Math 120b - Winter 2003-2004
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Final
Due: 17th March 2004

Instructions for this final

Except for question 4 and question 5, you may use the notes for this course, your solutions to course homeworks, and the course textbooks (Dummit and Foote, Hungerford and Lang) for reference. Please provide references for any quoted theorems. You may not use the Internet or other books for reference. Please do not discuss this exam with other students or faculty without contacting me first.

Question 4 is on the additive version of Hilbert's Theorem 90. Please do not read Chapter V, section 7 of Hungerford or Chapters VI.6 through VI.10 of Lang before or during this final.

Question 5 is on Galois theory and cyclotomic extensions. Please do not refer to *any* materials (books, notes, marked courseworks for instance) during this question. This question is closed-book.

Do not spend more than one hour on any one question.

Each question is worth an equal number of marks, and partial credit will be given for incomplete answers.

Hand this exam in to my office (Sloan 358) or to my pigeonhole in a sealed envelope before 6pm on the 17th of March 2004.

Let L be an extension of the field K.

1. Show that every K-homomorphism $L \to L$ is an isomorphism if the following conditions are satisfied:

   (a) L is algebraically closed,

   (b) L has finite transcendence degree over K.

   Show that both of these conditions are necessary.

2. (a) Show that the only automorphism of $\mathbf{R}$ over $\mathbf{Q}$ is the identity. (You may assume that $\mathbf{Q}$ is dense in $\mathbf{R}$.) A possible strategy is as follows:

      i. Let $\sigma \in \mathrm{Aut}_{\mathbf{Q}}(\mathbf{R})$. Show that for all $a, b \in \mathbf{R}$ it follows that $a < b \Rightarrow \sigma(a) < \sigma(b)$.

      ii. Prove that σ is continuous.

      iii. Show that a continuous map on $\mathbf{R}$ which is the identity on $\mathbf{Q}$ is the identity.

   (b) What are the automorphisms of $\mathbf{C}$ over $\mathbf{R}$? Over $\mathbf{Q}$?

3. Let K be a subfield of $\mathbf{C}$ which is maximal with respect to the property that $\sqrt{3} \notin K$.

   (a) Prove that K exists, and that $\mathbf{C}$ is algebraic over K.

   (b) Prove that every finite extension of K inside $\mathbf{C}$ is Galois, with Galois group a cyclic 2-group.

   (c) Hence prove that $[\mathbf{C} : K]$ is countably infinite.

   (d) Which parts of this proof would work for any algebraic irrational number, not just $\sqrt{3}$?

4. Let L/K be a Galois extension, and let σ be an element of the Galois group $G = G(L/K)$.

   (a) Using the linear independence of characters, show that there exists $\alpha \in L$ with $\mathrm{Tr}_{L/K}(\alpha) \neq 0$.

   (b) Suppose that $\alpha \in K$ is of the form $\beta - \sigma(\beta)$, for some $\beta \in L$. Show that $\mathrm{Tr}_{L/K}(\alpha) = 0$.

   (c) Now suppose that G is cyclic and is generated by σ. Suppose that $\alpha \in K$ has $\mathrm{Tr}_{L/K}(\alpha) = 0$. Prove that $\alpha = \beta - \sigma(\beta)$ for some $\beta \in L$.

   (d) Hence prove that $H^1(G, L) = 0$, where G is acting on the additive group of L.

5. Let L/K be a finite Galois extension, with Galois group G.

   (a) State the Fundamental Theorem of Galois Theory.

   (b) Let $L = \mathbf{Q}(\sqrt{13}, \sqrt{17}, \sqrt{19})$ and let $K = \mathbf{Q}$. Write down all of the fields H such that $K \subseteq H \subseteq L$, and draw a diagram which shows these fields. You may assume that $[L : \mathbf{Q}] = 8$.

   (c) Let $K = \mathbf{Q}$, and let L be the cyclotomic field $\mathbf{Q}(\zeta_n)$. Write down the Galois group of L/K.

   (d) Now let $L = \mathbf{Q}(\zeta_{19})$. Write down all of the subfields of L, and draw a diagram which shows all of them.

   (e) What needs to be modified in the Fundamental Theorem of Galois Theory if L/K is an infinite extension?