Math 120b - Winter 2003-2004
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Homework set 4
Due: 13th February 2004

Let L be an extension of the field K.

1. Suppose that for all $z \in L$ either $z \in K$ or z is transcendental over K. Is L/K a purely transcendental extension? Give a proof or exhibit a counterexample.

2. Let $K = \mathbf{C}(t)$, where t is an indeterminate. Calculate the Galois groups over K of the splitting fields of
   (a) $Y^3 + Y + t$
   (b) $Y^3 + tY + 1$
   (c) $Y^3 + t^2Y - t^3$.

3. Let G be a finite abelian group.
   (a) Show that there is an abelian extension of $\mathbf{Q}$ with Galois group G.
   (b) Let K be a finite extension of $\mathbf{Q}$. Show that there are infinitely many extensions of K with Galois group G.
   (c) If $K/\mathbf{Q}$ is finite, then K contains only a finite number of roots of unity.
   (d) Show that there are infinitely many nonzero integers a and b such that $-(4a^3 + 27b^2)$ is a square in $\mathbf{Z}$.

4. Compute the Galois group of $Y^4 - 5$ over $\mathbf{Q}$, $\mathbf{Q}(\sqrt{5})$, $\mathbf{Q}(\sqrt{-5})$, $\mathbf{Q}(i)$. Choose another irreducible degree 4 polynomial over $\mathbf{Q}$ and compute its Galois group over $\mathbf{Q}$ and some quadratic extensions of $\mathbf{Q}$.

5. Let $n > 2$ and let ζ be a primitive nth root of unity. Prove that

$$\left[\mathbf{Q}(\zeta + \zeta^{-1} : \mathbf{Q}\right] = \frac{\varphi(n)}{2}.$$