Journal of Number Theory 170 (2017) 347–367

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Diophantine inequalities involving a prime and an almost-prime

Liyang Yang

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, PR China

ARTICLE INFO

Article history:
Received 24 August 2015
Received in revised form 24 January 2016
Accepted 13 May 2016
Available online 24 August 2016
Communicated by R.C. Vaughan

Keywords:
Diophantine inequalities
Sieve method
Almost-primes

ABSTRACT

We prove that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{131}},$$

where λ_0 is an arbitrary real number and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 \neq 0$ and $0 > \frac{\lambda_1}{\lambda_2}$ not in $\mathbb{Q}$. This improves a result by Harman.

1. Introduction

In Diophantine Approximation, a classical theorem of Kronecker ([2], Theorem 440) indicates that there are infinitely many solutions in positive integers n_1, n_2 of

$$|\lambda_0 + \lambda_1 n_1 + \lambda_2 n_2| < 3 \left(\max \left\{ \frac{n_1}{\lambda_2}, \frac{n_2}{\lambda_1} \right\} \right)^{-1},$$

where $\frac{\lambda_1}{\lambda_2}$ is irrational and λ_0 is an arbitrary real number.

E-mail address: yly12@mails.tsinghua.edu.cn.

http://dx.doi.org/10.1016/j.jnt.2016.05.008
0022-314X/© 2016 Published by Elsevier Inc.

The case where n_1 and n_2 are both primes is of great interest and remains open to date ([6,7]). The first approximation in this direction has been given by Vaughan [8] who proved that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_4| < p^{-1/600000},$$

where and henceforth in this paper the letter p denotes a prime and P_r a number with at most r prime factors. Harman [4] proved that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\tau}, \tag{1}$$

with $\tau = \frac{1}{300}$.

In this paper, we will improve Harman's result by showing that in (1) one can actually take $\tau = \frac{1}{131}$. The main result of this paper will be the following theorem.

Theorem 1. *For $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ with $\frac{\lambda_1}{\lambda_2}$ both negative and irrational, there are infinitely many solutions of*

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{131}}.$$

2. Notation and outline of the method

2.1. Notation

We shall use η and ε for arbitrary small positive numbers; and sometimes they may be slightly different in context just for simplicity.

We write $\lfloor x \rfloor$ for the largest integer not exceeding x. We write $\|x\|$ for the distance from x to a nearest integer and $\lceil x \rfloor$ for the nearest integer to x when $\|x\| \neq \frac{1}{2}$. Clearly we may assume that $\lambda_1 > 0$ and $\lambda_2 = -1$. Let $\frac{a}{q}$ be a convergence to the continued fraction for λ_1 and assume q to be quite large in terms of λ_0, λ_1 and λ_1^{-1}; let X be a large number such that $q \asymp X^{\frac{1}{3}+\rho+\eta}$. Trivially, one can write $\lambda_0 = \frac{b}{q} + \gamma$ with $|\gamma| < \frac{1}{q}$.

As in [4], we assume that q is so large that $\min\{\frac{a}{q}, \frac{q}{a}\} > X^{-\frac{\rho}{4}}$ and $aX + b < qX^{1+\frac{\eta}{4}}$. In this paper, p, $p_i, i = 1, 2, \ldots$ represent primes; $\sum^\flat$ indicates that the summation is only over square-free numbers. For convenience, we shall denote

$$e(x) := \exp(2\pi i x), \qquad \xi := X^{-\rho}, \quad \text{where } \rho \text{ is a positive number;}$$
$$P(z) := \prod_{p \le z} p, \qquad Y := \lfloor 3\xi^{-1} X^{\eta} \rfloor.$$

2.2. The weighted sieve

Essentially, if we use the same method as in [4] but with a parameterized weight to optimize the result, we will obtain that $\tau = \frac{1}{147}$ is admissible as mentioned in Section 6.

However, we will show in Theorem 11 that some terms in the resulting sums can be estimated more efficiently by using a 2-dimensional sieve, rather than using the linear sieve only. The 2-dimensional sieve helps us sieve primes in a much larger range, which will give a better result. We then work out the restrictions of those parameters both from main terms and error terms explicitly, and then figure out the optimal results from them, which can be done by *Mathematica 9*.

As it points out in [4] it suffices to show that the number of solutions of

$$\left|\frac{b}{q}+\frac{pa}{q}-P_3\right|<\frac{X^{-\rho}}{2}$$

tends to infinity with X. Here $p<X$, $P_3<\frac{aX+b}{q}$. Hence, we will work with the set

$$\mathcal{A}:=\left\{\left\lceil\frac{b+pa}{q}\right\rceil : p\leqslant X, \left\|\frac{b+pa}{q}\right\|<\frac{\xi}{2}\right\}.$$

Here we list all notation used in the sieve method:

$$\mathcal{H}_r:=\{n\in\mathcal{H}: r\mid n\},\quad \text{for any finite set of positive integers } \mathcal{H};$$

$$\mathfrak{N}(\beta):=\Bigg\{p_1p_2p_3p_4: X^{\beta}\leqslant p_1<2X^{\beta}, p_1\leqslant p_2\leqslant\left(\frac{aX+b}{qp_1}\right)^{\frac{1}{3}},$$

$$p_2\leqslant p_3\leqslant\left(\frac{aX+b}{qp_1p_2}\right)^{\frac{1}{2}}, X^{\frac{\alpha}{4}}\leqslant p_4\leqslant\frac{aX+b}{qp_1p_2p_3}\Bigg\};$$

$$\mathcal{A}(\beta)^*:=\left\{n: n\leqslant X, \left\|\frac{b+pa}{q}\right\|<\frac{\xi}{2}, \left\lceil\frac{b+pa}{q}\right\rceil\in\mathfrak{N}(\beta)\right\};$$

$$\mathfrak{P}_r:=\{n\in\mathbb{N}: n \text{ has at most } r \text{ prime divisors}\};$$

$$R_d:=\#\mathcal{A}_d-\frac{\pi(X)\xi}{d};$$

$$\mathcal{S}:=\sum_{n\in\mathcal{A}\cap\mathfrak{P}_3}1;\qquad w_p:=1-\frac{u\log p}{\log X};$$

$$\mathcal{W}(\mathcal{A},u,\lambda):=\sideset{}{^\flat}\sum_{\substack{s\in\mathcal{A}\\ \left(s,P\left(X^{\frac{\alpha}{4}}\right)\right)=1}}\left(1-\lambda\sum_{\substack{X^{\frac{\alpha}{4}}\leqslant p\leqslant X^{\frac{1}{u}}\\ p\mid s}}w_p\right)+\sum_{p\geqslant X^{\frac{\alpha}{4}}}\sum_{\substack{h\in\mathcal{A}\\ p^2\mid h}}1;$$

$$\mathcal{S}(\mathcal{A}(\beta)^*,z):=\sum_{\beta}\sum_{\substack{n\in\mathcal{A}(\beta)^*\\ (n,P(z))=1}}1;$$

where $0<\alpha\leqslant 4\beta\leqslant 1$, both are undetermined parameters.

Assume that $0 < u < \frac{4}{\alpha}$ and we define

$$\mathcal{J}(\lambda) := \mathcal{W}(\mathcal{A}, u, \lambda) - \lambda \mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2}-\eta}).$$

For simplicity, we shall denote $z := X^{\frac{\alpha}{4}}$, $y := X^{\frac{1}{u}}$, where $u > \alpha^{-1}$.

Lemma 2. *We have*

$$\mathcal{S} \geq \mathcal{J}(\lambda) \quad \textit{if and only if} \quad \lambda^{-1} < 5 - u. \tag{2}$$

Proof. To achieve that, we need some crucial results from [4] and [3].

Notice that

$$\mathcal{S} = \sideset{}{^\flat}\sum_{\substack{s \in \mathcal{A} \cap \mathfrak{P}_3 \\ \left(s, P\left(X^{\frac{\alpha}{4}}\right)\right) = 1}} 1 + O\left(X^{1-\frac{\alpha}{4}}\right),$$

thus we only need the following inequality:

$$\sideset{}{^\flat}\sum_{\substack{s \in \mathcal{A} \setminus \mathfrak{P}_3 \\ \left(s, P\left(X^{\frac{\alpha}{4}}\right)\right) = 1}} 1 \leqslant \lambda \sideset{}{^\flat}\sum_{\substack{s \in \mathcal{A} \\ \left(s, P\left(X^{\frac{\alpha}{4}}\right)\right) = 1}} \sum_{X^{\frac{\alpha}{4}} \leqslant p \leqslant X^{\frac{1}{u}}} \left(1 - \frac{u \log p}{\log X}\right)$$
$$+ \lambda \sum_{\beta} \sum_{\substack{n \in \mathcal{A}(\beta)^* \\ \left(n, P\left(X^{\frac{1}{2}-\eta}\right)\right) = 1}} 1 + O\left(X^{1-\frac{\alpha}{4}}\right),$$

with the assumption that $0 < \rho < \frac{\alpha}{4}$. To this end, we divide it into two cases:

Case 1: $s \in \mathcal{A} \setminus \mathfrak{P}_4$, so that s has at least 5 prime factors. Set $s = \prod_{i=1}^{m} p_i$, where $X^{\frac{\alpha}{4}} \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_m$, $m \geqslant 5$. We may assume $p_k \leqslant X^{\frac{1}{u}} < p_{k+1}$, then by assumption we have $X^{1+\eta} \gg X^{\frac{m-k}{u}} \prod_{i=1}^{k} p_i$. So we only need that

$$\lambda^{-1} < k - \frac{u \log X^{1+\eta-\frac{m-k}{u}}}{\log X} = m - u - u\eta,$$

i.e. we should have

$$\lambda^{-1} < 5 - u - u\eta < 5 - u.$$

Case 2: $s \in \mathcal{A} \cap \mathfrak{P}_4$, set $s = \prod_{i=1}^{4} p_i$, where $X^{\frac{\alpha}{4}} \leqslant p_1 \leqslant p_2 \leqslant p_3 \leqslant p_4$. As above, we may assume $p_k \leqslant X^{\frac{1}{u}} < p_{k+1}$, then by assumption we have $X^{1+\eta} \gg X^{\frac{4-k}{u}} \prod_{i=1}^{k} p_i$. So we only need that

$$\lambda^{-1} < k - \frac{u \log X^{1+\eta-\frac{4-k}{u}}}{\log X} = 5 - u - u\eta < 5 - u,$$

henceforth we shall assume that

$$\lambda^{-1} < 5 - u.$$

Conversely, were the assumption true, the inequality above would hold since we can take η to be arbitrary small and thus $\mathcal{J}(\lambda)$ would be a well-defined weighted sieve for this problem, that is, $\mathcal{S} \gg \mathcal{J}(\lambda)$. □

Therefore, we have

Corollary 3. *For $\lambda^{-1} < 5 - u$, if*

$$\mathcal{J}(\lambda) := \mathcal{W}(\mathcal{A}, u, \lambda) - \lambda \mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2}-\eta}) \gg \frac{\pi(X)\xi}{\log X},$$

then Theorem 1 holds with $\tau = \rho$.

In the following sections, we will prove that $\mathcal{J}(\lambda) \gg \frac{\pi(X)\xi}{\log X}$ and we can take $\rho = \frac{1}{131}$.

3. Some auxiliary lemmas

Lemma 4. *For any $x \geq 2$, we have*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right);$$

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + c + O\left(\frac{1}{\log x}\right),$$

where c is an absolute constant.

Remark. These two estimates are usually called Mertens formulas.

Lemma 5 *([5]). Let $\delta_0 < \frac{1}{2}$ be a small positive constant and $\chi(t)$ be the characteristic function of interval $(-\delta_0, \delta_0)$ extended to be periodic with period 1, then there exist $A(t)$, $B(t)$ such that*

$$A(t) \leq \chi(t) \leq B(t),$$

where $A(t)$, $B(t)$ can be written as

$$A(t) := 2\delta_0 - (N+1)^{-1} + \sum_{1 \le |n| \le N} A_n e(nt),$$

$$B(t) := 2\delta_0 + (N+1)^{-1} + \sum_{1 \le |n| \le N} B_n e(nt),$$

with coefficients A_n, B_n *satisfying* $\max\{|A_n|, |B_n|\} \ll \delta_0$, *for* $1 \le |n| \le N$.

4. Estimates for exponential sums

Our main goal in this section is to prove that

$$\sum_{d \leqslant X^\alpha} \frac{\xi}{d} \max_{N \leqslant X} \sum_{l=1}^{dY} \left| \sum_{n \leqslant N} \Lambda(n) e\left(\frac{anl}{dq}\right) \right| \ll \xi \pi(X) X^{-\eta} \tag{3}$$

with α as large as possible.

However, the lemmas in [4] can only give the result without taking *max* between the two sums. We should point out that with some slight modifications of the proof in [4] we will be able to prove (3).

This is a generalization of [4], Lemma 3:

Lemma 6. *Suppose* $X, M \geqslant 1$, $\delta > 0$, $\mathcal{M}$ *a set of* $\leqslant T$ *integer points* (l, m) *with* $M \leqslant m < 2M$, λ_{lm} *real numbers for* $(l, m) \in \mathcal{M}$, *and* $\{a_n\}$ *a sequence of complex numbers, then*

$$\sum_{(l,m) \in \mathcal{M}} \max_{N \leqslant X} \left| \sum_{mn \leqslant N} a_n e(\lambda_{lm} n) \right|^2 \ll D_\delta \log^3(2TX) \left(\frac{X}{M} + \delta^{-1} \right) \sum_{n \leqslant X/M} |a_n|^2,$$

where

$$D_\delta = \max_{(l,m) \in \mathcal{M}} \# \left\{ (l', m') \in \mathcal{M} : \| \lambda_{lm} - \lambda_{l'm'} \| < \delta \right\}.$$

Proof. Define

$$\delta(\beta) := \begin{cases} 1, & \text{if } 0 \leqslant \beta \leqslant \gamma, \\ 0, & \text{otherwise}, \end{cases}$$

which is a truncation function. Then we have

$$\delta(\beta) = \int_{-A}^{A} e^{i\beta t} \frac{\sin \gamma t}{\pi t} \mathrm{d}t + O\left(\frac{1}{A|\gamma - \beta|} \right)$$

as in the proof of Lemma 2 of [10]. Here we take $A = 2TX$, $\gamma_{lm} = \log\left(N_{lm} + \frac{1}{2}\right)$, for $(l, m) \in \mathcal{M}$, where

$$N_{lm} = \max\left\{ n_0 \in X : \max_{N \leqslant X} \left| \sum_{mn \leqslant N} a_n e(\lambda_{lm} n) \right|^2 = \left| \sum_{mn \leqslant n_0} a_n e(\lambda_{lm} n) \right|^2 \right\}.$$

Then we have

$$\begin{aligned}
&\sum_{(l,m)\in\mathcal{M}} \max_{N\leqslant X} \left| \sum_{mn\leqslant N} a_n e(\lambda_{lm} n) \right|^2 = \sum_{(l,m)\in\mathcal{M}} \left| \sum_{mn\leqslant N_{lm}} a_n e(\lambda_{lm} n) \right|^2 \\
&\leq \sum_{(l,m)\in\mathcal{M}} \left(\int_{-A}^{A} \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right| \cdot \left| \frac{\sin \gamma_{lm} t}{\pi t} \right| \mathrm{d}t \right)^2 \\
&\qquad + O\left(\sum_{(l,m)\in\mathcal{M}} \left(\sum_n |a_n| \frac{1}{A \log \frac{N_{lm}+1/2}{mn}} \right)^2 \right) \\
&\ll \sum_{(l,m)\in\mathcal{M}} \int_{-A}^{A} \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right|^2 \cdot \min\{\gamma_{lm}, \frac{1}{|t|}\} \mathrm{d}t \cdot \log A \\
&\qquad + O\left(\sum_{(l,m)\in\mathcal{M}} \left(\sum_n |a_n| \frac{1}{A \log \frac{N_{lm}+1/2}{N_{lm}}} \right)^2 \right) \\
&\ll \log A \cdot \int_{-A}^{A} \left(\sum_{(l,m)\in\mathcal{M}} \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right|^2 \right) \cdot \min\{\log X, \frac{1}{|t|}\} \mathrm{d}t \\
&\qquad + O\left(\sum_{(l,m)\in\mathcal{M}} \left(\sum_n |a_n| \frac{1}{A \log \frac{X+1/2}{X}} \right)^2 \right) \\
&\ll \log A \cdot \left(\sum_{(l,m)\in\mathcal{M}} \left| \sum_n a_n n^{it} e(\lambda_{lm} n) \right|^2 \right) \cdot \int_{-A}^{A} \min\{\log X, \frac{1}{|t|}\} \mathrm{d}t \\
&\qquad + O\left(\sum_{(l,m)\in\mathcal{M}} \left(\sum_n |a_n| \frac{1}{A \log \frac{X+1/2}{X}} \right)^2 \right) \\
&\ll D_\delta \log^2(2TX) \left(\frac{X}{M} + \delta^{-1} \right) \left(\int_0^1 \log X \mathrm{d}t + \int_1^A \frac{1}{t} \mathrm{d}t \right) \sum_n |a_n|^2
\end{aligned}$$

$$\ll D_\delta \log^3(2TX)\left(\frac{X}{M}+\delta^{-1}\right)\sum_n |a_n|^2,$$

where the last step comes from [4], Lemma 3. □

This is a generalization of [4], Lemma 5:

Lemma 7. *Suppose $\varepsilon > 0$, $X > R$, $J, M \geqslant 1$, $1 < q \leqslant X^{\frac{3}{4}}$, $\log|a| \ll \log X$, $(a,q) = 1$, then*

$$\sum_{r\sim R}\max_{N\leqslant X}\sum_{j\sim J}\sum_{m\sim M}\left|\sum_{mn\leqslant N} e\left(\frac{ajmn}{rq}\right)\right| \ll X^{\varepsilon}\left(\frac{JX}{q}+RJM+qR^2\right).$$

Proof. By lemma 3 of [9] we obtain

$$\sum_{r\sim R}\max_{N\leqslant X}\sum_{j\sim J}\sum_{m\sim M}\left|\sum_{mn\leqslant N} e\left(\frac{ajmn}{rq}\right)\right| \\ \ll \log X (JM)^{\frac{\varepsilon}{3}}\sum_{r\sim R}\left(\frac{JX\cdot(r,a)}{rq}+JM+qR\right).$$

Hence, it follows from the same estimates in lemma 5 of [4]. □

This is a generalization of [4], Lemma 7:

Lemma 8. *Suppose that $\varepsilon > 0$, $X \geqslant R$, $L, M \geqslant 1$, $1 < q \leqslant X$, $(a,q) = 1$ and $a \asymp q$, $\max\left\{\frac{LM}{qR}, \frac{qM}{X}\right\} < 1$, $a_n, b_m \ll X^{\varepsilon}$. Then*

$$\sum_{r\sim R}\max_{N\leqslant X}\sum_{l\sim L}\left|\sum_{m\sim M} b_m \sum_{mn\leqslant N} a_n e\left(\frac{lmna}{qr}\right)\right| \\ \ll X^{1+3\varepsilon} R\left(L+\frac{R}{M}\right)\left(\frac{M}{X}+\frac{1}{MRL+R^2}\right)^{\frac{1}{2}}.$$

Proof. The proof is essentially the same as that of lemma 7 of [4], with lemma 3 of [4] replaced by Lemma 6 above. □

This is a generalization of [4], Lemma 8:

Lemma 9. *Suppose that $X, R, L \geqslant 1$, $a \asymp q$, $(a,q) = 1$, $\varepsilon > 0$ and $\frac{TX^{\frac{1}{3}}}{R} < q < X^{\frac{2}{3}}$, where $T = \max\{L, R\}$. Then we have*

$$\sum_{r\sim R}\max_{N\leqslant X}\sum_{l\sim L}\left|\sum_{n\leqslant N}\Lambda(n)e\left(\frac{anl}{rq}\right)\right| \ll X^{\varepsilon}\left(X^{\frac{2}{3}}TR+X^{\frac{11}{12}}(TR)^{\frac{1}{2}}\right).$$

Proof. Using Vaughan's identity we split the inner sum above into $\ll \log N$ sums of the form

$$\sum_{m\sim M}\sum_{mn\leqslant N} a_n b_m e\left(\frac{nalm}{dq}\right),$$

with either

(I) $a_n = 1$ or $\log n$, $M < X^{\frac{2}{3}}$, $b_m \ll X^{\varepsilon}$, or

(II) $a_n, b_m \ll X^{\varepsilon}$, $X^{\frac{1}{6}} < M < X^{\frac{1}{3}}$.

Sums of type (I) can be handled by Lemma 7 and sums of type (II) by Lemma 8 and the estimate above follows. □

Corollary 10. *We have*

$$\sum_{d\leqslant X^{\alpha}}\frac{\xi}{d}\max_{N\leqslant X}\sum_{l=1}^{dY}\left|\sum_{n\leqslant N}\Lambda(n)e\left(\frac{anl}{dq}\right)\right| \ll \xi\pi(X)X^{-\eta}.$$

5. Sieve estimates

Let f_1, F_1 and F_2 be the limit functions occurred in Beta-Sieve, which are given by the following definition:

$$\begin{aligned} f_1(s) &= A_1 s^{-1}\log(s-1) \text{ for } 2\leqslant s\leqslant 4;\\ F_1(s) &= A_1 s^{-1} \text{ for } s\leqslant 3;\\ F_2(s) &= A_2 s^{-2} \text{ for } s\leqslant \beta_2+1,\end{aligned}$$

where $A_1 = 2e^{-\gamma}$, $\beta_2 = 4.8333\cdots$, $A_2 = 43.496\cdots$ are defined in [1], Chapter 11.

In this section, we will prove the following theorem, which improves [4], Lemma 1:

Theorem 11. *We have*

$$\mathcal{W}(\mathcal{A},u,\lambda) \geqslant \xi\pi(X)V(z)\left(f_1(4)-\lambda\left(\int_{\frac{\alpha}{4}}^{\delta}\left(\frac{1}{s}-u\right)F_1\left(\frac{4(\alpha-s)}{\alpha}\right)\mathrm{d}s\right.\right. \tag{4}$$

$$\left.\left.+\frac{4e^{-\gamma}}{\alpha}\int_{\delta}^{\frac{1}{u}}\left(\frac{1}{s}-u\right)F_2\left(\frac{4(\frac{2}{3}-\rho-s)}{\alpha}\right)\mathrm{d}s\right)+o(1)\right),$$

where $V(z) = \frac{e^{-\gamma}}{\log z}\big(1+o(1)\big)$*, and* δ *is an arbitrary constant in* $\left[\frac{\alpha}{4},\frac{1}{u}\right]$.

To this end, we need the following lemmas.

Lemma 12. *We have*

$$\mathcal{S}(\mathcal{A}, z) \geqslant \xi\pi(X)V(z)\big(f_1(4) + o(1)\big).$$

Proof. Take $M \asymp \frac{dX^\eta}{\xi}$ in Lemma 5 then we have

$$\begin{aligned}\#\mathcal{A}_d = \sum_{\substack{p\leqslant X \\ d\left|\left\lceil\frac{ap+b}{q}\right\rfloor \\ \left\|\frac{ap+b}{q}\right\|<\frac{\xi}{2}}} 1 = \sum_{\substack{p\leqslant X \\ \left\|\frac{ap+b}{dq}\right\|<\frac{\xi}{2d}}} 1 = \sum_{p\leqslant X} \chi\left(\frac{ap+b}{dq}\right) \\ = \frac{\pi(x)\xi}{d} + E(\mathcal{A}_d) + O\left(\frac{\xi\pi(X)X^{-\eta}}{d}\right),\end{aligned}$$

where

$$\sum_{p\leqslant X}\sum_{1\leq|l|\leq M} a_l e\left(\frac{(ap+b)l}{qd}\right) \leq E(\mathcal{A}_d) \leq \sum_{p\leqslant X}\sum_{1\leq|l|\leq M} b_l e\left(\frac{(ap+b)l}{qd}\right)$$

with $|a_l| + |b_l| \ll \frac{\xi}{d}, \forall 1 \leq |l| \leq M$.

Therefore, by partial summation we have

$$\begin{aligned}E(\mathcal{A}_d) &\ll \max_{N\leqslant X} \frac{1}{\log X} \sum_{1\leq|l|\leq M} (|a_l| + |b_l|) \left|\sum_{n\leqslant N} \Lambda(n)e\left(\frac{anl}{qd}\right)\right| \\ &\ll \max_{N\leqslant X} \frac{\xi}{d} \sum_{l=1}^{dY} \left|\sum_{n\leqslant N} \Lambda(n)e\left(\frac{anl}{qd}\right)\right|,\end{aligned}$$

hence the density function of sequence $\mathcal{A}$ is $g_1(d) = \frac{1}{d}$; and thus, by Jurkat–Richert's theorem, we obtain

$$\begin{aligned}\mathcal{S}(\mathcal{A}, z) \geqslant{}& \xi\pi(X)V(z)\big(f_1(4) + o(1)\big) \\ &+ O\left(\xi\pi(X)X^{-\eta} + \sum_{d\leqslant X^\alpha} \frac{\xi}{d} \max_{N\leqslant X} \sum_{i=1}^{dY} \left|\sum_{n\leqslant N} \Lambda(n)e\left(\frac{anl}{dq}\right)\right|\right).\end{aligned}$$

Then this lemma comes from Corollary 10 since $f_1(4) > 0$. □

Lemma 13. *If* $0 < \frac{\alpha}{4} < \delta < \alpha$, *let* $w = X^\delta$, *then*

$$\sum_{z \leqslant p < w} w_p \mathcal{S}(\mathcal{A}_p, z) \leqslant \xi \pi(X) V(z) \left(\int_{\frac{\alpha}{4}}^{\delta} \left(\frac{1}{s} - u \right) F_1 \left(\frac{4(\alpha - s)}{\alpha} \right) \mathrm{d}s + o(1) \right).$$

Proof. Corollary 10 shows that the level of distribution of $\mathcal{A}$ is X^α. Hence by Jurkat–Richert's theorem, we have

$$\mathcal{S}(\mathcal{A}_p, z) \leqslant \frac{\xi \pi(X) V(z)}{p} \left(F_1(s_p) + O\left((\log \frac{X^\alpha}{p})^{-\frac{1}{6}} \right) \right) + O\left(\sum_{d \leqslant X^\alpha / p} |R_{pd}| \right),$$

where

$$s_p = \frac{\log \frac{X^\alpha}{p}}{\log z}, \text{ and } R_{pd} \ll \frac{\xi}{pd} \max_{N \leqslant X} \sum_{l=1}^{pdY} \left| \sum_{n \leqslant N} A(n) e\left(\frac{anl}{pdq} \right) \right|.$$

Since

$$\begin{aligned} \sum_{z \leqslant p < w} w_p \sum_{d \leqslant X^\alpha / p} |R_{pd}| &\ll \sum_{z \leqslant p < w} \sum_{d \leqslant X^\alpha / p} |R_{pd}| \\ &\ll \sum_{d \leqslant X^\alpha} |R_d| \sum_{\substack{p \mid d \\ z \leqslant p < w}} 1 \\ &\ll X^{\frac{\eta}{2}} \sum_{d \leqslant X^\alpha} |R_d| \\ &\ll X^{\frac{\eta}{2}} \sum_{d \leqslant X^\alpha} \frac{\xi}{d} \max_{N \leqslant X} \sum_{l=1}^{dY} \left| \sum_{n \leqslant N} \Lambda(n) e\left(\frac{anl}{dq} \right) \right| \ll X^{1 - \frac{\eta}{2}} \xi, \end{aligned}$$

we obtain

$$\sum_{z \leqslant p < w} w_p \mathcal{S}(\mathcal{A}_p, z) \leqslant \xi \pi(X) V(z) \left(\sum_{z \leqslant p < w} \frac{w_p}{p} F_1(s_p) + o(1) \right).$$

By Mertens formula we have

$$\sum_{t' \leqslant p < t} (1 - \frac{\log p}{\log y}) = \log \frac{\log t}{\log t'} - \frac{\log t - \log t'}{\log y} + R(t', t),$$

where

$$R(t', t) \ll \frac{1}{t'}, \qquad \text{for any } t' < t.$$

Notice that F_1 is bounded and increasing, so we obtain that

$$\begin{aligned}\sum_{z\leqslant p<w}\frac{w_p}{p}F_1(s_p) &= \int_z^w F_1\left(\frac{\log X^\alpha/t}{\log z}\right)\mathrm{d}\sum_{z\leqslant p<t}(1-\frac{\log p}{\log y}) \\ &= \int_z^w F_1\left(\frac{\log X^\alpha/t}{\log z}\right)\mathrm{d}\left(\log\frac{\log t}{\log z}-\frac{\log t-\log z}{\log y}+R(z,t)\right) \\ &= \int_z^w \frac{1}{t}\left(\frac{1}{\log t}-\frac{1}{\log y}\right)F_1\left(\frac{\log X^\alpha/t}{\log z}\right)\mathrm{d}t + R(z,t)F_1\left(\frac{\log X^\alpha/t}{\log z}\right)\Big|_z^w \\ &\quad + O\left(\int_z^w \frac{1}{z}\mathrm{d}F_1\left(\frac{\log X^\alpha/t}{\log z}\right))\right) \\ &= \int_{\frac{\alpha}{4}}^{\delta}\left(\frac{1}{s}-u\right)F_1\left(\frac{4(\alpha-s)}{\alpha}\right)\mathrm{d}s + O(\frac{1}{z}).\end{aligned}$$

Therefore, we have

$$\sum_{z\leqslant p<w} w_p\mathcal{S}(\mathcal{A}_p,z)\leqslant \xi\pi(X)V(z)\left(\int_{\frac{\alpha}{4}}^{\delta}\left(\frac{1}{s}-u\right)F_1\left(\frac{4(\alpha-s)}{\alpha}\right)\mathrm{d}s+o(1)\right).$$

This completes the proof. □

Define

$$\widetilde{\mathcal{A}} := \left\{n\left\lceil\frac{an+b}{q}\right\rfloor : n\in[z,X], p \mid \left\lceil\frac{an+b}{q}\right\rfloor, \left\|\frac{an+b}{q}\right\| < \frac{1}{2}\xi\right\},$$

then we have the following auxiliary lemma.

Lemma 14. *For $d \mid P(z)$, we have*

$$\#\widetilde{\mathcal{A}}_d = \frac{X\xi}{p}g_2(d) + E(X;p,d),$$

where

$$g_2(d) := \prod_{p\mid d}\left(\frac{2}{p}-\frac{1}{p^2}\right),$$

$$E(X;p,d) \ll q\xi\tau(d) + \frac{X}{pdq}\sum_{d=d_1d_2}(q,d_1)(a,pd_2).$$

Proof. Define

$$J := \{j : |j+b| \leqslant \frac{1}{2}q\xi\}.$$

$$\begin{aligned}
\#\widetilde{\mathcal{A}}_d &= \sum_{|j|\leqslant\frac{1}{2}q\xi} \sum_{\substack{n\in[z,X] \\ an+b\equiv j \pmod{pq} \\ n(an+b)\equiv jn \pmod{dq}}} 1 = \sum_{j\in J} \sum_{\substack{n\in[z,X] \\ an\equiv j \pmod{pq} \\ an^2\equiv jn \pmod{dq}}} 1 \\
&= \sum_{j\in J} \sum_{\substack{n\in[z,X] \\ an-j\equiv 0 \pmod{pq} \\ n(an-j)\equiv 0 \pmod{dq}}} 1 = \sum_{j\in J} \sum_{d=d_1d_2} \sum_{\substack{n\in[z,X] \\ an-j\equiv 0 \pmod{pq} \\ n(an-j)\equiv 0 \pmod{dq} \\ (n,d)=d_1}} 1 \\
&= \sum_{d=d_1d_2} \sum_{j\in J} \sum_{\substack{n\in[z/d_1,X/d_1] \\ ad_1n-j\equiv 0 \pmod{pq} \\ n(ad_1n-j)\equiv 0 \pmod{d_2q} \\ (n,d_2)=1}} 1 = \sum_{d=d_1d_2} \sum_{j\in J} \sum_{\substack{n\in[z/d_1,X/d_1] \\ ad_1n-j\equiv 0 \pmod{pd_2q} \\ (n,d_2)=1}} 1 \\
&= \sum_{d=d_1d_2} \sum_{\substack{j\in J \\ (ad_1,pd_2q)|j}} \left(\frac{\varphi(d_2)}{d_2} \cdot \frac{X-z}{pdq}(ad_1,pd_2q) + O(1) \right) \\
&= \sum_{d=d_1d_2} \left(\frac{q\xi}{(ad_1,pd_2q)} + O(1) \right) \left(\frac{\varphi(d_2)}{d_2} \cdot \frac{X-z}{pdq}(ad_1,pd_2q) + O(1) \right) \\
&= \sum_{d=d_1d_2} \frac{\varphi(d_2)}{d_2} \cdot \frac{(X-z)\xi}{pd} + O\left(\sum_{d=d_1d_2} \frac{q\xi}{(ad_1,pd_2q)} + \sum_{d=d_1d_2} \frac{X\cdot(ad_1,pd_2q)}{pdq} \right),
\end{aligned}$$

and thus lemma follows by noting that $(a,pd_2q)(d_1,pd_2q) \le (q,d_1)(a,pd_2)$. □

Hence $\widetilde{\mathcal{A}}$ has a density function $g_2(d)$ with

$$\begin{aligned}
V_2(z) &:= \prod_{p\leqslant z} \big(1 - g_2(p)\big) = \prod_{p\leqslant z} \left(1 - \frac{2}{p} + \frac{1}{p^2}\right) \\
&= \frac{e^{-2\gamma}}{\log^2 z}\big(1+o(1)\big) \qquad \text{by Mertens estimate.}
\end{aligned}$$

We will use Beta-Sieve theory to $\widetilde{\mathcal{A}}$ to obtain an upper bound with a larger exponent of level of distribution. To this end, we shall compute its dimension as follows:

$$\sum_{p\le v} g_2(p)\log p = 2\sum_{p\le v} \left(\frac{\log p}{p} - \frac{\log p}{2p^2}\right) = 2\log v + O(1), \qquad \text{for any } v \ge 2.$$

Therefore, the sieve dimension is 2. Denote by θ_2 the exponent of level of distribution of $\widetilde{\mathcal{A}}$.

Lemma 15. *Assuming $w \leqslant p \leqslant y$ and p is a prime number, where $w = X^{\delta}$, $\frac{\alpha}{4} \leqslant \delta \leqslant \alpha$, then we have*

$$\mathcal{S}(\mathcal{A}_p, z) \leqslant \frac{X\xi}{p} V_2(z)\left(F_2(s_p') + O\left((\log X)^{-\frac{1}{6}}\right)\right) + \sum_{d \leq \frac{X^{\theta_2}}{p}} E(X; p, d),$$

where

$$s_p' := \frac{\log\left(X^{\theta_2}/p\right)}{\log z}.$$

Proof. We have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}_p, z) &= \sum_{\substack{n \in \mathcal{A}_p \\ (n, P(z))=1}} 1 \\
&= \#\{p' : z \leqslant p' \leqslant X, p \mid \left\lceil \frac{ap'+b}{q} \right\rfloor, \left\| \frac{ap'+b}{q} \right\| < \frac{1}{2}\xi, \left(p' \left\lceil \frac{ap'+b}{q} \right\rfloor, P(z)\right) = 1\} \\
&\quad + \#\{p' : p' < z, p \mid \left\lceil \frac{ap'+b}{q} \right\rfloor, \left\| \frac{ap'+b}{q} \right\| < \frac{1}{2}\xi, \left(p' \left\lceil \frac{ap'+b}{q} \right\rfloor, P(z)\right) = 1\} \\
&\leqslant \#\{n : z \leqslant n \leqslant X, p \mid \left\lceil \frac{an+b}{q} \right\rfloor, \left\| \frac{an+b}{q} \right\| < \frac{1}{2}\xi, \left(n \left\lceil \frac{an+b}{q} \right\rfloor, P(z)\right) = 1\} \\
&\quad + O\big(\xi\pi(z)\big) \\
&= \mathcal{S}(\widetilde{\mathcal{A}}, z) + O\big(\xi\pi(z)\big).
\end{aligned}$$

We now meet a sifting problem of dimension two. By Beta-Sieve theory we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}_p, z) &\leqslant \mathcal{S}(\widetilde{\mathcal{A}}, z) + O\big(\xi\pi(z)\big) \\
&\leqslant \frac{X\xi}{p} V_2(z)\left(F_2(s_p') + O\left(\left(\log \frac{X^{\theta_2}}{p}\right)^{-\frac{1}{6}}\right)\right) + \sum_{d \leqslant X^{\theta_2}/p} E(X; p, d) \\
&\quad + O\big(\xi\pi(z)\big) \\
&= \frac{X\xi}{p} V_2(z)\left(F_2(s_p') + O\left((\log X)^{-\frac{1}{6}}\right)\right) + \sum_{d \leqslant X^{\theta_2}/p} E(X; p, d)
\end{aligned}$$

and the last inequality holds because

$$\xi\pi(z) \ll X\xi V_2(z)(\log X)^{-\frac{1}{6}}.$$

This completes the proof. □

Lemma 16. *If* $\frac{\alpha}{4} \leqslant \delta \leqslant \frac{1}{u} < \alpha$*, let* $w = X^{\delta}$*, then*

$$\sum_{w \leqslant p \leq y} w_p \mathcal{S}(\mathcal{A}_p, z) \leqslant \xi\pi(X)V(z)\left(\frac{4e^{-\gamma}}{\alpha}\int_{\delta}^{\frac{1}{u}}\left(\frac{1}{s}-u\right)F_2\left(\frac{4(\theta_2-s)}{\alpha}\right)\mathrm{d}s + o(1)\right) + O\left(q\xi X^{\theta_2+\varepsilon}+\frac{X^{1+\varepsilon}}{q}\right).$$

Proof. From Lemma 15 we obtain

$$\begin{aligned}\sum_{w \leqslant p \leq y} w_p \mathcal{S}(\mathcal{A}_p, z) &\leqslant X\xi V_2(z)\left(\sum_{w\leqslant p\leq y}\frac{w_p}{p}F_2(s_p') + O\left((\log X)^{-\frac{1}{6}}\sum_{w\leqslant p\leq y}\frac{1}{p}\right)\right) \\ &\quad + E_{\mathcal{A}}(X; w, y) \\ &= X\xi V_2(z)\left(\sum_{w\leqslant p\leq y}\frac{w_p}{p}F_2(s_p') + o(1)\right) + E_{\mathcal{A}}(X; w, y),\end{aligned}$$

where

$$E_{\mathcal{A}}(X; w, y) := \sum_{w\leqslant p\leq y} w_p \sum_{d \leq \frac{X^{\theta_2}}{p}} E(X; p, d).$$

Use the same method in Lemma 13 to handle $\sum_{w\leqslant p\leq y}\frac{w_p}{p}F_2(s_p')$ and we obtain that

$$\sum_{w\leqslant p\leq y}\frac{w_p}{p}F_2(s_p') = \int_{\delta}^{\frac{1}{u}}\left(\frac{1}{s}-u\right)F_2\left(\frac{4(\theta_2-s)}{\alpha}\right)\mathrm{d}s + o(1).$$

As for $E_{\mathcal{A}}(X; w, y)$, noting that for any $0 < \theta_2 < 1$, and for any $1 \leqslant B \leqslant X^{\theta_2}/p$, we have

$$\sum_{d\sim B}(k,d) = \sum_{c|k}\sum_{\substack{d\sim B\\ c=(k,d)}} c = \sum_{c|k} c \sum_{\substack{d\sim Bc^{-1}\\ (d,kc^{-1})=1}} 1 \ll \sum_{c|k} B = B\tau(k),$$

by Abel transformation,

$$\sum_{d\sim B}\frac{(k,d)}{d} \ll \tau(k),$$

which illustrates

$$\sum_{d\leqslant X^{\theta_2}/p}\frac{(k,d)}{d} \ll \tau(k)\log X. \tag{5}$$

Hence we conclude that

$$E_{\mathcal{A}}(X;w,y) \le \sum_{w\le p\le y}\sum_{d\le X^{\theta_2}/p}|E(X;p,d)|$$
$$\ll \sum_{w\le p\le y}\sum_{d\leqslant X^{\theta_2}/p}\left(q\xi\tau(d)+\frac{X}{pq}\sum_{d=d_1d_2}\frac{(a,pd_2)(q,d_1)}{d_2d_1}\right)$$
$$\ll q\xi\sum_{w\le p\le y}\frac{X^{\theta_2+\varepsilon}}{p}+\frac{X}{q}\sum_{w\le p\le y}\sum_{d_1d_2\leqslant X^{\theta_2}/p}\frac{(a,pd_2)}{pd_2}\cdot\frac{(q,d_1)}{d_1}.$$

Noticing that (5), the lemma follows immediately. □

Proof of Theorem 11. We have

$$\sum_{p\geqslant X^{\frac{\alpha}{4}}}\sum_{h\in\mathcal{A}_{p^2}}1 \ll \sum_{p\geqslant X^{\frac{\alpha}{4}}}\frac{\pi(x)\xi}{p^2} \ll \frac{\pi(x)\xi}{X^{\frac{\alpha}{4}}} \ll X^{1-\eta}\xi = o\left(\xi\pi(X)V(z)\right).$$

Therefore, it comes from Lemma 12, Lemma 13 and Lemma 16 that

$$\mathcal{W}(\mathcal{A},u,\lambda)=\mathcal{S}(\mathcal{A},z)-\lambda\sum_{z\le p\le y}w_p\mathcal{S}(\mathcal{A}_p,p)+\sum_{p\geqslant X^{\frac{\alpha}{4}}}\sum_{\substack{h\in\mathcal{A}\\ p^2|h}}1$$
$$=\mathcal{S}(\mathcal{A},z)-\lambda\sum_{z\le p< w}w_p\mathcal{S}(\mathcal{A}_p,p)-\lambda\sum_{w\le p\le y}w_p\mathcal{S}(\mathcal{A}_p,p)+\sum_{p\geqslant X^{\frac{\alpha}{4}}}\sum_{\substack{h\in\mathcal{A}\\ p^2|h}}1$$
$$\ge \xi\pi(X)V(z)\left(f_1(4)-\lambda\left(\int_{\frac{\alpha}{4}}^{\delta}\left(\frac{1}{s}-u\right)F_1\left(\frac{4(\alpha-s)}{\alpha}\right)\mathrm{d}s\right.\right.$$
$$\left.\left.+\frac{4e^{-\gamma}}{\alpha}\int_{\delta}^{\frac{1}{u}}\left(\frac{1}{s}-u\right)F_2\left(\frac{4(\theta_2-s)}{\alpha}\right)\mathrm{d}s\right)+o(1)\right)$$
$$+O\left(q\xi X^{\theta_2+\varepsilon}+\frac{X^{1+\varepsilon}}{q}\right).$$

Obviously, we have $\frac{X^{1+\varepsilon}}{q} \ll \xi\pi(X)^{1-\varepsilon}$ since $q \asymp q^{\frac{1}{3}+\rho+\eta}$. Hence, to prove Theorem 11, it suffices to take θ_2 such that

$$q\xi X^{\theta_2+\varepsilon} \ll \xi\pi(X)^{1-\varepsilon},$$

which gives that $\theta_2 \le \frac{2}{3}-\rho-3\varepsilon$.

Thus, take $\theta_2=\frac{2}{3}-\rho-3\varepsilon$ and by the continuity of F_2 we obtain Theorem 11. □

Remark. The lemma 1 in [4] is the special case of Theorem 11 where $\delta = \frac{1}{u}$.

Lemma 17. *We have*

$$\mathcal{S}\left(\mathcal{A}(\beta)^*, X^{\frac{1}{2}-\eta}\right) \leqslant \left(4\wp(\alpha,\rho) + o(1)\right)\frac{\xi\pi(X)}{\log X} + O\left(X^{\varepsilon}\sum_{r\leqslant X^{\nu}} |R_r^*(\beta)|\right),$$

where

$$\sum_{r\leqslant X^{\nu}} |R_r^*(\beta)| \ll \xi\pi(X)X^{-\frac{\eta}{2}}, \tag{6}$$

with $\nu = \frac{1-\beta}{2} - \rho - 2\eta$ *and* $\wp(\alpha,\rho)$ *is defined by*

$$\wp(\alpha,\rho) := \int_{\frac{\alpha}{4}}^{\frac{1}{4}} \frac{\mathrm{d}u_1}{u_1(1-u_1-2\rho)} \int_{u_1}^{\frac{1-u_1}{3}} \frac{\mathrm{d}u_2}{u_2} \int_{u_2}^{\frac{1-u_1-u_2}{2}} \frac{\mathrm{d}u_3}{u_3(1-u_1-u_2-u_3)}. \tag{7}$$

Proof. This follows from [3], Theorem 8.3 and [4]. □

Remark. We shall use (6) to give some restrictions in Theorem 19.

6. Optimization of the parameters and proof of Theorem 1

6.1. Restrictions of the parameters

Theorem 18. *The restriction from the main terms is given by*

$$\begin{cases} \left|\dfrac{4}{3} + \alpha - 2\rho\right| \leqslant \dfrac{3A_2\alpha}{2}, \\ (5-\alpha^{-1})\log 3 > \log\dfrac{4\delta_0}{\alpha} + \dfrac{A_2}{2e^{\gamma}}\displaystyle\int_{\delta_0}^{\alpha} \frac{\alpha - s}{s\left(\frac{2}{3}-\rho-s\right)^2}\mathrm{d}s + 2\alpha\wp(\alpha,\rho), \end{cases}$$

where

$$\delta_0 = \frac{2}{3} - \rho - \frac{1}{2}\left(\frac{A_2}{2} - \left(\frac{A_2^2}{4} - 2A_2\left(\frac{2}{3}-\rho-\alpha\right)\right)^{\frac{1}{2}}\right). \tag{8}$$

Proof. By Corollary 3 it suffices to restrict λ such that

$$\mathcal{W}(\mathcal{A}, u, \lambda) > \lambda\mathcal{S}(\mathcal{A}(\beta)^*, X^{\frac{1}{2}-\eta}), \quad \text{and } \lambda^{-1} < 5 - u,$$

which gives by Theorem 11 and Lemma 6 that

$$\lambda^{-1} f_1(4) > \int_{\frac{\alpha}{4}}^{\delta} \left(\frac{1}{s} - u\right) F_1\left(\frac{4(\alpha - s)}{\alpha}\right) \mathrm{d}s + \frac{4e^{-\gamma}}{\alpha} \int_{\delta}^{\frac{1}{u}} \left(\frac{1}{s} - u\right) F_2\left(\frac{4(\frac{2}{3} - \rho - s)}{\alpha}\right) \mathrm{d}s + \alpha e^{\gamma} \wp(\alpha, \rho).$$

We assume $u = \frac{1+\eta}{\alpha}$ as in [4], then by continuity it suffices to have

$$\left(5 - \alpha^{-1}\right) f_1(4) > F_\alpha(\delta) + e^{\gamma} \wp(\alpha, \rho),$$

where

$$F_\alpha(\delta) := \int_{\frac{\alpha}{4}}^{\delta} \left(\frac{1}{s} - \frac{1}{\alpha}\right) F_1\left(\frac{4(\alpha - s)}{\alpha}\right) \mathrm{d}s + \frac{4e^{-\gamma}}{\alpha} \int_{\delta}^{\alpha} \left(\frac{1}{s} - \frac{1}{\alpha}\right) F_2\left(\frac{4(\frac{2}{3} - \rho - s)}{\alpha}\right) \mathrm{d}s.$$

We want to take $\delta = \delta_0$ such that

$$F'_\alpha(\delta)|_{\delta=\delta_0} = 0 \text{ and } 0 < \delta_0 < \alpha \leqslant \frac{1}{3}. \tag{9}$$

We assume that

$$\frac{4(\frac{2}{3} - \rho - \delta)}{\alpha} \leqslant \beta_2 + 1. \tag{10}$$

Then we conclude that

$$F_1\left(\frac{4(\alpha - \delta_0)}{\alpha}\right) = \frac{4e^{-\gamma}}{\alpha} F_2\left(\frac{4(\frac{2}{3} - \rho - \delta_0)}{\alpha}\right),$$

that is,

$$\frac{2}{\alpha - \delta_0} = \frac{A_2}{(\frac{2}{3} - \rho - \delta_0)^2},$$

which deduces that

$$\delta_0 = \frac{2}{3} - \rho - \frac{1}{2}\left(\frac{A_2}{2} - \left(\frac{A_2^2}{4} - 2A_2\left(\frac{2}{3} - \rho - \alpha\right)\right)^{\frac{1}{2}}\right).$$

It's easy to check that δ_0 satisfies assumption (9) and (10), and $\delta = \delta_0$ is the point where $F_\alpha(\delta)$ takes its minimum in $\delta \in (0,1)$. Hence, take $\delta = \delta_0$, which is the optimal choice, and we have $\left(5-\alpha^{-1}\right) f_1(4) > F_\alpha(\delta_0) + e^\gamma \wp(\alpha,\rho)$, i.e.

$$(5-\alpha^{-1})\log 3 > \log\frac{4\delta_0}{\alpha} + \frac{A_2}{2e^\gamma}\int\limits_{\delta_0}^{\alpha}\frac{\alpha - s}{s\left(\frac{2}{3}-\rho-s\right)^2}\mathrm{d}s + 2\alpha\wp(\alpha,\rho).$$

On the other hand, by the assumptions of Theorem 11, we shall restrict δ_0 such that $\delta_0 \geqslant \frac{\alpha}{4}$ which gives

$$\rho \leqslant \frac{2}{3} - \frac{\alpha}{4} - \frac{1}{2}\left(\frac{A_2}{2} - \left(\frac{A_2^2}{4} - 2A_2\left(\frac{2}{3}-\rho-\alpha\right)\right)^{\frac{1}{2}}\right),$$

that is,

$$\left|\frac{4}{3}+\alpha-2\rho\right| \leqslant \frac{3A_2\alpha}{2}.$$

So we complete the proof. □

Theorem 19. *The restrictions from the error terms are given as the following inequation systems:*

$$\begin{cases} 0 < \rho < \frac{\alpha}{4}, \\ \alpha + \rho < \frac{1}{3}, \ \alpha > 0. \end{cases}$$

Proof. In Corollary 10 and Lemma 9 above, where we show that

$$\sum_{r\leqslant X^\alpha}\frac{1}{r}\max_{N\leqslant X}\sum_{l=1}^{rY}\left|\sum_{n\leqslant N}\Lambda(n)e\left(\frac{\alpha n l}{rq}\right)\right| \ll \pi(X)X^{-\eta},$$

with $Y \asymp X^{\rho+\eta}$, we have to make sure that all the parameters satisfy the assumptions of those lemmas.

Divide the intervals into dyadic segments and thus we have the following estimation:

$$\begin{aligned}\sum_i\sum_{i_j} X^\varepsilon\left(X^{\frac{2}{3}}T_i + X^{\frac{11}{12}}\left(\frac{T_i}{R_i}\right)^{\frac{1}{2}}\right) &\ll X^{2\varepsilon}X^{\frac{2}{3}+\alpha+\rho+\eta} + X^{\frac{11}{12}}\sum_i\sum_{i_j}X^{\frac{\rho+\eta}{2}} \\ &\ll X^{2\varepsilon}X^{\frac{2}{3}+\alpha+\rho+\eta} + X^{\frac{11}{12}+\varepsilon+\frac{\rho+\eta}{2}},\end{aligned}$$

where $L_{i_j} \leqslant R_iY \ll R_iX^{\rho+\eta}$, $T_i \ll R_iX^{\rho+\eta}$ and for simplicity we omit the precise range of i and j, actually, only the bound $i, j \ll \log X$ matters.

Therefore, we get our restrictions as below:

$$\begin{cases} 2\varepsilon + \frac{2}{3} + \alpha + \rho + \eta < 1 - \eta, \\ \frac{11}{12} + \varepsilon + \frac{\rho+\eta}{2} < 1 - \eta, \end{cases}$$

i.e.

$$\begin{cases} \alpha + \rho < \frac{1}{3}, \\ \rho < \frac{1}{6}. \end{cases}$$

Now let's consider another estimation from (6). By assumption, we have $X^{\rho+\beta+\eta} < X^{\rho+\frac{1}{4}+\eta} < q < X^{\frac{3}{4}-\eta} < X^{1-\beta-\eta}$. Additionally, by Lemma 9, there should be

$$\xi X^{1+3\varepsilon} \sum_i \sum_{i_j} \left(L_{i_j} + \frac{R_i}{X^\beta} \right) \left(X^{\frac{\beta-1}{2}} + X^{-\beta} R_i^{-\frac{1}{2}} \left(L_{i_j} + \frac{R_i}{X^\beta} \right)^{-\frac{1}{2}} \right) \ll \xi \pi(X) X^{-\eta}.$$

While

$$\xi X^{\frac{\beta+1}{2}+3\varepsilon} \sum_i \sum_{i_j} \left(L_{i_j} + \frac{R_i}{X^\beta} \right)$$
$$\ll \xi X^{\frac{\beta+1}{2}+4\varepsilon} \left(X^{\frac{1-\beta}{2}-\rho-2\eta} + X^{-\beta} X^{\frac{1-\beta}{2}-2\eta} \right) \ll \xi X^{1-\rho-2\eta+4\varepsilon} + \xi X^{1-\beta-2\eta},$$

and

$$\xi X^{1+3\varepsilon} \sum_i \sum_{i_j} \left(X^{-\frac{\beta}{2}} R_i^{-\frac{1}{2}} \left(L_{i_j} + \frac{R_i}{X^\beta} \right)^{\frac{1}{2}} \right)$$
$$\ll \xi X^{1+3\varepsilon} \sum_i \sum_{i_j} \left(X^{-\frac{\beta}{2}} R_i^{-\frac{1}{2}} L_{i_j}^{\frac{1}{2}} + X^{-\beta} \right) \ll \xi X^{1+4\varepsilon+\frac{\rho+\eta}{2}-\frac{\beta}{2}},$$

so it suffices to have the restriction: $1+4\varepsilon+\frac{\rho+\eta}{2}-\frac{\beta}{2} < 1-\frac{\eta}{2}-\varepsilon$, which could be deduced by the condition: $\beta > \rho \Leftarrow \frac{\alpha}{4} > \rho$.

In short, we obtain the restrictions

$$\begin{cases} 0 < \rho < \min\left\{\frac{1}{6}, \frac{\alpha}{4}\right\} = \frac{\alpha}{4}, \\ \alpha + \rho < \frac{1}{3}, \ \alpha > 0. \end{cases}$$

This completes the proof. $\square$

6.2. Proof of the main theorem

Combine all the restrictions from Theorem 18 and Theorem 19 as follows:

$$\begin{cases} \alpha + \rho < \dfrac{1}{3}, \ \alpha > 0, \\ (5 - \alpha^{-1}) \log 3 > \log \dfrac{4\delta_0}{\alpha} + \dfrac{A_2}{2e^{\gamma}} \displaystyle\int_{\delta_0}^{\alpha} \dfrac{\alpha - s}{s\left(\frac{2}{3} - \rho - s\right)^2} \mathrm{d}s + 2\alpha\wp(\alpha, \rho), \end{cases}$$

where $\wp(\alpha, \rho)$ is defined in (7), and δ_0 is defined in (8), and $A_2 = 43.496\cdots$ is defined in [1], Chapter 11.

Take $\alpha = \frac{1}{3} - \rho - 10^{-8}$, then insert this into the above conditions, with the help of the software *Mathematica 9*, we can obtain that $\rho < \frac{1}{130}$; however, by calculation we find that $\rho = \frac{1}{131}$ satisfies the restrictions above and thus we have proven that there are infinitely many solutions of

$$|\lambda_0 + \lambda_1 p + \lambda_2 P_3| < p^{-\frac{1}{131}}.$$

Remark. If we just take $\delta = \alpha$ as Harman did in [4], then we have $\tau < \frac{1}{146}$ and we can take $\tau = \frac{1}{147}$.

Acknowledgments

I am grateful to Professor Yingchun Cai for his suggesting this problem for me. I also thank Ping Xi for his kind help. His valuable suggestions and comments should be greatly acknowledged. Sincere thanks are also due to Jinzhao Pan for helping me type the first draft. The numerical computations in this paper are based on the Mathematica codes of Bingyi Chen, and I thank him for his kind help.

References

[1] J. Friedlander, H. Iwaniec, Opera de cribro, Amer. Math. Soc., vol. 57, 2010.
[2] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth edition, University Press, Oxford, 1979.
[3] H. Halberstam, H.E. Richert, Sieve Methods, Academic Press, London, 1974.
[4] G. Harman, Diophantine approximation with a prime and an almost-prime, J. Lond. Math. Soc. (2) 29 (1984) 13–22.
[5] H.L. Montgomery, The analytic principle of the large sieve, Bull. Amer. Math. Soc. 84 (1978) 547–567.
[6] K. Ramachandra, Two remarks in prime number theory, Bull. Soc. Math. France 105 (1977) 433–437.
[7] S. Srinivasan, A note on $|\alpha p - q|$, Acta Arith. 41 (1982) 15–18.
[8] R.C. Vaughan, Diophantine approximation by prime numbers, Proc. Lond. Math. Soc. (3) 33 (1976) 177–192.
[9] R.C. Vaughan, On the distribution of αp modulo 1, Mathematika 24 (1977) 135–141.
[10] R.C. Vaughan, An elementary method in prime number theory, Acta Arith. 37 (1980) 111–115.