Math 120c - Spring 2003-2004
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Homework set 3
Due: 26th April 2004

Let R be a ring, with $1 \in R$.

1. Let B be a commutative ring and let A be a subring of B such that B is integral over A.
    (a) Suppose that B is an integral domain. Show that $J(B) = 0$ if and only if $J(A) = 0$.
    (b) A commutative ring R is called a *Jacobson ring* if every prime ideal of R is an intersection of maximal ideals. Show that B is a Jacobson ring if and only if A is a Jacobson ring.
2. Let $R = C(\mathbf{R})$ be the ring of continuous real-valued functions on $\mathbf{R}$. Show that R is not Noetherian.
3. Let M be an R-module and let M_1 and M_2 be R-modules such that $M = M_1 + M_2$. Suppose that M and $M_1 \cap M_2$ are finitely generated R-modules. Show that both M_1 and M_2 are finitely generated R-modules.
4. Let R be Noetherian and let I be an ideal of R.
    (a) For any $x \in R$ which is not a zero divisor, show that there exists a positive integer k such that
    $$xy \in I^n \Rightarrow y \in I^{n-k} \text{ for all } n \geq k.$$
    (b) For any ideal J of R, there exists a positive integer n such that
    $$I^n \cup J \subseteq IJ.$$
5. Let G be a finite group of automorphisms of a ring B. Let $A = B^G$. Suppose that the order of G is n and that $1/n \in B$. Prove that if B is Noetherian then A is Noetherian also.