Math 120b - Winter 2003-2004
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Homework set 5
Due: 20th Febrary 2004

Let L be an extension of the field K.

1. Let $\alpha = \sqrt[4]{2}$. Use the trace from $\mathbf{Q}(\alpha)$ to $\mathbf{Q}$ to show that $\sqrt{3} \notin \mathbf{Q}(\alpha)$. (Hint: write $\sqrt{3} = a + b\alpha + c\alpha^2 + d\alpha^3$ and show that a, b, c are all zero. Why does this imply a contradiction?)

2. Let F be the extension of $\mathbf{Q}$ given by the splitting field of the family of polynomials
$$\{X^2 - p\}, \text{ where } p \text{ is prime.}$$
Compute the Galois group $G = \mathrm{Gal}(F/\mathbf{Q})$ as a topological group. What are the open subgroups of G? Show that there exist subgroups of finite index in G which are *not* open.

3. Let K be a a field of q elements, and let $L = K(X)$ where X is an indeterminate. Let $G = \mathrm{Aut}_K(L)$. Show that
   (a) The order of G is $q^3 - q$.
   (b) The fixed field of G is equal to $K(Y)$ where
   $$Y = \frac{(X^{q^2} - X)^{q+1}}{(X^q - X)^{q^2+1}}.$$
   (c) Let H_1 be the subgroup of G consisting of automorphisms of the form $X \mapsto aX + b$. Show that H_1 has fixed field $K(T)$ where
   $$T = (X^q - X)^{q-1}.$$

4. (a) Using Zorn's Lemma, show that every field extension has a transcendence base.
   (b) Show that every algebraically independent subset of K is contained in a transcendence base.