Math 120c
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Coursework 5
Due: 12th May 2003

1. Let K be a finite extension of $\mathbf{Q}$ with $\mathbf{Q}$-basis $\{\alpha_1, \ldots, \alpha_n\}$. Let $\sigma_1, \ldots, \sigma_n$ denote the distinct embeddings of K into $\overline{\mathbf{Q}}$. Define

$$d = d(\alpha_1, \ldots, \alpha_n) = (\det(\sigma_i(\alpha_j))^2.$$

Show that $d \neq 0$ and that d is independent of the choice of basis of $K/\mathbf{Q}$, up to an element of $(\mathbf{Q}^*)^2$.

2. Let F be the extension of $\mathbf{Q}$ given by the splitting field of the family of polynomials

$$\{X^2 - p\}, \text{ where } p \text{ is prime.}$$

Compute the Galois group $G = \mathrm{Gal}(F/\mathbf{Q})$ as a topological group. What are the open subgroups of G? Show that there exist subgroups of finite index in G which are *not* open.

3. Let K be a a field of q elements, and let $L = K(X)$ where X is an indeterminate. Let $G = \mathrm{Aut}_K(L)$. Show that

   (a) The order of G is $q^3 - q$.

   (b) The fixed field of G is equal to $K(Y)$ where

   $$Y = \frac{(X^{q^2} - X)^{q+1}}{(X^q - X)^{q^2+1}}.$$

   (c) Let H_1 be the subgroup of G consisting of automorphisms of the form $X \mapsto aX + b$. Show that H_1 has fixed field $K(T)$ where

   $$T = (X^q - X)^{q-1}.$$