Math 160a - Fall 2002
Lloyd Kilford

Homework set 6
Due: 14th November 2002

1. Show that all imaginary quadratic fields have only finitely many units, and determine these explicitly.

2. Prove that any positive definite binary integral quadratic form of discriminant 5 is equivalent over $\mathbf{Z}$ to either $x^2 + 5y^2$ or $2x^2 + 2xy + 3y^2$. Show that these two forms are not equivalent to one another over $\mathbf{Z}$.

3. Show that there are infinitely many units in $\mathbf{Q}(\sqrt{2})$, and hence exhibit three solutions to the equation

$$x^2 - 2y^2 = 1,\ x, y \in \mathbf{Z}.$$

   Hint: find a unit in $\mathbf{Q}(\sqrt{2})$ which is not ± 1, and show that it is not a root of unity.

4. Let $\mathbf{A}$ be the adeles of $\mathbf{Q}$ and let $\mathbf{A}^f$ be the subring of the adeles whose real coordinate is 0. We see that $\mathbf{Q}$ embeds diagonally into $\mathbf{A}^f$. Prove that the image of $\mathbf{Q}$ is dense in $\mathbf{A}^f$.

5. Let S be a finite set of places of $\mathbf{Q}$, and let $\mathbf{A}_S$ be the subring of $\mathbf{A}$ consisting of the elements of $\mathbf{A}$ whose v-coordinate is 0 for all $v \in S$. Show that, if S is not empty, then $\mathbf{Q}$ is dense in $\mathbf{A}$.

6. Let m be a squarefree integer. Show that the fields $\mathbf{Q}(\sqrt{m})$ are pairwise distinct, by considering the equation $\sqrt{m} = a + b\sqrt{n}$ or otherwise.