Math 120c
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Coursework 1
Due: 11th April 2003

1. Let K be an integral domain of finite cardinality. Show that K is a field.
2. Let K be a field and let L be a finite-dimensional K-algebra. If L is an integral domain, then show that L is a field.
3. Show that $\overline{\mathbf{Q}}$ has infinite degree over $\mathbf{Q}$.
4. Write down and prove a universal property satisfied by the composition of two fields (inside a third field). [Hint: it's nearly the *definition*].
5. Let p be a prime number and let
$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + 1.$$
Use the Eisenstein criterion to show that this is irreducible.
Let $L = \mathbf{Q}(\zeta_p)$, where ζ_p is a root of $\Phi_p(x)$. Show that ζ_p is a p^{th} root of unity and that all p^{th} roots of unity are contained in L.
6. Are $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ isomorphic? (Give a proof or provide a counterexample).
7. Show that, if L/K is an algebraic extension of fields, then $L = \cup L_i$, where L_i/K is a finite extension for all i.
8. Let L/K be an algebraic extension of fields. Suppose that L'/L is a finite extension. Show that there is a field $K' \subset L'$ such that K'/K is finite and $L' = LK'$.