Math 120c
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Coursework 4
Due: 5th May 2003

1. Compute the Galois groups of
   (a) $x^3 - 15$ over $\mathbf{Q}$,
   (b) $x^3 - 21x + 7$ over $\mathbf{Q}$,
   (c) $x^3 - 15$ over $\mathbf{Q}(\sqrt{3})$,
   (d) $x^4 - 13$ over $\mathbf{Q}, \mathbf{Q}(\sqrt{13}), \mathbf{Q}(\sqrt{-13}),$ and $\mathbf{Q}(i)$.

2. Let $K = \mathbf{C}(t)$, where t is an indeterminate. Calculate the Galois groups over K of the splitting fields of
   (a) $X^3 + X + t$,
   (b) $X^3 + tX + 1$,
   (c) $X^3 + t^2X - t^3$.

3. Let $f(x) = x^4 + ax^2 + b \in \mathbf{Q}[x]$. Let $\pm\alpha$ and $\pm\beta$ be the roots of f, let K be the splitting field over $\mathbf{Q}$ and let G be the Galois group.

   Show that

   (a) G is isomorphic to the dihedral group of order 8.
   (b) $G \cong \mathbf{Z}/4\mathbf{Z}$ if and only if
   $$\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \in \mathbf{Q}.$$
   (c) $G \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ if and only if either $\alpha\beta \in \mathbf{Q}$ or $\alpha^2 - \beta^2 \in \mathbf{Q}$.