Math 120b - Winter 2003-2004
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Homework set 1
Due: 23rd January 2004

Let L be an extension of the field K.

1. If $\alpha \in L$ is algebraic of odd degree over K, then prove that α^2 is also algebraic of odd degree over K, and that $K(\alpha) = K(\alpha^2)$.

2. Let L/K be an algebraic extension, and suppose that L'/L is finite. Show that there is a field $K' \subseteq L'$ such that K'/K is finite and $L' = LK'$.

3. Let p and q be distinct prime numbers. Show that $\mathbf{Q}(\sqrt{p})$ and $\mathbf{Q}(\sqrt{q})$ are isomorphic as vector spaces, but not as fields.

4. Let L and M be intermediate fields of the extension $K \subseteq N$, of finite dimension over K. Assume that $[LM : K] = [L : K][M : K]$.

   (a) Prove that $L \cap M = K$.

   (b) Prove the converse in the case when $[L : K]$ or $[M : K]$ is 2.

   (c) Find an example where $L \cap M = K$, $[L : K] = [M : K] = 3$, but $[LM : K] < 9$. (Consider a real and a nonreal cube root of 2).

5. Find a splitting field K of $X^{p^m} - 1 \in \mathbf{F}_p$. What is $[K : \mathbf{F}_p]$?

6. What is $\left[\mathbf{Q}(\sqrt{5 + 3\sqrt{3}}) : \mathbf{Q}\right]$?