserts that $s=\beta^+$ must be a pole of one of the meromorphic functions $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ because $I^+(\beta^++\epsilon)\to+\infty$ as $\epsilon\to 0^+$. However, this contradicts the fact that $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ is entire. Therefore, we must have that $\beta^+\leq\beta^-$.

In fact, If $\beta^-<\beta$, then the left hand side of (20) converges absolutely in the region $\mathrm{Re}(s)>(\beta+\beta^-)/2$, defining a holomorphic function in that region. However, the right hand side of (20) has at least a pole in the region $\mathrm{Re}(s)>(\beta+\beta^-)/2$, giving a contradiction. Hence $\beta^-\geq\beta\geq 1/2$.

Since $I_{\leq 2}^-(s)$ is a Dirichlet series with negative coefficient, then $I_{\leq 2}^-(s)$ has a pole at $s=\beta^-$. Together with $\beta^-\geq\beta\geq 1/2$ we conclude that

$$\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v\in\mathcal{C},\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0}}\frac{a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\log q_v}{q_v^{\beta}}+\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^2\in\mathcal{C},\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0}}\frac{a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\log q_v}{q_v^{2\beta}}=-\infty. \tag{21}$$

Note that when $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0$, $-1\leq a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\leq 8$. Note also that $\beta\geq 1/2$. Hence, if Theorem A did not hold, then the first sum in (21) must converge at $\beta=1/2$.. Then one has

$$\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^2\in\mathcal{C},\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0\\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0}}\frac{a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\log q_v}{q_v^{2\beta}}+\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^2\in\mathcal{C},\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0\\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\log q_v}{q_v^{2\beta}}=-\infty. \tag{22}$$

Note that when $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0$, $|a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})|\leq 8$. Then the first sum in (22) converge since we assume that Theorem A did not hold. Therefore, the second sum in (22) must diverge at $\beta=1/2$. Hence, we have

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}'\\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0\\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\log q_v}{q_v}+\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v\in\mathcal{C},\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0\\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\log q_v}{q_v}=-\infty, \tag{23}$$

where $\mathcal{C}'$ is nontrivial but $\mathcal{C}'$ has order two.

Case 1. If the first sum in the left hand side of (23) converge, then the second sum must diverge.

By Lemma 12, we have $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})^2+a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\geq 0$, supposing that $a_{\mathfrak{p}_v}(\pi,Ad)\geq 0$. Then in conjunction with (23) we obtain that

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}\\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0,\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})^2\log q_v}{q_v}=+\infty. \tag{24}$$

If $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\leq 1$, then $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})^2\leq a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})$. Implying that

$$\sum_{\substack{v\in S^{ur}(\pi)\\ \mathfrak{p}_v^2\in\mathcal{C},\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\log q_v}{q_v}=+\infty.$$

Thus we must have $\beta^+\geq 2\beta$. So $\beta^-\geq 2\beta\geq 1$, contradicting with our assumption that (1) did not hold.

If $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})>1$,then $|a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})|\leq 8$ since $a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0$, which implies that π_v is tempered. Hence by (24) one has

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}\\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0,\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{8a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\log q_v}{q_v}=+\infty. \tag{25}$$

Again, it follows from (25) that $\beta^+\geq 2\beta\geq 1$. Same argument as before applies. In all, Theorem A holds in this case.

Case 2. If the second sum in the left hand side of (23) converge, then the first sum must diverge, namely, one has in this case that

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}'\\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0,\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\log q_v}{q_v}=-\infty. \tag{26}$$

Again, by Lemma 12, we have $a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})^2+a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})\geq 0$, assuming $a_{\mathfrak{p}_v}(\pi,Ad)\geq 0$. Then in conjunction with (26) we obtain that

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}'\\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0,\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq0}}\frac{a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})^2\log q_v}{q_v}=+\infty. \tag{27}$$

Similar analysis on (27) as before will lead to that

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}'\\ a_{\mathfrak{p}_v^2}(\pi,\mathrm{Ad})<0,\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\geq 0}} \frac{a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\log q_v}{q_v}=+\infty. \tag{28}$$

Substituting (28) into (20) with $\mathcal{C}$ replaced with $\mathcal{C}'$, then the $I^+(s)$-term blow up at $s=1$. However, since each $L^{S^{ur}(\pi)}(s,\pi,\mathrm{Ad}\otimes\chi)$ is holomorhic and nonvanishing at $s=1$, the right hand side of (20) is regular at $s=1$. Therefore, necessarily $I_{\leq 2}^{-1}(1)=-\infty$, forcing that

$$\sum_{\substack{v\in S^{ur}(\pi),\ \mathfrak{p}_v\in\mathcal{C}'\\ a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})<0}} \frac{a_{\mathfrak{p}_v}(\pi,\mathrm{Ad})\log q_v}{q_v}=-\infty.$$

Then the condition (1) follows once more. Thus our assumption does not hold, which implies that the first part of Theorem A holds.

Suppose further that Q_π, the arithmetic conductor of π, is cube-free, then by Theorem A of [18], π cannot be self-twist. Hence the second part of Theorem A follows. □

5. Proof of Theorem B and Theorem C

5.1. Proof of Theorem B. We may assume q_2 is square-free. Define the partial L-function associated to q_2 to be

$$L^{(q_2)}(s,\pi,\mathrm{Ad}\otimes\chi)=L(s,\pi,\mathrm{Ad}\otimes\chi)\cdot\prod_{p\mid q_2}\frac{\prod_{i=1}^3\prod_{j=1}^3(1-\alpha_{p,i}\chi(p)\alpha_{p,j}^{-1}p^{-s})}{1-\chi(p)p^{-s}},$$

where $\{\alpha_{p,1},\alpha_{p,2},\alpha_{p,3}\}$ is the Satake parameter od π_p defined via local Langlands correspondence. Note that $\alpha_{p,i}$ might vanish when π_p is ramified.

Let $x\geq 1$ be such that $a_n(\pi,\mathrm{Ad})\geq 0$ for any $(n,q_2)=1$, and $n\leq x$. Such an x exists since $a_1(\pi,\mathrm{Ad})\geq |a_1(\pi)|^2-1=0$. Then we can set

$$S_1(x):=\sum_{n\leq x,\ (n,q_2)=1} a_n(\pi,\mathrm{Ad})\exp\left(-\frac{n}{x}\right). \tag{29}$$

Proposition 16. *Let notation be as before. Assume that π is not self-twist. Let $A>0$ be any positive constant. Then*

$$S_1(x)\ll_\epsilon C_\pi^{1+2\epsilon}q_2^{15/14-\epsilon}. \tag{30}$$

Proof. Consider the inverse Mellin transform of the Gamma function

$$\exp(-y)=\lim_{T\to+\infty}\frac{1}{2\pi}\int_{-T}^{T} y^{-\sigma-it}\Gamma(\sigma+it)dt=:\frac{1}{2\pi i}\int_{(\sigma)} y^{-s}\Gamma(s)ds, \tag{31}$$

where $\sigma>0$, and y^{-s} is taken on the principal branch. Combining (29) and (31) to see

$$\begin{aligned} S_1(x)=&\sum_{n\leq x,\ (n,q_2)=1} a_n(\pi,\mathrm{Ad})\cdot\frac{1}{2\pi i}\int_{(2)} n^{-s}x^s\Gamma(s)ds\\ =&\frac{1}{2\pi i}\int_{(2)}\sum_{n\leq x,\ (n,q_2)=1}\frac{a_n(\pi,\mathrm{Ad})}{n^s}\cdot x^s\Gamma(s)ds\\ =&\frac{1}{2\pi i}\int_{(2)} L^{(q_2)}(s,\pi,\mathrm{Ad})\cdot x^s\Gamma(s)ds, \end{aligned}$$

where the interchange of integral and summation is guaranteed by the absolute convergence of Dirichlet series of $L^{(q_2)}(s,\pi,\mathrm{Ad}\otimes\chi)$ as well as the rapidly decaying of $\Gamma(s)$ along $\mathrm{Re}(s)=2$.

Let $\epsilon>0$. By Lemma 9 and Theorem 6, one can apply Cauchy's theorem to shift contour, which implies that

$$\begin{aligned} S_1(x) =& \frac{1}{2\pi i}\int_{(\epsilon)} L^{(q_2)}(s,\pi,\mathrm{Ad})\cdot x^s\Gamma(s)ds \\ =& \frac{1}{2\pi i}\int_{(\epsilon)} x^s\Gamma(s)L(s,\pi,\mathrm{Ad})\cdot\prod_{p\mid q_2}\frac{\prod_{i=1}^3\prod_{j=1}^3(1-\alpha_{p,i}\chi(p)\alpha_{p,j}^{-1}p^{-s})}{1-\chi(p)p^{-s}}ds \\ \ll& \prod_{p\mid q_2}\frac{\prod_{i=1}^3\prod_{j=1}^3(1+|\alpha_{p,i}\alpha_{p,j}^{-1}|p^{-\epsilon})}{1+p^{\epsilon}}\cdot\int_{(\epsilon)}|L(s,\pi,\mathrm{Ad}\otimes\chi)|\cdot|\Gamma(s)|ds. \end{aligned}$$

If $p\in\mathcal{R}_3(\pi;\mathbb{Q})$, then clearly $|\alpha_{p,1}|=|\alpha_{p,2}|=|\alpha_{p,3}|=1$. In this case we have

$$\prod_{p\mid q_2}\frac{\prod_{i=1}^3\prod_{j=1}^3(1+|\alpha_{p,i}\alpha_{p,j}^{-1}|p^{-\epsilon})}{1+p^{\epsilon}}\ll\prod_{p\mid q_2}(1+p^{-\epsilon})^8\ll 2^{8\Omega(q_2)},$$

where $\Omega(n)$ denotes the number of prime factors of n, counting without multiplicity.

While if $p\notin\mathcal{R}_3(\pi;\mathbb{Q})$, then according to discussion in Section 3, one can write $|\alpha_{p,1}|=p^t$, $|\alpha_{p,2}|=p^{-t}$, for some $t>0$; and $|\alpha_{p,3}|=1$. Then by work in [2], one has $t\leq 5/14$. Now the Satake parameters associated to $\mathrm{Ad}\,\pi_p\otimes\chi$ are given by the matrix $\mathrm{diag}\{1,1,p^{2t},p^{-2t},p^tu\overline{v},p^t\overline{u}v,p^{-t}u\overline{v},p^{-t}\overline{u}v,\}\in GL(8,\mathbb{C})$, where $u,v\in\mathbb{C}$ such that $|u|=|v|=1$. Hence in this case, one has

$$\prod_{p\mid q_2}\frac{\prod_{i=1}^3\prod_{j=1}^3(1+|\alpha_{p,i}\alpha_{p,j}^{-1}|p^{-\epsilon})}{1+p^{\epsilon}}\ll\prod_{p\mid q_2}\frac{3}{2}\cdot(1+p^{\frac{5}{14}-\epsilon})(1+p^{\frac{10}{14}-\epsilon})\ll 2^{\Omega(q_2)}q_2^{\frac{15}{14}-\epsilon}.$$

Note also that $2^{8\Omega(q_2)}\ll q_2^{\frac{20}{\log\log(2+q_2)}}$ (e.g. ref. [19]). Hence, combing the above cases we obtain, up to replacing ϵ with $\epsilon/2$, that

$$\prod_{p\mid q_2}\frac{\prod_{i=1}^3\prod_{j=1}^3(1+|\alpha_{p,i}\alpha_{p,j}^{-1}|p^{-\epsilon})}{1+p^{\epsilon}}\ll_\epsilon q_2^{\frac{15}{14}-\epsilon}. \tag{32}$$

Then (30) simply follows from (32) and Proposition 9. □

Proposition 17. *Let notation be as before. Then*

$$S_1(x)\gg\frac{x^{1/4}}{\log x}-\Omega(q_2). \tag{33}$$

Proof. Note that if $p\equiv 1 \mod q$, then $p^k\equiv 1 \mod q$ for any positive integer k. Thus we have

$$\begin{aligned} S_1(x)\geq&\sum_{k=1}^4\sum_{p^k\leq x,\ p\nmid q_2}a_{p^k}(\pi,\mathrm{Ad})\exp\left(-\frac{p^k}{x}\right) \\ \geq&\frac{1}{e}\sum_{p\leq x^{1/4},\ p\nmid q_2}\sum_{k=1}^4 a_{p^k}(\pi,\mathrm{Ad}). \end{aligned}$$

Note that if π_p is non-tempered, then $a_{p^k}(\pi,\mathrm{Ad})$ for any $k\leq 1$. Thus Lemma 11 implies $\sum_{k=1}^4 a_{p^k}(\pi,\mathrm{Ad})\geq 16/3$. If π_p is tempered, noting that $a_{p^3}(\pi,\mathrm{Ad})\geq 0$ by our assumption, then Lemma 13 implies $\sum_{k=1}^4 a_{p^k}(\pi,\mathrm{Ad})\geq 2$. Putting the cases together (33) follows. □

Proof of Theorem B. Assume that $x \gg q_2^{\epsilon}$. Note that $\Omega(q_2) \ll \log q_2$. Then from (33) we see that $S_1(x) \gg x^{1/4}/\log x$. Then in conjunction with Stirling formula and Proposition 16 one would have a contradiction supposing $x \gg_\epsilon C_\pi^{4+\epsilon} q_2^{10/7-\epsilon}$. Thus Theorem B follows. □

5.2. **Proof of Theorem C.** Let $\delta = \delta(q_1, q_2) < 1/4$ be a positive small constant depending only on q_1 and q_2. Let ω be a smooth non-negative function, defined by

$$\omega(x) = \begin{cases} \exp(-x^{-1/16} - (1-x)^{-1/16}) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

Let $x > 0$ be such that $a_n(\pi, \mathrm{Ad}) \geq 0$ for any $n \equiv 1 \mod q_1$, $(n, q_2) = 1$ and $n \leq x$. Then we can set

$$S_2(x) := \sum_{\substack{n \equiv 1 \mod q_1 \\ n \geq 1 \ (n,q_2)=1}} a_n(\pi, \mathrm{Ad}) \omega\left(\frac{n}{x}\right). \tag{35}$$

Proposition 18. *Let notation be as before. Let $A > 0$ be a positive constant. Then*

$$S_2(x) \ll_{A,\epsilon} x^{-A} (C_\pi q_1^4)^{2A+1} q_2^{8A+\epsilon}. \tag{36}$$

Proof. Since $\omega(x)$ is rapidly decaying, its Mellin transform

$$W(s) = \int_0^\infty x^{s-1} \omega(x) dx$$

is analytic. One then see, from partial integration, that

$$|W(\sigma + it)| \ll_k \prod_{j=0}^{k} \frac{1}{|j + \sigma + it|},$$

for any positive integer A, where $\sigma, t \in \mathbb{R}$. Substituting the expression of $\omega(x)$ as the Mellin inversion of $W(s)$ into (35) to get

$$\begin{aligned} S_2(x) :=& \frac{1}{2\pi i} \sum_{\substack{n \equiv 1 \mod q_1 \\ n \geq 1, \ (n,q_2)=1}} a_n(\pi, \mathrm{Ad}) \int_{(2)} W(s) \left(\frac{n}{x}\right)^{-s} ds \\ =& \frac{1}{2\pi i} \cdot \frac{1}{\varphi(q_1)} \sum_{\chi \mod q_1} \int_{(2)} L^{(q_2)}(s, \pi, \mathrm{Ad} \otimes \chi) W(s) x^s ds. \end{aligned}$$

Since we assume that $3 \nmid q_1$, $\pi \otimes \chi$ cannot be isomorphic to π, implying by Theorem 6 that $L^{(q_2)}(s, \pi, \mathrm{Ad} \otimes \chi)$ is entire. Let $s = \sigma + it$. Then following discussions on temperedness of π_p as in the proof of Proposition 16 we then deduce that

$$L(s; q_2) := \prod_{p|q_2} \frac{\prod_{i=1}^3 \prod_{j=1}^3 (1 - \alpha_{p,i} \chi(p) \alpha_{p,j}^{-1} p^{-s})}{1 - \chi(p) p^{-s}} \ll \prod_{p|q_2} \max_{|t| \leq 5/14} L_p(\sigma; t), \tag{37}$$

where $L_p(\sigma; t) = (1 + p^{-\sigma})^2 (1 + p^{t-\sigma})^2 (1 + p^{-t-\sigma})^2 (1 + p^{2t-\sigma})(1 + p^{-2t-\sigma})$.

Moreover, by Proposition 9 and functional equation (2) we have

$$L^{(q_2)}(s, \pi, \mathrm{Ad} \otimes \chi) \ll_\epsilon \begin{cases} C(\pi, \mathrm{Ad} \otimes \chi)^{\frac{1-\sigma}{2} + \epsilon} & \text{if } 0 \leq \sigma \leq 1, \\ C(\pi, \mathrm{Ad} \otimes \chi)^{1 - \frac{\sigma}{2} + \epsilon} & \text{if } \sigma < 0. \end{cases} \tag{38}$$

So $L^{(q_2)}(s, \pi, \mathrm{Ad} \otimes \chi)$ is slowly increasing. Recall that both $W(s)$ and the partial L-function $L^{(q_2)}(s, \pi, \mathrm{Ad} \otimes \chi)$ are entire, and $W(s)$ is rapidly decreasing. Therefore, it follows from (37) and (38) that we can shift the contour to the vertical line $\mathrm{Re}(s) = -A$, where $A \geq 0$, getting

$$S_2(x) = \frac{1}{2\pi i} \cdot \frac{1}{\varphi(q_1)} \sum_{\chi \mod q_1} \int_{(-A)} L^{(q_2)}(s, \pi, \mathrm{Ad} \otimes \chi) W(s) x^s ds. \tag{39}$$

Now one can substitute the functional equation (2) into (39) to get

$$\begin{aligned} S_2(x) =& \frac{1}{2\pi i}\cdot\frac{1}{\varphi(q_1)}\sum_{\chi \mod q_1}\int_{(-A)} L(s,\pi,\mathrm{Ad}\otimes\chi)L(s;q_2)W(s)x^s ds \\ =& \frac{1}{2\pi\varphi(q_1)i}\sum_{\chi \mod q_1}\int_{(-A)} \epsilon Q^{1/2-s}L(1-s,\pi,\mathrm{Ad}\otimes\overline{\chi})L(s;q_2)G(s)W(s)x^s ds, \end{aligned}$$

where $\epsilon=\epsilon(\mathrm{Ad}\otimes\chi)$, $Q=Q_{\pi,\mathrm{Ad}\otimes\chi}$, and

$$G(s)=L_\infty(1-s,\pi,\mathrm{Ad}\otimes\overline{\chi})/L_\infty(s,\pi,\mathrm{Ad}\otimes\chi).$$

Let $A\geq 3$. Then $L(1-s,\pi,\mathrm{Ad}\otimes\overline{\chi})\ll 1$ with the implied constant absolute. Thus we have, assuming $A\geq 3$, that

$$S_2(x)\ll \frac{x^{-A}}{\varphi(q_1)}\sum_{\chi \mod q_1}\int_{(-A)} Q_{\pi,\mathrm{Ad}\otimes\chi}^{1/2+A}|L(s;q_2)G(s)W(s)|\cdot|ds|.$$

By Stirling formula, we have $Q_{\pi,\mathrm{Ad}\otimes\chi}^{1/2+A}G(s)\ll C(\pi,\mathrm{Ad}\otimes\chi)^{1/2+A}$. Also, when $\mathrm{Re}(s)=-A$, we have by (37), that

$$\begin{aligned} |L(s;q_2)|\ll& \prod_{p|q_2}\max_{|t|\leq 5/14}(1+p^A)^2(1+p^{t+A})^2(1+p^{-t+A})^2(1+p^{2t+A})(1+p^{-2t+A}) \\ \ll_\epsilon& q_2^\epsilon\prod_{p|q_2}p^{8A}=q_2^{8A+\epsilon}. \end{aligned}$$

Therefore, we can put the above estimates together to deduce that

$$\begin{aligned} S_2(x)\ll_\epsilon& \frac{q_2^{8A+\epsilon}}{\varphi(q_1)x^A}\sum_{\chi \mod q_1}C(\pi,\mathrm{Ad}\otimes\chi)^{1/2+A}\int_{(-A)}|W(s)|\cdot|ds| \\ \ll_\epsilon& \frac{q_2^{8A+\epsilon}}{\varphi(q_1)x^A}\sum_{\chi \mod q_1}C(\pi,\mathrm{Ad}\otimes\chi)^{1/2+A}\int_{-\infty}^{\infty}\frac{dt}{|A-it|\cdot|A-1-it|} \\ \ll_\epsilon& \frac{q_2^{8A+\epsilon}}{x^A}\max_{\chi \mod q_1}C(\pi,\mathrm{Ad}\otimes\chi)^{1/2+A}\ll_\epsilon \frac{q_2^{8A+\epsilon}}{x^A}(C_\pi q_1^4)^{1+2A}. \end{aligned}$$

Thus (36) follows. □

Proposition 19. *There exists a computable constant $L\geq 13$ such that if $x\geq q_1^L+q_1^7\log^9 q_2$, one can find some $\delta=\delta(q_1,q_2)\in(0,1/4)$ and an $n^*\in[\delta x,(1-\delta)x]$ satisfying the following conditions:*

- $(n^*,q_2)=1$ *and* $n^*\equiv 1 \mod q$,
- $a_{n^*}(\pi,\mathrm{Ad})\geq x^{-1}$.

Proof. Let $\mathcal{P}$ be the set of primes p such that $p\equiv 1 \mod q_1$, $p\nmid q_2$, and $p\leq (1-\delta)x^{1/4}$. Note that $p^k\equiv 1 \mod q_1$ for any $1\leq k\leq 4$, if $p\in\mathcal{P}$. Hence by our assumption, $a_{p^k}(\pi,\mathrm{Ad})\geq 0$. If π_p is non-tempered, then by Lemma 11 one has

$$\sum_{k=1}^{4}a_{p^k}(\pi,\mathrm{Ad})\geq a_p(\pi,\mathrm{Ad})+a_{p^2}(\pi,\mathrm{Ad})\geq 16/3. \tag{40}$$

On the other hand, if π_p is tempered, then by Lemma 13 one has

$$\sum_{k=1}^{4}a_{p^k}(\pi,\mathrm{Ad})\geq 2. \tag{41}$$

Thus, combining (40) and (41) with pigeonhole principle, one conclude that for any prime $p\in\mathcal{P}$, there exists some integer $1\leq k\leq 4$ such that $a_{p^k}(\pi,\mathrm{Ad})\geq 1/2$. Note that k may well depend on p. We then denote by k_p the smallest integer such that $a_{p^{k_p}}(\pi,\mathrm{Ad})\geq 1/2$, for any $p\in\mathcal{P}$.

Let $\mathcal{P}=\{p_0,\cdots,p_m\}$, with $p_0^{k_{p_0}}<\cdots<p_m^{k_{p_m}}$. If $p_m^{k_{p_m}}\geq\delta x$, then we can simply take $n_0=p_m^{k_{p_m}}$, noting that $p_m^{k_{p_m}}\leq p_m^4\leq(1-\delta)^4x\leq(1-\delta)x$. Otherwise, $p_m^{k_{p_m}}<\delta x$. Set $\mathcal{P}_l=\{p_0,\cdots,p_l\}$, $0\leq l\leq m$. Therefore, there exists some integer $r\geq 0$, such that

$$p_l^{k_{p_l}}<\frac{\delta x}{p_m^{k_{p_m}}\cdots p_{l+1}^{k_{p_{l+1}}}}, \tag{42}$$

holds for any $r\leq l\leq m$, but (42) does not hold for $l=r$, where we denote by $p_{-1}=x$. If $r=0$, one shall have that

$$p_0p_1\cdots p_m\leq p_0^{k_{p_0}}p_1^{k_{p_1}}\cdots p_m^{k_{p_m}}<\delta x. \tag{43}$$

Let L_1 be the computable constant defined from an explicit log-free zero density of Dirichlet L-functions associated to characters mod q_1. Then by Corollary 18.8 in [9] one has a quantitative version of Linnik's problem:

$$\sum_{\substack{n\leq x\\ n\equiv 1\mod q}}\Lambda(n)\gg\frac{x}{\varphi(q_1)\sqrt{q_1}},$$

as long as $x\geq q_1^{L_1}$. Therefore, taking $L=L_1+12$, we then have

$$\begin{aligned}\sum_{p\in\mathcal{P}}\log p=&\sum_{\substack{n\equiv 1\mod q_1\\ n\leq(1-\delta)x^{1/4},\ (n,q_2)=1}}\Lambda(n)-\sum_{k\geq 2}\sum_{\substack{p^k\equiv 1\mod q\\ p^k\leq(1-\delta)x^{1/4},\ p\nmid q_2}}\log p\\ \geq&\sum_{\substack{n\equiv 1\mod q_1\\ n\leq(1-\delta)x^{1/4}}}\Lambda(n)-\sum_{k\geq 2}\sum_{\substack{p^k\equiv 1\mod q\\ p^k\leq(1-\delta)x^{1/4}}}\log p-\sum_{\substack{p\equiv 1\mod q\\ p\leq(1-\delta)x^{1/4},\ p\mid q_2}}\log p\\ \gg&\sum_{\substack{n\equiv 1\mod q_1\\ n\leq(1-\delta)x^{1/4}}}\Lambda(n)+O(x^{1/8}\log x)+O((\log q_2)^2)\\ \gg&\frac{x^{1/4}}{q_1{}^{3/2}}+O(x^{1/8}\log x)+O((\log q_2)^2).\end{aligned}$$

Assume $x\geq q_1^L+q_1^7\log^9 q_2$. Then one then sees from the above estimate that

$$\log(p_0p_1\cdots p_m)=\sum_{p\in\mathcal{P}}\log p\gg\frac{x^{1/4}}{q_1{}^{3/2}}+O(x^{1/8}\log x)+O((\log q_2)^2)\gg\log^2 x,$$

which contradicts (43). Hence, $r\geq 1$, which means that $p_0^{k_{p_0}}p_m^{k_{p_m}}\cdots p_r^{k_{p_r}}\geq\delta x$.

Let S be the set of integers of the form $\prod_{i=0}^m p_i^{u_i}$, such that $\prod_{i=0}^m p_i^{u_i}<\delta x$, where $p_i\in\mathcal{P}$ and $u_i\in\{0,k_{p_i}\}$. Let $n_0\in S$ be such that $\Omega(n_0)$ is minimal among all $\Omega(n)$, where $n\in S$. Then $\Omega(n_0)\leq m$, since $r\geq 1$. Then there must be an integer $t\geq 0$ such that $p_t\nmid n_0$. Let t_0 be the minimum of such t's. Then we have

Claim 20. *We have* $p_{t_0}\ll q_1^{L-8}+q_1^{3/2}\log^3 q_2$.

Now we take $\delta=\delta(q_1,q_2)=\left[1+(q_1^{L-8}+q_1^{3/2}\log^3 q_2)^4\right]^{-1}$. One then has

$$p_{t_0}^{k_{p_{t_0}}}\delta\leq p_{t_0}^4\delta\leq 1-\delta. \tag{44}$$

Also, by definition of t_0, $n_0p_{t_0}^{k_{p_{t_0}}}\geq\delta x$; and $n_0<\delta x$. Hence, in conjunction with (44) one concludes that

$$n_0p_{t_0}^{k_{p_{t_0}}}<p_{t_0}^{k_{p_{t_0}}}\delta x\leq(1-\delta)x.$$

Take $n^* = n_0 p_{t_0}^{k_{p_{t_0}}}$. Then we have $\delta x \le n^* \le (1-\delta)x$, $n^* \equiv 1 \mod q_1$ and $(n^*, q_2) = 1$. Moreover, since the Dirichlet coefficient is a multiplicative function, we have

$$a_{n^*}(\pi, \mathrm{Ad}) = \prod_{p \mid n^*} a_{p^{k_p}}(\pi, \mathrm{Ad}) \ge 2^{-\Omega(n^*)} \ge 2^{-\frac{\log x}{\log 2}} = x^{-1}.$$

□

Proof of Claim 20. We may assume $t_0 \ge 1$. By minimality of t_0, we shall have $p_0^{k_{p_0}} \cdots p_{t_0-1}^{k_{p_{t_0-1}}} < p_{t_0}^{k_{p_{t_0}}}$. Therefore, we have, by the range of each k_{p_j}, that

$$\sum_{p \in \mathcal{P},\ p < p_{t_0}} \log p = \sum_{j=0}^{t_0-1} \log p_j < 4 \log p_{t_0}. \tag{45}$$

Let $y \ge q_1^{L-8} + q_1^{3/2} \log^3 q_2$. Then a similar argument to the one used to show $r \ge 1$ leads to

$$\sum_{p \in \mathcal{P},\ p < y} \log p \gg \log^2 y. \tag{46}$$

Therefore, one concludes from (45) and (46) that $p_{t_0} \ll q_1^{L-8} + q_1^{3/2} \log^3 q_2$. □

Corollary 21. *Let notation be as before. Then*

$$S_2(x) \gg x^{-1} \exp(-4(q_1^{L-8} + q_1^{3/2} \log^3 q_2)^{1/4}). \tag{47}$$

Proof. Let $x \in (0,1)$. Then by definition (ref. (34)) we have

$$\omega'(x) = \frac{1}{16}\omega(x) \cdot \left\{x^{-17/16} - (1-x)^{-17/16}\right\}.$$

Hence $\omega(x)$ is increasing in the interval $(0, 1/2]$ and is decreasing in the interval $[1/2, 1)$. Note that $n^* \in [\delta x, (1-\delta)x]$. Therefore, we have

$$\omega(n^*/x) \ge \omega(\delta) = \exp(-\delta^{-1/16} - (1-\delta)^{-1/16}).$$

Recall that $\delta = \delta(q_1, q_2) = \left[1 + (q_1^{L-8} + q_1^{3/2} \log^3 q_2)^4\right]^{-1}$. Therefore, we have

$$\omega(n^*/x) \ge \exp(-4(q_1^{L-8} + q_1^{3/2} \log^3 q_2)^{1/4}). \tag{48}$$

Thus (47) follows from Proposition 19 and (48). □

Proof of Theorem C. Combining Proposition 18 and Corollary 21 we have that

$$x^{A-1} \exp(-4(q_1^{L-8} + q_1^{3/2} \log^3 q_2)^{1/4}) \ll_\epsilon (C_\pi q_1^4)^{2A+1} q_2^{8A+\epsilon}. \tag{49}$$

Thus one can take $A \ge 3$ to be large enough and deduce that

$$x \ll_\epsilon (C_\pi q_1^4)^{2+\epsilon} q_2^{8+\epsilon}.$$

Since we assume $x \ge q_1^L + q_1^7 \log^9 q_2$ in Proposition 19, we then conclude that when $x \ll_\epsilon (C_\pi q_1^4)^{2+\epsilon} q_2^{8+\epsilon} + q_1^L$, there must be some $n \equiv 1 \mod q_1$, $(n, q_2) = 1$ and $n \ll_\epsilon (C_\pi q_1^4)^{2+\epsilon} q_2^{8+\epsilon} + q_1^L$, such that $a_n(\pi, \mathrm{Ad}) \ge 0$. Thus Theorem C follows. □

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