ARCHIMEDEAN ZETA INTEGRALS OF QUATERNION ALGEBRAS

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Abstract. In this note, we will compute explicitly archimedean L-functions associated to irreducible representations of $\mathbb{H}^\times$.

Contents

1. Introduction

Let $\mathbb{H} = B_{-1,-1}/\mathbb{R} = \mathbb{R}+\mathbb{R}\boldsymbol{i}+\mathbb{R}\boldsymbol{j}+\mathbb{R}\boldsymbol{k}$ with $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = \boldsymbol{k}$. We have the embedding $\mathbb{R}_+^\times \to \mathbb{H}^\times$, $t \mapsto (t,0,0,0)$. Let $Z_\mathbb{H}$ be the image of this map, then in fact $Z_\mathbb{H}$ is equal to the center of $\mathbb{H}^\times$. Let $N : \mathbb{H}^\times \to \mathbb{R}_+^\times$ be the norm map given by $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto q\bar{q} = a^2 + b^2 + c^2 + d^2$. Set $\mathbb{H}^1 = \{q \in \mathbb{H}^\times : N(q) = 1\}$. Consider the embedding $\xi : \mathbb{H}^\times \hookrightarrow GL_2(\mathbb{C})$ defined by

$$q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto \begin{pmatrix} a+b\boldsymbol{i} & c+d\boldsymbol{i} \\ -c+d\boldsymbol{i} & a-b\boldsymbol{i} \end{pmatrix}.$$

One then has $\xi(\mathbb{H}^1) = SU(2)$. We will henceforth identify $\mathbb{H}^1$ with $SU(2)$ via this embedding ξ.

For $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathbb{H}$, set $\psi(q) = e^{4\pi a i}$, which is a nontrivial additive character of $\mathbb{H}$. Let $\psi_{q'}(q) = \psi(\sqrt{N(q')}q)$, for any $q' \in \mathbb{H}^\times$ and $q \in \mathbb{H}$. Then $\psi_{q'}$ is also a nontrivial additive character of $\mathbb{H}$. Moreover, any nontrivial additive character on $\mathbb{H}$ is of the form $\psi_{q'}$ for some $q' \in \mathbb{H}^\times$. For any $q' \in \mathbb{H}^\times$, let $|q'| = \sqrt{N(q')}$. Denote by $\mathcal{S}_0(\mathbb{H};\psi_{q'})$ the space

$$\left\{q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto Q(a,b,c,d)e^{-\pi|q'|N(q)} : \ q \in \mathbb{H},\ Q \in \mathbb{C}[x_0,x_1,x_2,x_3]\right\},$$

Clearly $\mathcal{S}_0(\mathbb{H};\psi_{q'})$ is a subspace of $\mathcal{S}(\mathbb{H})$, the space of Schwartz functions on $\mathbb{H}$. Define the Fourier transform with respect to ψ_q, $q \in \mathbb{H}$, on $\mathcal{S}_0(\mathbb{H};\psi_q)$ via

$$\Phi \mapsto \left(q \mapsto \widehat{\Phi}(x) = \int_\mathbb{H} \Phi(y)\psi_q(-xy)dy\right),$$

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where $x \in \mathbb{H}$, $dy = dy_1dy_2dy_3dy_4$ for $y = y_1 + y_2\boldsymbol{i} + y_3\boldsymbol{j} + y_4\boldsymbol{k} \in \mathbb{H}$ is the Lebesgue measure on $\mathbb{R}^4$, which is a Haar measure. It is well defined and one can check that $\mathcal{S}_0(\mathbb{H}; \psi_q)$ is invariant under the Fourier transform with respect to ψ_q.

Let $q \in \mathbb{H}^\times$. Let (σ, V) be a representation of $\mathbb{H}^\times$ and let $s \in \mathbb{C}$. Define the Zeta integral map $Z_{s,\sigma,q}$ from $\mathcal{S}_0(\mathbb{H}; \psi_q)$ to $\mathrm{End}(V)$ by

$$\Phi \mapsto Z_{s,\sigma}(\Phi) := \int_{\mathbb{H}^\times} \Phi(x)\sigma(x)(\det x)^s d^*x \triangleq Z(s, \Phi, \sigma; q).$$

Foe simplicity, we will write $\mathcal{S}_0(\mathbb{H}; \psi)$ for $\mathcal{S}_0(\mathbb{H}; \psi_1)$; and write $Z(s, \Phi, \sigma)$ for the zeta integral $Z(s, \Phi, \sigma; 1)$. Godement and Jacquet showed that (ref. Theorem 8.7 in [1]) for irreducible representation of $\mathbb{H}^\times$, the following holds

(a): There exists some $\beta > 0$ such that for any $\Phi \in \mathcal{S}_0(\mathbb{H}; \psi_q)$, the zeta integral $Z(s, \Phi, \sigma)$ converges normally in $\mathrm{Re}(s) > \beta$.

(b): There is a meromorphic function $L_\infty(s, \sigma)$ such that $Z(s, \Phi, \sigma; q)/L_\infty(s, \sigma)$ has the form $Q(s)|q|^{-sn/2}$, where $Q(s) \in \mathbb{C}[s]$ is a polynomial.

(c): One may choose some constants $c_{i,j}$ and basis e_i of V and some functions $\Phi_k \in \mathcal{S}_0(\mathbb{H}; \psi_q)$ in such a manner that

$$\sum_k \sum_i \sum_j c_{i,j} \cdot \langle Z(s, \Phi_k, \sigma; q)e_i, e_j\rangle = |q|^{-sn/2}.$$

However, Godement and Jacquet did not show the existence of such an Euler factor $L_\infty(s, \sigma)$ in their book. Usually people define the archimedean Euler factor through the local Langlands correspondence. Moreover, it seems that there is no reference available computing $L_\infty(s, \sigma)$ for general σ via zeta integrals. In this note, we shall compute explicitly $L_\infty(s, \sigma)$ to complete Godement and Jacquet's proof. Our main result is the following:

Theorem 1. *Let σ be an n-dimensional irreducible representation of $\mathbb{H}^1$, extended to $\mathbb{H}^\times$ by making $Z_\mathbb{H}$ act trivially. Then $L_\infty(s, \sigma)$ exists in the sense of loc. cit., given explicitly by*

$$L_\infty(s, \sigma) = \pi^{-s-\frac{n-1}{2}}\Gamma\left(s + \frac{n-1}{2}\right). \tag{1}$$

In particular, $L_\infty(s, \sigma)$ depends only on $\dim \sigma$.

Remark. The case where σ is a general representation follows from Theorem 1 and the fact that $L_\infty(s, \sigma \otimes |\cdot|^\lambda) = L_\infty(s + \lambda, \sigma)$.

2. Representations on Homogeneous Polynomials

Let $n \geq 2$ be an integer. Let V_n be the linear complex vector space of dimension n of polynomials of homogeneous degree $n - 1$ in the coordinates z_1 and z_2. Let $V = \bigoplus_{n\geq 2} V_n$. We get a representation of $SU(2)$ on V by defining the action

$$\rho(g)Q(z_1, z_2) = Q(z_1', z_2'),\ Q \in V,\ g \in SU(2),$$

where $(z_1', z_2') = (z_1, z_2)\,{}^tg^{-1}$. Let σ_n be defined by $\sigma_n(g)Q(z_1, z_2) = \rho(g)Q(z_1, z_2)$, $\forall$ $Q \in V_n$, $g \in SU(2)$. We thus obtain a map $SU(2) \longrightarrow GL_n(\mathbb{C})$ given by $g \mapsto \sigma_n(g)$. It is easy to verify that this map is a homomorphism. In fact, any irreducible n-dimensional representation of $SU(2)$ is isomorphism to (σ_n, V_n).

When $n = 2$, the representation (σ_2, V_1) is exactly the standard representation of $SU(2)$ on $\mathbb{C}^2$. We can equip $V_1 \simeq \mathbb{C}^2$ with the inner product $\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \bar{z}_1w_1 + \bar{z}_2w_2$, where $\boldsymbol{z} = (z_1, z_2)$, $\boldsymbol{w} = (w_1, w_2)$. One can check that $\langle\cdot, \cdot\rangle$ is $SU(2)$-invariant.

Let $\mathcal{D} = \{\boldsymbol{z} \in \mathbb{C}^2 : \langle \boldsymbol{z}, \boldsymbol{z} \rangle \leq 1\}$ be the solid three-sphere embedded into $\mathbb{C}^2$. Generally, we can equip V_n with the following inner product:

$$(Q_1, Q_2) = \int_{\mathcal{D}} \overline{Q_1(\boldsymbol{z})} Q_2(\boldsymbol{z}) d\mu(\boldsymbol{z}),$$

where Q_1, $Q_2 \in V_n$ and $d\mu(\boldsymbol{z}) = dz_1 d\bar{z}_1 dz_2 d\bar{z}_2$.

Let $n \geq 2$ be an integer. Denote by $e_j(\boldsymbol{z}) = z_1^j z_2^{n-1-j}$, $1 \leq j \leq n-1$. Then clearly $\{e_j\}_{j=0}^{n-1}$ forms a basis for V_n.

Lemma 2. *Let notation be as above. Then we have*

$$(e_j, e_k) = \frac{4\pi^2 \delta_{j,k}}{n(n+1)} \binom{n-1}{j}^{-1}. \tag{2}$$

Proof. Let $z_j = r_j e^{i\alpha_j}$, $r_j \geq 0$, $0 \leq \alpha_i < 2\pi$, $1 \leq j \leq 2$. Then $dz_j d\bar{z}_j = 2r_j dr_j d\alpha_j$. Thus one has that (e_j, e_k) is equal to

$$4 \iint_{r_i^2 + r_2^2 \leq 1} \int_0^{2\pi} \int_0^{2\pi} r_1^{j+k+1} r_2^{2n-j-k-1} e^{i(k-j)\alpha_1} e^{i(j-k)\alpha_2} d\alpha_1 d\alpha_2 dr_1 dr_2$$

$$= 4\pi^2 \delta_{j,k} \int_0^1 r_2^{n-1-\frac{j+k}{2}} dr_2 \int_0^{1-r_2} r_1^{\frac{j+k}{2}} dr_1 = \frac{4\pi^2 \delta_{j,k}}{j+1} \int_0^1 r_2^{n-1-\frac{j+k}{2}} (1-r_2)^{\frac{j+k}{2}+1} dr_2$$

$$= \frac{4\pi^2 \delta_{j,k}}{j+1} \frac{(n-j-1)!(j+1)!}{(n+1)!} = \frac{4\pi^2 \delta_{j,k}}{n(n+1)} \binom{n-1}{j}^{-1}.$$

□

3. Polarized Haar Measure on $\mathbb{H}^1$

The measure $d\mu(\boldsymbol{z}) = dz_1 d\bar{z}_1 dz_2 d\bar{z}_2$ is invariant under $SU(2)$-transformations. Indeed, under $\boldsymbol{z} \mapsto \rho(g)\boldsymbol{z}$ we have $dz_1 dz_2 \mapsto \det g \cdot dz_1 dz_2 = dz_1 dz_2$. Note that the group $SU(2) = \partial\mathcal{D} \subset \mathbb{C}^2$. Inside $\mathbb{C}^2$ we have spherical coordinates $(r, \theta, \alpha, \beta)$ given by setting $z_1 = r\cos\theta e^{i\alpha}$ and $z_2 = r\sin\theta e^{i\beta}$, where $r \geq 0$, $0 \leq \theta \leq \pi/2$, $0 \leq \alpha, \beta < 2\pi$. Then $SU(2)$ corresponds to the subset where $r = 1$. Then the polynomials $e_k(\boldsymbol{z})$ restrict to the functions $E_k(\theta, \alpha, \beta)$ on $SU(2)$ defined by

$$E_k(\theta, \alpha, \beta) = (\cos\theta)^k (\sin\theta)^{n-1-k} e^{ik\alpha} e^{i(n-1-k)\beta}.$$

Note that Peter-Weyl theorem assets that integrable functions on $SU(2)$ can be approximated by the functions $E_k(\theta, \alpha, \beta)$. Then functions on $SU(2)$ can be extended to those on $\mathcal{D}$ by decomposing a function into its V_n components and then multiplying with the radical coordinate r. For functions on $\mathcal{D}$ we have an $SU(2)$-invariant measure. Hence we obtain an invariant measure on $SU(2)$ by restricting the one on $\mathcal{D}$ to its boundary. Note that an $SU(2)$ transformation only changes the angles in a function and not its r-dependence. Hence when we split off the radical part of the measure $d\mu(\boldsymbol{z}) = rdr\mu(\theta, \alpha, \beta)d\theta d\alpha d\beta$, we have that

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2\pi} f(r, \theta, \alpha, \beta)\mu(\theta, \alpha, \beta) d\alpha d\beta d\theta r^3 dr$$

in invariant under rotating the angles by means of an $SU(2)$-transformation. The $SU(2)$-invariance means

$$\int_{\mathcal{D}} (\rho(g)f)(z_1, z_2) d\mu(\boldsymbol{z}) = \int_{\mathcal{D}} f(z_1, z_2) d\mu(\boldsymbol{z}), \ f \in L^1(\mathcal{D}; \mathbb{C}).$$

Since $SU(2)$ only acts on the radical part of the functions, we have that

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2\pi} f(r, \theta, \alpha, \beta)\mu(\theta, \alpha, \beta) d\alpha d\beta d\theta$$

is invariant under $SU(2)$-transformations for each r. Hence also for $r=1$, which gives back the original function f on $SU(2)$.

Lemma 3. *Let notation be as before. Then we have that*

$$\mu(\theta,\alpha,\beta)d\theta d\alpha d\beta=\sin 2\theta d\theta d\alpha d\beta. \tag{3}$$

Proof. Substituting the polar coordinates of $\boldsymbol{z}$, we then have $d\mu(\boldsymbol{z})=dz_1d\bar{z}_1dz_2d\bar{z}_2=r^3|\det J|d\theta d\alpha d\beta$, where

$$r^3J=\begin{pmatrix}\frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial \theta} & \frac{\partial z_1}{\partial \alpha} & \frac{\partial z_1}{\partial \beta}\\ \frac{\partial \bar{z}_1}{\partial r} & \frac{\partial \bar{z}_1}{\partial \theta} & \frac{\partial \bar{z}_1}{\partial \alpha} & \frac{\partial \bar{z}_1}{\partial \beta}\\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial \theta} & \frac{\partial z_2}{\partial \alpha} & \frac{\partial z_2}{\partial \beta}\\ \frac{\partial \bar{z}_2}{\partial r} & \frac{\partial \bar{z}_2}{\partial \theta} & \frac{\partial \bar{z}_2}{\partial \alpha} & \frac{\partial \bar{z}_2}{\partial \beta}\end{pmatrix}. \tag{4}$$

Then (3) follows from a straightforward computation of the determinant of (4) and the definition: $dz_1d\bar{z}_1dz_2d\bar{z}_2=rdr\mu(\theta,\alpha,\beta)d\theta d\alpha d\beta$. □

Since we have identified $\mathbb{H}^1$ with $SU(2)$ via the embedding ξ, we may pullback the spherical coordinates (θ,α,β) to $\mathbb{H}^1$. We will also use (θ,α,β) to indicate the polarized Haar measure on $\mathbb{H}^1$, via identifying $L^1(\mathbb{H}^1)$ with $L^1(SU(2))$ and

$$\int_{\mathbb{H}^1}f(q)d^*q=\int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}f(\cos\theta e^{-i\alpha}+\sin\theta e^{i\beta}\boldsymbol{j})\sin 2\theta d\theta d\alpha d\beta,$$

where we identify $\boldsymbol{i}$ with the complex number $i=\sqrt{-1}$.

4. Proof of Theorem 1

In this section, we shall show, for an irreducible representation σ of $\mathbb{H}^\times$, the existence of such a meromorphic factor $E(s,\sigma)$ that the function $Z(s,\Phi,\sigma;q)/E(s,\sigma)$ has the form $Q(s)|q|^{-sn/2}$ for any $q\in\mathbb{H}^\times$, where $Q(s)\in\mathbb{C}[s]$ is a polynomial. We will construct such an $E(s,\sigma)$ by explicit computation via the auxiliary results established in Section 2 and Section 3.

Note that $\mathcal{S}_0(\mathbb{H};\psi_q)\xrightarrow{\sim}\mathcal{S}_0(\mathbb{H};\psi)$, $(x\mapsto\Phi(x))\mapsto\left(x\mapsto\Phi_0(x)=\Phi(q^{-1}x)\right)$. Moreover, $Z(s,\Phi,\sigma;q)=N(q)^{-s}\sigma(q)^{-1}Z(s,\Phi_0,\sigma)$. Therefore, we may just consider the zeta integral $Z(s,\Phi,\sigma)$ with Φ running over $\mathcal{S}_0(\mathbb{H};\psi)$. Henceforward we will take $\Phi(x)$ to be of the form $P(x)e^{-\pi N(x)}$, where $P(x)$ is a polynomial with complex coefficients in components of $x\in\mathbb{H}^\times$.

4.1. The case where $\dim\sigma=1$.

If σ is linear, we may assume, up to a suitable shifting, that σ is trivial. Denote by $E(s,1)=\pi^{-s}\Gamma(s)$. Let $x\in\mathbb{H}^\times$, write $x=x_1+x_2\boldsymbol{i}+x_3\boldsymbol{j}+x_4\boldsymbol{k}$, where $x_l\in\mathbb{R}$, $1\le l\le 4$. Let $Q(x)=Q(x_1,x_2,x_3,x_4)\in\mathbb{C}[x_1,x_2,x_3,x_4]$ be a monomial. Set $\Phi_Q(x)=Q(x)e^{-\pi N(x)}\in\mathcal{S}_0(\mathbb{H};\psi)$. Let $r=\sqrt{N(x)}$. Write $x'=r^{-1}x\in\mathbb{H}^1$, and $x'_l=r^{-1}x_l$, $1\le l\le 4$. Then $Q(x)=Q(x_1,x_2,x_3,x_4)=Q(rx'_1,rx'_2,rx'_3,rx'_4)=r^{\deg Q}Q(x'_1,x'_2,x'_3,x'_4)$. Thus we can compute $Z(s,\Phi_Q,1)$ directly as

$$\begin{aligned}\int_{\mathbb{H}^\times}Q(x)e^{-\pi N(x)}(\det x)^sd^*x=&\int_{\mathbb{H}^1}Q(x'_1,x'_2,x'_3,x'_4)d^*x'\int_{\mathbb{R}_+^\times}e^{-\pi r^2}r^{2s+\deg Q}d^\times r^2\\ =&2\int_{\mathbb{H}^1}Q(x'_1,x'_2,x'_3,x'_4)d^*x'\int_0^\infty e^{-\pi r^2}r^{2s+\deg Q-1}dr\\ =&\pi^{-s-\frac{\deg Q}{2}}\Gamma\left(s+\frac{\deg Q}{2}\right)\int_{\mathbb{H}^1}Q(x'_1,x'_2,x'_3,x'_4)d^*x'.\end{aligned}$$

If $\deg Q$ is odd, then $Q(-x'_1,-x'_2,-x'_3,-x'_4)=-Q(x'_1,x'_2,x'_3,x'_4)$. Hence

$$I=\int_{\mathbb{H}^1}Q(x'_1,x'_2,x'_3,x'_4)d^*x'=\int_{\mathbb{H}^1}Q(-x'_1,-x'_2,-x'_3,-x'_4)d^*x'=-I,$$

leading to that $I = 0$. Therefore, $Z(s, \Phi_Q, 1) = 0$ if $\deg Q$ is odd. In particular, $Z(s, \Phi_Q, 1)/E(s, 1) \in \mathbb{C}[s]$.

Assume that $\deg Q$ is even. Then one also has $Z(s, \Phi_Q, 1)/E(s, 1) \in \mathbb{C}[s]$, since

$$\Gamma\left(s + \frac{\deg Q}{2}\right) = \Gamma(s) \prod_{j=0}^{\frac{1}{2}\deg Q} (s + j).$$

Therefore, we always have that $Z(s, \Phi_Q, 1)/E(s, 1) \in \mathbb{C}[s]$, for any monomial $Q \in \mathbb{C}[x_1, x_2, x_3, x_4]$. Since monomials form a basis of $\mathbb{C}[x_1, x_2, x_3, x_4]$, and the zeta integral map is linear, then $Z(s, \Phi, 1)/E(s, 1) \in \mathbb{C}[s]$, for any $\Phi \in \mathcal{S}_0(\mathbb{H}; \psi)$. The above computation also shows that when Q is a suitable constant and $\Phi = \Phi_Q$, then $Z(s, \Phi, 1) = E(s, 1)$.

4.2. **The case where** $\dim \sigma \geq 2$**.** If σ is nonlinear, we may assume it is an n-dimensional irreducible representation of $\mathbb{H}^1$, extended to $\mathbb{H}^\times$ by making $Z_{\mathbb{H}}$ act trivially. Since we have identified $\mathbb{H}^1$ with $SU(2)$, we can think σ as an n-dimensional irreducible representation of $SU(2)$, realized on the space V_n, the linear complex vector space of dimension n of polynomials of homogeneous degree $n-1$ in the coordinates z_1 and z_2. Also, we can take the spherical coordinate system (θ, α, β) on $SU(2)$ to describe the identification $\xi : \mathbb{H}^1 \hookrightarrow SU(2)$ defined by

$$q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix} = \begin{pmatrix} \cos\theta e^{i\alpha} & \sin\theta e^{i\beta} \\ -\sin\theta e^{-i\beta} & \cos\theta e^{-i\alpha} \end{pmatrix}.$$

Then $q = \cos\theta e^{i\alpha} + \sin\theta e^{i\beta}\boldsymbol{j}$, where we identify $\boldsymbol{i}$ with the complex number $i = \sqrt{-1}$. Then under the above identification we have

$$q^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\theta e^{-i\alpha} & -\sin\theta e^{i\beta} \\ \sin\theta e^{-i\beta} & \cos\theta e^{i\alpha} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}, \tag{5}$$

where $z_1' = \cos\theta e^{-i\alpha} z_1 - \sin\theta e^{i\beta} z_2$ and $z_2' = \sin\theta e^{-i\beta} z_1 + \cos\theta e^{i\alpha} z_2$. Recall that for $n \geq 2$ be an integer, $e_j(\boldsymbol{z}) = z_1^j z_2^{n-1-j}$, $1 \leq j \leq n-1$. Then clearly $\{e_j\}_{j=0}^{n-1}$ forms a basis for V_n. We see that $\rho(q)e_k(\boldsymbol{z})$ is equal to

$$(\cos\theta e^{-i\alpha} z_1 - \sin\theta e^{i\beta} z_2)^k (\sin\theta e^{-i\beta} z_1 + \cos\theta e^{i\alpha} z_2)^{n-1-k} = \sum_{l=0}^{n-1} c_l e_l(\boldsymbol{z}),$$

where the coefficients c_l are given by

$$\sum_{m+j=l} \binom{k}{m}\binom{n-1-k}{j} (\cos\theta e^{-i\alpha})^m (\sin\theta e^{-i\beta})^j (-\sin\theta e^{i\beta})^{k-m} (\cos\theta e^{i\alpha})^{n-1-k-j},$$

which, after a straightforward computation, becomes

$$\sum_{m+j=l} (-1)^{k-m} \binom{k}{m}\binom{n-1-k}{j} (\cos\theta)^{\lambda_m} (\sin\theta)^{\lambda_m'} e^{i(n-1-k-l)\alpha} e^{i(k-l)\beta}, \tag{6}$$

where $\lambda_m = n - 1 - k - j + m$ and $\lambda_m' = k - m + j$. Let $x \in \mathbb{H}^\times$, write $x = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}$, where $x_l \in \mathbb{R}$, $1 \leq l \leq 4$. Let $Q(x) = Q(x_1, x_2, x_3, x_4) = x_1^{d_1} x_2^{d_2} x_3^{d_3} x_4^{d_4} \in \mathbb{C}[x_1, x_2, x_3, x_4]$ be a monomial. Let $d = d_1 + d_2 + d_3 + d_4$ be the degree of Q. Set $\Phi_Q(x) = Q(x)e^{-\pi N(x)} \in \mathcal{S}_0(\mathbb{H}; \psi)$. Let $r = \sqrt{N(x)}$. Write $x' = r^{-1}x \in \mathbb{H}^1$, and $x_l' = r^{-1}x_l$, $1 \leq l \leq 4$. Then $Q(x) = Q(x_1, x_2, x_3, x_4) = Q(rx_1', rx_2', rx_3', rx_4') = r^d Q(x_1', x_2', x_3', x_4')$. Since $Z(s, \Phi_Q, \sigma) \in \mathrm{End}(V_n)$, we may consider its matrix representation and compute $Z(s, \Phi_Q, \sigma)$ by computing each matrix coefficient $(Z(s,\Phi_Q,\sigma)e_k(\boldsymbol{z}),e_l(\boldsymbol{z}))$, which is equal to

$$\begin{aligned}&\int_{\mathbb{H}^\times}(\sigma(q)e_k(\boldsymbol{z}),e_l(\boldsymbol{z}))Q(x)e^{-\pi N(x)}(\det x)^s d^*x\\ =&\sum_{j=0}^{n-1}\int_{\mathbb{H}^1}c_j\cdot(e_j(\boldsymbol{z}),e_l(\boldsymbol{z}))Q(x_1',x_2',x_3',x_4')d^*x'\int_{\mathbb{R}_+^\times}e^{-\pi r^2}r^{2s+d}d^\times r^2\\ =&\frac{8\pi^2}{n(n+1)}\binom{n-1}{l}^{-1}\int_{\mathbb{H}^1}c_l(x')Q(x_1',x_2',x_3',x_4')d^*x'\int_0^\infty e^{-\pi r^2}r^{2s+d-1}dr\\ =&\frac{8\pi^{-s-\frac{d}{2}+2}}{n(n+1)}\binom{n-1}{l}^{-1}\Gamma\left(s+\frac{d}{2}\right)\int_{\mathbb{H}^1}c_l(x')Q(x_1',x_2',x_3',x_4')d^*x'.\end{aligned}$$

Now we can apply spherical coordinates to the above integral, by writing

$$Q(x_1',x_2',x_3',x_4')=(\cos\theta e^{i\alpha})^{d_1}(\sin\theta e^{i\beta})^{d_2}(\sin\theta e^{-i\beta})^{d_3}(\cos\theta e^{-i\alpha})^{d_4},$$

which is equal to $(\cos\theta)^{d_1+d_4}(\sin\theta)^{d_2+d_3}e^{-i(d_4-d_1)\alpha}e^{-i(d_3-d_2)\beta}$. Combine this with (6) we see that $\int_{\mathbb{H}^1}c_j(x')Q(x_1',x_2',x_3',x_4')d^*x'$ is equal to the sum over $m+j=l$ of $(-1)^{k-m}\binom{k}{m}\binom{n-1-k}{j}$ multiplying the integral

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\pi}(\cos\theta)^{\gamma_m}(\sin\theta)^{\gamma_m'}e^{i(n-1-k-l-d_4+d_1)\alpha}e^{i(k-l-d_3+d_2)\beta}\sin 2\theta d\theta d\alpha d\beta,$$

where $\gamma_m=\lambda_m+d_1+d_4=n-1-k-j+m+d_1+d_4$ and $\gamma_m'=\lambda_m'+d_2+d_3=k-m+j+d_2+d_3$. Denote the above integral by $J_{k,l}(m)$. Then $J_{k,l}(m)=0$ otherwise

$$\begin{cases}d_4-d_1=n-1-k-l,\\ d_3-d_2=l-k.\end{cases}\tag{7}$$

Substitute the relations (7) into the integral with respect to θ to see that it equals

$$\begin{aligned}&\sum_{m+j=l}(-1)^{k-m}\binom{k}{m}\binom{n-1-k}{j}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\gamma_m}(\sin\theta)^{\gamma_m'}\sin 2\theta d\theta\\ =&\sum_{m+j=l}(-1)^{k-m}\binom{k}{m}\binom{n-1-k}{j}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2m+2d_4}(\sin\theta)^{2j+2d_2}\sin 2\theta d\theta\\ =&\sum_{m+j=l}(-1)^{k-m}\binom{k}{m}\binom{n-1-k}{j}\int_0^{\frac{\pi}{2}}\left(1-(\sin\theta)^2\right)^{m+d_4}(\sin\theta)^{2j+2d_2}d(\sin\theta)^2\\ =&\sum_{m+j=l}(-1)^{k-m}\binom{k}{m}\binom{n-1-k}{j}\cdot\frac{(m+d_4)!(j+d_2)!}{(l+d_2+d_4+1)!}\\ =&\frac{1}{l+d_2+d_4+1}\sum_{m=0}^{l}(-1)^{k-m}\binom{k}{m}\binom{n-1-k}{l-m}\cdot\binom{l+d_2+d_4}{m+d_4}^{-1},\end{aligned}$$

where $d_2\geq\max\{k-l,0\}$ and $d_4\geq\max\{n-1-k-l,0\}$. Denote by $J_{k,l}(Q)$ the last expression. If $k=0$, we have by (7) that $d_3+d_4-d_1-d_2=n-1$. Hence $d\geq n-1$, where the identity holds if and only if $d_1=d_2=0$ and $d_3+d_4=n-1$. If $l=0$, we have by (7) that $d_2+d_4-d_1-d_3=n-1$. Hence $d\geq n-1$, where the identity holds if and only if $d_1=d_3=0$ and $d_2+d_4=n-1$. Otherwise,we have

Lemma 4. *Let notation be as before. Let $k\geq1$, $l\geq1$, $d_2\geq\max\{k-l,0\}$ and $d_4\geq\max\{n-1-k-l,0\}$. Then $J_{k,l}(Q)$ is vanishing unless $d_2+d_4\geq n-1-l$.*

Proof. It could be checked by straightforward computation that Lemma 4 holds for $n \le 5$. To do the general case, we first show that $J_{k,l}(Q) = 0$ if $d_2 + d_4 = n - 2 - l$ by induction on k and the identity

$$\binom{m_1}{m_2} = \binom{m_1 - 1}{m_2} + \binom{m_1 - 1}{m_2 - 1}. \tag{8}$$

The remaining cases come from induction on n and again the identity (8). □

When $d_2 + d_4 \ge n - 1 - l$, we have by (7) that $d = 2d_2 + 2d_4 + 2l + 1 - n \ge n - 1$. Note that one has, by the previous computation, that

$$(Z(s, \Phi_Q, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z})) = \frac{32\pi^4}{n(n+1)} \binom{n-1}{l}^{-1} J_{k,l}(Q) \cdot E_Q(s, \sigma), \tag{9}$$

where $E_Q(s, \sigma) = \pi^{-s-\frac{d}{2}} \Gamma\left(s + \frac{d}{2}\right)$. Then one always has that

$$(Z(s, \Phi_Q, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z})) = 0, \;\; d = \deg Q < n - 1.$$

Let $d \ge n-1$. Note that, by (7), $d = 2(d_2 + d_4) + 2l + 1 - n$. Then $\Gamma(s + d/2) = \Gamma(s + t + (n-1)/2) \in \Gamma(s + (n-1)/2)\mathbb{C}[s]$, where $t = d_2 + d_4 + l + 1 - n \ge 0$. Hence if we denote by $E(s, \sigma) = \pi^{-s-(n-1)/2}\Gamma(s + (n-1)/2)$, which is always nonvanishing, then $(Z(s, \Phi, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z}))/E(s, \sigma) \in \mathbb{C}[s]$, for any $\Phi \in \mathcal{S}_0(\mathbb{H}; \psi)$. Let $(d_1', d_2', d_3', d_4') \in \mathbb{N}_{\ge 0}^4$ satisfy the following system of equations

$$\begin{cases} d_4 - d_1 = n - 1 - k - l \\ d_3 - d_2 = l - k \\ d_2 + d_4 = n - 1 - l \end{cases}$$

and $Q^\circ(x_1', x_2', x_3', x_4') = (\cos\theta e^{i\alpha})^{d_1'}(\sin\theta e^{i\beta})^{d_2'}(\sin\theta e^{-i\beta})^{d_3'}(\cos\theta e^{-i\alpha})^{d_4'}$. Then by the above discussion, there exists some constant $c = c(n; k, l)$ such that if we take $Q(x) = Q^\circ(x)$ and $\Phi = \Phi_Q$, then $(Z(s, \Phi, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z})) = E(s, \sigma)$.

Remark. Such a quadruple $(d_1', d_2', d_3', d_4') \in \mathbb{N}_{\ge 0}^4$ alway exists. For example, one can take $(d_1', d_2', d_3', d_4') = (\min\{k, l\}, k - \min\{k, l\}, l - \min\{k, l\}, n - 1 - k - l + \min\{k, l\})$.

4.3. $L_\infty(s, \sigma)$ **as a g.c.d.** For any $\Phi \in \mathcal{S}_0(\mathbb{H}; \psi)$, denote by $\Phi'(x) = \frac{d}{dt}\Phi(xe^{-t}) \mid_{t=0}$, $\forall\, x \in \mathbb{H}^\times$. Then clearly $\Phi' \in \mathcal{S}_0(\mathbb{H}; \psi)$. Now when $\mathrm{Re}(s)$ is large enough,

$$Z(s, \Phi, \sigma) = \int_{\mathbb{H}^\times} \Phi(x)\sigma(x)(\det x)^s d^*x = \int_{\mathbb{H}^\times} \Phi(xe^{-t})\sigma(x)(\det xe^{-t})^s d^*x.$$

Taking the derivative of this relation for $t = 0$ we get a relation of the type

$$a(s + b)Z(s, \Phi, \sigma) + Z(s, \Phi', \sigma) = 0,$$

where a and b are constants and $a \ne 0$. This shows that the sub-vector space of $\mathbb{C}[x]$ spanned by the polynomials $\{(Z(s, \Phi, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z}))/E(s, \sigma) : \; \Phi \in \mathcal{S}_0(\mathbb{H}; \psi)\}$ is in fact an ideal for any $0 \le k, l \le n - 1$. Since we have shown that one can take suitable $\Phi \in \mathcal{S}_0(\mathbb{H}; \psi)$ to make $(Z(s, \Phi, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z}))/E(s, \sigma) = 1$, this ideal is trivial. Therefore, we have, for any $0 \le k, l \le n - 1$, that

$$\left\{(Z(s, \Phi, \sigma)e_k(\boldsymbol{z}), e_l(\boldsymbol{z})) : \; \Phi \in \mathcal{S}_0(\mathbb{H}; \psi)\right\} = E(s, \sigma)\mathbb{C}[s].$$

Now we take $L_\infty(s, \sigma) = E(s, \sigma)$ and Theorem 1 follows.

5. Some Explicit Examples

In this section, we shall give some explicit descriptions of $Z(s,\Phi,\sigma)$ in the case $\dim\sigma \le 4$ by computing its matrix representation. Since we have done the case $\dim\sigma = 1$ in Section (4.1), we will assume $2 \le n \le 3$ henceforward.

Case 1: If $\dim\sigma = 2$. Then σ is the standard representation. So naturally it has the matrix representation

$$(\sigma, V_2) \longrightarrow GL_2(\mathbb{C}),\ \sigma(q) = \begin{pmatrix} \cos\theta e^{i\alpha} & \sin\theta e^{i\beta} \\ -\sin\theta e^{-i\beta} & \cos\theta e^{-i\alpha} \end{pmatrix},$$

where $q = \cos\theta e^{i\alpha} + \sin\theta e^{i\beta}\boldsymbol{j}$, where we identify $\boldsymbol{i}$ with the complex number $i = \sqrt{-1}$. Then $Z(s,\Phi,\sigma)$ is equal to

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\pi} \begin{pmatrix} \cos\theta e^{i\alpha} & \sin\theta e^{i\beta} \\ -\sin\theta e^{-i\beta} & \cos\theta e^{-i\alpha} \end{pmatrix} \Phi(r,\theta,\alpha,\beta)\sin 2\theta d\theta d\alpha d\beta \int_0^\infty e^{-\pi r^2} r^{2s-1} dr.$$

Case 2: If $\dim\sigma = 3$. Let $q = \cos\theta e^{i\alpha} + \sin\theta e^{i\beta}\boldsymbol{j}$. Then by (5) we have

$$\begin{aligned}
\sigma(q)z_1^2 &= (\cos\theta e^{-i\alpha}z_1 - \sin\theta e^{i\beta}z_2)^2 \\
&= (\cos\theta)^2 e^{-2i\alpha}z_1^2 - 2\cos\theta\sin\theta e^{-i\alpha}e^{i\beta}z_1z_2 + (\sin\theta)^2 e^{2i\beta}z_2^2; \\
\sigma(q)z_2^2 &= (\sin\theta e^{-i\beta}z_1 + \cos\theta e^{i\alpha}z_2)^2 \\
&= (\sin\theta)^2 e^{-2i\beta}z_1^2 - 2\cos\theta\sin\theta e^{i\alpha}e^{-i\beta}z_1z_2 + (\cos\theta)^2 e^{2i\alpha}z_2^2; \\
\sigma(q)z_1z_2 &= (\cos\theta e^{-i\alpha}z_1 - \sin\theta e^{i\beta}z_2)(\sin\theta e^{-i\beta}z_1 + \cos\theta e^{i\alpha}z_2) \\
&= \cos\theta\sin\theta e^{-i\alpha}e^{-i\beta}z_1^2 - \cos 2\theta z_1z_2 - \cos\theta\sin\theta e^{i\alpha}e^{i\beta}z_2^2;
\end{aligned}$$

Therefore, the matrix representation of σ is

$$\sigma(q) = \begin{pmatrix} (\cos\theta)^2 e^{-2i\alpha} & -\sin 2\theta e^{-i\alpha}e^{i\beta} & (\sin\theta)^2 e^{2i\beta} \\ \cos\theta\sin\theta e^{-i\alpha}e^{-i\beta} & -\cos 2\theta & -\cos\theta\sin\theta e^{i\alpha}e^{i\beta} \\ (\sin\theta)^2 e^{-2i\beta} & -\sin 2\theta e^{i\alpha}e^{-i\beta} & (\cos\theta)^2 e^{2i\alpha} \end{pmatrix}.$$

Then $Z(s,\Phi_Q,\sigma)$ factorizes as a product of $c\pi^{-s-d/2}\Gamma(s+d/2)$ and

$$\iiint \begin{pmatrix} (\cos\theta)^2 e^{-2i\alpha} & -\sin 2\theta e^{-i\alpha}e^{i\beta} & (\sin\theta)^2 e^{2i\beta} \\ \cos\theta\sin\theta e^{-i\alpha-i\beta} & -\cos 2\theta & -\cos\theta\sin\theta e^{i\alpha+i\beta} \\ (\sin\theta)^2 e^{-2i\beta} & -\sin 2\theta e^{i\alpha}e^{-i\beta} & (\cos\theta)^2 e^{2i\alpha} \end{pmatrix} Q\sin 2\theta d\theta d\alpha d\beta,$$

where $Q = Q(\theta,\alpha,\beta) = (\cos\theta e^{i\alpha})^{d_1}(\sin\theta e^{i\beta})^{d_2}(\sin\theta e^{-i\beta})^{d_3}(\cos\theta e^{-i\alpha})^{d_4}$, $d_i \ge 0, 1 \le i \le 4$ and $d = d_1+d_2+d_3+d_4$. One can see clearly that whenever $d < 2$ the above triple integral vanishes. Hence to make $Z(s,\Phi_Q,\sigma) \ne 0$, then $\deg Q = d \ge 2$. Such a monomial Q can be chosen explicitly (ref. Section 4.2).

References

[1] R. Godement, H. Jacquet. *Zeta functions of simple algebras.*[M]. Springer, 2006.

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