On the events satisfying the sure-thing principle*

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Abstract

Given a preference relation $\succsim$ over Anscombe-Aumann acts, we study the collection of events for which the preference satisfies Savage's P2 axiom. We show that this collection is a σ-algebra whenever $\succsim$ is monotone, suitably continuous, and satisfies the independence axiom over constant acts. We provide a relatively self-contained proof, which by and large builds on Gul and Pesendorfer (2014) and Grant, Liu, and Yang (2024).

1 Definitions

We use the same notation and terminology as in Denti and Pomatto (2022). We consider a set Ω of *states of the world*, endowed with a σ-algebra $\mathcal{S}$ of *events*, and a set X of *consequences*. The latter is assumed to be a convex subset of a Hausdorff topological vector space, endowed with the Borel σ-algebra. Acts are measurable functions from Ω to X. We restrict attention to the collection $\mathfrak{F}$ of acts f for which there exists a finite set $Y \subseteq X$ such that f takes values in the convex hull of Y. We call a sequence (f_n) in $\mathfrak{F}$ *bounded* if there exists a finite set $Y \subseteq X$ such that each f_n takes values in the convex hull of Y.

We write x for the constant act f such that $f(\omega) = x$ for all $\omega \in \Omega$. Given $f, g \in \mathfrak{F}$ and $\alpha \in [0, 1]$, we denote by $\alpha f + (1 - \alpha)g$ the act in $\mathfrak{F}$ that takes value $\alpha f(\omega) + (1 - \alpha)g(\omega)$ in each state ω. Given acts f and g and event A, fAg is the act that coincides with f on A and with g on A^c (note that if $f, g \in \mathfrak{F}$, then fAg belongs to $\mathfrak{F}$ as well).

1.1 Preferences

We study a binary relation $\succsim$ over $\mathfrak{F}$. We denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$, respectively. An event A is *null* if $fAh \sim gAh$ for all $f, g, h \in \mathfrak{F}$.

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1.2 Axioms and the sure-thing principle

We impose the following axioms for $\succsim$.

Axiom 1. *The preference $\succsim$ is complete, transitive, and non-trivial.*

Axiom 2. *For all $f, g \in \mathfrak{F}$, if $f(\omega) \succsim g(\omega)$ for all ω, then $f \succsim g$.*

Axiom 3. *If (f_n) and (g_n) are bounded sequences in $\mathfrak{F}$ that converge pointwise to f and g, respectively, and $f_n \succsim g_n$ for every n, then $f \succsim g$.*

An event A *satisfies the sure-thing principle* if the following conditions hold for all $f, g, h, h' \in \mathfrak{F}$:

(i). If $fAh \succsim gAh$, then $fAh' \succsim gAh'$.

(ii). If $hAf \succsim hAg$, then $h'Af \succsim h'Ag$.

In words, an event A satisfies the sure-thing principle if both A and its complement satisfy Savage's postulate P2. We denote by $\mathcal{S}_{\text{stp}}$ the family of all such events. The properties of the collection $\mathcal{S}_{\text{stp}}$ were originally studied by Gul and Pesendorfer (2014), who referred to such events as *ideal*.

The next axiom is the standard independence axiom over constant acts.

Axiom 4. *For all $x, y, z \in X$ and $\alpha \in [0, 1]$, if $x \succsim y$ then $\alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z$.*

Axioms 1–3 correspond to the first three axioms in Denti and Pomatto (2022). If the fourth axiom from that paper were imposed, then Axiom 4 would be implied by Axioms 1–3.

2 Main Result

We can now state the main result of this note.

Theorem 1. *Let $\succsim$ satisfy axioms 1–4. Then $\mathcal{S}_{\text{stp}}$ is a σ-algebra.*

The result is established by Gul and Pesendorfer (2014) and Grant, Liu, and Yang (2024) under slightly different hypotheses. Our proof is divided into two steps. The first step shows that $\mathcal{S}_{\text{stp}}$ is an algebra, largely following the approach of Gul and Pesendorfer. However, we do not assume that $\succsim$ satisfies Savage's P4 axiom over events that satisfy the sure-thing principle. Additionally, instead of relying on a theorem by Gorman (1968), we directly use the results on functional equations that Gorman applies in his paper.

To establish that $\mathcal{S}_{\text{stp}}$ is a σ-algebra, we then prove that $\mathcal{S}_{\text{stp}}$ is a monotone class, meaning that if (A_n) is a sequence in $\mathcal{S}_{\text{stp}}$ satisfying $A_n \uparrow A$ or $A_n \downarrow A$, then $A \in \mathcal{S}_{\text{stp}}$. Here the difference is in our assumption of pointwise continuity, which is more stringent than the continuity conditions considered by Gul and Pesendorfer (2014) and Grant, Liu, and Yang

(2024), but simplifies the analysis. Our proof of this step is novel and relies on Debreu's classic theorem on additively separable representations (Debreu, 1960).

We also offer a shorter proof that $\mathcal{S}_{\text{stp}}$ is a monotone class under a strengthening of Axiom 2. This alternative proof does not rely on Debreu's theorem.

3 Proof of Theorem 1

3.1 Functional equations

The next theorem is a characterization of generalized bisymmetric functional equations.

Theorem 2 (Maksa, 1999)**.** *Let $U_{11}, U_{12}, U_{21}, U_{22}$ be intervals in $\mathbb{R}$ of positive length. Let*

$$F_1 \colon U_{11} \times U_{12} \to \mathbb{R}, \quad F_2 \colon U_{21} \times U_{22} \to \mathbb{R}, \quad G_1 \colon U_{11} \times U_{21} \to \mathbb{R}, \quad G_2 \colon U_{12} \times U_{22} \to \mathbb{R},$$

be functions that are strictly increasing and continuous in each argument. Let

$$F_1(U_{11} \times U_{12}) = J_1, \quad F_2(U_{21} \times U_{22}) = J_2, \quad G_1(U_{11} \times U_{21}) = I_1, \quad G_2(U_{12} \times U_{22}) = I_2,$$

and suppose $F \colon I_1 \times I_2 \to \mathbb{R}$ and $G \colon J_1 \times J_2 \to \mathbb{R}$ are functions that are strictly increasing and continuous in each argument. Then, the functional equation

$$G(F_1(u_{11}, u_{12}), F_2(u_{21}, u_{22})) = F(G_1(u_{11}, u_{21}), G_2(u_{12}, u_{22}))$$

holds for all $(u_{11}, u_{12}, u_{21}, u_{22}) \in U_{11} \times U_{12} \times U_{21} \times U_{22}$ if and only if there are strictly increasing and continuous functions

$$\varphi \colon G(J_1 \times J_2) \to \mathbb{R} \quad \textit{and} \quad \varphi_{ij} \colon U_{ij} \to \mathbb{R} \quad \textit{with} \quad i, j = 1, 2$$

such that for all $(u_{11}, u_{12}, u_{21}, u_{22})$,

$$\varphi(G(F_1(u_{11}, u_{12}), F_2(u_{21}, u_{22}))) = \varphi_{11}(u_{11}) + \varphi_{12}(u_{12}) + \varphi_{21}(u_{21}) + \varphi_{22}(u_{22}). \tag{1}$$

The next result characterizes the solution of generalized associative functional equations.

Theorem 3 (Maksa, 2005)**.** *Let U_1, U_2, U_3 be intervals in $\mathbb{R}$ of positive length. Let*

$$G \colon U_1 \times U_2 \to \mathbb{R}, \quad K \colon U_2 \times U_3 \to \mathbb{R}, \quad F \colon G(U_1, U_2) \times U_3 \to \mathbb{R}, \quad H \colon U_1 \times K(U_2, U_3) \to \mathbb{R}$$

be functions that are strictly increasing and continuous in each argument. Then, the functional equation

$$F(G(u_1, u_2), u_3) = H(u_1, K(u_2, u_3))$$

holds for all $(u_1, u_2, u_3) \in U_1 \times U_2 \times U_3$ if and only if there exist strictly increasing and continuous functions

$$\varphi \colon F(G(U_1, U_2) \times U_3) \to \mathbb{R} \quad \textit{and} \quad \varphi_i \colon U_i \to \mathbb{R} \quad \textit{with} \quad i = 1, 2, 3$$

such that for all (u_1, u_2, u_3),

$$\varphi(F(G(u_1, u_2), u_3)) = \varphi_1(u_1) + \varphi_2(u_2) + \varphi_3(u_3). \tag{2}$$

The implications (1) and (2) will play a key role in the proof that $\mathcal{S}_{\text{stp}}$ is an algebra.

3.2 Debreu's theorem

Let $U \subseteq \mathbb{R}$ be an interval and I a finite index set. We denote by U^I the set of functions from I to U. Given $\xi, \zeta \in U^I$ and $E \subseteq I$, we denote by $\xi E \zeta$ the element of U^I defined as $(\xi E \zeta)(i) = \xi(i)$ if $i \in E$, and $(\xi E \zeta)(i) = \zeta(i)$ if $i \notin E$.

Let $\succsim^I$ be a binary relation on U^I. It is *continuous* if for every $\zeta \in U^I$, the sets $\{\xi \in U^I : \xi \succsim^I \zeta\}$ and $\{\xi \in U^I : \zeta \succsim^I \xi\}$ are closed relative to U^I. It is *separable* if for every $\xi, \zeta, \psi, \chi \in U^I$ and $E \subseteq I$, $\xi E \psi \succsim^I \zeta E \psi$ implies $\xi E \chi \succsim^I \zeta E \chi$. An index i is *null* if $a\{i\}\xi \sim^I b\{i\}\xi$ for all $a, b \in U$ and $\xi \in U^I$.

Theorem 4 (Debreu, 1960)**.** *Let $\succsim^I$ be a binary relation on U^I, and assume at least 3 indexes are non-null. Then the following statements are equivalent:*

(i) *The binary relation $\succsim^I$ is complete, transitive, continuous, and separable;*

(ii) *For every $i \in I$ there is a continuous function $W_i \colon U \to \mathbb{R}$ such that $\succsim^I$ is represented by*

$$W(\xi) = \sum_{i \in I} W_i(\xi(i)).$$

Moreover, if W and $\tilde{W}$ are two such representations, then there are $\alpha > 0$ and β_i for every $i \in I$ such that $\tilde{W}_i = \alpha W + \beta_i$ for every $i \in I$.

Debreu's theorem will be crucial to show that $\mathcal{S}_{\text{stp}}$ is a monotone class.

3.3 Preliminary results

From now on we denote by $\succsim$ a binary relation over $\mathfrak{F}$ that satisfies Axioms 1-4. The next result is similar to Lemma 6 in Denti and Pomatto (2022). For completeness, we include a proof.

Lemma 1. *The following conditions hold:*

(i) *For all acts f, g, h in $\mathfrak{F}$ the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed.*

(ii) *There exists an affine function $u \colon X \to \mathbb{R}$ representing $\succsim$ on X.*

Proof. (i). Let (α_n) be a sequence in $[0,1]$ converging to α. Let $f_n = \alpha_n f + (1-\alpha_n) g$. The sequence (f_n) is bounded and converges to $\alpha f + (1-\alpha) g$. The result follows from Axiom 3.

(ii). The claim is an application of the mixture space theorem (Herstein and Milnor, 1953), together with (i) and Axioms 1 and 4. □

The next result appears as Lemma 7 in Denti and Pomatto (2022). Given a nonempty set $U \subseteq \mathbb{R}$ and σ-algebra $\mathcal{T} \subseteq \mathcal{S}$, we denote by $B(\mathcal{T}, U)$ the space of $\mathcal{T}$-measurable bounded functions $\xi \colon \Omega \to \mathbb{R}$ that satisfy $\xi(\Omega) \subseteq U$. We denote by $B_b(\mathcal{T}, U)$ the set of all $\xi \in B(\mathcal{T}, U)$ for which there exist $a, b \in U$ that satisfy $a \geq \xi \geq b$.

Lemma 2. *For every σ-algebra $\mathcal{T} \subseteq \mathcal{S}$ and affine function $u\colon X \to \mathbb{R}$,*

$$B_b(\mathcal{T}, u(X)) = \{u(f) : f \in \mathfrak{F} \text{ and } f \text{ is } \mathcal{T}\text{-measurable}\}.$$

Proof. Let $f \in \mathfrak{F}$ be $\mathcal{T}$-measurable and let $Y \subseteq X$ be a polytope such that $f(\Omega) \subseteq Y$. Since u is affine, it is continuous on Y and $\min u(Y) \leq u(f) \leq \max u(Y)$. Moreover, the function $u(f)$ is $\mathcal{T}$-measurable.[1] It follows that $u(f)$ belongs to $B_b(\mathcal{T}, u(X))$. In the opposite direction, let $\xi \in B_b(\mathcal{T}, u(X))$ and $u(x) \geq \xi \geq u(y)$ for some $x, y \in X$. If $u(x) = u(y)$, take $f = x$. If instead $u(x) > u(y)$, take $\zeta = \frac{\xi - u(y)}{u(x) - u(y)}$ and $f = \zeta x + (1 - \zeta) y$. The function f takes values in the convex hull of x and y, is $\mathcal{T}$-measurable, and satisfies $u(f) = \xi$. □

Using Lemma 1(ii), we fix a non-constant affine function u representing $\succsim$ on X, and define $U = u(X)$. Since u is affine, U is an interval. We assume $[0, 1] \in U$, without loss of generality.

An immediate implication of Lemma 2 is that for every $f \in \mathfrak{F}$ there exist $x, y \in X$ such that $x \succsim f(\omega) \succsim y$ for every ω; by Lemma 1, we can find $\alpha \in [0, 1]$ such that $\alpha x + (1 - \alpha) y \sim f$. For every act f we fix a consequence $c(f) \in X$ that satisfies $f \sim c(f)$. Without loss of generality, $c(x) = x$ for all $x \in X$.

The next result is similar to Lemma 8 in Denti and Pomatto (2022).

Lemma 3. *The following conditions hold:*

(i) *If a sequence (f_n) is bounded and $f_n \to f$ pointwise, then $u(c(f_n)) \to u(c(f))$.*

(ii) *If a sequence (f_n) is such that $u(x) \geq \sup_n u(f_n)$ and $\inf_n u(f_n) \geq u(y)$ for some $x, y \in X$, and $u(f_n) \to u(f)$ pointwise, then $u(c(f_n)) \to u(c(f))$.*

Proof. (i). Choose $x, y \in X$ such that $x \succsim f_n(\omega) \succsim y$ for all n and ω. By Axiom 3 this implies $x \succsim f \succsim y$ as well. By Lemma 1(i) we can choose $\alpha_n \in [0, 1]$ such that $f_n \sim \alpha_n x + (1 - \alpha_n) y$. Possibly passing to a subsequence, we can assume that $\alpha_n \to \alpha$ for some $\alpha \in [0, 1]$. It follows from Axiom 3 that $f \sim \alpha x + (1 - \alpha) y$, i.e., $u(c(f)) = \alpha u(x) + (1 - \alpha) u(y)$. Thus

$$u(c(f_n)) = \alpha_n u(x) + (1 - \alpha_n) u(y) \to \alpha u(x) + (1 - \alpha) u(y) = u(c(f)).$$

The computations used the affinity of u.

(ii). If $u(x) = u(y)$, then Axiom 2 implies $f_n \sim x \sim f$ for every n. So suppose $u(x) > u(y)$. Since $u(f_n) \to u(f)$ pointwise, and $x \succsim f_n(\omega) \succsim y$ for all n and all ω, then Axiom 3 implies $x \succsim f(\omega) \succsim y$ for all ω as well. As in the proof of Lemma 2, take $\xi_n \in B_b(\mathcal{S}, [0, 1])$ and

[1] Denote by $\mathcal{B}(X)$ the Borel σ-algebra on X, and denote by $\mathcal{B}(Y)$ the Borel σ-algebra on Y generated by the Euclidean topology (being Y a polytope). Theorem 5.21 and Lemma 4.20 in Aliprantis and Border (2006) imply $\mathcal{B}(Y) = \mathcal{B}(X) \cap Y$. Let $E = \{x \in Y : u(x) \geq t\}$. Then $\{\omega \in \Omega : u(f(\omega)) \geq t\} = f^{-1}(E)$. Since u is affine, it is continuous on Y. Hence $E \in \mathcal{B}(Y)$. The set Y is a closed subset of X (Aliprantis and Border, 2006, Corollary 5.22) and thus $Y \in \mathcal{B}(X)$. It follows that $E \in \mathcal{B}(X)$, and thus $f^{-1}(E) \in \mathcal{T}$. Since t is arbitrary, it follows that $u(f)$ is $\mathcal{T}$-measurable.

$\xi \in B_b(\mathcal{S}, [0,1])$ such that $u(f_n) = \xi_n u(x) + (1-\xi_n)u(y)$ and $u(f) = \xi u(x) + (1-\xi)u(y)$. Define $g_n = \xi_n x + (1-\xi_n)y$ and $g = \xi x + (1-\xi)y$. Observe that $u(f_n) = u(g_n)$ and $u(f) = u(g)$: it follows from Axiom 2 that $u(c(f_n)) = u(c(g_n))$ and $u(c(f)) = u(c(g))$. In addition, $u(f_n) \to u(f)$ pointwise implies $g_n \to g$ pointwise. The sequence (g_n) is bounded. The desired result then follows from (i) above. $\square$

One more definition is necessary. Let $\mathcal{S}^l_{\text{stp}}$ be the collection of events $A \in \mathcal{S}$ such that for all f, g, h and h' in $\mathfrak{F}$, $fAh \succsim gAh$ implies $fAh' \succsim gAh'$. Gul and Pesendorfer (2014) call $\mathcal{S}^l_{\text{stp}}$ the collection of *left ideals*. Observe that $A \in \mathcal{S}_{\text{stp}}$ if and only if $A \in \mathcal{S}^l_{\text{stp}}$ and $A^c \in \mathcal{S}^l_{\text{stp}}$. For every non-null event $A \in \mathcal{S}^l_{\text{stp}}$, let $\succsim_A$ be the binary relation on $\mathfrak{F}$ defined as

$$f \succsim_A g \quad \text{if} \quad fAh \succsim gAh \text{ for some } h \in \mathfrak{F}.$$

Because $A \in \mathcal{S}^l_{\text{stp}}$, the choice of h is irrelevant and the relation $\succsim_A$ is well defined. We denote by $\sim_A$ and $\succ_A$ the symmetric and asymmetric parts of $\succsim_A$, respectively. Observe that $f \sim_A g$ if and only if $fAh \sim gAh$ for some h, and $f \succ_A g$ if and only if $fAh \succ gAh$ for some h.

Lemma 4. *For every $f \in \mathfrak{F}$ and $A \in \mathcal{S}^l_{\text{stp}}$, there is $x \in X$ such that $x \sim_A f$.*

Proof. By Lemma 2 there exist $x, y \in X$ such that $x \succsim f(\omega) \succsim y$ for all ω. By Axiom 2, $xAh \succsim fAh \succsim yAh$, and by Lemma 1(i) there exists $\alpha \in [0,1]$ such that $(\alpha x + (1-\alpha)y)Ah = \alpha(xAh) + (1-\alpha)(yAh) \sim fAh$. Thus, $\alpha x + (1-\alpha)y$ satisfies $\alpha x + (1-\alpha)y \sim_A f$. $\square$

Thanks to Lemma 4, for every $f \in \mathfrak{F}$ and $A \in \mathcal{S}^l_{\text{stp}}$, we can fix a consequence $c(f|A) \in X$ that satisfies $f \sim_A c(f|A)$. Without loss of generality, $c(f|\Omega) = c(f)$.

The next lemma collects a few known properties of null events. Given a pair of events A and B, let $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$ be their symmetric difference.

Lemma 5. *For all $A, B \in \mathcal{S}$, the following properties holds:*

(i). *If A is null and $B \subseteq A$, then B is null.*

(ii). *If A and B are null, then $A \cup B$ is null.*

(iii). *If $A \triangle B$ is null, then $fAg \sim fBg$ for all $f, g \in \mathfrak{F}$.*

Proof. (i). Assume A is null. For all $f, g, h \in \mathfrak{F}$, we obtain

$$fBh = (fBh)Ah \sim (gBh)Ah = gBh.$$

Thus, B is also null.

(ii). Assume A and B are null. For all $f, g, h \in \mathfrak{F}$, we obtain

$$f(A \cup B)h = fA(fBh) \sim gA(fBh) = (gAf)B(gAh) \sim gB(gAh) = g(A \cup B)h.$$

Thus, $A \cup B$ is also null.

(iii). Assume $A \triangle B$ is null. For all $f, g \in \mathfrak{F}$, we obtain

$$fAg = (fAg)(A \triangle B)(f(A \cap B)g) \sim (gAf)(A \triangle B)(f(A \cap B)g) = fBg.$$

Thus, $fAg \sim fBg$. $\square$

3.4 $\mathcal{S}_{\text{stp}}$ is an algebra

Lemma 6. *If $A \in \mathcal{S}_{\text{stp}}$, then $A^c \in \mathcal{S}_{\text{stp}}$. If $A \in \mathcal{S}_{\text{stp}}$ and $A \triangle B$ is null, then $B \in \mathcal{S}_{\text{stp}}$.*

Proof. The first implication is trivial. To prove the second implication, suppose $A \in \mathcal{S}_{\text{stp}}$ and $A \triangle B$ is null. Let f, g, h in $\mathfrak{F}$ be such that $fBh \succsim gBh$. Being $A \triangle B$ null, we have $fBh \sim fAh$, $gBh \sim gAh$, $fBh' \sim fAh'$, and $gBh' \sim gAh'$ (Lemma 5). Thus $fBh \succsim gBh$ implies $fAh \succsim gAh$, which implies $fAh' \succsim gAh'$ (being $A \in \mathcal{S}_{\text{stp}}$), which in turn implies $fBh' \succsim gBh'$. Since $A \triangle B = A^c \triangle B^c$ and $A^c \in \mathcal{S}_{\text{stp}}$, we conclude that $B \in \mathcal{S}_{\text{stp}}$. □

An implication of Lemma 6 is that null events satisfy the sure-thing principle. To see this, suppose A is null. Since $\Omega \in \mathcal{S}_{\text{stp}}$ and $A = \Omega \triangle A^c$, then $A^c \in \mathcal{S}_{\text{stp}}$, which implies $A \in \mathcal{S}_{\text{stp}}$.

Lemma 7. *If $A, B \in \mathcal{S}^l_{\text{stp}}$ then $A \cap B \in \mathcal{S}^l_{\text{stp}}$.*

Proof. We follow Gul and Pesendorfer (2014). Suppose $f(A \cap B)h \succsim g(A \cap B)h$, that is $(fAh)Bh \succsim (gAh)Bh$. Because $B \in \mathcal{S}^l_{\text{stp}}$, then

$$(fAh)Bh' \succsim (gAh)Bh'. \tag{3}$$

Note that $(fAh)Bh' = (fBh')A(hBh')$ and $(gAh)Bh' = (gBh')A(hBh')$. Since $A \in \mathcal{S}^l_{\text{stp}}$, we obtain from (3) that $(fBh')Ah' \succsim (gBh')Ah'$. Equivalently, $f(A \cap B)h' \succsim g(A \cap B)h'$. This concludes the proof that $A \cap B \in \mathcal{S}^l_{\text{stp}}$. □

Let x^0 satisfy $u(x^0) = 0$. For every $A \in \mathcal{S}^l_{\text{stp}}$, we define the set

$$U_A = \{u(c(fAx^0)) : f \in \mathfrak{F}\}.$$

Lemma 8. *For every $A \in \mathcal{S}^l_{\text{stp}}$, the set U_A satisfies the following properties:*

(i) $U_A = \{u(c(xAx^0)) : x \in X\}$.

(ii) *U_A is an interval.*

(iii) *If A is null, then $U_A = \{0\}$.*

(iv) *If A is not null, then U_A has positive length.*

Proof. (i). For every $f \in \mathfrak{F}$, $u(c(fAx^0)) = u(c(xAx^0))$ for $x = c(f|A)$. Thus, $U_A = \{u(c(xAx^0)) : x \in X\}$.

(ii). Let (x_n) and (y_n) be sequences such that $u(x_n) \uparrow \sup U$, $u(y_n) \downarrow \inf U$, and $u(y_1) \leq u(x_1)$. If $\sup U = \max U$, we choose (x_n) such that $u(x_n) = \max U$ for all n. If $\inf U = \min U$, we choose (y_n) such that $u(y_n) = \min U$ for all n. For every n and $\alpha \in [0, 1]$, let $\phi_n(\alpha) = u(c((\alpha x_n + (1-\alpha) y_n) A x^0))$. Lemma 3(i) implies that the function $\phi_n : [0, 1] \to U_A$ is continuous. Since ϕ_n is continuous, then $\phi_n([0, 1])$ is an interval. We claim that $\phi_n([0, 1]) \subseteq \phi_{n+1}([0, 1])$ for all n. To see this, note that

$$x_{n+1} \succsim \alpha x_n + (1-\alpha) y_n \succsim y_{n+1}$$

because $u(x_{n+1}) \geq u(x)$, $u(y_{n+1}) \leq u(y)$, and u is affine. Then, Axiom 2 implies that

$$x_{n+1}Ax^0 \succsim (\alpha x_n + (1-\alpha)y_n)Ax^0 \succsim y_{n+1}Ax^0.$$

By Lemma 1(i), there exists $\beta \in [0,1]$ such that

$$(\beta x_{n+1} + (1-\beta)y_{n+1})Ax^0 \sim (\alpha x_n + (1-\alpha)y_n)Ax^0.$$

Thus, $\phi_n(\alpha) = \phi_{n+1}(\beta)$. This shows that $\phi_n([0,1]) \subseteq \phi_{n+1}([0,1])$. It follows that the set $V_A = \bigcup_n \phi_n([0,1])$ is a union of an increasing sequence of intervals, and therefore it is an interval.

We claim that $V_A = U_A$. Clearly, $V_A \subseteq U_A$. Now take any $x \in X$. Then there exists n sufficiently large such that $x_n \succsim x \succsim y_n$. Reasoning as above, there exists $\alpha \in [0,1]$ such that

$$(\alpha x_n + (1-\alpha)y_n)Ax^0 \sim xAx^0.$$

We deduce that $\phi_n(\alpha) = u(c(xAx^0))$. This demonstrates that $V_A \subseteq U_A$. Hence, $U_A = V_A$, and so U_A is an interval.

(iii). If A is null, then for every $f \in \mathfrak{F}$ we have $fAx^0 \sim x^0$ and therefore $U_A = \{0\}$.

(iv). Being A not null, there are acts f, g, h in $\mathfrak{F}$ such that $fAh \succ gAh$, which implies $fAx^0 \succ gAx^0$ (being $A \in \mathcal{S}^l_{\text{stp}}$). Thus U_A contains at least two distinct elements. □

For every $A \in \mathcal{S}_{\text{stp}}$, we define the function $\Psi_A \colon U_A \times U_{A^c} \to U$ by

$$\Psi_A(u(c(fAx^0)), u(c(x^0Ag))) = u(c(fAg)).$$

To see that Ψ_A is well defined, suppose f, f', g, g' in $\mathfrak{F}$ are so that $u(c(fAx^0)) = u(c(f'Ax^0))$ and $u(c(x^0Ag)) = u(c(x^0Ag'))$. Then $fAx^0 \sim f'Ax^0$ and $x^0Ag \sim x^0Ag'$. Since $A \in \mathcal{S}_{\text{stp}}$, we obtain $fAg \sim f'Ag$ and $f'Ag \sim f'Ag'$. Thus, $c(fAg) \sim c(f'Ag')$, and hence $u(c(fAg)) = u(c(f'Ag'))$.

Lemma 9. *The function Ψ_A is strictly increasing and continuous in each argument. Moreover, it satisfies $\Psi_A(0, b) = b$ for all $b \in U_{A^c}$.*

Proof. We show that Ψ_A is continuous and strictly increasing in the first argument: the proof for the second argument is analogous. Towards this goal, we fix an arbitrary value for the second argument, say $u(c(x^0Ag))$.

Let $f, f' \in \mathfrak{F}$ be such that $u(c(fAx^0)) > u(c(f'Ax^0))$. Thus, $fAx^0 \succ f'Ax^0$. Since $A \in \mathcal{S}_{\text{stp}}$, we deduce that $fAg \succ f'Ag$—that is, $u(c(fAg)) > u(c(f'Ag))$. We conclude that Φ_A is strictly increasing in the first argument.

We now prove continuity. Let (f_n) be a sequence in $\mathfrak{F}$ such that $u(c(f_nAx^0)) \to u(c(fAx^0))$ for some $f \in \mathfrak{F}$. By Lemma 8(i), it is enough to consider the case where $f_n = x_nAx^0$ and $f = xAx^0$ for a sequence (x_n) and a consequence x in X. By way of contradiction, suppose there exist a subsequence (x_{n_k}) and an $\varepsilon > 0$ such that $u(c(x_{n_k}Ag)) \geq u(c(xAg)) + \varepsilon$ for all k.

The same argument used in the proof of Lemma 8(ii) yields that the set $\{u(c(yAg)) : y \in X\}$ is an interval. Hence there exists $y \in X$ such that $u(c(yAg)) = u(c(xAg)) + \varepsilon$. We obtain that for every k,

$$x_{n_k}Ag \succsim yAg \succ xAg,$$

and thus $x_{n_k}Ax^0 \succsim yAx^0 \succ xAx^0$ because $A \in \mathcal{S}_{\text{stp}}$. It follows that $u(c(x_{n_k}Ax^0)) \geq u(c(yAx^0)) > u(c(xAx^0))$ for every k: a contradiction. We deduce: $\limsup_{n\to\infty} u(c(x_nAg) \leq u(c(xAg))$. An analogous argument shows that $\liminf_{n\to\infty} u(c(x_nAg) \geq u(c(xAg))$. Hence, $u(c(x_nAg)) \to u(c(xAg))$. This concludes the proof of continuity.

It remains to prove the last claim. First note that $0 \in U_A$, since $0 = u(c(x^0Ax^0))$. If $b = u(c(x^0Ag))$, then

$$\Psi_A(0, b) = \Psi_A(u(c(x^0Ax^0)), u(c(x^0Ag))) = u(c(x^0Ag)) = b.$$

□

Lemma 10. *For all $A, B \in \mathcal{S}_{\text{stp}}$, every $f \in \mathfrak{F}$ satisfies $u(c(f)) = \Psi_A(a, b)$, where*

$$a = \Psi_B\Big(u(c(f(A \cap B)x^0)), u(c(f(A \cap B^c)x^0))\Big),$$
$$b = \Psi_B\Big(u(c(f(A^c \cap B)x^0)), u(c(f(A^c \cap B^c)x^0))\Big).$$

Proof. The definition of Ψ_A yields $u(c(f)) = \Psi_A(u(c(fAx^0)), u(c(x^0Af)))$. By applying the definition of Ψ_B we obtain

$$\begin{aligned} u(c(fAx^0)) &= \Psi_B\Big(u(c((fAx^0)Bx^0)), u(c(x^0B(fAx^0)))\Big) \\ &= \Psi_B\Big(u(c(f(A \cap B)x^0)), u(c(f(A \cap B^c)x^0))\Big). \end{aligned}$$

In the same way,

$$\begin{aligned} u(c(x^0Af)) &= \Psi_B\Big(u(c((fA^cx^0)Bx^0)), u(c(x^0B(fA^cx^0)))\Big) \\ &= \Psi_B\Big(u(c(f(A^c \cap B)x^0)), u(c(f(A^c \cap B^c)x^0))\Big). \end{aligned}$$

Thus, $u(c(f)) = \Psi_A(a, b)$ as claimed. □

Lemma 11. *For every $A, B \in \mathcal{S}_{\text{stp}}$, we have the functional equation*

$$\Psi_A(\Psi_B(a, b), \Psi_B(d, e)) = \Psi_B(\Psi_A(a, d), \Psi_A(b, e)))$$

for all $a \in U_{A\cap B}$, $b \in U_{A\cap B^c}$, $d \in U_{A^c\cap B}$, and $e \in U_{A^c\cap B^c}$.

Note that since $A, B \in \mathcal{S}_{\text{stp}}$, then $A^c, B^c \in \mathcal{S}_{\text{stp}}$. Lemma 7 implies $A \cap B$, $A \cap B^c$, $A^c \cap B$, and $A^c \cap B^c$ belong to $\mathcal{S}^l_{\text{stp}}$.

Proof. The sets $A \cap B$, $A \cap B^c$, $A^c \cap B$, and $A^c \cap B^c$ form a partition of Ω. Therefore, we can choose $f \in \mathfrak{F}$ such that $u(c(f(A \cap B)x^0)) = a$, $u(c(f(A \cap B^c)x^0)) = b$, $u(c(f(A^c \cap B)x^0)) = d$, and $u(c(f(A^c \cap B^c)x^0)) = e$. Lemma 10 implies

$$u(c(f)) = \Psi_A(\Psi_B(a, b), \Psi_B(d, e)).$$

Inverting the roles of A and B, we obtain

$$u(c(f)) = \Psi_B(\Psi_A(a, d), \Psi_A(b, e))),$$

as desired. □

Lemma 12. *If $A, B \in \mathcal{S}_{\text{stp}}$ then $A \cup B \in \mathcal{S}_{\text{stp}}$.*

Proof. Lemma 6 implies it is without loss of generality to assume that $A \cap B^c$, $A^c \cap B$, and $A^c \cap B^c$ are not null. Since $A^c, B^c \in \mathcal{S}_{\text{stp}} \subseteq \mathcal{S}^l_{\text{stp}}$, then $(A \cup B)^c = A^c \cap B^c \in \mathcal{S}^l_{\text{stp}}$ by Lemma 7. It is therefore enough to show that $A \cup B \in \mathcal{S}^l_{\text{stp}}$. We divide the proof in two cases depending on $A \cap B$ being null or not.

Case 1: $A \cap B$ is not null. From Lemma 11 we obtain the bisymmetry functional equation

$$\Psi_A(\Psi_B(a, b), \Psi_B(d, e)) = \Psi_B(\Psi_A(a, d), \Psi_A(b, e))).$$

for all $a \in U_{A\cap B}$, $b \in U_{A\cap B^c}$, $d \in U_{A^c\cap B}$, and $e \in U_{A^c\cap B^c}$. These four sets are intervals of positive length (Lemma 8). The functions Ψ_A and Ψ_B are continuous and strictly increasing in each argument (Lemma 9). By applying Theorem 2 and Lemma 10 we obtain that there are strictly increasing functions $\psi\colon U \to \mathbb{R}$, $\psi_{A\cap B}\colon U_{A\cap B} \to \mathbb{R}$, $\psi_{A\cap B^c}\colon U_{A\cap B^c} \to \mathbb{R}$, $\psi_{A^c\cap B}\colon U_{A^c\cap B} \to \mathbb{R}$, and $\psi_{A^c\cap B^c}\colon X_{A^c\cap B^c} \to \mathbb{R}$ such that for every act f

$$\psi(u(c(f))) = \psi_{A\cap B}(a) + \psi_{A\cap B^c}(b) + \psi_{A^c\cap B}(d) + \psi_{A^c\cap B^c}(e)$$

where

$$a = u(c(f(A \cap B)x^0)), \quad b = u(c(f(A \cap B^c)x^0)),$$
$$d = u(c(f(A^c \cap B)x^0)), \quad e = u(c(f(A^c \cap B^c)x^0)).$$

We now show this implies $A \cup B \in \mathcal{S}^l_{\text{stp}}$. Indeed, suppose $f(A \cup B)h' \succsim g(A \cup B)h'$, and define:

$$f' = f(A \cup B)h', \quad g' = g(A \cup B)h', \quad f'' = f(A \cup B)h'', \quad g'' = g(A \cup B)h''.$$

Note that $f'(A^c \cap B^c)x^0 = h'(A^c \cap B^c)x^0 = g'(A^c \cap B^c)x^0$. Thus

$$\begin{aligned}\psi(u(c(f'))) - \psi(u(c(g'))) = {}& \psi_{A\cap B}(u(c(f'(A \cap B)x^0))) - \psi_{A\cap B}(u(c(g'(A \cap B)x^0))) \\ & + \psi_{A\cap B^c}(u(c(f'(A \cap B^c)x^0))) - \psi_{A\cap B^c}(u(c(g'(A \cap B^c)x^0))) \\ & + \psi_{A^c\cap B}(u(c(f'(A^c \cap B)x^0))) - \psi_{A^c\cap B}(u(c(g'(A^c \cap B)x^0))).\end{aligned}$$

Observe that $f'Ex^0 = f''Ex^0$ and $g'Ex^0 = g''Ex^0$ for all $E \in \{A \cap B, A \cap B^c, A^c \cap B\}$. It follows that

$$\psi(u(c(f'))) - \psi(u(c(g'))) = \psi(u(c(f''))) - \psi(u(c(g''))),$$

and thus $f'' \succsim g''$, being $f' \succsim g'$ and ψ strictly increasing. Since h' and h'' are arbitrary, we conclude that $A \cup B \in \mathcal{S}^l_{\text{stp}}$.

Case 2: $A \cap B$ is null. By Lemma 8 we have $U_{A\cap B} = \{0\}$. Therefore, by Lemma 9, $\Psi_B(a, b) = b$ for all $(a, b) \in U_{A\cap B} \times U_{A\cap B^c}$. Applying this fact to Lemma 11 yields the associativity functional equation

$$\Psi_A(b, \Psi_B(d, e)) = \Psi_B(d, \Psi_A(b, e)))$$

for all $b \in U_{A\cap B^c}$, $d \in U_{A^c\cap B}$, and $e \in U_{A^c\cap B^c}$. The sets $U_{A\cap B^c}$, $U_{A^c\cap B}$, and $U_{A^c\cap B^c}$ are intervals of positive length (Lemma 8). The functions Ψ_A and Ψ_B are continuous and strictly increasing in each argument (Lemma 9). Theorem 3 and Lemma 10 yield that there are strictly increasing functions $\psi\colon U \to \mathbb{R}$, $\psi_{A\cap B^c}\colon U_{A\cap B^c} \to \mathbb{R}$, $\psi_{A^c\cap B}\colon U_{A^c\cap B} \to \mathbb{R}$, and $\psi_{A^c\cap B^c}\colon U_{A^c\cap B^c} \to \mathbb{R}$ such that for every act f

$$\psi(u(c(f))) = \psi_{A\cap B^c}(b) + \psi_{A^c\cap B}(d) + \psi_{A^c\cap B^c}(e),$$

where $b = u(c(f(A \cap B^c)x^0))$, $d = u(c(f(A^c \cap B)x^0)$, and $e = u(c(f(A^c \cap B^c)x^0))$. By same argument used in the first part of the proof we conclude that $A \cup B \in \mathcal{S}^l_{\text{stp}}$. □

Lemmas 6 and 12 imply that $\mathcal{S}_{\text{stp}}$ is an algebra.

3.5 $\mathcal{S}_{\text{stp}}$ is a monotone class

A finite partition $\pi = \{A_1, \ldots, A_n\}$ of Ω is *essential* if $n \geq 3$, each A_i belongs to $\mathcal{S}_{\text{stp}}$, and at least three sets in π are non-null. We divide the proof that $\mathcal{S}_{\text{stp}}$ is a monotone class into two cases, depending on whether an essential partition exists or not.

Lemma 13. *Suppose there does not exist an essential partition. For every $A, B \in \mathcal{S}_{\text{stp}}$, if A, A^c, B and B^c are non-null, then $A \triangle B$ or $A \triangle B^c$ are null.*

Proof. We show that if $A \triangle B$ is non-null, then $A \triangle B^c$ must be null. So, assume $A \triangle B$ is non-null. Then, $A \setminus B$ or $B \setminus A$ must be non-null (Lemma 5). Assume $A \setminus B$ is non-null (the case where $B \setminus A$ is null is analogous, hence omitted). Consider the following partitions of Ω: $\{A \setminus B, A \cap B, A^c\}$ and $\{A \setminus B, B, (A \cup B)^c\}$. Since $A, B \in \mathcal{S}_{\text{stp}}$ and $\mathcal{S}_{\text{stp}}$ is an algebra, the cells of these partitions are members of $\mathcal{S}_{\text{stp}}$. Since the sets $A \setminus B$, A^c, and B are not null, it must be that $(A \cup B)^c$ and $A \cap B$ are null events; otherwise, an essential partition would exist. We conclude that $A \triangle B^c = (A \cap B) \cup (A \cup B)^c$ is null (Lemma 5). □

The next lemma shows that in the case where an essential partition does not exist, $\mathcal{S}_{\text{stp}}$ is a σ-algebra.

Lemma 14. *Suppose there does not exist an essential partition. If (A_n) is a sequence in $\mathcal{S}_{\text{stp}}$, and $A_n \uparrow A$ or $A_n \downarrow A$, then $A \in \mathcal{S}_{\text{stp}}$.*

Proof. Suppose there exists a subsequence (A_{n_k}) where each A_{n_k} is null. For all acts f, g, h we have $fA_{n_k}h \sim gA_{n_k}h$. Since the two sequences of acts $(fA_{n_k}h)$ and $(gA_{n_k}h)$ are bounded, Axiom 3 implies $fAh \sim gAh$. We conclude that A is null, and in particular that it belongs to $\mathcal{S}_{\text{stp}}$ (Lemma 6). Now suppose there exists a subsequence (A_{n_k}) where each $A^c_{n_k}$ is null. If $A_n \downarrow A$ then $A^c_{n_k} \uparrow A^c$ and thus A^c is null. Thus, $A^c \in \mathcal{S}_{\text{stp}}$ (Lemma 6). Since $\mathcal{S}_{\text{stp}}$ is an algebra, then $A \in \mathcal{S}_{\text{stp}}$. The same argument applies to the case where $A_n \uparrow A$.

We consider the remaining case: there is N such that for all $n \geq N$, both A_n and A^c_n are non-null. Without loss of generality, suppose $N = 1$. By Lemma 13, $A_1 \triangle A_n$ is null or $A_1 \triangle A^c_n$ is null. Since $A_1 \subseteq A_n$ or $A_n \supseteq A_1$, it must be that $A_1 \triangle A_n$ is null. Thus, by Axiom 3, we obtain that $A_1 \triangle A$ is null. It follows from $A_1 \in \mathcal{S}_{\text{stp}}$ that $A \in \mathcal{S}_{\text{stp}}$ (Lemma 6). □

From now on we assume there exists an essential partition, and denote by Π the collection of such partitions. In addition, we denote by $\mathcal{A}^\pi$ the algebra of events generated by π.

Lemma 15. *Let $\pi \in \Pi$, and let σ be a finite partition of Ω whose cells are in $\mathcal{S}_{\text{stp}}$. Then, the meet of π and σ belong to Π.*

Proof. Let ρ be the meet of π and σ. Since $\mathcal{S}_{\text{stp}}$ is an algebra, then $\rho \subseteq \mathcal{S}_{\text{stp}}$. Suppose $A \in \pi$ is nonnull. Since $\mathcal{A}^\pi \subseteq \mathcal{A}^\rho$, then A is a union of subsets in ρ. At least one of these subsets must be non-null. This follows from the fact that a finite union of disjoint null sets is null. Since π contains at least 3 non-null events, we conclude that the same is true for ρ. Therefore, ρ is an essential partition. □

A corollary of this result is that for every $A \in \mathcal{S}_{\text{stp}}$, there is $\rho \in \Pi$ such that $A \in \mathcal{A}^\rho$. For each $\pi \in \Pi$, let $\mathfrak{F}^\pi \subseteq \mathfrak{F}$ be the set of acts that are measurable with respect to π. To ease notation, we denote by $f(A)$ the consequence that $f \in \mathfrak{F}^\pi$ takes on $A \in \pi$.

Similar results to the following two lemmas can be found in Wakker and Zank (1999).

Lemma 16. *For every $\pi \in \Pi$, the preference relation $\succsim$ admits on $\mathfrak{F}^\pi$ the representation*

$$V^\pi(f) = \sum_{A \in \pi} W^\pi_A(u(f(A)))$$

where $W^\pi_A \colon U \to \mathbb{R}$ is a continuous function for every $A \in \pi$. Moreover, if V^π and $\tilde{V}^\pi$ are two such representations, then there are $\alpha^\pi > 0$ and $\beta^\pi_A \in \mathbb{R}$ for every $A \in \mathcal{A}^\pi$ such that $\tilde{W}^\pi_A = \alpha^\pi W^\pi_A + \beta^\pi_A$ for every $A \in \mathcal{A}^\pi$.

Proof. The proof is an application of Theorem 4. To apply Debreu's result, we will treat π as an index set—the cells of the partition are the indexes. We denote by U^π be the set of functions from π into U.

Le $\succsim^\pi$ be the preference relation defined on U^π as $\xi \succsim^\pi \zeta$ if $f \succsim g$ for any two acts in $\mathfrak{F}^\pi$ such that $u(f(A)) = \xi(A)$ and $u(g(A)) = \zeta(A)$ for every $A \in \pi$.

The relation $\succsim^\pi$ is well defined. To see this, suppose $u(f(A)) = u(f'(A)) = \xi(A)$ and $u(g(A)) = u(g'(A)) = \zeta(A)$ for every $A \in \pi$. This implies $f(\omega) \sim f'(\omega)$ and $g(\omega) \sim g'(\omega)$ for every ω, and thus $f \sim f'$ and $g \sim g'$ by Axiom 2. Hence, $f \succsim g$ if and only if $f' \succsim g'$. This demonstrates that $\succsim^\pi$ is well defined.

Next we verify $\succsim^\pi$ satisfies the conditions of Theorem 4. It is clear that $\succsim^\pi$ is complete and transitive. It is also continuous. Indeed, let ξ, ζ be elements of U^π and (ξ_n) be a sequence in U^π such that $\xi_n(A) \to \xi(A)$ for every $A \in \pi$ and $\xi_n \succsim^\pi \zeta$ for every n. Given $A \in \pi$, since U is an interval and the sequence of real numbers $(\xi_n(A))$ is convergent, there exist $a, b \in U$ such that $a \geq \xi_n(A) \geq b$ for every n. Since π is finite, there exist $a, b \in U$ such that $a \geq \xi_n \geq b$. Let (f_n) be a sequence of acts in $\mathfrak{F}^\pi$ such that $u(f_n(A)) = \xi_n(A)$ for all n and $A \in \pi$. Moreover, let $f, g \in \mathfrak{F}^\pi$ such that $u(f(A)) = \xi(A)$ and $u(g(A)) = \zeta(A)$ for all $A \in \pi$. Lemma 3(ii) yields $u(c(f_n)) \to u(c(f))$. In addition, we have $f_n \succsim g$ and thus $u(c(f_n)) \geq u(c(g))$ for all n. It follows that $u(c(f)) \geq u(c(g))$. Thus $f \succsim g$ and $\xi \succsim^\pi \zeta$. We conclude that $\succsim^\pi$ is continuous.

Now we show that π has at least three non-null indexes. Suppose $A \in \pi$ is a non-null event. Then $fAh \succ gAh$ for some acts f, g and h. In other words, $f \succ_A g$. Since $c(f|A) \sim_A f$ and $c(g|A) \sim_A g$, we obtain $c(f|A) \succ_A c(g|A)$. Taking any $z \in X$, we get $c(f|A)Az \succ c(g|A)Az$. This shows that the "index" A is not null. Because π is essential, π has at least three non-null indexes.

The fact that $\mathcal{S}_{\text{stp}}$ is an algebra and $A \in \mathcal{S}_{\text{stp}}$ for all $A \in \pi$ yields that each $A \in \mathcal{A}^\pi$ satisfies the sure-thing principle, which implies $\succsim^\pi$ is separable.

We conclude that $\succsim^\pi$ satisfies all the conditions of Theorem 4. Thus, for every $A \in \pi$ there exists a continuous function $W^\pi_A \colon U \to \mathbb{R}$ such that $\succsim^\pi$ is represented by

$$W^\pi(\xi) = \sum_{A \in \pi} W^\pi_A(\xi(A)).$$

Moreover, if W^π and $\tilde{W}^\pi$ are two such representations, then there are $\alpha^\pi > 0$ and $\beta^\pi_A \in \mathbb{R}$ for every $A \in \mathcal{A}^\pi$ such that $\tilde{W}^\pi_A = \alpha^\pi W^\pi_A + \beta^\pi_A$ for every $A \in \mathcal{A}^\pi$.

We obtain, in particular, that $\succsim$ is represented on $\mathfrak{F}^\pi$ by

$$V^\pi(f) = \sum_{A \in \pi} W^\pi_A(u(f(A))).$$

The uniqueness properties of this representation follow from the uniqueness properties of the representation of $\succsim^\pi$. □

From now on, for each $\pi \in \Pi$ we fix a representation V^π as in Lemma 16. Without loss, the representation is normalized so that $W^\pi_A(0) = 0$ for every $A \in \pi$, and $\sum_{A \in \pi} W^\pi_A(1) = 1$. For each non-empty $A \in \mathcal{A}^\pi$, we define W^π_A as

$$W^\pi_A = \sum_{B \in \pi : B \subseteq A} W^\pi_B.$$

If $A = \emptyset$, we define $W^\pi_A = 0$. For every $A \in \mathcal{A}^\pi$, the function $W^\pi_A : U \to \mathbb{R}$ is continuous. By construction we have that if A and B in $\mathcal{A}^\pi$ are disjoint, then $W^\pi_{A\cup B} = W^\pi_A + W^\pi_B$.

Lemma 17. *If $\pi, \sigma \in \Pi$ and $A \in \mathcal{A}^\pi \cap \mathcal{A}^\sigma$, then $W^\pi_A = W^\sigma_A$.*

Proof. Let ρ be the meet of σ and π. By Lemma 15, ρ is essential. Since $\mathcal{A}^\pi \subseteq \mathcal{A}^\rho$, then both V^π and V^ρ represent $\succsim$ on $\mathfrak{F}^\pi$. By the uniqueness properties of such representations, there are $\alpha^\pi > 0$ and $\beta^\pi_A \in \mathbb{R}$ for every $A \in \pi$ such that $W^\rho_A = \alpha^\pi W^\pi_A + \beta^\pi_A$ for every $A \in \pi$. Since $W^\rho_A(0) = W^\pi_A(0) = 0$, then $\beta^\pi_A = 0$. Since $\sum_{A\in\pi} W^\pi_A(1) = 1 = \sum_{A\in\pi} W^\rho_A(1)$, then $\alpha^\pi = 1$. It follows that $W^\rho_A = W^\pi_A$ for all $A \in \pi$. Thus, $W^\rho_A = W^\pi_A$ for all $A \in \mathcal{A}^\pi$. By applying the same argument to σ, we obtain $W^\rho_A = W^\sigma_A$ for all $A \in \mathcal{A}^\sigma$. We conclude that $W^\pi_A = W^\sigma_A$ for all $A \in \mathcal{A}^\pi \cap \mathcal{A}^\sigma$. □

We can therefore drop the superscript and denote each function W^π_A as W_A. Moreover, letting $\mathfrak{F}_\Pi = \bigcup\{\mathfrak{F}^\pi : \pi \in \Pi\}$ we can define V on $\mathfrak{F}_\Pi$ as

$$V(f) = \sum_{x\in f(\Omega)} W_{f^{-1}(x)}(u(x)).$$

The map V represents $\succsim$ on $\mathfrak{F}_\Pi$.

Lemma 18. *For $f, h \in \mathfrak{F}$ and $A \in \mathcal{S}_{\text{stp}}$,*

$$V(c(fAh)) = W_A(u(c(f|A))) + W_{A^c}(u(c(h|A^c))).$$

Proof. Define $g = c(f|A)Ac(h|A^c)$. Since $A \in \mathcal{S}_{\text{stp}}$, then $A \in \mathcal{A}(\pi)$ for some $\pi \in \Pi$ (Lemma 15), and thus $g \in \mathfrak{F}_\Pi$. From the fact that $c(f|A) \sim_A f$ and $c(h|A^c) \sim_{A^c} h$ we obtain $fAh \sim c(f|A)Ah \sim g$. Thus

$$V(c(fAh)) = V(g) = W_A(u(c(f|A))) + W_{A^c}(u(c(h|A^c))).$$

□

Recall that x^0 is a consequence such that $u(x^0) = 0$.

Lemma 19. *Let (A_n) be a sequence in $\mathcal{S}_{\text{stp}}$ with $A_n \downarrow A$ or $A_n \uparrow A$. Then, for all $f, h \in \mathfrak{F}$,*

$$V(c(fAh)) = V(c(fAx^0)) + V(c(x^0Ah)).$$

Proof. Lemma 3(i) implies $u(c(fA_nx^0)) \to u(c(fAx^0))$. Thus, by the continuity of W_Ω,

$$V(c(fAx^0)) = W_\Omega(u(c(fAx^0))) = \lim_{n\to\infty} W_\Omega(u(c(fA_nx^0))) = \lim_{n\to\infty} V(c(fA_nx^0)).$$

Lemma 18 yields

$$V(c(fA_nx^0)) = W_{A_n}(u(c(f|A_n))) + W_{A_n^c}(0) = W_{A_n}(u(c(f|A_n))).$$

We conclude that

$$V(c(fAx^0)) = \lim_{n\to\infty} W_{A_n}(u(c(f|A_n))). \tag{4}$$

The same argument, applied to A^c and h, implies

$$V(c(x^0Ah)) = \lim_{n\to\infty} W_{A_n^c}(u(c(h|A_n^c))). \tag{5}$$

Lemma 3(i) yields $u(c(fA_nh)) \to u(c(fAh))$. Therefore, using the continuity of W_Ω,

$$V(fAh) = W_\Omega(u(c(fAh))) = \lim_{n\to\infty} W_\Omega(u(c(fA_nh))) = \lim_{n\to\infty} V(fA_nh).$$

This implies:

$$\begin{aligned} V(fAh) &= \lim_{n\to\infty} V(fA_nh) \\ &= \lim_{n\to\infty} \left(W_{A_n}(c(f|A_n)) + W_{A_n^c}(c(h|A_n^c))\right) \\ &= V(c(fAx^0)) + V(c(x^0Ah)). \end{aligned}$$

where the second equality follows from Lemma 18 and third equality from (4) and (5). □

We can now conclude the proof that $\mathcal{S}_{\text{stp}}$ is a monotone class. Suppose (A_n) is a sequence in $\mathcal{S}_{\text{stp}}$ and $A_n \uparrow A$ or $A_n \downarrow A$. Lemma 19 shows that $fAh \succsim gAh$ holds if and only if $V(c(fAx^0)) \geq V(c(gAx^0))$, that is if and only if $fAx^0 \succsim gAx^0$. Similarly, $hAf \succsim hAg$ if and only if $x^0Af \succsim x^0Ag$. It follows that A satisfies the sure-thing principle.

3.6 $\mathcal{S}_{\text{stp}}$ is a monotone class: alternative approach

We consider a strengthening of Axiom 2:

Axiom 5. *If $f(\omega) \succsim g(\omega)$ for all ω, then $f \succsim g$; if in addition $f(\omega) \succ g(\omega)$ for all ω in a non-null set A, then $f \succ g$.*

Axiom 5 is satisfied by the smooth identifiable model of Denti and Pomatto (2022). The axiom has the following implication:

Lemma 20. *If $f(\omega) \succsim g(\omega)$ for all ω outside of a null set A, then $f \succsim g$; if in addition $f(\omega) \succ g(\omega)$ for all ω in a non-null set B, then $f \succ g$.*

Proof. Assume that $f(\omega) \succsim g(\omega)$ for all ω outside of a null set A. Since A is null, $f \sim gAf$. Moreover, $gAf \succsim g$ by Axiom 5. Transitivity implies that $f \succsim g$. Suppose in addition that $f(\omega) \succ g(\omega)$ for all ω in a non-null set B. Since B is not null and A is null, the set $B \cap A^c$ is not null. Thus, $gAf \succ g$ by Axiom 5. Given that $f \sim gAf$, we obtain $f \succ g$. □

Utilizing Axiom 5, we now provide a shorter proof that $\mathcal{S}_{\text{stp}}$ is a monotone class. This proof does not use Debreu's theorem.

Let (A_n) is a sequence in $\mathcal{S}_{\text{stp}}$ such that $A_n \uparrow A$ or $A_n \downarrow A$. We desire to show that $A \in \mathcal{S}_{\text{stp}}$. If A is null, then $A \in \mathcal{S}_{\text{stp}}$ by Lemma 6; we assume throughout that A is not null.

Take $f, g, h, h' \in \mathfrak{F}$ with $fAh \succsim gAh$. Let B and C be the events defined by

$$B = \left\{ \omega \in A : u(f(\omega)) < \sup_{x \in X} u(x) \right\},$$
$$C = \left\{ \omega \in A : u(g(\omega)) > \inf_{x \in X} u(x) \right\}.$$

To demonstrate that $fAh' \succsim gAh'$, we distinguish between two cases based on whether B and C are null or non-null.

Suppose first that B and C are both null. Then, $B \cup C$ is null. It follows from Lemma 20 that $fAh \succ gAh$, that is, $u(c(fAh)) > u(c(gAh))$. By Lemma 3(i), $u(c(fA_nh)) > u(c(gA_nh))$ for all n sufficiently large. Since each $A_n \in \mathcal{S}_{\text{stp}}$, we deduce that $fA_nh' \succsim gA_nh'$ for all n sufficiently large. It follows from Axiom 3 that $fAh' \succsim gAh'$.

Suppose now that B or C is non-null. For short, suppose B is non-null (the case in which C is non-null is analogous). Take $x \in X$ such that $u(f(\omega)) \leq u(x)$ for all $\omega \in A$ and $u(f(\omega)) < u(x)$ for all $\omega \in B$. Given any $\alpha \in (0, 1)$, consider the act $f_\alpha = \alpha f + (1 - \alpha)x$. Notice that for all ω,

$$u(f_\alpha(\omega)) = \alpha u(f(\omega)) + (1 - \alpha)u(x).$$

Hence, since B is non-null, $f_\alpha Ah \succ fAh$ by Axiom 2. Since in addition $fAh \succsim gAh$, we have $f_\alpha Ah \succ gAh$. Reasoning as above, we deduce that $f_\alpha Ah' \succsim gAh'$. Since this is true for every α, it follows from Axiom 3 that $fAh' \succsim gAh'$.

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